Synthetic (metric) methods in General Relativity and Lorentzian geometry

Part II: Applications

Working Seminar "Mathematical Physics"
University of Regensburg

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Lorentzian pre-length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau\colon X \times X \to [0,\infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- smooth spacetimes (M,g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\}$
- finite directed graphs

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2. *timelike, causal sectional curvature bounds* via triangle comparison (Kunzinger-S. '18)
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Inextendibility of spacetimes

joint work with Grant, Kunzinger AGAG 2019

Inextendibility of spacetimes

When is a spacetime *maximal?* (i.e. no *isometric embedding* into larger spacetime)

Theorem

X *strongly causal* Lorentzian length space s.t. all *inextendible timelike geodesics* have *infinite τ-length*, then X is *inextendible as a regular Lorentzian length space*

Corollary

(M, g) *strongly causal, smooth* and *timelike geodesically complete* spacetime, then (M, g) is *inextendible as a regular Lorentzian length space*

being non-regular is related to (timelike/causal) *curvature unbounded*

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Generalized cones as Lorentzian length spaces: Causality, curvature, and singularity theorems

joint work with S. Alexander, M. Graf, M. Kunzinger, Comm. Anal. Geom. to appear, 2021

Generalized cones (1/2)

Definition (Generalized cones)

Given a metric space (X, d), an open interval I, and a continuous function $f : I \to (0, \infty)$, we call $Y = I \times_f X$ a generalized cone or warped product with one-dim. base and f *warping function*.

$\gamma = (\alpha, \beta) : J \to Y$ absolutely continuous (w.r.t. the product topology on $I \times X$) then α, β AC and $\dot{\alpha}$ metric derivative v_β of β exist almost everywhere

$$v_\beta(t) := \lim_{s\to 0} \frac{d(\beta(t+s), \beta(t))}{|s|}, \quad \text{satisfies } L(\beta) = \int_J v_\beta$$

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$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \end{cases} \quad \text{if} \quad -\dot{\alpha}^2 + (f \circ \alpha)^2 v_\beta^2 \quad \begin{cases} < 0 \\ = 0 \\ \leq 0\,, \end{cases}$$

a.e. and γ is *future/past directed causal* if $\dot{\alpha} > 0$ or $\dot{\alpha} < 0$ a.e.

length of a causal curve $L(\gamma) := \int_a^b \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2 v_\beta^2}$

For this talk: (X, d) locally compact geodesic length space

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Causal structure of generalized cones (1/2)

$\ll$, $\leq$, $I^{\pm}$, $J^{\pm}$ and the Lorentzian time separation τ as usual

$\gamma = (\alpha, \beta) : [a, b] \to Y$ future directed causal, maximizing $\rightsquigarrow$

- fiber component β minimizing in (X, d)
- converse (β min. $\implies$ γ max.) does not hold! (problem: right parametrization of α)
- Fiber independence: base component α depends *only* on length of β (independent of β, (X, d) otherwise)
  I.e., take any other (X', d') and maximizing curve β' with $L^d(\beta) = L^{d'}(\beta')$ and $v_\beta = v_{\beta'}$, then $\gamma' = (\alpha, \beta')$ is maximizing causal/timelike for $I \times_f X'$ $\implies$ $\tau(p, q)$ depends only on $p_0, q_0, d_X(\bar{p}, \bar{q})$ (and f)!

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- I^+ open and push-up? (Not necessarily in low regularity, cf. bubbling examples by Chrusciel & Grant '12)
- $h_{p_0} : (a_{p_0}, b_{p_0}) \to I$ as the unique maximal solution of the ODE

$$\frac{d}{ds} h_{p_0} = f \circ h_{p_0}, \qquad h_{p_0}(0) = p_0$$

  Then: h_{p_0} is strictly increasing, bijective and C^1 and

$$I^+((p_0, \bar{p})) = \{(q_0, \bar{q}) \in Y : d(\bar{p}, \bar{q}) < b_{p_0} \text{ and } q_0 > h_{p_0}(d(\bar{p}, \bar{q}))\}$$

- In particular, I^+ open and push-up holds
- $\partial I^+((p_0, \bar{p})) = \partial J^+((p_0, \bar{p}))$ = continuous graph over $B_{b_{p_0}}(\bar{p}) \subseteq X$ & its height depends only on the distance to $\bar{p}$!
- If X is geodesic: $J^+ = I^+ \cup \partial I^+$

$\implies Y = I \times_f X$ is a Lorentzian pre-length space

(+ more work $\implies$ Y is a regular strongly causal Lorentzian length space)

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- In particular, I^+ open and push-up holds
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- If X is geodesic: $J^+ = I^+ \cup \partial I^+$

$\implies$ $Y = I \times_f X$ is a Lorentzian pre-length space

(+ more work $\implies$ Y is a regular strongly causal Lorentzian length space)

Causal structure of generalized cones (2/2)

- I^+ open and push-up? (Not necessarily in low regularity, cf. bubbling examples by Chrusciel & Grant '12)
- $h_{p_0}\colon (a_{p_0}, b_{p_0}) \to I$ as the unique maximal solution of the ODE

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Curvature bounds for generalized cones

Theorem

X curvature bounded below (above) by K, $I \times_f \mathbb{M}^2(K)$ timelike curvature bounded below (above) by K', then $Y = I \times_f X$ timelike curvature bounded below (above) by K'

Special case: f smooth, then $I \times_f \mathbb{M}^2(K)$ smooth Lorentzian manifold $\rightsquigarrow$ sectional curvatures easily computable

$f'' - K'f \leq 0$ and X curv. bounded below by $K = \sup K' f^2 - (f')^2 \implies$ $I \times_f X$ timelike curvature bounded below by K'

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Singularity theorems

incomplete timelike geodesics (i.e., inextendible timelike geodesics of finite length) in generalized cone? if $f \to 0$ in finite time ($I \neq (-\infty, \infty)$)
properties of f from timelike curvature bounds?

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(iii) $K = 0$ and f non-constant, then $a > -\infty$ or $b < \infty$; hence Y past or future *timelike geodesically incomplete*

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Sketch of the proof

- Assume not, then $\exists K' > K, J \subset I$ s.t. $f'' > K' f$ on J
- Look at $Y' := J \times_f \mathbb{R} \rightsquigarrow$ smooth, has TL sectional curvature $\mathcal{R} = \frac{f''}{f} > K' \implies Y'$ has TL curv. bounded above by $K' > K$
- Fiber independence $\implies$ for timelike comparison triangles $\Delta' \in Y'$ with $x'z'$-side perpendicular to $\mathbb{R}$ and $\Delta \in Y$ with xz-side "perpendicular" to X (and $q' \in y'z'$, $q \in yz$):

$$\tau_{Y'}(x', q') = \tau_Y(x, q)$$

- Since Y has TL curv. bounded below by K,

$$\tau_Y(x, q) \leq \tau_{\mathbb{L}^2(K)}(x''', q''')$$

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A Lorentzian analog for Hausdorff dimension and measure

joint work with Robert J. McCann, preprint 2021

Hausdorff measures and dimension

Definition

(X,d) metric space, $A \subseteq X$, $\delta > 0$, $N \in [0,\infty)$

$$\mathcal{H}^N_\delta(A) := \inf\{c_N \sum_i \operatorname{diam}(A_i)^N : A \subseteq \bigcup_i A_i, \operatorname{diam}(A_i) \leq \delta\}$$

N-dimensional Hausdorff measure $\mathcal{H}^N(A) := \sup_{\delta>0} \mathcal{H}^N_\delta(A)$

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Hausdorff dimension $\dim^H(A) := \inf\{N \geq 0 : \mathcal{H}^N(A) = 0\}$

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Ricci-limit spaces and (R)CD(K, N)-spaces (1/2)

Theorem (Cheeger-Colding 1997)

$(M_n, g_n, p_n)_n$ sequence of pointed (complete, connected) Riemannian mf., same dim. N, *Ricci curvature* uniformly bounded below, $(M_n, g_n, p_n) \to (X, d, p)$ pointed Gromov-Hausdorff $\Rightarrow$ either

1. $\mathrm{vol}^{g_n}(B_1^{M_n}(p_n)) \to 0$ (collapsed) or
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(X, d, m) *metric measure space*, synthetic lower Ricci curvature bounded by $K \in \mathbb{R}$ and dimension bounded above by $N \rightsquigarrow$ CD(K, N) (using optimal transport, convexity/concavity of functionals on the space of probability measures) (Lott-Villani 2009, Sturm 2006)

Riemannian condition $\rightsquigarrow$ RCD(K, N)-spaces (Ambrosio-Gigli-Savaré 2014)

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1. $\mathrm{vol}^{g_n}(B_1^{M_n}(p_n)) \to 0$ (collapsed) or
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(X, d, m) *metric measure space*, synthetic lower Ricci curvature bounded by $K \in \mathbb{R}$ and dimension bounded above by $N \rightsquigarrow$ CD(K, N) (using optimal transport, convexity/concavity of functionals on the space of probability measures) (Lott-Villani 2009, Sturm 2006)

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Theorem (Brué-Semola 2019)

(X, d, m) metric measure space satisfying RCD(K, N) $(K \in \mathbb{R}, N \in (1, \infty)) \Rightarrow \exists k \in \mathbb{N} : 1 \leq k \leq N$ s.t. $m|_R$ *a.c. wrt* $\mathcal{H}^k$ and $m(X \backslash R) = 0$

Bakry-Émery Ricci tensor: $\mathrm{Ric}_V^N := \mathrm{Ric} + \mathrm{Hess}(V) - \frac{dV \otimes dV}{N-n}$ amounts to replacing $d\mathrm{vol}^g$ by $e^{-V} d\mathrm{vol}^g \rightsquigarrow$ generalization to non-smooth setting $e^{-V} d\mathcal{H}^N$

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Lorentzian analog of Hausdorff measures

Definition

X set, $\leq$ preorder on X, $\tau\colon X\times X\to[0,\infty]$, $J(x,y):=J^+(x)\cap J^-(y)$

$$\rho^N(J(x,y)) := \omega_N \tau(x,y)^N$$

$\omega_N := \frac{\pi^{\frac{N-1}{2}}}{N\,\Gamma(\frac{N+1}{2})2^{N-1}}$, Γ Euler's gamma function, $N\in[0,\infty)$

$\mathbb{N}\ni N\geq 2$: $\rho^N(J(x,y))=$ *vol. CD* in N-dim Minkowski w eq. time-sep.

Definition

X as above, d metric on X, $A\subseteq X$, $\delta>0$, $N\in[0,\infty)$

$$\mathcal{V}^N_\delta(A) := \inf\{\sum_i \rho^N(J_i) : A\subseteq\bigcup_i J_i, \operatorname{diam}(J_i)\leq\delta, J_i=J(x_i,y_i)\}$$

N-dimensional Lorentzian measure $\mathcal{V}^N(A):=\sup_{\delta>0}\mathcal{V}^N_\delta(A)$

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Synthetic dimension

Definition

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space, $A \subseteq X$, the *synthetic dimension of A* is

$$\dim^\tau(A) := \inf\{N \geq 0 : \mathcal{V}^N(A) < \infty\}$$

Proposition

X locally d-uniform ($\tau = o(1)$) Lorentzian pre-length space, $A \subseteq X$
$N = \dim^\tau(A)$ if and only if $\forall k < N < K$: $\mathcal{V}^k(A) = \infty$, $\mathcal{V}^K(A) = 0$; thus

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One-dimensional measure versus length

Null curves are zero-dimensional

$\gamma\colon [a,b] \to X$ future directed *null* curve in strongly causal Lorentzian pre-length space: $\dim^\tau(\gamma([a,b])) = 0$

Proposition

$\gamma\colon [a,b] \to X$ f.d. *causal* curve, X strongly causal: $\mathcal{V}^1(\gamma([a,b])) \leq L_\tau(\gamma)$; all causal diamonds $J(x,y)$ *closed* (e.g. X is globally hyperbolic), then

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Countable sets are zero dimensional and measured by their cardinality

X strongly causal, $N \in [0,\infty)$; additionally in case $N > 0$ assume $\forall x \in X$, $\forall U$ nhd. of x $\exists x^\pm \in U$ s.t. $x^- < x < x^+$, $x^- \not\ll x \not\ll x^+$: $A \subseteq X$ *countable*, then $\mathcal{V}^N(A) = 0$ for $N > 0$; and $A \subseteq X$ *arbitrary* then $\mathcal{V}^0(A) = |A|$ (cardinality of A)

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Dimension and measure of Minkowski subspaces

Lemma

restriction of $\mathcal{V}^k$ to spacelike subspace of Minkowski spacetime $\mathbb{R}^n_1$ with algebraic dimension k is *positive multiple of Hausdorff measure*

Linear *null* hypersurfaces have geometric *codimension two*

Lemma

$n \geq 2$, $S \subset \mathbb{R}^n_1$ *null* subspace of algebraic dimension $k \neq n$, then $\dim^\tau(S) = k - 1$ and Lorentzian measure splits as $\mathcal{V}^{k-1} = c\,\mathcal{H}^{k-1} \times \mathcal{H}^0$ on $S = R \times \mathbb{R}\nu$, where R spacelike subspace of S, $\nu \in S$ null vector

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Compatibility for continuous spacetimes

Theorem

(M, g) continuous, strongly causal, causally plain spacetime of dim n

- $\mathcal{V}^n = \mathrm{vol}^g$
- $\dim^\tau(M) = n$

- use appropriate *cylindrical neighborhoods*
- machinery of *Federer: Geometric measure theory 1969*
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Doubling measures in metric spaces

Definition

(X, d) metric space, Borel measure μ on X is *doubling* if $\exists C \geq 0$: $\forall x \in X$, $r > 0$

1. $0 < \mu(B_r(x)) < \infty$
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Theorem (well-known)

(X, d) metric space, μ doubling measure with doubling constant $C \Rightarrow$

$$\dim^H(X) \leq \log_2(C)$$

Theorem (Sturm 2006)

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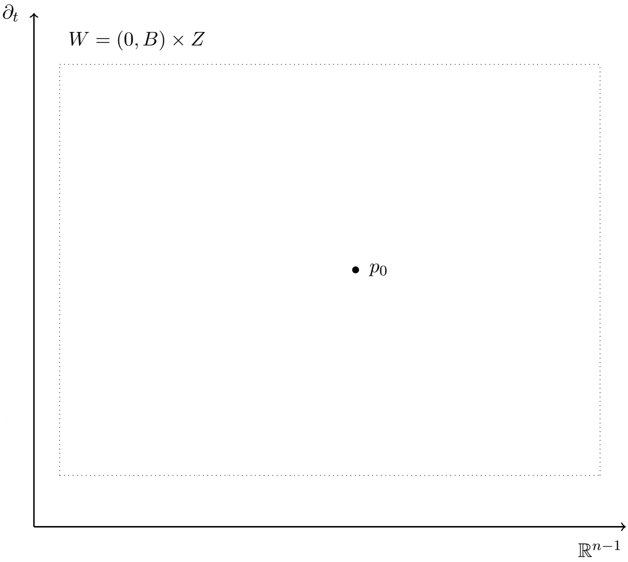
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $J(p, q) \cap J(u, v) \neq \emptyset$ $\Rightarrow J(u, v) \subseteq J(\hat{p}, \hat{q}, W)$
6. W arb. small, g.h.



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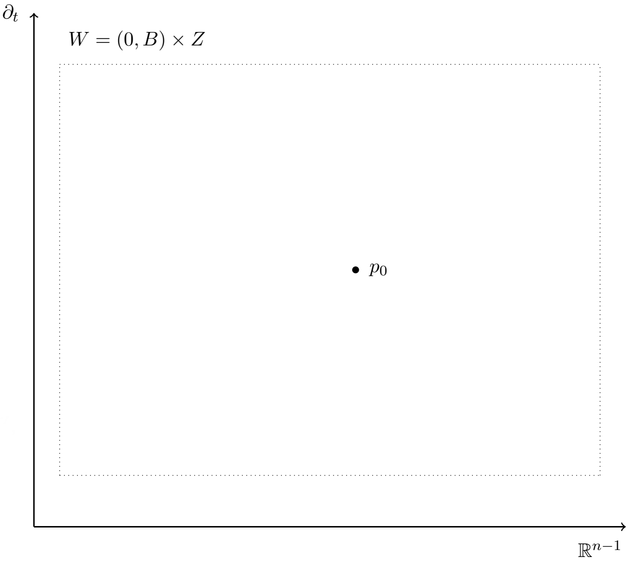

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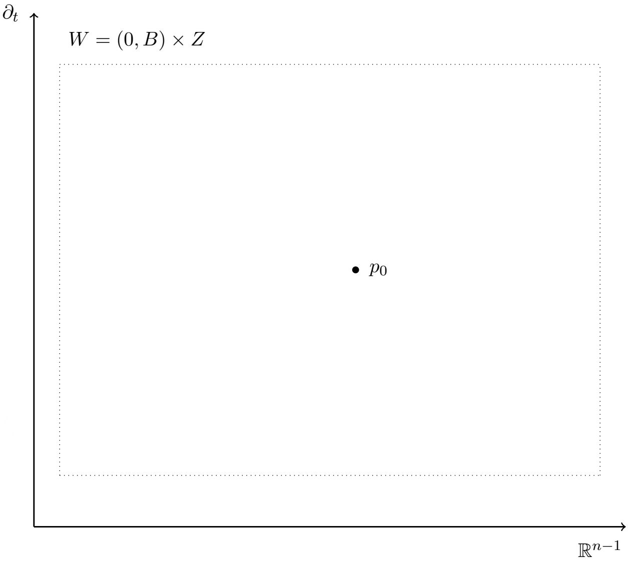

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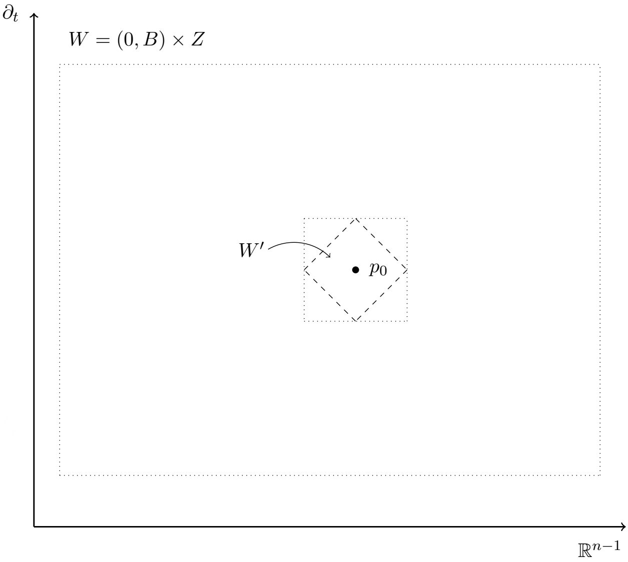

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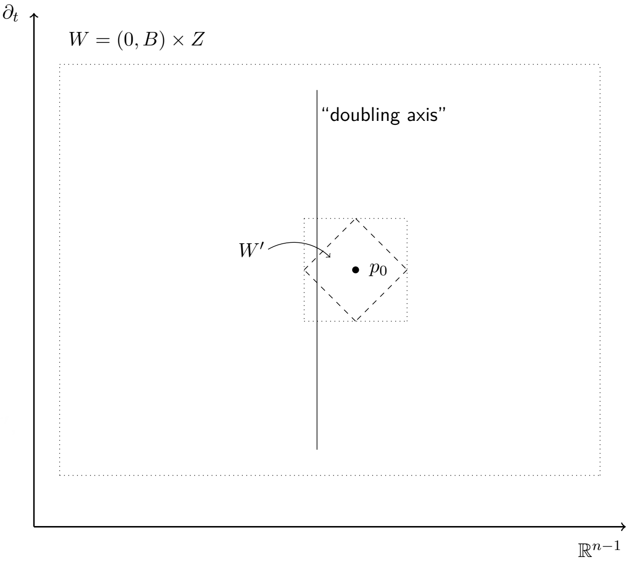

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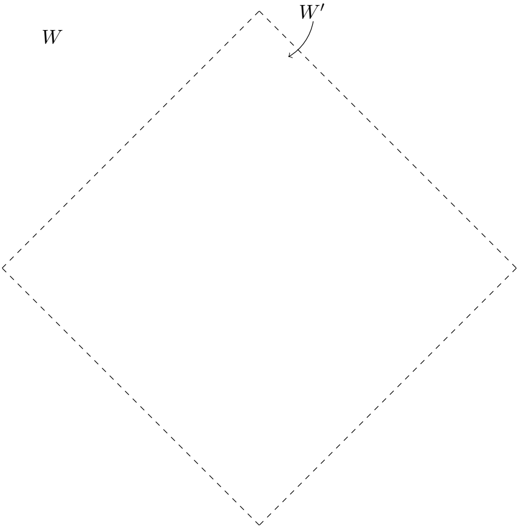

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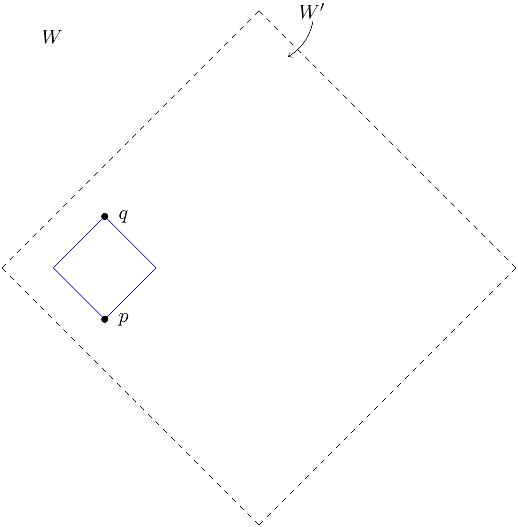

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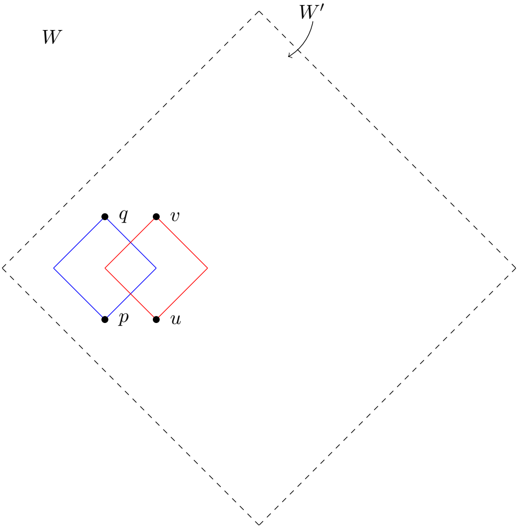

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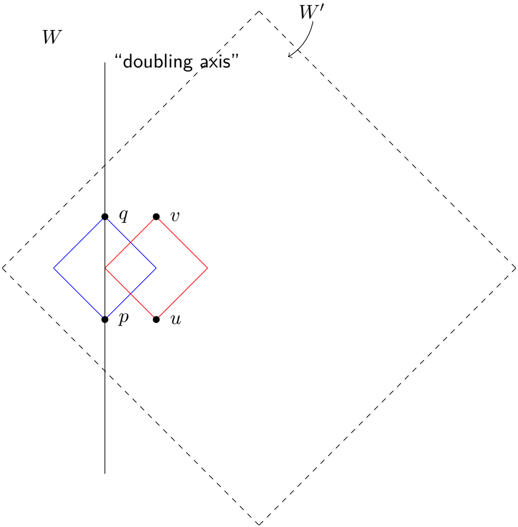

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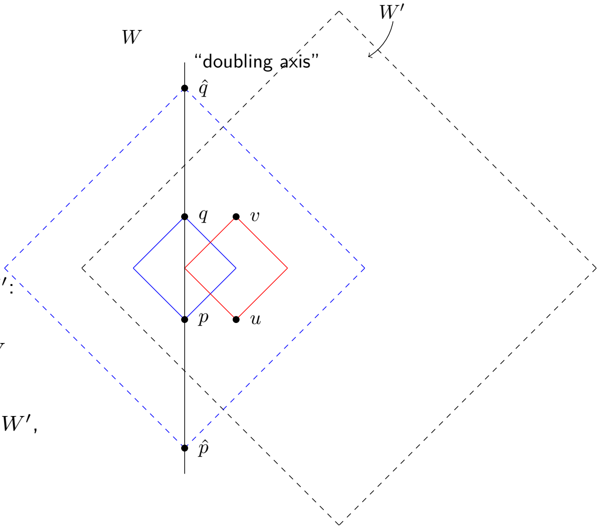

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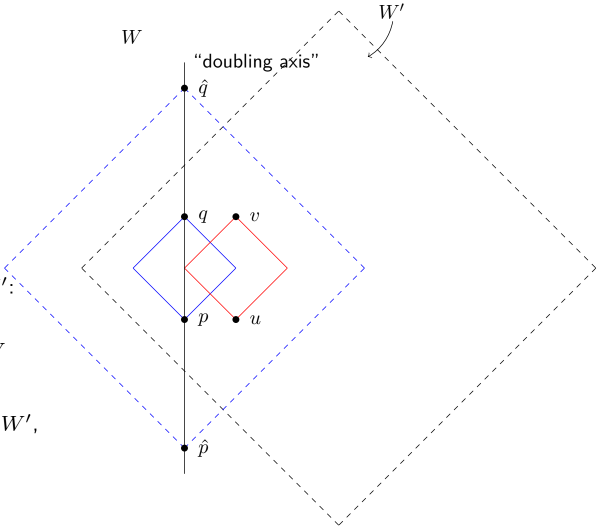

Doubling of causal diamonds in cont. spacetimes (2/2)

cylindrical nhd. W: $\mathrm{vol}^g(J(\hat{p},\hat{q},W)) \leq L\ \mathrm{vol}^g(J(p,q))$

Definition

Borel measure m on M is *loc. causally doubling* if $\forall$ cyl. nhds. (W', W) $\exists L \geq 1$:

1. $\forall p = (t,x), q = (s,x) \in W'$: $m(J(\hat{p},\hat{q},W)) \leq L\, m(J(p,q))$
2. $0 < m(J(p,q,W)) < \infty$ ($p, q \in W$ with $p \ll q$)
3. $m(\overline{W}) < \infty$

Theorem

(M,g) cont., causally plain, strongly causal spacetime; m *loc. causally doubling measure*, *loc. doubling constant* L on all suff. small cyl. nhds $\Rightarrow$

$$\dim(M) = \dim^\tau(M) \leq \log_{1+2\lambda}(L)$$

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$B_r^\tau(x) := \{y \in X : \tau(x,y) < r\}$, $E_r := E \cap \overline{B_r^\tau(x)}$

Lemma

glob. hyp. locally causally closed measured Lorentzian length space sat. $\mathsf{wTCD}^e_p(K,N)$ ($K \in \mathbb{R}$, $N \in [1,\infty)$, $p \in (0,1)$) $\Rightarrow \exists L = L(K,N) \geq 1$: $\forall x_0 \in X$, $E \subseteq I^+(x_0) \cup \{x_0\}$ comp., τ-star-shaped wrt x_0, $r > 0$ small

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- measures on *hypersurfaces*
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