SYMMETRIC CRITICALITY PRINCIPLE
(SCP)

[R. Palais 1979]

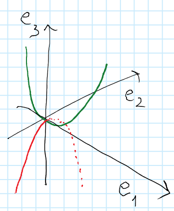
- find critical points
- useful & suprising
- but often "used" w/o justification

TODAY:
- EXAMPLES
- ⤳ DEFINITIONS
- ⤳ why suprising ?
- WARNING: (SCP) does NOT always hold
- (SCP) holds
  - for Riemannian G-mfds
  - for G-mfds if G compact Lie gp

- how (SCP) entered our research

(– very briefly: other versions of (SCP))

$\boxed{1a}$ $f : \mathcal{M} := \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$
$\qquad (x,y) \longmapsto x^2 - y^2$



$\rightsquigarrow \quad f(x,-y) = f(x,y)$

symmetric subset $\Sigma := \mathbb{R} \times \{0\} \subset \mathcal{M}$

minimize $f|_\Sigma$: $\quad f(x,0) = x^2 \geq f(0,0) = 0$

$\rightsquigarrow \quad f_x(x,0)\Big|_{(x,0)=(0,0)} = 2x\Big|_{(0,0)} = 0$

$\rightsquigarrow \quad (0,0) \in \mathrm{Crit}(f|_\Sigma)$

but also: $\quad \nabla f(x,y)\Big|_{(x,y)=(0,0)} = \begin{pmatrix} 2x \\ -2y \end{pmatrix}\Big|_{(0,0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ (SCP) holds here :

$$\mathcal{C}rit(f|_\Sigma) \subset \mathcal{C}rit(f) \cap \Sigma$$

$\boxed{1b}$ $f : \mathcal{M} := B_1 \times B_2 \xrightarrow{C^1} \mathbb{R}$

w/ $f(x,-y) = f(x,y) \ \forall (x,y) \in \mathcal{M}$

group $G := \{Id, R\}$ $(\cong \mathbb{Z}_2)$

$\circ$	Id	R
Id	Id	R
R	R	Id

$\rightsquigarrow$ representation $\tau : G \times \mathcal{M} \xrightarrow{C^1} \mathcal{M}$

$$(g,(x,y)) \longmapsto \tau_g(x,y) := \begin{cases} (x,y) & \text{if } g = Id \\ (x,-y) & \text{if } g = R \end{cases}$$

w/ $\tau_{g \circ h}(x,y) = \tau_g(\tau_h(x,y)) \ \forall g,h \in G, (x,y) \in \mathcal{M}$

w/ $\tau_g : \mathcal{M} \xrightarrow{C^1} \mathcal{M}$ diffeo $\forall g \in G$

$\rightsquigarrow$ DEF: $\mathcal{M}$ is a G-manifold.

here: f is G-invariant $:\Leftrightarrow$ $\boxed{f \circ \tau_g = f \ \forall g \in G \quad (1)}$

$\Sigma := B_1 \times \{0\}$ G-symmetric subset

$:\Leftrightarrow \tau_g(p) = p \ \forall p \in \Sigma$

here:

$$df_{(x,y)} : T_{(x,y)}\mathcal{M} \cong T_x B_1 \oplus T_y B_2 \cong B_1 \oplus B_2 \longrightarrow T_{f(x,y)}\mathbb{R} \cong \mathbb{R}$$

$$\rightsquigarrow df_{(x,y)} = df_{(x,y)}\big|_{B_1 \oplus \{0\}} + df_{(x,y)}\big|_{\{0\} \oplus B_2} =: d_1 f_{(x,y)} + d_2 f_{(x,y)}$$

$$\rightsquigarrow d_2 f_{(x,y)} \overset{(1)}{=} d_2 (f \circ \tau_R)_{(x,y)} = d_2 f_{(x,-y)} \circ \underbrace{d_2(\tau_R)_{(x,y)}}_{=-1}$$

$$\Rightarrow d_2 f_{(x,0)} = 0 \Rightarrow \boxed{d_2 f|_\Sigma = 0}.$$

$\Rightarrow \quad d_2 f_{(x,0)} = 0 \quad \Rightarrow \quad \boxed{d_2 f\big|_{\Sigma} = 0}.$

If $(x,0) \in \mathrm{Crit}(f|_{\Sigma})$ $(\Longleftrightarrow \; d_1 f_{(x,0)} = 0)$

then $(x,0) \in \mathrm{Crit}(f) \cap \Sigma$ $(\Leftrightarrow \; df_{(x,0)} = 0)$,

i.e., (SCP) holds here.

$\boxed{2}$ $f : \tilde{\mathcal{M}} := C^2(\bar{A}) \longrightarrow \mathbb{R}$, $A := B_{r_2} \setminus \bar{B}_{r_1}(0) \subset \mathbb{R}^n$

$$u \longmapsto \int_A |\nabla u|^2 \, dx$$

$G := SO(n)$ w/ $\tau_g(u)(x) := u(g^T x)$, $g \in G$

$\rightsquigarrow$ f is G-invariant $\Longleftrightarrow$ $f \circ \tau_g = f$

w/ $\{ g^T x : g \in G \} = \partial B_{|x|}(0) \quad \forall x \in \bar{A}$

$\rightsquigarrow$ $\tilde{\Sigma} := \{ u \in \tilde{\mathcal{M}} : \exists v \in C^2([r_1, r_2]) : u(x) = v(|x|), x \in \bar{A} \}$

restrict to $\mathcal{M} := \{ u \in \tilde{\mathcal{M}} : u\big|_{\partial B_{r_i}(0)} \overset{!}{=} c_i, \; i = 1,2 \}$

for given bdy data $c_1, c_2 \in \mathbb{R}$.

$\rightsquigarrow$

$\mathcal{M}$ smooth submanifold of $\tilde{\mathcal{M}}$

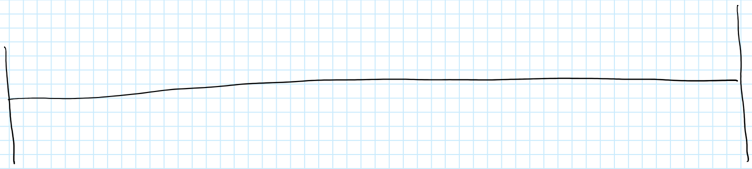
$\rightsquigarrow$ set $\Sigma := \tilde{\Sigma} \cap \mathcal{M}$ as G-symmetric subset

& compute $f\big|_{\Sigma}(v) \sim \int_{r_1}^{r_2} v'(r)^2 \, r^{n-1} \, dr$

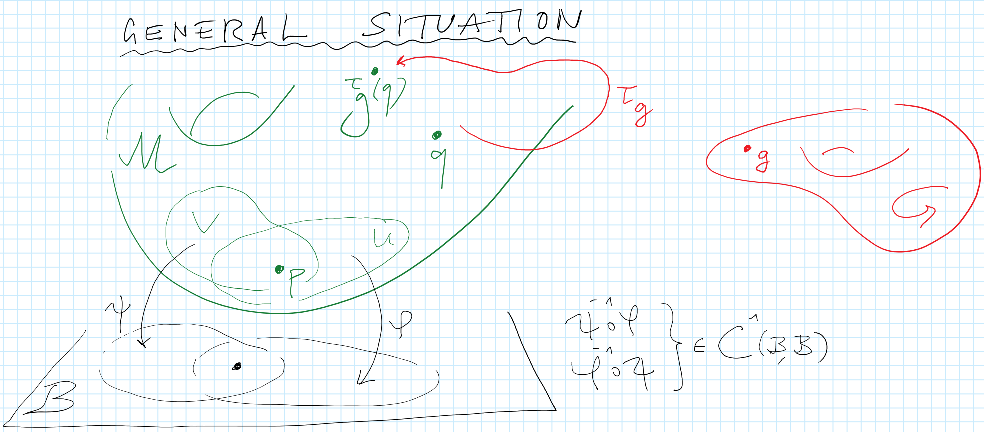
w/ Euler-Lagrange eq. $\underbrace{v''(r) + \frac{n-1}{r} v'(r) = 0}$

$$\hat{=} \Delta u$$

$\rightsquigarrow$ solutions $$v(r) = \begin{cases} a + b \log r & \text{f. } n = 2 \\ a + \dfrac{b}{r^{n-2}} & \text{f. } n \geq 3 \end{cases}.$$



GENERAL SITUATION



$$\left.\begin{matrix} \psi^{-1} \circ \varphi \\ \varphi^{-1} \circ \psi \end{matrix}\right\} \in C^1(\mathcal{B}, \mathcal{B})$$

EXAMPLE of MFDs

$\boxed{3}$ discrete sets $=$ 0-dim mfds

$\boxed{4}$ $\mathcal{B}$ = Banach space

$\boxed{5}$ $\Omega^{\text{open}} \subset \mathcal{B}$

$\boxed{6}$ $F : \mathcal{M} \xrightarrow{C^1} \mathcal{N}$, $y \in \mathcal{N}$ regular value

$\Rightarrow$ $F^{-1}(y)$ is C^1-submfd of $\mathcal{M}$

(RECALL: $y \in \mathcal{N}$ regular value $:\Longleftrightarrow$ $dF_x : T_x\mathcal{M} \to T_{F(x)}\mathcal{N}$ surjective

& $\ker dF_x$ split $T_x\mathcal{M}$

$\forall x \in F^{-1}(y)$)

(RECALL:

$$T_x\mathcal{M} = \ker dF_x \oplus X$$

w/ continuous projections $P_{\ker dF_x}, P_X$)

RMKs 1. If f G-invariant $\Leftrightarrow$ $f \circ \tau_g = f$

$$\Rightarrow \quad df_{\tau_g(p)} \circ \underbrace{d(\tau_g)_p}_{\text{isomorphism}} = df_p$$

$\forall g \in G$

$$\Rightarrow \quad \tau_g(\mathrm{Crit}(f)) \subset \mathrm{Crit}(f)$$

potentially many critical pts !

but

$$\forall\, p \in \Sigma : \quad \# \{\tau_g(p) : g \in G\} = 1$$

2. Assume $\Sigma \subset \mathcal{M}$ submfd : Crit$(f|_\Sigma)$ vs. Crit(f)

$$\Sigma \xhookrightarrow{i} \mathcal{M} \xrightarrow{f} \mathbb{R} : \quad f|_\Sigma = f \circ i$$

$$\& \quad di_p : T_p\Sigma \underset{\text{inclusion}}{\hookrightarrow} T_{i(p)}\mathcal{M} = T_p\mathcal{M}, \quad p \in \Sigma.$$

$$\Rightarrow \quad d(f|_\Sigma)_p = df_{\underset{=p}{i(p)}} \circ di_p = df_p\big|_{T_p\Sigma}$$

$$\Rightarrow \quad p \in \mathrm{Crit}(f|_\Sigma) \Leftrightarrow df_p[w] = 0 \quad \forall w \in T_p\Sigma$$

$\Rightarrow$ IF (SCP) HOLDS, then $df_p[v] = 0 \quad \forall v \in T_p\mathcal{M}$

WARNING: (SCP) does NOT always hold !

$\boxed{\exists}$ closed Cantor set $C \subset \mathcal{M}^{\text{compact}}$

$\Rightarrow \exists \; h \in C^\infty(\mathcal{M}) \; : \quad C = h^{-1}(0)$

[Whitney]

& fix $p \in C$

& find smooth vector field Y on $\mathcal{M}$

w/ $\quad p = Y^{-1}(0)$

& set

$$X := h\,Y$$

THEN global flow $\tau_t : \mathcal{M} \xrightarrow{\text{diffeo}} \mathcal{M}, \; t \in G := \mathbb{R}$

(w/ $\quad \tau_t \circ \tau_s(q) = \tau_{t+s}(q)$

$\qquad \tau_0(q) = q \quad)$

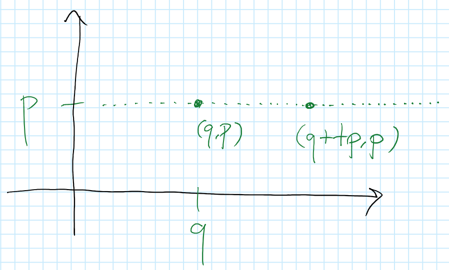
induces $\quad G-$manifold $\quad \mathcal{M}$

w/ G-symmetric subset $\quad \Sigma = C \neq$ mfd !

$\Rightarrow \cancel{\text{(SCP)}}$

$\boxed{8}$ $\quad \mathcal{M} := \mathbb{R}^2, \quad G := \mathbb{R}, \quad \tau_t(q,p) := (q+tp, p)$

$$\rightsquigarrow \quad \Sigma := \{ (q,p) \in \mathcal{M} : p = 0 \}$$



IF $\quad f : \mathcal{M} \xrightarrow{C^1} \mathbb{R}$

is $G-$invariant

$\Leftrightarrow \quad f \circ \tau_t = f$

$\Leftrightarrow \quad f(q,p) = \varphi(p)$

for some $\varphi \in C^1(\mathbb{R})$

$\Rightarrow \quad f\big|_\Sigma = \varphi(0) = \text{const.} \quad \Rightarrow \boxed{\mathrm{Crit}(f|_\Sigma) = \Sigma}$

BUT: $df_{(q,p)} = \varphi'(p) \Rightarrow df_{(q,0)} = \varphi'(0)$

$\Rightarrow$ IF $\varphi'(0) \neq 0 \Rightarrow \boxed{\mathrm{Crit}(f) \cap \Sigma = \emptyset}$

$\Rightarrow \cancel{(SCP)}$

POSITIVE STATEMENTS

THM 1 [Palais '79]

(SCP) holds for Riemannian G-mfds

i.e. $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ G-mfd w/

$\tau_g : \mathcal{M} \longrightarrow \mathcal{M}$ isometry $\forall g \in G$

$$:\Leftrightarrow \left\langle d(\tau_g)_p[v], d(\tau_g)_p[w] \right\rangle_{\tau_g(p)} = \langle v, w \rangle_p \quad \forall v, w \in T_p\mathcal{M}$$

$\boxed{9}$ $\mathcal{M} := \mathcal{H}$ = real Hilbert space

w/ $\tau_g : \mathcal{H} \xrightarrow{\text{unitary}} \mathcal{H} \quad \forall g \in G$

RMK. One can generalize THM 1 to paracompact Hilbert mfds $\mathcal{M}$ & compact G

but more generally:

THM 2 [Palais '79]

If G is compact Lie group

THEN (SCP) holds $\forall$ G-mfds M.

$\boxed{10}$ finite or countably infinite groups G are Lie groups

$\boxed{11}$ $GL(n, \mathbb{R})$ smooth submfd of $M(n,\mathbb{R})$ is Lie group as well as $SL(n,\mathbb{R})$ and $SO(n)$ compact

$\boxed{12}$ $(\mathbb{R}, +)$, $(\mathbb{R}^n, +)$, $(\mathbb{R}\setminus\{0\}, \cdot)$, $((0,\infty), \cdot)$ are Lie groups

$\boxed{13}$ $\mathbb{S}^1$, $\mathbb{S}^3$ are Lie groups

but not $\mathbb{S}^2$

(IN FACT: NO OTHER SPHERE IS!)