Paracausal deformations of globally hyperbolic spacetimes and their applications in AQFT

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Framework and scope

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- Paracausal deformations of globally hyperbolic metrics;

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Based on a recent paper with V.Moretti and S.Murro:
Paracausal deformations of Lorentzian metric and geometric Møller isomorphisms in algebraic quantum field theory. (2021).

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- **Causal sets** $J^\pm(A) = A \cup$ points in M reachable by future/past directed smooth causal curves.

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- *an isometry* $h = -\beta^2 dt^2 + h_t$, h_t *family of Riemannian metrics on the slices.*

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g_χ **will be important later!**

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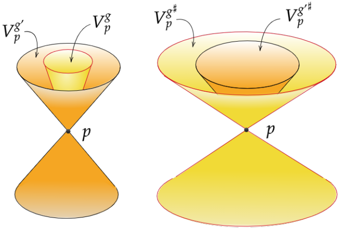

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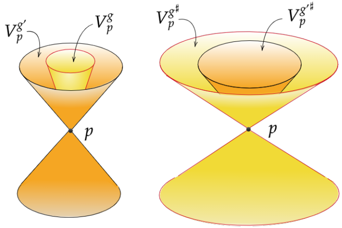

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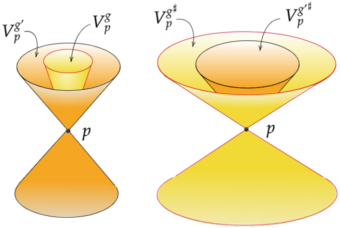

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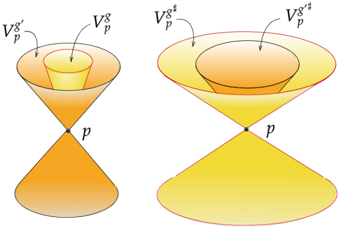

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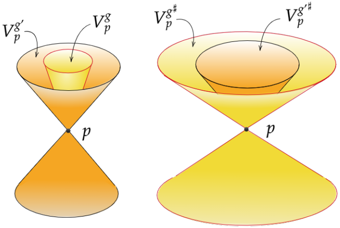

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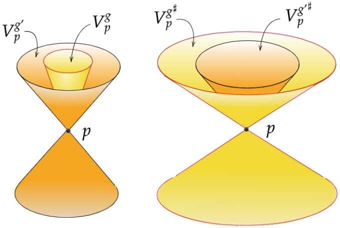

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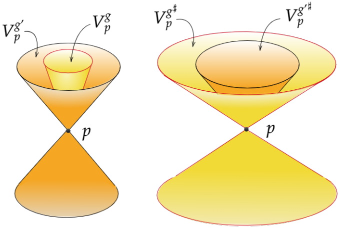

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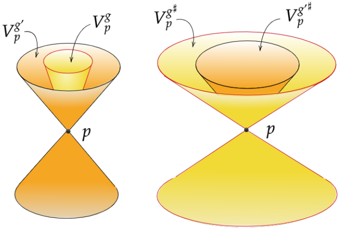

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Therefore Σ is g' spacelike $\implies$ Σ is g spacelike.
g timelike curves are g' timelike, so they intersect Σ once $\implies$ Σ is g Cauchy.

The paracausal relation I

Definition (Paracausal relation)

$g, g' \in \mathcal{GH}_{\mathrm{M}}$

g is **paracausally related** to g ($g \simeq g'$) if there is a finite sequence $g = g, g', \ldots, g_N = g' \in \mathcal{GH}_{\mathrm{M}}$ such that, for $k = 0, \ldots, N-1$,

(i) $g_k \preceq g_{k+1}$ or $g_{k+1} \preceq g_k$

(ii) (a) if $g_k \preceq g_{k+1}$, then $V_p^{g_k +} \subset V_p^{g_{k+1} +}$ for all $p \in \mathrm{M}$,
(b) if $g_{k+1} \preceq g_k$, then $V_p^{g_{k+1} +} \subset V_p^{g_p +}$ for all $p \in \mathrm{M}$.

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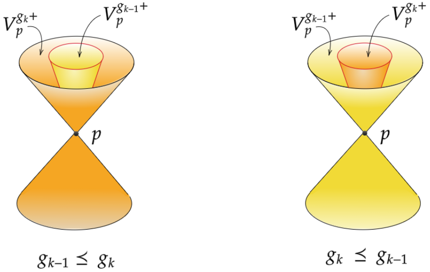

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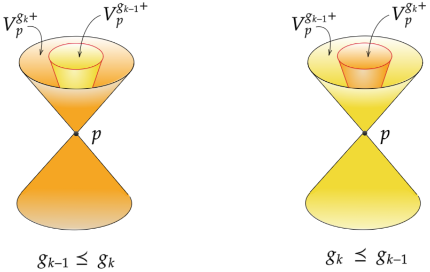

At each step the future cones of one metric are included in the future cones of the other metric!

Paracausally related metrics: example

$\mathsf{M} = \mathbb{R}^{n+1}$
$\eta_0 = -dt \otimes dt + \sum_{i=1}^{n+1} dx^i \otimes dx^i$
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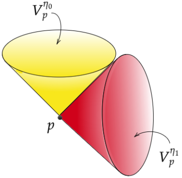

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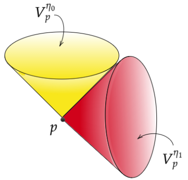
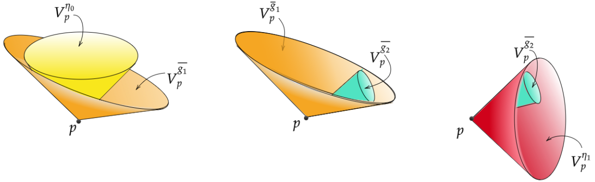


The partially ordered (P.O.) sequence relating $\eta_0 \simeq \eta_1$:



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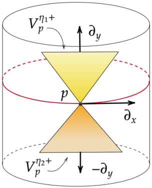

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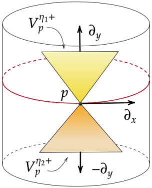

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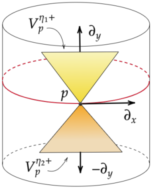

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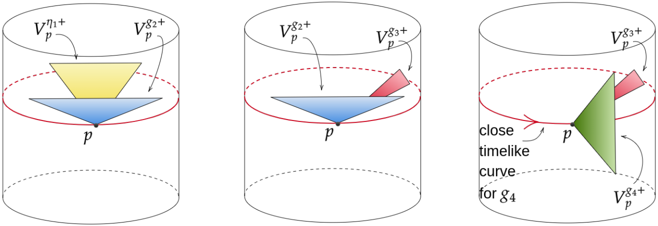

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Equivalent characterization

$g \simeq g' \iff \exists$ a sequence $\{g_i\} \subset \mathcal{GH}(\mathrm{M})$ such that $V_P^{g_i+} \cap V_P^{g_{i+1}+} \neq \emptyset \; \forall P \in \mathrm{M}$.

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Easiest example of n.h.o: Klein-Gordon operator $K = \Box_g + m^2$ on the trivial bundle.

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The kernel is characterized by the **causal propagator**

$$\mathsf{G} := \mathsf{G}^{+}|_{\Gamma_c(\mathsf{E})} - \mathsf{G}^{-}|_{\Gamma_c(\mathsf{E})} : \Gamma_c(\mathsf{E}) \to \Gamma(\mathsf{E})\,.$$

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Let us fix a differentiable manifold M and two different globally hyperbolic metrics g and g'.

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Gluing spacetimes!: $g \preceq g' \implies g_\chi \in \mathcal{GH}(\mathrm{M}) \implies \mathrm{N}_\chi$ Green-hyp on (M, g_χ).

The Møller map for ordered metrics

Møller maps

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Symplectic forms preserved:

$$\sigma^{\mathsf{N}'}_{g'}(\mathsf{R}\Psi, \mathsf{R}\Phi) = \sigma^{\mathsf{N}}_{g}(\Psi, \Phi) \quad \text{for every } \Psi, \Phi \in \mathrm{Ker}^{g}_{sc}(\mathsf{N}).$$

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Causal propagators

The Møller operator **intertwines the causal propagators**.

$$R G_N R^{\dagger_{gg'}} = G_{N'}$$

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- **CCR $*$-algebras**: $\mathcal{A} = \frac{\bigoplus_n Ker^g_{sc}(\mathsf{N})^{\otimes n}}{\Psi_\psi \otimes \Phi_\phi - \Phi_\phi \otimes \Psi_\psi - \sigma^{\mathsf{N}}_g(\psi, \phi)\mathrm{Id}}$;

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- The isomorphism preserves **Hadamard states**

$$WF(\omega_2) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \setminus \{0\} | (x, k_x) \sim_{\parallel} (y, k_y), k_x \rhd 0\}.$$

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Thanks for the attention!