Synthetic (metric) methods in General Relativity and Lorentzian geometry

Part I: Lorentzian length spaces

Working Seminar "Mathematical Physics"
University of Regensburg

Clemens Sämann
Faculty of Mathematics
University of Vienna, Austria

Nov 4, 2021

Introduction (1/2)

Why a *synthetic* approach to Lorentzian geometry?

- need for *low regularity* (of the metric): PDE point-of-view, physically relevant models (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- separate main concepts and derived notions of the causal structure
- minimal framework for *causality* and *(timelike/causal)* curvature bounds with continuous metrics
- notion of *(timelike/causal) curvature bounds* without a Lorentzian metric
- possible applications to Quantum Gravity (no Lorentzian metric): *causal fermion systems*, causal sets, etc.

Regularity class (of the metric) where causality theory is fine: $\mathcal{C}^{1,1}$, i.e., first derivative is (locally) Lipschitz continuous

Introduction (1/2)

Why a *synthetic* approach to Lorentzian geometry?

- need for *low regularity* (of the metric): PDE point-of-view, physically relevant models (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- separate main concepts and derived notions of the causal structure
- minimal framework for *causality* and *(timelike/causal)* curvature bounds with continuous metrics
- notion of *(timelike/causal) curvature bounds* without a Lorentzian metric
- possible applications to Quantum Gravity (no Lorentzian metric): *causal fermion systems*, causal sets, etc.

Regularity class (of the metric) where causality theory is fine: $C^{1,1}$, i.e., first derivative is (locally) Lipschitz continuous

Introduction (1/2)

Why a *synthetic* approach to Lorentzian geometry?

- need for *low regularity* (of the metric): PDE point-of-view, physically relevant models (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- separate main concepts and derived notions of the causal structure
- minimal framework for *causality* and *(timelike/causal)* curvature bounds with continuous metrics
- notion of *(timelike/causal) curvature bounds* without a Lorentzian metric
- possible applications to Quantum Gravity (no Lorentzian metric): *causal fermion systems*, causal sets, etc.

Regularity class (of the metric) where causality theory is fine: $\mathcal{C}^{1,1}$, i.e., first derivative is (locally) Lipschitz continuous

Introduction (1/2)

Why a *synthetic* approach to Lorentzian geometry?

- need for *low regularity* (of the metric): PDE point-of-view, physically relevant models (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- separate main concepts and derived notions of the causal structure
- minimal framework for *causality* and *(timelike/causal)* curvature bounds with continuous metrics
- notion of *(timelike/causal) curvature bounds* without a Lorentzian metric
- possible applications to Quantum Gravity (no Lorentzian metric): *causal fermion systems*, causal sets, etc.

Regularity class (of the metric) where causality theory is fine: $C^{1,1}$, i.e., first derivative is (locally) Lipschitz continuous

Introduction (1/2)

Why a *synthetic* approach to Lorentzian geometry?

- need for *low regularity* (of the metric): PDE point-of-view, physically relevant models (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- separate main concepts and derived notions of the causal structure
- minimal framework for *causality* and *(timelike/causal)* curvature bounds with continuous metrics
- notion of *(timelike/causal) curvature bounds* without a Lorentzian metric
- possible applications to Quantum Gravity (no Lorentzian metric): *causal fermion systems*, causal sets, etc.

Regularity class (of the metric) where causality theory is fine: $\mathcal{C}^{1,1}$, i.e., first derivative is (locally) Lipschitz continuous

Introduction (1/2)

Why a *synthetic* approach to Lorentzian geometry?

- need for *low regularity* (of the metric): PDE point-of-view, physically relevant models (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- separate main concepts and derived notions of the causal structure
- minimal framework for *causality* and *(timelike/causal)* curvature bounds with continuous metrics
- notion of *(timelike/causal) curvature bounds* without a Lorentzian metric
- possible applications to Quantum Gravity (no Lorentzian metric): *causal fermion systems*, causal sets, etc.

Regularity class (of the metric) where causality theory is fine: $\mathcal{C}^{1,1}$, i.e., first derivative is (locally) Lipschitz continuous

Introduction (1/2)

Why a *synthetic* approach to Lorentzian geometry?

- need for *low regularity* (of the metric): PDE point-of-view, physically relevant models (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- separate main concepts and derived notions of the causal structure
- minimal framework for *causality* and *(timelike/causal)* curvature bounds with continuous metrics
- notion of *(timelike/causal) curvature bounds* without a Lorentzian metric
- possible applications to Quantum Gravity (no Lorentzian metric): *causal fermion systems*, causal sets, etc.

Regularity class (of the metric) where causality theory is fine: $\mathcal{C}^{1,1}$, i.e., first derivative is (locally) Lipschitz continuous

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

⇝ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Introduction (2/2)

Kronheimer and Penrose (1967): *On the structure of causal spaces*

- abstract approach to *causality theory*
- allow more *general spaces* than manifolds
- develop *causality conditions* in GR and popularize *Alexandrov topology*
- *reconstruction* of causality relations from others
- applications to *causal boundary*, certain approaches to *Quantum Gravity* (causal sets, etc.)

going beyond causality: include (abstract) *time-separation function* (cf. Busemann 1967)

$\rightsquigarrow$ *synthetic* approach to Lorentzian geometry

- synthetic *sectional curvature* bounds via *triangle comparison* (Kunzinger, S. 2018)
- synthetic *Ricci curvature* bounds via *optimal transport* (Cavalletti, Mondino 2020)
- non-smooth spacetimes; inextendibility; singularity theorems

Semi-Riemannian curvature bounds and triangles

Theorem (Toponogov)

(smooth) Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall \triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d(p,q) \geq \bar{d}(\bar{p},\bar{q}) \qquad (d(p,q) \leq \bar{d}(\bar{p},\bar{q}))$$

Definition

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if *spacelike* sectional curvatures $\geq K$ ($\leq$) and *timelike* sectional curvatures $\leq K$ ($\geq$)

Theorem (Alexander, Bishop 2008)

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall$ geodesic $\triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d_{\text{signed}}(p,q) \geq \bar{d}_{\text{signed}}(\bar{p},\bar{q}) \qquad (d_{\text{signed}}(p,q) \leq \bar{d}_{\text{signed}}(\bar{p},\bar{q}))$$

Semi-Riemannian curvature bounds and triangles

Theorem (Toponogov)

(smooth) Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall \triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d(p,q) \geq \bar{d}(\bar{p},\bar{q}) \qquad (d(p,q) \leq \bar{d}(\bar{p},\bar{q}))$$

Definition

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if *spacelike* sectional curvatures $\geq K$ ($\leq$) and *timelike* sectional curvatures $\leq K$ ($\geq$)

Theorem (Alexander, Bishop 2008)

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall$ geodesic $\triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d_{\text{signed}}(p,q) \geq \bar{d}_{\text{signed}}(\bar{p},\bar{q}) \qquad (d_{\text{signed}}(p,q) \leq \bar{d}_{\text{signed}}(\bar{p},\bar{q}))$$

Semi-Riemannian curvature bounds and triangles

Theorem (Toponogov)

(smooth) Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall \triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d(p,q) \geq \bar{d}(\bar{p},\bar{q}) \qquad (d(p,q) \leq \bar{d}(\bar{p},\bar{q}))$$

Definition

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if *spacelike* sectional curvatures $\geq K$ ($\leq$) and *timelike* sectional curvatures $\leq K$ ($\geq$)

Theorem (Alexander, Bishop 2008)

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall$ geodesic $\triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d_{\text{signed}}(p,q) \geq \bar{d}_{\text{signed}}(\bar{p},\bar{q}) \qquad (d_{\text{signed}}(p,q) \leq \bar{d}_{\text{signed}}(\bar{p},\bar{q}))$$

(Metric) length spaces

(X, d) a metric space, $\gamma\colon [a,b] \to X$ a continuous curve (path)

Definition

the *length* of γ is

$$L_d(\gamma) := \sup\{\sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})) : N \in \mathbb{N}, a \leq t_0 < t_1 < \ldots < t_N \leq b\},$$

the *length metric $\hat{d}$ associated to d* is defined as

$$\hat{d}(x,y) := \inf\{L_d(\lambda) : \lambda \text{ path connecting } x \text{ and } y\} \quad (x, y \in X)$$

Definition

a metric space (X, d) is a *length space* if $d = \hat{d}$, i.e.,

$$d(x,y) = \inf\{L_d(\lambda) : \lambda \text{ path connecting } x \text{ and } y\}$$

(Metric) length spaces

(X,d) a metric space, $\gamma\colon [a,b] \to X$ a continuous curve (path)

Definition

the *length* of γ is

$$L_d(\gamma) := \sup\{\sum_{i=0}^{N-1} d(\gamma(t_i),\gamma(t_{i+1})) : N \in \mathbb{N}, a \leq t_0 < t_1 < \ldots < t_N \leq b\}\,,$$

the *length metric* $\hat{d}$ *associated to* d is defined as

$$\hat{d}(x,y) := \inf\{L_d(\lambda) : \lambda \text{ path connecting } x \text{ and } y\} \quad (x,y \in X)$$

Definition

a metric space (X,d) is a *length space* if $d = \hat{d}$, i.e.,

$$d(x,y) = \inf\{L_d(\lambda) : \lambda \text{ path connecting } x \text{ and } y\}$$

(Metric) length spaces

(X, d) a metric space, $\gamma\colon [a,b] \to X$ a continuous curve (path)

Definition

the *length* of γ is

$$L_d(\gamma) := \sup\{\sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})) : N \in \mathbb{N}, a \leq t_0 < t_1 < \ldots < t_N \leq b\},$$

the *length metric $\hat{d}$ associated to d* is defined as

$$\hat{d}(x,y) := \inf\{L_d(\lambda) : \lambda \text{ path connecting } x \text{ and } y\} \quad (x, y \in X)$$

Definition

a metric space (X, d) is a *length space* if $d = \hat{d}$, i.e.,

$$d(x,y) = \inf\{L_d(\lambda) : \lambda \text{ path connecting } x \text{ and } y\}$$

(Metric) length spaces

(X, d) a metric space, $\gamma\colon [a, b] \to X$ a continuous curve (path)

Definition

the *length* of γ is

$$L_d(\gamma) := \sup\{\sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})) : N \in \mathbb{N}, a \leq t_0 < t_1 < \ldots < t_N \leq b\},$$

the *length metric* $\hat{d}$ *associated to* d is defined as

$$\hat{d}(x, y) := \inf\{L_d(\lambda) : \lambda \text{ path connecting } x \text{ and } y\} \quad (x, y \in X)$$

Definition

a metric space (X, d) is a *length space* if $d = \hat{d}$, i.e.,

$$d(x, y) = \inf\{L_d(\lambda) : \lambda \text{ path connecting } x \text{ and } y\}$$

Curvature bounds for length spaces

$k \in \mathbb{R}$, M_k^{Riem} 2D (Riemannian) model space of constant curvature k
(X, d) a length space

Definition

(X, d) has *curvature bounded from below by k / from above by k* if $\forall x \in X$ $\exists$ nhd. U of X s.t. $\forall \triangle abc$ in U and any point p on the side $\bar{ac}$

$$d(p,b) \geq \bar{d}(\bar{p},\bar{b}) \quad / \quad d(p,b) \leq \bar{d}(\bar{p},\bar{b})$$

$\bar{d}$ metric of M_k^{Riem}, $\triangle\bar{a}\bar{b}\bar{c}$ comparison triangle in M_k^{Riem} (i.e., with the same side lengths)

no differential structure, curvature tensor etc.
$\rightsquigarrow$ *Lorentzian* setting? Causal relations and time separation function
Alexander, Bishop 2008: *triangle comparison* characterizes semi-Riemannian *sectional curvature* bounds

Curvature bounds for length spaces

$k \in \mathbb{R}$, M_k^{Riem} 2D (Riemannian) model space of constant curvature k
(X, d) a length space

Definition

(X, d) has *curvature bounded from below by k / from above by k* if $\forall x \in X$ $\exists$ nhd. U of X s.t. $\forall \triangle abc$ in U and any point p on the side $\bar{ac}$

$$d(p, b) \geq \bar{d}(\bar{p}, \bar{b}) \quad / \quad d(p, b) \leq \bar{d}(\bar{p}, \bar{b})$$

$\bar{d}$ metric of M_k^{Riem}, $\triangle \bar{a}\bar{b}\bar{c}$ comparison triangle in M_k^{Riem} (i.e., with the same side lengths)

no differential structure, curvature tensor etc.
$\rightsquigarrow$ *Lorentzian* setting? Causal relations and time separation function
Alexander, Bishop 2008: *triangle comparison* characterizes semi-Riemannian *sectional curvature* bounds

Curvature bounds for length spaces

$k \in \mathbb{R}$, M_k^{Riem} 2D (Riemannian) model space of constant curvature k
(X, d) a length space

Definition

(X, d) has *curvature bounded from below by k / from above by k* if $\forall x \in X\ \exists$ nhd. U of X s.t. $\forall \triangle abc$ in U and any point p on the side $\bar{ac}$

$$d(p,b) \geq \bar{d}(\bar{p},\bar{b}) \quad / \quad d(p,b) \leq \bar{d}(\bar{p},\bar{b})$$

$\bar{d}$ metric of M_k^{Riem}, $\triangle\bar{a}\bar{b}\bar{c}$ comparison triangle in M_k^{Riem} (i.e., with the same side lengths)

no differential structure, curvature tensor etc.

$\rightsquigarrow$ *Lorentzian* setting? Causal relations and time separation function

Alexander, Bishop 2008: *triangle comparison* characterizes semi-Riemannian *sectional curvature* bounds

Curvature bounds for length spaces

$k \in \mathbb{R}$, M_k^{Riem} 2D (Riemannian) model space of constant curvature k
(X, d) a length space

Definition

(X, d) has *curvature bounded from below by k / from above by k* if $\forall x \in X$ $\exists$ nhd. U of X s.t. $\forall \triangle abc$ in U and any point p on the side $\bar{ac}$

$$d(p, b) \geq \bar{d}(\bar{p}, \bar{b}) \quad / \quad d(p, b) \leq \bar{d}(\bar{p}, \bar{b})$$

$\bar{d}$ metric of M_k^{Riem}, $\triangle \bar{a}\bar{b}\bar{c}$ comparison triangle in M_k^{Riem} (i.e., with the same side lengths)

no differential structure, curvature tensor etc.

$\rightsquigarrow$ *Lorentzian* setting? Causal relations and time separation function
Alexander, Bishop 2008: *triangle comparison* characterizes semi-Riemannian *sectional curvature* bounds

Curvature bounds for length spaces

$k \in \mathbb{R}$, M_k^{Riem} 2D (Riemannian) model space of constant curvature k
(X, d) a length space

Definition

(X, d) has *curvature bounded from below by k / from above by k* if $\forall x \in X$ $\exists$ nhd. U of X s.t. $\forall \triangle abc$ in U and any point p on the side $\bar{ac}$

$$d(p, b) \geq \bar{d}(\bar{p}, \bar{b}) \quad / \quad d(p, b) \leq \bar{d}(\bar{p}, \bar{b})$$

$\bar{d}$ metric of M_k^{Riem}, $\triangle \bar{a}\bar{b}\bar{c}$ comparison triangle in M_k^{Riem} (i.e., with the same side lengths)

no differential structure, curvature tensor etc.
$\rightsquigarrow$ *Lorentzian* setting? Causal relations and time separation function
Alexander, Bishop 2008: *triangle comparison* characterizes semi-Riemannian *sectional curvature* bounds

Curvature bounds for length spaces

$k \in \mathbb{R}$, M_k^{Riem} 2D (Riemannian) model space of constant curvature k
(X, d) a length space

Definition

(X, d) has *curvature bounded from below by k / from above by k* if $\forall x \in X$ $\exists$ nhd. U of X s.t. $\forall \triangle abc$ in U and any point p on the side $\bar{ac}$

$$d(p, b) \geq \bar{d}(\bar{p}, \bar{b}) \quad / \quad d(p, b) \leq \bar{d}(\bar{p}, \bar{b})$$

$\bar{d}$ metric of M_k^{Riem}, $\triangle \bar{a}\bar{b}\bar{c}$ comparison triangle in M_k^{Riem} (i.e., with the same side lengths)

no differential structure, curvature tensor etc.
$\rightsquigarrow$ *Lorentzian* setting? Causal relations and time separation function
Alexander, Bishop 2008: *triangle comparison* characterizes semi-Riemannian *sectional curvature* bounds

Curvature bounds for length spaces

$k \in \mathbb{R}$, M_k^{Riem} 2D (Riemannian) model space of constant curvature k
(X, d) a length space

Definition

(X, d) has *curvature bounded from below by k / from above by k* if $\forall x \in X$ $\exists$ nhd. U of X s.t. $\forall \triangle abc$ in U and any point p on the side $\bar{ac}$

$$d(p, b) \geq \bar{d}(\bar{p}, \bar{b}) \quad / \quad d(p, b) \leq \bar{d}(\bar{p}, \bar{b})$$

$\bar{d}$ metric of M_k^{Riem}, $\triangle \bar{a}\bar{b}\bar{c}$ comparison triangle in M_k^{Riem} (i.e., with the same side lengths)

no differential structure, curvature tensor etc.
$\rightsquigarrow$ *Lorentzian* setting? Causal relations and time separation function
Alexander, Bishop 2008: *triangle comparison* characterizes semi-Riemannian *sectional curvature* bounds

Review of Lorentzian causality in spacetimes (1/2)

Definition

(M, g) *Lorentzian manifold*, M smooth, connected manifold, g Lorentzian metric, i.e., a symmetric, non-degenerate $(0, 2)$ tensor with signature $(-, +, +, +, \ldots)$, usually g smooth

Definition

(M, g) spacetime, $v \in T_pM$ is

$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \\ \textit{spacelike} \end{cases} \quad \text{if} \quad g_p(v, v) \begin{cases} < 0 \\ = 0 \text{ and } v \neq 0 \\ \leq 0 \text{ and } v \neq 0 \\ > 0 \text{ or } v = 0 \end{cases}$$

analogously for curves into M of sufficient regularity

length of a curve γ: $L_g(\gamma) := \int_a^b \sqrt{|g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))|}\,\mathrm{d}s$

Review of Lorentzian causality in spacetimes (1/2)

Definition

(M,g) *Lorentzian manifold*, M smooth, connected manifold, g Lorentzian metric, i.e., a symmetric, non-degenerate $(0,2)$ tensor with signature $(-,+,+,+,\dots)$, usually g smooth

Definition

(M,g) spacetime, $v \in T_pM$ is

$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \\ \textit{spacelike} \end{cases} \quad \text{if} \quad g_p(v,v) \quad \begin{cases} < 0 \\ = 0 \text{ and } v \neq 0 \\ \leq 0 \text{ and } v \neq 0 \\ > 0 \text{ or } v = 0 \end{cases}$$

analogously for curves into M of sufficient regularity

length of a curve γ: $L_g(\gamma) := \int_a^b \sqrt{|g_{\gamma(s)}(\dot\gamma(s),\dot\gamma(s))|}\,\mathrm{d}s$

Review of Lorentzian causality in spacetimes (1/2)

Definition

(M,g) *Lorentzian manifold*, M smooth, connected manifold, g Lorentzian metric, i.e., a symmetric, non-degenerate $(0,2)$ tensor with signature $(-,+,+,+,\dots)$, usually g smooth

Definition

(M,g) spacetime, $v \in T_pM$ is

$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \\ \textit{spacelike} \end{cases} \quad \text{if} \quad g_p(v,v) \quad \begin{cases} < 0 \\ = 0 \text{ and } v \neq 0 \\ \leq 0 \text{ and } v \neq 0 \\ > 0 \text{ or } v = 0 \end{cases}$$

analogously for curves into M of sufficient regularity

length of a curve γ: $L_g(\gamma) := \int_a^b \sqrt{|g_{\gamma(s)}(\dot\gamma(s),\dot\gamma(s))|}\,\mathrm{d}s$

Review of Lorentzian causality in spacetimes (1/2)

Definition

(M, g) *Lorentzian manifold*, M smooth, connected manifold, g Lorentzian metric, i.e., a symmetric, non-degenerate $(0, 2)$ tensor with signature $(-, +, +, +, \dots)$, usually g smooth

Definition

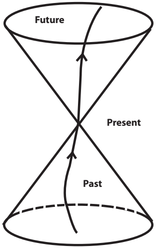
(M, g) spacetime, $v \in T_pM$ is

$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \\ \textit{spacelike} \end{cases} \quad \text{if} \quad g_p(v, v) \quad \begin{cases} < 0 \\ = 0 \text{ and } v \neq 0 \\ \leq 0 \text{ and } v \neq 0 \\ > 0 \text{ or } v = 0 \end{cases}$$

analogously for curves into M of sufficient regularity

length of a curve γ: $L_g(\gamma) := \int_a^b \sqrt{|g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))|}\,\mathrm{d}s$

Review of Lorentzian causality in spacetimes (2/2)



Definition

(M,g) *spacetime*: (M,g) Lorentzian manifold, *time-oriented*, i.e., $\exists$ timelike vectorfield T

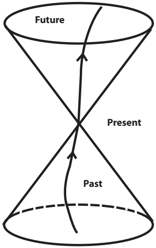
Definition

$v \in T_pM$ is *future directed* if $g_p(v,T(p)) < 0$

analogously for curves

Causal relations: $p \ll q :\Leftrightarrow \exists$ f.d. timelike curve from p to q

$p \leq q :\Leftrightarrow \exists$ f.d. causal curve from p to q or $p = q$



Definition

(M,g) *spacetime*: (M,g) Lorentzian manifold, *time-oriented*, i.e., $\exists$ timelike vectorfield T

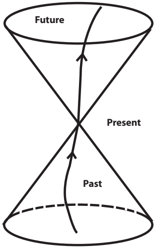
Definition

$v \in T_pM$ is *future directed* if $g_p(v, T(p)) < 0$

analogously for curves

Causal relations: $p \ll q :\Leftrightarrow \exists$ f.d. timelike curve from p to q
$p \leq q :\Leftrightarrow \exists$ f.d. causal curve from p to q or $p = q$

Review of Lorentzian causality in spacetimes (2/2)



Definition

(M,g) *spacetime*: (M,g) Lorentzian manifold, *time-oriented*, i.e., $\exists$ timelike vectorfield T

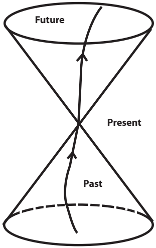
Definition

$v \in T_pM$ is *future directed* if $g_p(v, T(p)) < 0$

analogously for curves

Causal relations: $p \ll q :\Leftrightarrow \exists$ f.d. timelike curve from p to q

$p \leq q :\Leftrightarrow \exists$ f.d. causal curve from p to q or $p = q$



Definition

(M, g) *spacetime*: (M, g) Lorentzian manifold, *time-oriented*, i.e., $\exists$ timelike vectorfield T

Definition

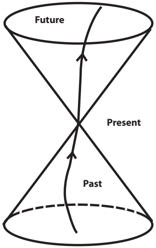
$v \in T_pM$ is *future directed* if $g_p(v, T(p)) < 0$

analogously for curves

Causal relations: $p \ll q :\Leftrightarrow \exists$ f.d. timelike curve from p to q

$p \leq q :\Leftrightarrow \exists$ f.d. causal curve from p to q or $p = q$

Review of Lorentzian causality in spacetimes (2/2)



Definition

(M, g) *spacetime*: (M, g) Lorentzian manifold, *time-oriented*, i.e., $\exists$ timelike vectorfield T

Definition

$v \in T_pM$ is *future directed* if $g_p(v, T(p)) < 0$

analogously for curves

Causal relations: $p \ll q :\Leftrightarrow \exists$ f.d. timelike curve from p to q

$p \leq q :\Leftrightarrow \exists$ f.d. causal curve from p to q or $p = q$

Causal spaces

slight generalization of Kronheimer and Penrose (1967)

Definition

$(X, \ll, \leq)$ is a *causal space* if X is a set, $\leq$ preorder on X and $\ll$ transitive relation contained in $\leq$

no "causality conditions" implicit

for $x, y \in X$

- $x < y :\Leftrightarrow x \leq y$ and $x \neq y$
- $I^+(x) := \{z \in X : x \ll z\}$ and $I^-(x) := \{z \in X : z \ll x\}$
- $J^+(x) := \{z \in X : x \leq z\}$ and $J^-(x) := \{z \in X : z \leq x\}$

Alexandrov topology on X via subbase $\{I^+(x) \cap I^-(y) : x, y \in X\}$

Causal spaces

slight generalization of Kronheimer and Penrose (1967)

Definition

$(X, \ll, \leq)$ is a *causal space* if X is a set, $\leq$ preorder on X and $\ll$ transitive relation contained in $\leq$

no "causality conditions" implicit

for $x, y \in X$

- $x < y :\Leftrightarrow x \leq y$ and $x \neq y$
- $I^+(x) := \{z \in X : x \ll z\}$ and $I^-(x) := \{z \in X : z \ll x\}$
- $J^+(x) := \{z \in X : x \leq z\}$ and $J^-(x) := \{z \in X : z \leq x\}$

Alexandrov topology on X via subbase $\{I^+(x) \cap I^-(y) : x, y \in X\}$

Causal spaces

slight generalization of Kronheimer and Penrose (1967)

Definition

$(X, \ll, \leq)$ is a *causal space* if X is a set, $\leq$ preorder on X and $\ll$ transitive relation contained in $\leq$

no "causality conditions" implicit

for $x, y \in X$

- $x < y :\Leftrightarrow x \leq y$ and $x \neq y$
- $I^+(x) := \{z \in X : x \ll z\}$ and $I^-(x) := \{z \in X : z \ll x\}$
- $J^+(x) := \{z \in X : x \leq z\}$ and $J^-(x) := \{z \in X : z \leq x\}$

Alexandrov topology on X via subbase $\{I^+(x) \cap I^-(y) : x, y \in X\}$

Causal spaces

slight generalization of Kronheimer and Penrose (1967)

Definition

$(X, \ll, \leq)$ is a *causal space* if X is a set, $\leq$ preorder on X and $\ll$ transitive relation contained in $\leq$

no "causality conditions" implicit

for $x, y \in X$

- $x < y :\Leftrightarrow x \leq y$ and $x \neq y$
- $I^+(x) := \{z \in X : x \ll z\}$ and $I^-(x) := \{z \in X : z \ll x\}$
- $J^+(x) := \{z \in X : x \leq z\}$ and $J^-(x) := \{z \in X : z \leq x\}$

Alexandrov topology on X via subbase $\{I^+(x) \cap I^-(y) : x, y \in X\}$

Causal spaces

slight generalization of Kronheimer and Penrose (1967)

Definition

$(X, \ll, \leq)$ is a *causal space* if X is a set, $\leq$ preorder on X and $\ll$ transitive relation contained in $\leq$

no "causality conditions" implicit

for $x, y \in X$

- $x < y :\Leftrightarrow x \leq y$ and $x \neq y$
- $I^+(x) := \{z \in X : x \ll z\}$ and $I^-(x) := \{z \in X : z \ll x\}$
- $J^+(x) := \{z \in X : x \leq z\}$ and $J^-(x) := \{z \in X : z \leq x\}$

Alexandrov topology on X via subbase $\{I^+(x) \cap I^-(y) : x, y \in X\}$

Causal spaces

slight generalization of Kronheimer and Penrose (1967)

Definition

$(X, \ll, \leq)$ is a *causal space* if X is a set, $\leq$ preorder on X and $\ll$ transitive relation contained in $\leq$

no "causality conditions" implicit

for $x, y \in X$

- $x < y :\Leftrightarrow x \leq y$ and $x \neq y$
- $I^+(x) := \{z \in X : x \ll z\}$ and $I^-(x) := \{z \in X : z \ll x\}$
- $J^+(x) := \{z \in X : x \leq z\}$ and $J^-(x) := \{z \in X : z \leq x\}$

Alexandrov topology on X via subbase $\{I^+(x) \cap I^-(y) : x, y \in X\}$

Causal spaces

slight generalization of Kronheimer and Penrose (1967)

Definition

$(X, \ll, \leq)$ is a *causal space* if X is a set, $\leq$ preorder on X and $\ll$ transitive relation contained in $\leq$

no "causality conditions" implicit

for $x, y \in X$

- $x < y :\Leftrightarrow x \leq y$ and $x \neq y$
- $I^+(x) := \{z \in X : x \ll z\}$ and $I^-(x) := \{z \in X : z \ll x\}$
- $J^+(x) := \{z \in X : x \leq z\}$ and $J^-(x) := \{z \in X : z \leq x\}$

Alexandrov topology on X via subbase $\{I^+(x) \cap I^-(y) : x, y \in X\}$

Lorentzian pre-length spaces (1/2)

$(X, \ll, \leq)$ a causal space, d metric on X, $\tau\colon X \times X \to [0, \infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z) \qquad (x \leq y \leq z),$$

and $\tau(x, y) = 0$ if $x \not\leq y$ and $\tau(x, y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- smooth spacetimes (M, g) with usual time separation function $\tau(p, q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\}$
- finite directed graphs

Lorentzian pre-length spaces (1/2)

$(X, \ll, \leq)$ a causal space, d metric on X, $\tau \colon X \times X \to [0, \infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- smooth spacetimes (M, g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\}$
- finite directed graphs

Lorentzian pre-length spaces (1/2)

$(X, \ll, \leq)$ a causal space, d metric on X, $\tau \colon X \times X \to [0, \infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- smooth spacetimes (M,g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\}$
- finite directed graphs

Lorentzian pre-length spaces (1/2)

$(X, \ll, \leq)$ a causal space, d metric on X, $\tau \colon X \times X \to [0, \infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- smooth spacetimes (M, g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\}$
- finite directed graphs

Lorentzian pre-length spaces (2/2)

Properties of Lorentzian pre length spaces

$(X, d, \ll, \leq, \tau)$ Lorentzian pre length space, then

- $I^{\pm}(x)$ open for all $x \in X$
- $\ll$ open in $X \times X$
- for every $x \in X$ either $\tau(x,x) = 0$ or $\tau(x,x) = \infty$
- the metric topology of d is finer than the Alexandrov topology

Next step: causal curves and their length

Lorentzian pre-length spaces (2/2)

Properties of Lorentzian pre length spaces

$(X, d, \ll, \leq, \tau)$ Lorentzian pre length space, then

- $I^{\pm}(x)$ open for all $x \in X$
- $\ll$ open in $X \times X$
- for every $x \in X$ either $\tau(x,x) = 0$ or $\tau(x,x) = \infty$
- the metric topology of d is finer than the Alexandrov topology

Next step: causal curves and their length

Lorentzian pre-length spaces (2/2)

Properties of Lorentzian pre length spaces

$(X, d, \ll, \leq, \tau)$ Lorentzian pre length space, then

- $I^{\pm}(x)$ open for all $x \in X$
- $\ll$ open in $X \times X$
- for every $x \in X$ either $\tau(x,x) = 0$ or $\tau(x,x) = \infty$
- the metric topology of d is finer than the Alexandrov topology

Next step: causal curves and their length

Lorentzian pre-length spaces (2/2)

Properties of Lorentzian pre length spaces

$(X, d, \ll, \leq, \tau)$ Lorentzian pre length space, then

- $I^{\pm}(x)$ open for all $x \in X$
- $\ll$ open in $X \times X$
- for every $x \in X$ either $\tau(x,x) = 0$ or $\tau(x,x) = \infty$
- the metric topology of d is finer than the Alexandrov topology

Next step: causal curves and their length

Lorentzian pre-length spaces (2/2)

Properties of Lorentzian pre length spaces

$(X, d, \ll, \leq, \tau)$ Lorentzian pre length space, then

- $I^{\pm}(x)$ open for all $x \in X$
- $\ll$ open in $X \times X$
- for every $x \in X$ either $\tau(x, x) = 0$ or $\tau(x, x) = \infty$
- the metric topology of d is finer than the Alexandrov topology

Next step: causal curves and their length

Causal curves (1/2)

Definition

$I \subseteq \mathbb{R}$ interval, $\gamma\colon I \to X$ non-constant is *future directed causal (timelike)* if γ locally Lipschitz continuous (wrt. d) and for $t_1, t_2 \in I$, $t_1 < t_2$: $\gamma(t_1) \leq \gamma(t_2)$ ($\gamma(t_1) \ll \gamma(t_2)$); analogously for *null* ($\gamma(t_1) \leq \gamma(t_2)$ and $\gamma(t_1) \not\ll \gamma(t_2)$) and *past directed* curves

- Lorentz cylinder $S^1_1 \times \mathbb{R}$: every non-constant locally Lipschitz curve is timelike and causal $\rightsquigarrow$ need causality conditions
- Minkowski spacetime $\mathbb{R}^3_1$: $t \mapsto (t, \cos(t), \sin(t))$ has null tangent but is timelike

Proposition (M. Kunzinger, C.S. 2018)

continuous, strongly causal spacetimes: *different notions* of causal curves *agree*

Causal curves (1/2)

Definition

$I \subseteq \mathbb{R}$ interval, $\gamma\colon I \to X$ non-constant is *future directed causal (timelike)* if γ locally Lipschitz continuous (wrt. d) and for $t_1, t_2 \in I$, $t_1 < t_2$: $\gamma(t_1) \leq \gamma(t_2)$ ($\gamma(t_1) \ll \gamma(t_2)$); analogously for *null* ($\gamma(t_1) \leq \gamma(t_2)$ and $\gamma(t_1) \not\ll \gamma(t_2)$) and *past directed* curves

- Lorentz cylinder $S^1_1 \times \mathbb{R}$: every non-constant locally Lipschitz curve is timelike and causal $\rightsquigarrow$ need causality conditions
- Minkowski spacetime $\mathbb{R}^3_1$: $t \mapsto (t, \cos(t), \sin(t))$ has null tangent but is timelike

Proposition (M. Kunzinger, C.S. 2018)

continuous, strongly causal spacetimes: *different notions* of causal curves *agree*

Causal curves (1/2)

Definition

$I \subseteq \mathbb{R}$ interval, $\gamma\colon I \to X$ non-constant is *future directed causal (timelike)* if γ locally Lipschitz continuous (wrt. d) and for $t_1, t_2 \in I$, $t_1 < t_2$: $\gamma(t_1) \leq \gamma(t_2)$ ($\gamma(t_1) \ll \gamma(t_2)$); analogously for *null* ($\gamma(t_1) \leq \gamma(t_2)$ and $\gamma(t_1) \not\ll \gamma(t_2)$) and *past directed* curves

- Lorentz cylinder $S^1_1 \times \mathbb{R}$: every non-constant locally Lipschitz curve is timelike and causal $\rightsquigarrow$ need causality conditions
- Minkowski spacetime $\mathbb{R}^3_1$: $t \mapsto (t, \cos(t), \sin(t))$ has null tangent but is timelike

Proposition (M. Kunzinger, C.S. 2018)

continuous, strongly causal spacetimes: *different notions* of causal curves *agree*

Causal curves (1/2)

Definition

$I \subseteq \mathbb{R}$ interval, $\gamma \colon I \to X$ non-constant is *future directed causal (timelike)* if γ locally Lipschitz continuous (wrt. d) and for $t_1, t_2 \in I$, $t_1 < t_2$: $\gamma(t_1) \leq \gamma(t_2)$ ($\gamma(t_1) \ll \gamma(t_2)$); analogously for *null* ($\gamma(t_1) \leq \gamma(t_2)$ and $\gamma(t_1) \not\ll \gamma(t_2)$) and *past directed* curves

- Lorentz cylinder $S^1_1 \times \mathbb{R}$: every non-constant locally Lipschitz curve is timelike and causal $\rightsquigarrow$ need causality conditions
- Minkowski spacetime $\mathbb{R}^3_1$: $t \mapsto (t, \cos(t), \sin(t))$ has null tangent but is timelike

Proposition (M. Kunzinger, C.S. 2018)

continuous, strongly causal spacetimes: *different notions* of causal curves *agree*

Causal curves (2/2)

Definition

$\gamma\colon [a,b] \to X$ f.d. causal, *τ-length* defined by

$$L_\tau(\gamma) := \inf\{\sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b\}$$

- L_τ *additive* and *invariant* under continuous and strictly increasing reparametrizations
- $\gamma\colon [a,b] \to X$ f.d. causal is *rectifiable* if $L_\tau(\gamma|_{[t_1,t_2]}) > 0$ for all $a \leq t_1 < t_2 \leq b$
- a *rectifiable* causal curve is *timelike*
- $t \mapsto (t, \cos(t), \sin(t))$ timelike but τ-length zero $\Rightarrow$ *non-rectifiable*

Proposition (M. Kunzinger, C.S. 2018)

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a *smooth* and *strongly causal* spacetime (M, g), then $L_\tau(\gamma) = L_g(\gamma)$

Causal curves (2/2)

Definition

$\gamma\colon [a,b] \to X$ f.d. causal, *τ-length* defined by

$$L_\tau(\gamma) := \inf\{\sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b\}$$

- L_τ *additive* and *invariant* under continuous and strictly increasing reparametrizations
- $\gamma\colon [a,b] \to X$ f.d. causal is *rectifiable* if $L_\tau(\gamma|_{[t_1,t_2]}) > 0$ for all $a \leq t_1 < t_2 \leq b$
- a *rectifiable* causal curve is *timelike*
- $t \mapsto (t, \cos(t), \sin(t))$ timelike but τ-length zero $\Rightarrow$ *non-rectifiable*

Proposition (M. Kunzinger, C.S. 2018)

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a *smooth* and *strongly causal* spacetime (M,g), then $L_\tau(\gamma) = L_g(\gamma)$

Causal curves (2/2)

Definition

$\gamma\colon [a,b] \to X$ f.d. causal, *τ-length* defined by

$$L_\tau(\gamma) := \inf\{\sum_{i=0}^{N-1} \tau(\gamma(t_i),\gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b\}$$

- L_τ *additive* and *invariant* under continuous and strictly increasing reparametrizations
- $\gamma\colon [a,b] \to X$ f.d. causal is *rectifiable* if $L_\tau(\gamma|_{[t_1,t_2]}) > 0$ for all $a \leq t_1 < t_2 \leq b$
- a *rectifiable* causal curve is *timelike*
- $t \mapsto (t,\cos(t),\sin(t))$ timelike but τ-length zero $\Rightarrow$ *non-rectifiable*

Proposition (M. Kunzinger, C.S. 2018)

$(M,d^h,\ll,\leq,\tau)$ the Lorentzian pre-length space induced by a *smooth* and *strongly causal* spacetime (M,g), then $L_\tau(\gamma) = L_g(\gamma)$

Causal curves (2/2)

Definition

$\gamma\colon [a,b] \to X$ f.d. causal, *τ-length* defined by

$$L_\tau(\gamma) := \inf\{\sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b\}$$

- L_τ *additive* and *invariant* under continuous and strictly increasing reparametrizations
- $\gamma\colon [a,b] \to X$ f.d. causal is *rectifiable* if $L_\tau(\gamma|_{[t_1,t_2]}) > 0$ for all $a \leq t_1 < t_2 \leq b$
- a *rectifiable* causal curve is *timelike*
- $t \mapsto (t, \cos(t), \sin(t))$ timelike but τ-length zero $\Rightarrow$ *non-rectifiable*

Proposition (M. Kunzinger, C.S. 2018)

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a *smooth* and *strongly causal* spacetime (M,g), then $L_\tau(\gamma) = L_g(\gamma)$

Causal curves (2/2)

Definition

$\gamma\colon [a,b] \to X$ f.d. causal, *τ-length* defined by

$$L_\tau(\gamma) := \inf\{\sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b\}$$

- L_τ *additive* and *invariant* under continuous and strictly increasing reparametrizations
- $\gamma\colon [a,b] \to X$ f.d. causal is *rectifiable* if $L_\tau(\gamma|_{[t_1,t_2]}) > 0$ for all $a \leq t_1 < t_2 \leq b$
- a *rectifiable* causal curve is *timelike*
- $t \mapsto (t, \cos(t), \sin(t))$ timelike but τ-length zero $\Rightarrow$ *non-rectifiable*

Proposition (M. Kunzinger, C.S. 2018)

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a *smooth* and *strongly causal* spacetime (M, g), then $L_\tau(\gamma) = L_g(\gamma)$

Causal curves (2/2)

Definition

$\gamma\colon [a,b] \to X$ f.d. causal, *τ-length* defined by

$$L_\tau(\gamma) := \inf\{ \sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b\}$$

- L_τ *additive* and *invariant* under continuous and strictly increasing reparametrizations
- $\gamma\colon [a,b] \to X$ f.d. causal is *rectifiable* if $L_\tau(\gamma|_{[t_1,t_2]}) > 0$ for all $a \leq t_1 < t_2 \leq b$
- a *rectifiable* causal curve is *timelike*
- $t \mapsto (t, \cos(t), \sin(t))$ timelike but τ-length zero $\Rightarrow$ *non-rectifiable*

Proposition (M. Kunzinger, C.S. 2018)

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a *smooth* and *strongly causal* spacetime (M, g), then $L_\tau(\gamma) = L_g(\gamma)$

Maximal causal curves

intrinsic notion of geodesics? $\rightsquigarrow$ *maximal causal curves*

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space

Definition

$\gamma\colon [a,b] \to X$ f.d. causal is *maximal* if $L_\tau(\gamma) = \tau(\gamma(a), \gamma(b))$

- *null* curves are *maximal*
- maximal curves are *maximal* on any *subinterval*
- *maximal* and *timelike* implies *rectifiable*

geodesic: locally maximizing curve (precise notion uses *localizability*)

Maximal causal curves

intrinsic notion of geodesics? $\rightsquigarrow$ *maximal causal curves*

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space

Definition

$\gamma\colon [a,b] \to X$ f.d. causal is *maximal* if $L_\tau(\gamma) = \tau(\gamma(a), \gamma(b))$

- *null* curves are *maximal*
- maximal curves are *maximal* on any *subinterval*
- *maximal* and *timelike* implies *rectifiable*

geodesic: locally maximizing curve (precise notion uses *localizability*)

Maximal causal curves

intrinsic notion of geodesics? $\rightsquigarrow$ *maximal causal curves*

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space

Definition

$\gamma\colon [a,b] \to X$ f.d. causal is *maximal* if $L_\tau(\gamma) = \tau(\gamma(a), \gamma(b))$

- *null* curves are *maximal*
- maximal curves are *maximal* on any *subinterval*
- *maximal* and *timelike* implies *rectifiable*

geodesic: locally maximizing curve (precise notion uses *localizability*)

Maximal causal curves

intrinsic notion of geodesics? $\rightsquigarrow$ *maximal causal curves*

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space

Definition

$\gamma\colon [a,b] \to X$ f.d. causal is *maximal* if $L_\tau(\gamma) = \tau(\gamma(a), \gamma(b))$

- *null* curves are *maximal*
- maximal curves are *maximal* on any *subinterval*
- *maximal* and *timelike* implies *rectifiable*

geodesic: locally maximizing curve (precise notion uses *localizability*)

Maximal causal curves

intrinsic notion of geodesics? $\rightsquigarrow$ *maximal causal curves*

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space

Definition

$\gamma \colon [a, b] \to X$ f.d. causal is *maximal* if $L_\tau(\gamma) = \tau(\gamma(a), \gamma(b))$

- *null* curves are *maximal*
- maximal curves are *maximal* on any *subinterval*
- *maximal* and *timelike* implies *rectifiable*

geodesic: locally maximizing curve (precise notion uses *localizability*)

Maximal causal curves

intrinsic notion of geodesics? $\rightsquigarrow$ *maximal causal curves*

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space

Definition

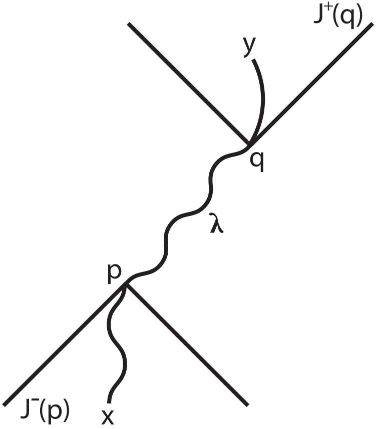
$\gamma\colon [a,b] \to X$ f.d. causal is *maximal* if $L_\tau(\gamma) = \tau(\gamma(a), \gamma(b))$

- *null* curves are *maximal*
- maximal curves are *maximal* on any *subinterval*
- *maximal* and *timelike* implies *rectifiable*

geodesic: locally maximizing curve (precise notion uses *localizability*)

Causal character of maximal curves

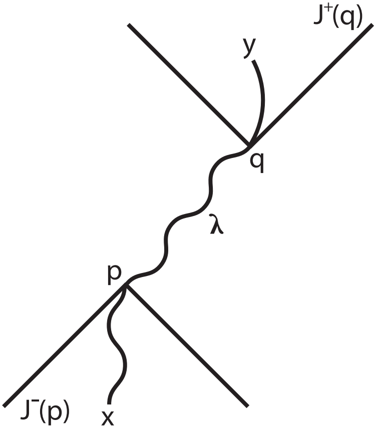
Minkowski space $\mathbb{R}^n_1$, λ f.d. causal connecting p, q;
$X = J^-(p) \cup J^+(q) \cup \lambda \rightsquigarrow$ *Lorentzian pre-length space*



λ null, $x \ll p$, $q \leq y \Rightarrow$ maximal curve from x to y *changes causal character*

Causal character of maximal curves

Minkowski space $\mathbb{R}^n_1$, λ f.d. causal connecting p, q;
$X = J^-(p) \cup J^+(q) \cup \lambda \rightsquigarrow$ *Lorentzian pre-length space*



λ null, $x \ll p$, $q \leq y \Rightarrow$ maximal curve from x to y *changes causal character*

Definition

causal space $(X, \ll, \leq)$ is

- *chronological* if $\ll$ is irreflexive, i.e., $x \not\ll x$ for all $x \in X$
- *causal* if $\leq$ is a partial order, i.e., $x \leq y$ and $y \leq x$ implies $x = y$

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is

- *non-totally imprisoning* if for every compact set $K \subseteq X$ $\exists C > 0$ s.t. the d-arclength of all causal curves contained in K is bounded by C
- *strongly causal* if the Alexandrov topology agrees with the metric topology
- *globally hyperbolic* if X is non-totally imprisoning and for every $x, y \in X$ the set $J^+(x) \cap J^-(y)$ is compact in X

Causality conditions (1/2)

Definition

causal space $(X, \ll, \leq)$ is

- *chronological* if $\ll$ is irreflexive, i.e., $x \not\ll x$ for all $x \in X$
- *causal* if $\leq$ is a partial order, i.e., $x \leq y$ and $y \leq x$ implies $x = y$

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is

- *non-totally imprisoning* if for every compact set $K \subseteq X$ $\exists C > 0$ s.t. the d-arclength of all causal curves contained in K is bounded by C
- *strongly causal* if the Alexandrov topology agrees with the metric topology
- *globally hyperbolic* if X is non-totally imprisoning and for every $x, y \in X$ the set $J^+(x) \cap J^-(y)$ is compact in X

Causality conditions (1/2)

Definition

causal space $(X, \ll, \leq)$ is

- *chronological* if $\ll$ is irreflexive, i.e., $x \not\ll x$ for all $x \in X$
- *causal* if $\leq$ is a partial order, i.e., $x \leq y$ and $y \leq x$ implies $x = y$

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is

- *non-totally imprisoning* if for every compact set $K \subseteq X$ $\exists C > 0$ s.t. the d-arclength of all causal curves contained in K is bounded by C
- *strongly causal* if the Alexandrov topology agrees with the metric topology
- *globally hyperbolic* if X is non-totally imprisoning and for every $x, y \in X$ the set $J^+(x) \cap J^-(y)$ is compact in X

Causality conditions (1/2)

Definition

causal space $(X, \ll, \leq)$ is

- *chronological* if $\ll$ is irreflexive, i.e., $x \not\ll x$ for all $x \in X$
- *causal* if $\leq$ is a partial order, i.e., $x \leq y$ and $y \leq x$ implies $x = y$

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is

- *non-totally imprisoning* if for every compact set $K \subseteq X$ $\exists C > 0$ s.t. the d-arclength of all causal curves contained in K is bounded by C
- *strongly causal* if the Alexandrov topology agrees with the metric topology
- *globally hyperbolic* if X is non-totally imprisoning and for every $x, y \in X$ the set $J^+(x) \cap J^-(y)$ is compact in X

Causality conditions (1/2)

Definition

causal space $(X, \ll, \leq)$ is

- *chronological* if $\ll$ is irreflexive, i.e., $x \not\ll x$ for all $x \in X$
- *causal* if $\leq$ is a partial order, i.e., $x \leq y$ and $y \leq x$ implies $x = y$

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is

- *non-totally imprisoning* if for every compact set $K \subseteq X$ $\exists C > 0$ s.t. the d-arclength of all causal curves contained in K is bounded by C
- *strongly causal* if the Alexandrov topology agrees with the metric topology
- *globally hyperbolic* if X is non-totally imprisoning and for every $x, y \in X$ the set $J^+(x) \cap J^-(y)$ is compact in X

Causality conditions (2/2)

Definition

causal space $(X, \ll, \leq)$ is *interpolative* if $x \ll y \Rightarrow \exists z \in X\colon\ x \ll z \ll y$ and $x \neq z \neq y$

Lemma

$(X, d, \ll, \leq, \tau)$ a Lorentzian pre-length space

- X causal and interpolative then chronological
- X chronological then τ is zero on the diagonal, i.e., $\tau(x, x) = 0$ for all $x \in X$
- X strongly causal, then $\forall x \in X$, for every nhd. U of x, $\exists V \subseteq U$ nhd. of x s.t. every causal curve with endpoints in V is contained in U (i.e., the usual definition of strong causality)

Causality conditions (2/2)

Definition

causal space $(X, \ll, \leq)$ is *interpolative* if $x \ll y \Rightarrow \exists z \in X\colon\ x \ll z \ll y$ and $x \neq z \neq y$

Lemma

$(X, d, \ll, \leq, \tau)$ a Lorentzian pre-length space

- X causal and interpolative then chronological
- X chronological then τ is zero on the diagonal, i.e., $\tau(x, x) = 0$ for all $x \in X$
- X strongly causal, then $\forall x \in X$, for every nhd. U of x, $\exists V \subseteq U$ nhd. of x s.t. every causal curve with endpoints in V is contained in U (i.e., the usual definition of strong causality)

Causality conditions (2/2)

Definition

causal space $(X, \ll, \leq)$ is *interpolative* if $x \ll y \Rightarrow \exists z \in X\colon\ x \ll z \ll y$ and $x \neq z \neq y$

Lemma

$(X, d, \ll, \leq, \tau)$ a Lorentzian pre-length space

- X causal and interpolative then chronological
- X chronological then τ is zero on the diagonal, i.e., $\tau(x,x) = 0$ for all $x \in X$
- X strongly causal, then $\forall x \in X$, for every nhd. U of x, $\exists V \subseteq U$ nhd. of x s.t. every causal curve with endpoints in V is contained in U (i.e., the usual definition of strong causality)

Causality conditions (2/2)

Definition

causal space $(X, \ll, \leq)$ is *interpolative* if $x \ll y \Rightarrow \exists z \in X\colon x \ll z \ll y$ and $x \neq z \neq y$

Lemma

$(X, d, \ll, \leq, \tau)$ a Lorentzian pre-length space

- X causal and interpolative then chronological
- X chronological then τ is zero on the diagonal, i.e., $\tau(x, x) = 0$ for all $x \in X$
- X strongly causal, then $\forall x \in X$, for every nhd. U of x, $\exists V \subseteq U$ nhd. of x s.t. every causal curve with endpoints in V is contained in U (i.e., the usual definition of strong causality)

Causal connectedness

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *causally path connected* if for $x \ll y$ $\exists$ f.d. timelike curve from x to y and for $x < y$ $\exists$ f.d. causal curve from x to y

Lemma

$(X, d, \ll, \leq, \tau)$ a causally path connected Lorentzian pre-length space

- X interpolative
- causal if and only if there are no closed causal curves
- chronological if and only if there are no closed timelike curves

Causal connectedness

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *causally path connected* if for $x \ll y$ $\exists$ f.d. timelike curve from x to y and for $x < y$ $\exists$ f.d. causal curve from x to y

Lemma

$(X, d, \ll, \leq, \tau)$ a causally path connected Lorentzian pre-length space

- X interpolative
- causal if and only if there are no closed causal curves
- chronological if and only if there are no closed timelike curves

Causal connectedness

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *causally path connected* if for $x \ll y$ $\exists$ f.d. timelike curve from x to y and for $x < y$ $\exists$ f.d. causal curve from x to y

Lemma

$(X, d, \ll, \leq, \tau)$ a causally path connected Lorentzian pre-length space

- X interpolative
- causal if and only if there are no closed causal curves
- chronological if and only if there are no closed timelike curves

Causal connectedness

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *causally path connected* if for $x \ll y$ $\exists$ f.d. timelike curve from x to y and for $x < y$ $\exists$ f.d. causal curve from x to y

Lemma

$(X, d, \ll, \leq, \tau)$ a causally path connected Lorentzian pre-length space

- X interpolative
- causal if and only if there are no closed causal curves
- chronological if and only if there are no closed timelike curves

Limit curves

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space, $x \in X$

Definition

nhd. U of x is *causally closed* if $\leq$ is closed in $\bar{U} \times \bar{U}$, i.e., $p_n, q_n \in U$ with $p_n \leq q_n$ and $p_n \to p \in \bar{U}$, $q_n \to q \in \bar{U}$, then $p \leq q$; X is *locally causally closed* if every point has a causally closed nhd.

Example: strongly causal spacetimes with continuous metrics

Theorem

$(\gamma_n)_n$ sequence of uniformly Lipschitz continuous, f.d. causal curves $\gamma_n \colon [a, b] \to X$; if d is proper and $(\gamma_n)_n$ accumulating at some point or $\gamma_n([a, b]) \subseteq K$, $K \subseteq X$ compact, then $\exists$ subsequence $(\gamma_{n_k})_k$ and a curve $\gamma \colon [a, b] \to X$ s.t. $\gamma_{n_k} \to \gamma$ uniformly, which is f.d. causal if non-constant

analogous results for *inextendible* causal curves

Limit curves

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space, $x \in X$

Definition

nhd. U of x is *causally closed* if $\leq$ is closed in $\bar{U} \times \bar{U}$, i.e., $p_n, q_n \in U$ with $p_n \leq q_n$ and $p_n \to p \in \bar{U}$, $q_n \to q \in \bar{U}$, then $p \leq q$; X is *locally causally closed* if every point has a causally closed nhd.

Example: strongly causal spacetimes with continuous metrics

Theorem

$(\gamma_n)_n$ sequence of uniformly Lipschitz continuous, f.d. causal curves $\gamma_n \colon [a, b] \to X$; if d is proper and $(\gamma_n)_n$ accumulating at some point or $\gamma_n([a, b]) \subseteq K$, $K \subseteq X$ compact, then $\exists$ subsequence $(\gamma_{n_k})_k$ and a curve $\gamma \colon [a, b] \to X$ s.t. $\gamma_{n_k} \to \gamma$ uniformly, which is f.d. causal if non-constant

analogous results for *inextendible* causal curves

Limit curves

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space, $x \in X$

Definition

nhd. U of x is *causally closed* if $\leq$ is closed in $\bar{U} \times \bar{U}$, i.e., $p_n, q_n \in U$ with $p_n \leq q_n$ and $p_n \to p \in \bar{U}$, $q_n \to q \in \bar{U}$, then $p \leq q$; X is *locally causally closed* if every point has a causally closed nhd.

Example: strongly causal spacetimes with continuous metrics

Theorem

$(\gamma_n)_n$ sequence of uniformly Lipschitz continuous, f.d. causal curves $\gamma_n \colon [a,b] \to X$; if d is proper and $(\gamma_n)_n$ accumulating at some point or $\gamma_n([a,b]) \subseteq K$, $K \subseteq X$ compact, then $\exists$ subsequence $(\gamma_{n_k})_k$ and a curve $\gamma \colon [a,b] \to X$ s.t. $\gamma_{n_k} \to \gamma$ uniformly, which is f.d. causal if non-constant

analogous results for *inextendible* causal curves

Limit curves

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space, $x \in X$

Definition

nhd. U of x is *causally closed* if $\leq$ is closed in $\bar{U} \times \bar{U}$, i.e., $p_n, q_n \in U$ with $p_n \leq q_n$ and $p_n \to p \in \bar{U}$, $q_n \to q \in \bar{U}$, then $p \leq q$; X is *locally causally closed* if every point has a causally closed nhd.

Example: strongly causal spacetimes with continuous metrics

Theorem

$(\gamma_n)_n$ sequence of uniformly Lipschitz continuous, f.d. causal curves $\gamma_n \colon [a, b] \to X$; if d is proper and $(\gamma_n)_n$ accumulating at some point or $\gamma_n([a, b]) \subseteq K$, $K \subseteq X$ compact, then $\exists$ subsequence $(\gamma_{n_k})_k$ and a curve $\gamma \colon [a, b] \to X$ s.t. $\gamma_{n_k} \to \gamma$ uniformly, which is f.d. causal if non-constant

analogous results for *inextendible* causal curves

Localizability (1/2)

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *localizable* if $\forall x \in X$ $\exists$ open nhd. (*localizing nhd.*) Ω_x of x with the following properties:

1. $\exists C > 0$ s.t. $L^d(\gamma) \leq C$ for all causal curves γ in Ω_x
2. $\exists \omega_x : \Omega_x \times \Omega_x \to [0, \infty)$ continuous s.t. $(\Omega_x, d|_{\Omega_x \times \Omega_x}, \ll|_{\Omega_x \times \Omega_x}, \leq|_{\Omega_x \times \Omega_x}, \omega_x)$ is a Lorentzian pre-length space with $I^{\pm}(y) \cap \Omega_x \neq \emptyset$ $\forall y \in \Omega_x$
3. $\forall p, q \in \Omega_x$ with $p < q$ $\exists$ Ω_x-maximal f.d. causal curve $\gamma_{p,q}$ from p to q with
$$L_\tau(\gamma_{p,q}) = \omega_x(p, q) \leq \tau(p, q)$$
4. if additionally $p, q \in \Omega_x$, $p \ll q$ implies $\gamma_{p,q}$ timelike and strictly longer than any f.d. causal curve from p to q with null segment, then X is *regularly localizable*

Localizability (1/2)

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *localizable* if $\forall x \in X\ \exists$ open nhd. (*localizing nhd.*) Ω_x of x with the following properties:

1. $\exists C > 0$ s.t. $L^d(\gamma) \leq C$ for all causal curves γ in Ω_x
2. $\exists \omega_x \colon \Omega_x \times \Omega_x \to [0, \infty)$ continuous s.t. $(\Omega_x, d|_{\Omega_x \times \Omega_x}, \ll|_{\Omega_x \times \Omega_x}, \leq|_{\Omega_x \times \Omega_x}, \omega_x)$ is a Lorentzian pre-length space with $I^\pm(y) \cap \Omega_x \neq \emptyset\ \forall y \in \Omega_x$
3. $\forall p, q \in \Omega_x$ with $p < q\ \exists\ \Omega_x$-maximal f.d. causal curve $\gamma_{p,q}$ from p to q with
$$L_\tau(\gamma_{p,q}) = \omega_x(p, q) \leq \tau(p, q)$$
4. if additionally $p, q \in \Omega_x$, $p \ll q$ implies $\gamma_{p,q}$ timelike and strictly longer than any f.d. causal curve from p to q with null segment, then X is *regularly localizable*

Localizability (1/2)

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *localizable* if $\forall x \in X$ $\exists$ open nhd. (*localizing nhd.*) Ω_x of x with the following properties:

1. $\exists C > 0$ s.t. $L^d(\gamma) \leq C$ for all causal curves γ in Ω_x
2. $\exists \omega_x \colon \Omega_x \times \Omega_x \to [0, \infty)$ continuous s.t. $(\Omega_x, d|_{\Omega_x \times \Omega_x}, \ll|_{\Omega_x \times \Omega_x}, \leq|_{\Omega_x \times \Omega_x}, \omega_x)$ is a Lorentzian pre-length space with $I^{\pm}(y) \cap \Omega_x \neq \emptyset$ $\forall y \in \Omega_x$
3. $\forall p, q \in \Omega_x$ with $p < q$ $\exists$ Ω_x-maximal f.d. causal curve $\gamma_{p,q}$ from p to q with
$$L_\tau(\gamma_{p,q}) = \omega_x(p, q) \leq \tau(p, q)$$
4. if additionally $p, q \in \Omega_x$, $p \ll q$ implies $\gamma_{p,q}$ timelike and strictly longer than any f.d. causal curve from p to q with null segment, then X is *regularly localizable*

Localizability (1/2)

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *localizable* if $\forall x \in X \ \exists$ open nhd. (*localizing nhd.*) Ω_x of x with the following properties:

1. $\exists C > 0$ s.t. $L^d(\gamma) \leq C$ for all causal curves γ in Ω_x
2. $\exists \omega_x \colon \Omega_x \times \Omega_x \to [0, \infty)$ continuous s.t. $(\Omega_x, d|_{\Omega_x \times \Omega_x}, \ll|_{\Omega_x \times \Omega_x}, \leq|_{\Omega_x \times \Omega_x}, \omega_x)$ is a Lorentzian pre-length space with $I^{\pm}(y) \cap \Omega_x \neq \emptyset \ \forall y \in \Omega_x$
3. $\forall p, q \in \Omega_x$ with $p < q \ \exists \ \Omega_x$-maximal f.d. causal curve $\gamma_{p,q}$ from p to q with

$$L_\tau(\gamma_{p,q}) = \omega_x(p, q) \leq \tau(p, q)$$

4. if additionally $p, q \in \Omega_x$, $p \ll q$ implies $\gamma_{p,q}$ timelike and strictly longer than any f.d. causal curve from p to q with null segment, then X is *regularly localizable*

Localizability (1/2)

Definition

Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is *localizable* if $\forall x \in X$ $\exists$ open nhd. (*localizing nhd.*) Ω_x of x with the following properties:

1. $\exists C > 0$ s.t. $L^d(\gamma) \leq C$ for all causal curves γ in Ω_x
2. $\exists \omega_x \colon \Omega_x \times \Omega_x \to [0, \infty)$ continuous s.t. $(\Omega_x, d|_{\Omega_x \times \Omega_x}, \ll|_{\Omega_x \times \Omega_x}, \leq|_{\Omega_x \times \Omega_x}, \omega_x)$ is a Lorentzian pre-length space with $I^{\pm}(y) \cap \Omega_x \neq \emptyset \ \forall y \in \Omega_x$
3. $\forall p, q \in \Omega_x$ with $p < q$ $\exists$ Ω_x-maximal f.d. causal curve $\gamma_{p,q}$ from p to q with
$$L_\tau(\gamma_{p,q}) = \omega_x(p, q) \leq \tau(p, q)$$
4. if additionally $p, q \in \Omega_x$, $p \ll q$ implies $\gamma_{p,q}$ timelike and strictly longer than any f.d. causal curve from p to q with null segment, then X is *regularly localizable*

Localizability (2/2)

Proposition

$(X, d, \ll, \leq, \tau)$ strongly causal, localizable Lorentzian pre-length space, then L_τ is upper semicontinuous, i.e., $(\gamma_n)_n$ sequence of f.d. causal curves on $[a, b]$ converging uniformly to a f.d. causal curve $\gamma\colon [a, b] \to X$, then

$$L_\tau(\gamma) \geq \limsup_n L_\tau(\gamma_n)$$

Theorem

$(X, d, \ll, \leq, \tau)$ regularly localizable Lorentzian pre-length space, then maximal causal curves have a causal character and (length increasing) push-up holds

Localizability (2/2)

Proposition

$(X, d, \ll, \leq, \tau)$ strongly causal, localizable Lorentzian pre-length space, then L_τ is upper semicontinuous, i.e., $(\gamma_n)_n$ sequence of f.d. causal curves on $[a,b]$ converging uniformly to a f.d. causal curve $\gamma\colon [a,b] \to X$, then

$$L_\tau(\gamma) \geq \limsup_n L_\tau(\gamma_n)$$

Theorem

$(X, d, \ll, \leq, \tau)$ regularly localizable Lorentzian pre-length space, then maximal causal curves have a causal character and (length increasing) push-up holds

Lorentzian length spaces

Definition

$(X, d, \ll, \leq, \tau)$ locally causally closed, causally path connected, localizable Lorentzian pre-length space; for $x, y \in X$ define

$$\mathcal{T}(x,y) := \sup\{L_\tau(\gamma) : \gamma \text{ f.d. causal from } x \text{ to } y\}\,,$$

if the set is not empty, otherwise $\mathcal{T}(x,y) := 0$
X is a *Lorentzian length space* if $\mathcal{T} = \tau$; if, in addition X is regularly localizing, then X is a *regular* Lorentzian length space

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a smooth and strongly causal spacetime (M, g) (since $L_\tau = L_g$) is a regular LLS

Theorem (M. Kunzinger, C.S. 2018)

(M, g) *continuous, strongly causal* and *causally plain* (no causal bubbles) spacetime, then the induced $(M, d^h, \ll, \leq, \tau)$ Lorentzian pre-length space is a *Lorentzian length space*

Lorentzian length spaces

Definition

$(X, d, \ll, \leq, \tau)$ locally causally closed, causally path connected, localizable Lorentzian pre-length space; for $x, y \in X$ define

$$\mathcal{T}(x,y) := \sup\{L_\tau(\gamma) : \gamma \text{ f.d. causal from } x \text{ to } y\}\,,$$

if the set is not empty, otherwise $\mathcal{T}(x,y) := 0$
X is a *Lorentzian length space* if $\mathcal{T} = \tau$; if, in addition X is regularly localizing, then X is a *regular* Lorentzian length space

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a smooth and strongly causal spacetime (M, g) (since $L_\tau = L_g$) is a regular LLS

Theorem (M. Kunzinger, C.S. 2018)

(M, g) *continuous, strongly causal* and *causally plain* (no causal bubbles) spacetime, then the induced $(M, d^h, \ll, \leq, \tau)$ Lorentzian pre-length space is a *Lorentzian length space*

Lorentzian length spaces

Definition

$(X, d, \ll, \leq, \tau)$ locally causally closed, causally path connected, localizable Lorentzian pre-length space; for $x, y \in X$ define

$$\mathcal{T}(x,y) := \sup\{L_\tau(\gamma) : \gamma \text{ f.d. causal from } x \text{ to } y\}\,,$$

if the set is not empty, otherwise $\mathcal{T}(x,y) := 0$
X is a *Lorentzian length space* if $\mathcal{T} = \tau$; if, in addition X is regularly localizing, then X is a *regular* Lorentzian length space

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a smooth and strongly causal spacetime (M, g) (since $L_\tau = L_g$) is a regular LLS

Theorem (M. Kunzinger, C.S. 2018)

(M, g) *continuous, strongly causal* and *causally plain* (no causal bubbles) spacetime, then the induced $(M, d^h, \ll, \leq, \tau)$ Lorentzian pre-length space is a *Lorentzian length space*

Lorentzian length spaces

Definition

$(X, d, \ll, \leq, \tau)$ locally causally closed, causally path connected, localizable Lorentzian pre-length space; for $x, y \in X$ define

$$\mathcal{T}(x,y) := \sup\{L_\tau(\gamma) : \gamma \text{ f.d. causal from } x \text{ to } y\}\,,$$

if the set is not empty, otherwise $\mathcal{T}(x,y) := 0$
X is a *Lorentzian length space* if $\mathcal{T} = \tau$; if, in addition X is regularly localizing, then X is a *regular* Lorentzian length space

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a smooth and strongly causal spacetime (M, g) (since $L_\tau = L_g$) is a regular LLS

Theorem (M. Kunzinger, C.S. 2018)

(M, g) *continuous, strongly causal* and *causally plain* (no causal bubbles) spacetime, then the induced $(M, d^h, \ll, \leq, \tau)$ Lorentzian pre-length space is a *Lorentzian length space*

Timelike triangles

Definition

timelike geodesic triangle in Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is triple $(x, y, z) \in X^3$ with $x \ll y \ll z$, $\tau(x, z) < \infty$ and s.t. sides are realized by f.d. causal curves

i.e., $\exists$ f.d. causal curves α, β, γ s.t. $L_\tau(\alpha) = \tau(x, y)$, $L_\tau(\beta) = \tau(y, z)$ and $L_\tau(\gamma) = \tau(x, z)$
$\rightsquigarrow \tau(x, y), \tau(y, z) < \infty$ and α, β, γ maximal

Timelike triangles

Definition

timelike geodesic triangle in Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is triple $(x, y, z) \in X^3$ with $x \ll y \ll z$, $\tau(x, z) < \infty$ and s.t. sides are realized by f.d. causal curves

i.e., $\exists$ f.d. causal curves α, β, γ s.t. $L_\tau(\alpha) = \tau(x, y)$, $L_\tau(\beta) = \tau(y, z)$ and $L_\tau(\gamma) = \tau(x, z)$

$\rightsquigarrow \tau(x, y), \tau(y, z) < \infty$ and α, β, γ maximal

Timelike triangles

Definition

timelike geodesic triangle in Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is triple $(x, y, z) \in X^3$ with $x \ll y \ll z$, $\tau(x, z) < \infty$ and s.t. sides are realized by f.d. causal curves

i.e., $\exists$ f.d. causal curves α, β, γ s.t. $L_\tau(\alpha) = \tau(x, y)$, $L_\tau(\beta) = \tau(y, z)$ and $L_\tau(\gamma) = \tau(x, z)$
$\rightsquigarrow \tau(x, y), \tau(y, z) < \infty$ and α, β, γ maximal

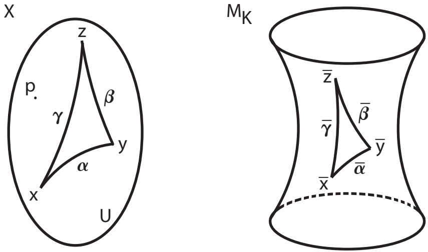
Timelike triangles

Definition

timelike geodesic triangle in Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is triple $(x, y, z) \in X^3$ with $x \ll y \ll z$, $\tau(x, z) < \infty$ and s.t. sides are realized by f.d. causal curves

i.e., $\exists$ f.d. causal curves α, β, γ s.t. $L_\tau(\alpha) = \tau(x, y)$, $L_\tau(\beta) = \tau(y, z)$ and $L_\tau(\gamma) = \tau(x, z)$

$\rightsquigarrow \tau(x, y), \tau(y, z) < \infty$ and α, β, γ maximal



Lorentzian model spaces of constant curvature

$K \in \mathbb{R}$

$$M_K = \begin{cases} \tilde{S}^2_1(r) & K = \frac{1}{r^2} \\ \mathbb{R}^2_1 & K = 0 \\ \tilde{H}^2_1(r) & K = -\frac{1}{r^2} \end{cases}$$

$\tilde{S}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudosphere*
$\mathbb{R}^2_1$ 2D *Minkowski space*
$\tilde{H}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudohyperbolic space*

Harris, Alexander and Bishop: for *small timelike triangles* there is a *unique* (up to isometry) *timelike triangle in M_K*

Lorentzian model spaces of constant curvature

$K \in \mathbb{R}$

$$M_K = \begin{cases} \tilde{S}^2_1(r) & K = \frac{1}{r^2} \\ \mathbb{R}^2_1 & K = 0 \\ \tilde{H}^2_1(r) & K = -\frac{1}{r^2} \end{cases}$$

$\tilde{S}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudosphere*
$\mathbb{R}^2_1$ 2D *Minkowski space*
$\tilde{H}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudohyperbolic space*

Harris, Alexander and Bishop: for *small timelike triangles* there is a *unique* (up to isometry) *timelike triangle in M_K*

Lorentzian model spaces of constant curvature

$K \in \mathbb{R}$

$$M_K = \begin{cases} \tilde{S}^2_1(r) & K = \frac{1}{r^2} \\ \mathbb{R}^2_1 & K = 0 \\ \tilde{H}^2_1(r) & K = -\frac{1}{r^2} \end{cases}$$

$\tilde{S}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudosphere*
$\mathbb{R}^2_1$ 2D *Minkowski space*
$\tilde{H}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudohyperbolic space*

Harris, Alexander and Bishop: for *small timelike triangles* there is a *unique* (up to isometry) *timelike triangle in M_K*

Lorentzian model spaces of constant curvature

$K \in \mathbb{R}$

$$M_K = \begin{cases} \tilde{S}^2_1(r) & K = \frac{1}{r^2} \\ \mathbb{R}^2_1 & K = 0 \\ \tilde{H}^2_1(r) & K = -\frac{1}{r^2} \end{cases}$$

$\tilde{S}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudosphere*
$\mathbb{R}^2_1$ 2D *Minkowski space*
$\tilde{H}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudohyperbolic space*

Harris, Alexander and Bishop: for *small timelike triangles* there is a *unique* (up to isometry) *timelike triangle in M_K*

Lorentzian model spaces of constant curvature

$K \in \mathbb{R}$

$$M_K = \begin{cases} \tilde{S}^2_1(r) & K = \frac{1}{r^2} \\ \mathbb{R}^2_1 & K = 0 \\ \tilde{H}^2_1(r) & K = -\frac{1}{r^2} \end{cases}$$

$\tilde{S}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudosphere*
$\mathbb{R}^2_1$ 2D *Minkowski space*
$\tilde{H}^2_1(r)$ *simply connected covering manifold* of 2D Lorentzian *pseudohyperbolic space*

Harris, Alexander and Bishop: for *small timelike triangles* there is a *unique* (up to isometry) *timelike triangle in* M_K

Timelike curvature bounds

Definition

Lorentzian pre-length space X has *timelike curvature bounded below (above)* by $K \in \mathbb{R}$ if all points in X have nhd. U s.t.:

1. $\tau|_{U \times U}$ *finite* and *continuous*
2. $x, y \in U$ with $x \ll y \Rightarrow \exists$ f.d. *maximal causal curve* in U from x to y
3. (x, y, z) *small timelike geodesic triangle* in U, $(\bar{x}, \bar{y}, \bar{z})$ *comparison triangle* of (x, y, z) in M_K, then for p, q points on the sides of (x, y, z) and $\bar{p}, \bar{q}$ *corresponding points* $(\bar{x}, \bar{y}, \bar{z})$:

$$\tau(p, q) \leq \bar{\tau}(\bar{p}, \bar{q}) \quad (\text{respectively } \tau(p, q) \geq \bar{\tau}(\bar{p}, \bar{q}))$$

Alexander, Bishop: *smooth Lorentzian manifold* with *sectional curvature bounds*

Timelike curvature bounds

Definition

Lorentzian pre-length space X has *timelike curvature bounded below (above)* by $K \in \mathbb{R}$ if all points in X have nhd. U s.t.:

1. $\tau|_{U\times U}$ *finite* and *continuous*
2. $x, y \in U$ with $x \ll y \Rightarrow \exists$ f.d. *maximal causal curve* in U from x to y
3. (x, y, z) *small timelike geodesic triangle* in U, $(\bar{x}, \bar{y}, \bar{z})$ *comparison triangle* of (x, y, z) in M_K, then for p, q points on the sides of (x, y, z) and $\bar{p}, \bar{q}$ *corresponding points* $(\bar{x}, \bar{y}, \bar{z})$:

$$\tau(p, q) \leq \bar{\tau}(\bar{p}, \bar{q}) \quad \text{(respectively } \tau(p, q) \geq \bar{\tau}(\bar{p}, \bar{q})\text{)}$$

Alexander, Bishop: *smooth Lorentzian manifold* with *sectional curvature bounds*

Timelike curvature bounds

Definition

Lorentzian pre-length space X has *timelike curvature bounded below (above)* by $K \in \mathbb{R}$ if all points in X have nhd. U s.t.:

1. $\tau|_{U\times U}$ *finite* and *continuous*
2. $x, y \in U$ with $x \ll y \Rightarrow \exists$ f.d. *maximal causal curve* in U from x to y
3. (x, y, z) *small timelike geodesic triangle* in U, $(\bar{x}, \bar{y}, \bar{z})$ *comparison triangle* of (x, y, z) in M_K, then for p, q points on the sides of (x, y, z) and $\bar{p}, \bar{q}$ *corresponding points* $(\bar{x}, \bar{y}, \bar{z})$:

$$\tau(p, q) \leq \bar{\tau}(\bar{p}, \bar{q}) \quad \text{(respectively } \tau(p, q) \geq \bar{\tau}(\bar{p}, \bar{q}))$$

Alexander, Bishop: *smooth Lorentzian manifold* with *sectional curvature bounds*

Timelike curvature bounds

Definition

Lorentzian pre-length space X has *timelike curvature bounded below (above)* by $K \in \mathbb{R}$ if all points in X have nhd. U s.t.:

1. $\tau|_{U\times U}$ *finite* and *continuous*
2. $x, y \in U$ with $x \ll y \Rightarrow \exists$ f.d. *maximal causal curve* in U from x to y
3. (x, y, z) *small timelike geodesic triangle* in U, $(\bar{x}, \bar{y}, \bar{z})$ *comparison triangle* of (x, y, z) in M_K, then for p, q points on the sides of (x, y, z) and $\bar{p}, \bar{q}$ *corresponding points* $(\bar{x}, \bar{y}, \bar{z})$:

$$\tau(p, q) \leq \bar{\tau}(\bar{p}, \bar{q}) \quad \text{(respectively } \tau(p, q) \geq \bar{\tau}(\bar{p}, \bar{q})\text{)}$$

Alexander, Bishop: *smooth Lorentzian manifold* with *sectional curvature bounds*

Branching of maximal curves

Alexandrov spaces with curvature bounded below: geodesics do not branch

Definition

X Lorentzian pre-length space, $\gamma\colon [a,b] \to X$ maximal curve; $x := \gamma(t)$, $t \in (a,b)$ is *branching point* of γ if $\exists$ maximal curves $\alpha, \beta\colon [a,c] \to X$ with $c > b$ and $\alpha|_{[a,t]} = \beta|_{[a,t]} = \gamma|_{[a,t]}$, $\alpha([t,c]) \cap \beta([t,c]) = \{x\}$

causal funnel: every maximal causal curve from $J^-(p)$ to $J^+(q)$ has q as branching point

Theorem (M. Kunzinger, C.S. 2018)

X *strongly causal* Lorentzian pre-length space with *timelike curvature bounded below* by some $K \in \mathbb{R}$ s.t. X regular or timelike locally uniquely geodesic, then *maximal timelike* curves do *not* have *timelike branching points*

Branching of maximal curves

Alexandrov spaces with curvature bounded below: geodesics do not branch

Definition

X Lorentzian pre-length space, $\gamma\colon [a,b] \to X$ maximal curve; $x := \gamma(t)$, $t \in (a,b)$ is *branching point* of γ if $\exists$ maximal curves $\alpha, \beta\colon [a,c] \to X$ with $c > b$ and $\alpha|_{[a,t]} = \beta|_{[a,t]} = \gamma|_{[a,t]}$, $\alpha([t,c]) \cap \beta([t,c]) = \{x\}$

causal funnel: every maximal causal curve from $J^-(p)$ to $J^+(q)$ has q as branching point

Theorem (M. Kunzinger, C.S. 2018)

X *strongly causal* Lorentzian pre-length space with *timelike curvature bounded below* by some $K \in \mathbb{R}$ s.t. X regular or timelike locally uniquely geodesic, then *maximal timelike* curves do *not* have *timelike branching points*

Branching of maximal curves

Alexandrov spaces with curvature bounded below: geodesics do not branch

Definition

X Lorentzian pre-length space, $\gamma\colon [a,b] \to X$ maximal curve; $x := \gamma(t)$, $t \in (a,b)$ is *branching point* of γ if $\exists$ maximal curves $\alpha, \beta\colon [a,c] \to X$ with $c > b$ and $\alpha|_{[a,t]} = \beta|_{[a,t]} = \gamma|_{[a,t]}$, $\alpha([t,c]) \cap \beta([t,c]) = \{x\}$

causal funnel: every maximal causal curve from $J^-(p)$ to $J^+(q)$ has q as branching point

Theorem (M. Kunzinger, C.S. 2018)

X *strongly causal* Lorentzian pre-length space with *timelike curvature bounded below* by some $K \in \mathbb{R}$ s.t. X regular or timelike locally uniquely geodesic, then *maximal timelike* curves do *not* have *timelike branching points*

Branching of maximal curves

Alexandrov spaces with curvature bounded below: geodesics do not branch

Definition

X Lorentzian pre-length space, $\gamma\colon [a,b] \to X$ maximal curve; $x := \gamma(t)$, $t \in (a,b)$ is *branching point* of γ if $\exists$ maximal curves $\alpha, \beta\colon [a,c] \to X$ with $c > b$ and $\alpha|_{[a,t]} = \beta|_{[a,t]} = \gamma|_{[a,t]}$, $\alpha([t,c]) \cap \beta([t,c]) = \{x\}$

causal funnel: every maximal causal curve from $J^-(p)$ to $J^+(q)$ has q as branching point

Theorem (M. Kunzinger, C.S. 2018)

X *strongly causal* Lorentzian pre-length space with *timelike curvature bounded below* by some $K \in \mathbb{R}$ s.t. X regular or timelike locally uniquely geodesic, then *maximal timelike* curves do *not* have *timelike branching points*

Curvature singularities

Definition

Lorentzian pre-length space X has *timelike (respectively causal) curvature unbounded below/above* if $\forall p \in X$ $\exists$ nhd. U s.t. τ finite and continuous on U and maximal timelike/causal curves exist in U but triangle comparison fails for every $K \in \mathbb{R} \rightsquigarrow X$ has *curvature singularity*

Curvature singularities

Definition

Lorentzian pre-length space X has *timelike (respectively causal) curvature unbounded below/above* if $\forall p \in X \; \exists$ nhd. U s.t. τ finite and continuous on U and maximal timelike/causal curves exist in U but triangle comparison fails for every $K \in \mathbb{R} \rightsquigarrow X$ has *curvature singularity*

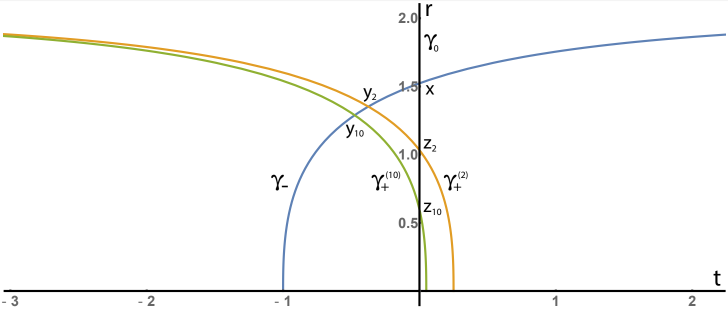


Figure: Schwarzschild has timelike curvature unbounded below

To be continued

To be continued tomorrow...

References

S. B. Alexander and R. L. Bishop.
Lorentz and semi-Riemannian spaces with Alexandrov curvature bounds. Comm. Anal. Geom., 16(2):251–282, 2008.

H. Busemann.
Timelike spaces. Dissertationes Math. Rozprawy Mat., vol. 53, 1967.

S. G. Harris.
A triangle comparison theorem for Lorentz manifolds. Indiana Univ. Math. J., 31(3):289–308, 1982.

E. H. Kronheimer and R. Penrose.
On the structure of causal spaces. Proc. Cambridge Philos. Soc., 63:481–501, 1967.

M. Kunzinger and S.
Lorentzian length spaces. Ann. Global Anal. Geom. 54, no. 3, 399–447, 2018.