Causality theory for closed cone structures and the distance formula

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Regensburg - Online, November 12, 2021

Some papers

This talk is based on the following works

- E. Minguzzi Causality theory for closed cone structures with applications, *Rev. Math. Phys.*, **31** (2019) 1930001
- D. Canarutto and E. Minguzzi. The distance formula in algebraic spacetime theories. *J. of Phys.: Conf. Ser.*, **1275** (2019) 012045.
- E. Minguzzi. Topological ordered spaces as a foundation for a quantum spacetime theory. *J. of Phys.: Conf. Ser.*, **442** (2013) 012034.
- E. Minguzzi. Convexity and quasi-uniformizability of closed preordered spaces. *Topol. Appl.*, **160** (2013) 965–978.
- E. Minguzzi. Normally preordered spaces and utilities. *Order*, **30** (2013) 137–150
- E. Minguzzi. K-causality coincides with stable causality. *Commun. Math. Phys.*, **290** (2009) 239–248.

and references therein.

Contents

Part I: Introduction to causality theory

In Einstein's general relativity spacetime
is a *differentiable manifold* M endowed with a metric

$$g = g_{\mu\nu}(x) \mathrm{d}x^\mu \mathrm{d}x^\nu$$

where $\{x^\mu\}$ are local coordinates. Here
g is *Lorentzian*, namely its signature is $(-, +, +, +)$.

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Dynamics is determined by the Einstein's equations

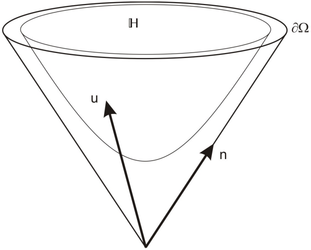
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

where $R_{\mu\nu}$ is the *Ricci tensor* and $T_{\mu\nu}$ is the *stress-energy tensor*.

At each $x \in M$ we have a *Lorentzian* bilinear form on T_xM as in special relativity

$$g_x = -(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2.$$

where y^μ are coordinates induced on T_xM.



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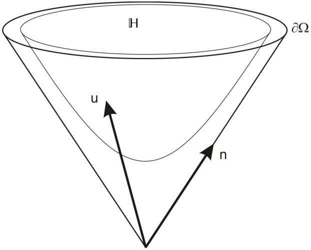
where y^μ are coordinates induced on T_xM. The cone of future *timelike vectors* is

$$\Omega_x = \{y \colon g_x(y,y) < 0,\ y^0 > 0\}$$

The set of future *lightlike vectors* is

$$\partial\Omega_x = \{y \colon g_x(y,y) = 0, y \neq 0\}$$



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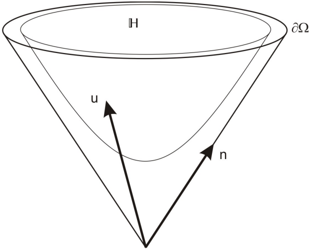
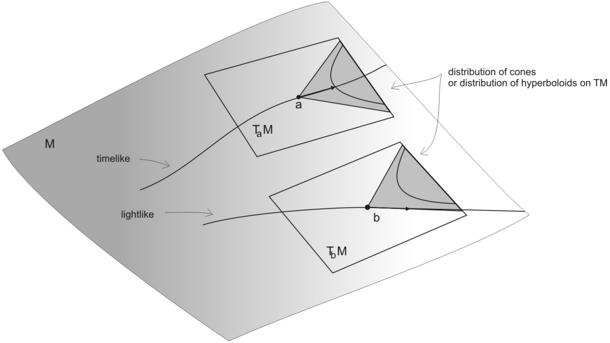


The set of future *causal vectors* is the union $C_x = \bar{\Omega}_x$. The *velocity space* of massive particles/observers is

$$\mathbb{H}_x = \{y \colon g_x(y,y) = -1\}.$$

This is the usual *hyperboloid*.
A spacetime differs from a Lorentzian manifold in that the selection of the future cone can be done so as to be continuous, i.e. it is *time orientable*.

Points on spacetime are called *events*. We have a distribution of causal cones $x \to C_x$, and a distribution of hyperboloids $x \to \mathbb{H}_x$.



A C^1 curve $x\colon t \mapsto x(t)$ is

- Timelike: if $g(\dot{x}, \dot{x}) < 0$, (massive particles),
- Lightlike: if $g(\dot{x}, \dot{x}) = 0$, (massless particles).

The *proper time* of a massive particle/observer is $\tau = \int_{x(t)} \sqrt{-g(\dot{x}, \dot{x})}\mathrm{d}t$.

Simple algebraic lemma

On a vector space of dimension $n \geq 3$ two Lorentzian bilinear forms η_1, η_2 *are proportional* if and only if they have the same causal cone C.

$$C_1 = C_2 \Leftrightarrow \exists a \in \mathbb{R} : \eta_1 = a^2 \eta_2.$$

Spacetime = (conic) causal order + spacetime measure

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Since the volume form induced by the metric is $\sqrt{-\det \eta_{\alpha\beta}}\, \mathrm{d}y^0 \wedge \cdots \wedge \mathrm{d}y^n$ it scales differently under conformal changes so

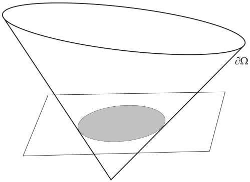
Corollary

Two spacetime metrics g_1 and g_2 *coincide* if and only if they induce the same distribution of causal cones $x \to C_x$ and the same volume form $\mathrm{d}\mu = \sqrt{-\det g}\, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$.

In other words the spacetime (M, g) of general relativity is nothing but a

- **spacetime measure + cone distribution**

where the cones are really *round*: they have ellipsoidal section according to the affine structure of the tangent space $T_x M$.



Causality theory for the most part focuses on the cone distribution, namely on conformal invariant properties.

Causality theory is the study of the global qualitative properties of the solutions $t \mapsto x(t)$, to the differential inclusion

$$\dot{x}(t) \in C_{x(t)},$$

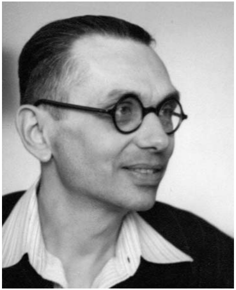
It focuses on the qualitative behavior of *causal curves* with a special attention to *causal geodesics*. It aims to answer questions such as:
According to general relativity

- Can closed timelike curves form? (grandfather paradox)
- Can closed causal curves form? (the novelist sends the text of his bestselling book to her younger self. The book exists but nobody has really written it.)
- Do continuous global increasing functions (time functions) exist? (they would prevent such pathologies)
- How much of the spacetime geometry is encoded in the family of such functions?

Answers to these questions help to clarify other problems such as

- Under what conditions geodesic singularities are unavoidable?

Einstein's equations impose very week constraints on causality.



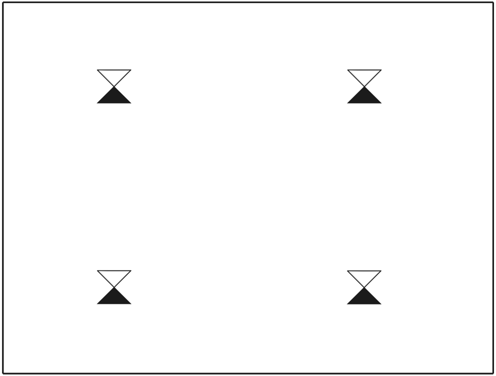
In 1949 Kurt Gödel found the following surprising solution: $M = \mathbb{R}^4$ and

$$g = \frac{1}{2\omega^2}[-(dt + e^x dz)^2 + dx^2 + dy^2 + \tfrac{1}{2}e^{2x}dz^2],$$

which is a solution for $\Lambda = -\omega^2$ and a stress-energy tensor $T_{\mu\nu}$ of dust type.
The problem is that through every point there passes a closed timelike curve.
An observer could go back in time.

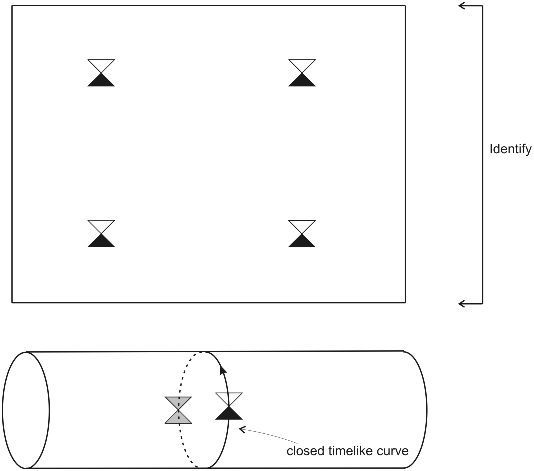
Minkowski spacetime

$M = \mathbb{R}^4$, $g = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.
In pictures we suppress 1 or 2 space dimensions.

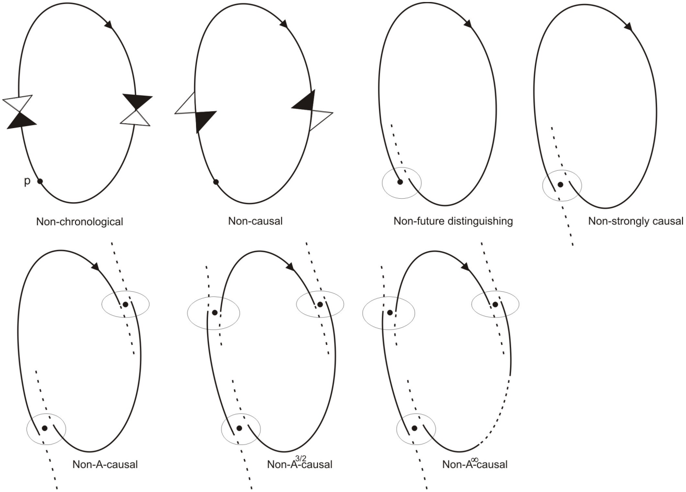


Non-chronological flat example

A spacetime of topology $S^1 \times \mathbb{R}^3$ which satisfies Einstein's equations in which there are closed timelike curves.



One can imagine spacetimes in which there are various types of *almost* causal curves.



This led to the introduction of some unpleasant hierarchy of stronger causality conditions.

Causal and chronological relation

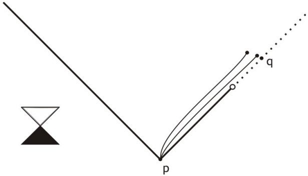
The causal and chronological relations are subsets of $M \times M$

$$I = \{(p,q) \in M \times M : \text{there is a timelike curve connecting } p \text{ to } q\},$$
$$J = \{(p,q) \in M \times M : \text{there is a causal curve connecting } p \text{ to } q, \text{ or } p = q\},$$

We shall also need the chronological and causal future of an event

$$I^+(p) = \{q \in M : \text{there is a timelike curve connecting } p \text{ to } q\},$$
$$J^+(p) = \{q \in M : \text{there is a causal curve connecting } p \text{ to } q, \text{ or } p = q\},$$

Timelike curve can be deformed remaining timelike, so the chronological relation I is open. Moreover $\bar{I} = \bar{J}$ but J is not closed in general. For instance, remove a point from Minkowski spacetime, here $(p,q) \in \partial J$.



This lack of closure is at the origin of the many pathologies illustrated previously.

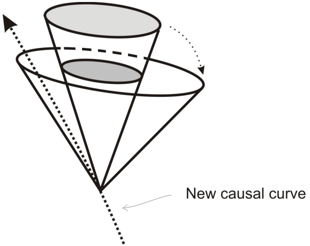
Stable causality

In 1968 Hawking introduced *stable causality*.

Definition

A spacetime is *stably causal* if the cones can be widened while preserving causality.

Notice that by enlarging the causal cones we introduce more causal curves so it is easier to violate causality.
Opening the cones:

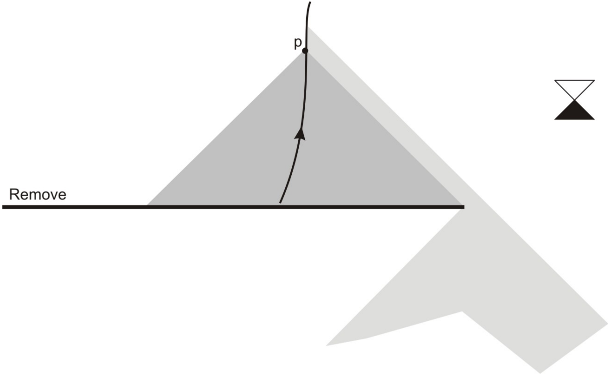


Some properties of stable causality

- it implies absence of almost closed causal curves,
- it implies existence of time.

A *time function* is a continuous function that increases over every causal curve (minus Lyapunov).

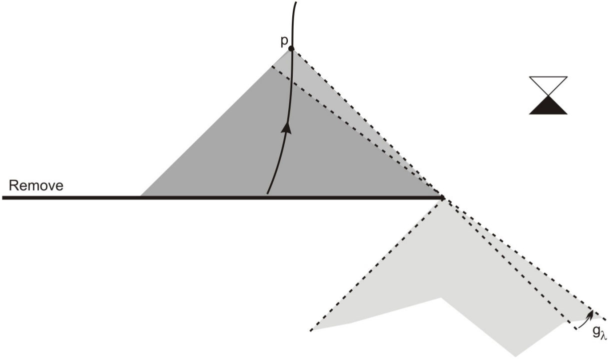
Volume functions were introduced by Geroch. They increase but might be discontinuous.



Taking an average in *stably causal* spacetimes solves the problem.
Let g_λ be a 1-parameter family of metrics with cones wider than g, μ a unit measure on M, then

$$t(p) = \int_1^2 \mu(I^-_{g_\lambda}(p))\mathrm{d}\lambda$$

is a time function. The converse holds: existence of time $\Leftrightarrow$ stable causality. The time function can be chosen smooth.



Seifert's (or stable) causality relation

In 1971 Seifert introduced the stable relation as follows

$$J_S = \bigcap_{g' > g} J_{g'}$$

where " $>$ " means "wider than".

Theorem

A spacetime is stably causal if and only if J_S is antisymmetric

$$(p, q) \in J_S \text{ and } (q, p) \in J_S \Rightarrow p = q.$$

The absence of closed causal curves (causality) is equivalent to the antisymmetry of J.

key property

The Seifert relation is *both* closed and transitive.

The K-relation

The K-relation is the *smallest* closed and transitive relation which contains J, thus $J \subset K \subset J_S$ (notion similar to the *Auslander prolongation* in dynamical system theory). One could claim that a natural causality condition is the antisymmetry of K, this is called K-causality.

Main problem

Is $J_S = K$? If not, is the antisymmetry of J_S (stable causality) equivalent to the antisymmetry of K (K-causality)?

The problem which goes back to Seifert (1971) was revived in 1996 by Sorkin, Woolgar and Low.

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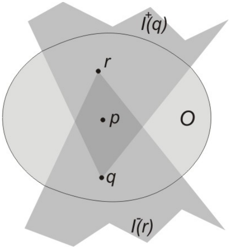
Theorem

Stable causality is equivalent to K-causality, and in this case $K = J_S$.

The difficulty was due to the fact that K is defined from abstract properties, so difficult to work with.
Recently (2018) Patrick Bernard and Stefan Suhr have given a different proof inspired by Conley theory for dynamical systems.

For many years it was believed that the most natural causality relation to work with is the chronological relation I. The reason was that I is at least open, so topologically nice, while J is not closed.
Moreover, under strong causality the topology of M can be deduced from I, namely the topology of M is the coarsest topology which makes I open.



My result proved that there is a natural closed relation in (stably causal) spacetime.

Part II: Time functions and causality. The topological approach.

Closed relation is much better than open relation

The point is to understand what is fundamental not only mathematically but physically. What survives at the deepest level where the C^1 manifold itself might not make sense?

A spacetime is given by three elements

Topology+(causal) order+measure

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The first two ingredients can be unified.

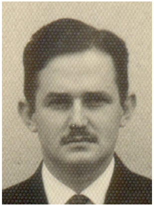
Nachbin's *closed ordered space*

A triple $(E, \leq, \mathscr{T})$ given by a topological space $(E, \mathscr{T})$ and a closed (partial) order over it.

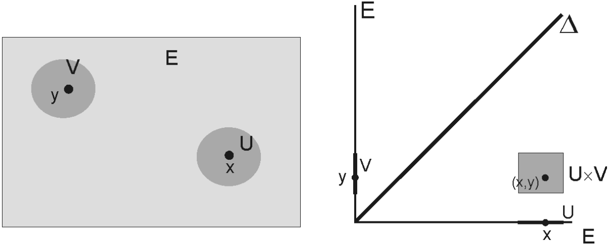
Nachbin (1965) developed the theory of closed ordered spaces in analogy with standard topology.
The standard topological theory is recovered for the ordered space $(E, \Delta, \mathscr{T})$ where

$$\Delta = \{(p, q) \in E \times E : p = q\}.$$

is the discrete order. Notice that on a *stably causal* spacetime $(M, J_S, \mathscr{T})$ is precisely this structure because J_S is *closed, transitive and a (partial) order.*



The Hausdorff condition is just the condition of closure for the relation Δ



So Hausdorff spaces are the simplest closed ordered spaces.

Some terminology

Order

- An order on a set E is a reflexive and transitive relation $R \subset E \times E$ on E which is antisymmetric "$x \le y$ and $y \le x \Rightarrow x = y$" (that is $R \cap R^{-1} = \Delta$ with $R^{-1} = \{(x, y) \colon (y, x) \in R\}$).

We shall write $x \le y$ for $(x, y) \in R$.

Increasing/decreasing hulls

Let $S \subset E$, the increasing and decreasing hulls are

$$i(S) = \{y \in E : x \le y \text{ for some } x \in S\},$$
$$d(S) = \{y \in E : y \le x \text{ for some } x \in S\}.$$

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Increasing/decreasing

Subsets S for which $i(S) = S$ are called *increasing*, while subsets for which $d(S) = S$ are called *decreasing*.

The complement of an increasing set is decreasing and the other way around.

Upper and lower topologies

- $\mathscr{U}$ is the topology generated by the open increasing sets,
- $\mathscr{L}$ is the topology generated by the open decreasing sets.

A closed ordered space is *convex* if $\mathscr{T} = \sup(\mathscr{U}, \mathscr{L})$.

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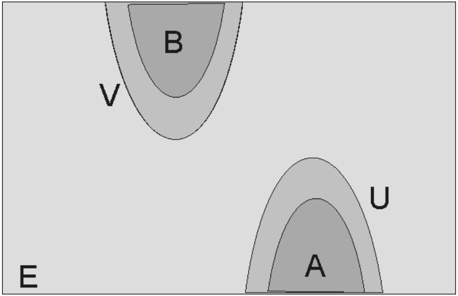
Isotone functions

A function between two preordered spaces $f : E \to E'$ is *isotone* if $x \le y \Rightarrow f(x) \le f(y)$.

Normally ordered space

E is a normally ordered space if it is a closed ordered space and for every pair of closed increasing set B, and closed decreasing set A such that $A \cap B = \emptyset$, there are an open increasing set V and an open decreasing set U such that, $A \subset U$, $B \subset V$, $U \cap V = \emptyset$.



Increasing and *decreasing* are the analog of *future* and *past* in relativity theory.

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Theorem (Nachbin's extension of Urysohn's lemma)

The topological ordered space $(E, \mathscr{T}, \leq)$ is normally ordered if and only if for any two disjoint closed subsets $A, B \subset E$, with A decreasing and B increasing, there exist on E a continuous isotone real-valued function f such that $f(x) = 0$ $(x \in A)$, $f(x) = 1$ $(x \in B)$, and $0 \leq f(x) \leq 1$ $(x \in E)$.

It follows

Order representability

$x \leq y \Leftrightarrow \forall f$ continuous and isotone, $f(x) \leq f(y)$

Completely regularly ordered space (Tychonoff-ordered space)

E is a *completely regularly ordered space* if

(i) The topology is the weak (initial) topology of the family of continuous isotone functions.

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Unification of topology and order

Quasi-Uniform Space $(E, \mathscr{U})$: A filter of the diagonal Δ, which respects composition of relations. Then $\bigcap \mathscr{U}$ is an order and we have also a topology obtained symmetrizing $\mathscr{U}$ and then proceeding in the usual way.

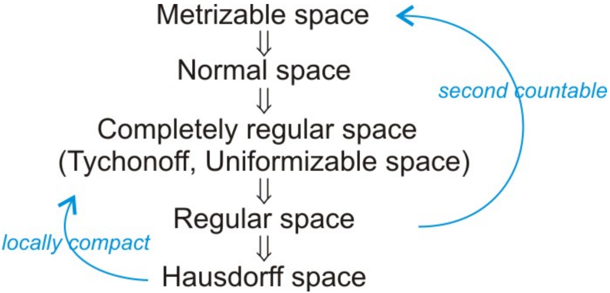
The quasi-uniformizable closed ordered spaces are precisely the completely regularly ordered space. They are also the closed ordered spaces that can be Nachbin compactified.

The standard separability ladder

Main objective

We want to show that $(M, J_S, \mathscr{T})$ is completely regularly ordered, namely that we can recover *both* topology and order from the time functions.

In standard (non-ordered) topology we can climb the ladder of separability conditions



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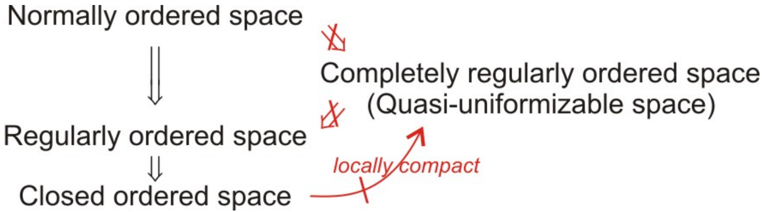
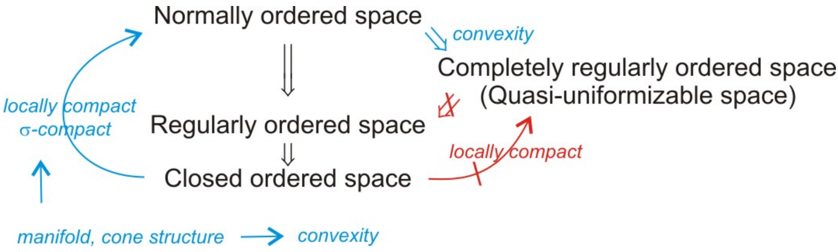
In ordered topology there are gaps



The property we want is isolated.

How to circumvent the gaps



So we proved

Theorem (M.2013)

$(M, J_S, \mathscr{T})$ is completely regularly ordered, namely we can recover *both* topology and order from the time functions.

Part III: Causality for cone distributions and low regularity

Purpose

Now we wish to answer the next questions:

Main problems

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The answer to (a) and (b) is that it is possible to weaken considerably the differentiability assumptions on the metric. How much? $C^{1,1}$?, Lipschitz? C^0? No, **upper semi-continuous**.

The answer to (c) is YES, with the non-round theory two important problems

(i) Proof of the Lorentzian version of Connes distance formula,

(ii) Complete characterization of Lorentzian manifolds embeddable in Minkowski.

What is (apparently) lost decreasing differentiability

[Differentiability assumptions] Results, meaningful objects

C^2 This is textbook causality theory where curvature makes sense,

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[Differentiability assumptions] Results, meaningful objects

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$C^{1,1}$ Generalized by Kunzinger et al. and myself (2014) with the proof that the exponential map is a local Lipeomorphism (Lipschitz with inverse Lipschitz), convex neighborhoods do exist and from here many results of causality theory follow. Notice that geodesics exist and are unique. The singularity theorems use the curvature so they require more work but it can be done (work by Graf, Grant, Kunzinger, Steinbauer, Stojkovic, Vickers).

What is (apparently) lost decreasing differentiability

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- usc Recently Bernard and Suhr (2017) have obtained characterizations of global hyperbolicity and stable causality using time functions and stability of global hyperbolicity.

Our results in this direction

We are able to generalize to the upper semi-continuous case much of causality theory, namely

- limit curve theorems,
- definition of all the levels of the causal ladder and its validity,
- validity of transverse ladder,
- define lightlike and causal geodesics,
- Avez-Seifert theorem (geodesic connectedness of globally hyperbolic spacetimes),
- Fermat's principle,
- usual results on domains of dependence, including the fact that horismos and Cauchy horizons are generated by lightlike geodesics,
- Penrose, Hawking, Hawking and Penrose's singularity theorems (in a causality sense),
- Characterizations of global hyperbolicity or stable causality through time functions,
- stability of global hyperbolicity.

In fact we do not use the roundness of the cone, so these results hold for Lorentz-Finsler theories.

Why care of low differentiability

(i) PDE studies of Einstein's equations show that it is useful to consider metrics of low differentiability (e.g. bounded L^2 curvature),

(ii) We don't want the singularity theorems to signal just a decrease in the regularity of the metric,

(iii) Physically speaking exploring the mathematical limits of our theory of gravity might tell us something on the very nature of gravity.

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Why care of even upper semi-continuity

(a) It turns out to be the natural assumption for the validity of most results. Assuming better differentiability properties might obscure part of the theory. For instance, since the chronological relation can hardly be defined in this case, it clarifies that such relation is almost irrelevant, what matters is the causal relation.

(b) Physics presents ourselves with this case: think of light propagation in a media with a discontinuous refractive index, say at the surface of two different continua (say air and glass). We have a discontinuous distribution of light cones (not to be confused with gravity cones). How do we choose the cone at the interface? We choose the smaller refractive index, or the larger speed of light, this way the distribution is upper semi-continuous. Why this choice? Because, the theory works in this case (we get Fermat's principle).

M is a connected, Hausdorff, second countable, paracompact manifold of dimension $n+1$.

Cone structure (M, C)

A *cone structure* is a multivalued map $x \mapsto C_x$, where $C_x \subset T_x M \backslash 0$ is a closed sharp convex non-empty cone.

Let $\mathbb{S}^n \subset TU$, $U \subset M$, be the unit sphere bundle induced by a local coordinate chart. Define $\hat{C}_q := C_q \cap \mathbb{S}^n$. Since $\mathbb{S}^n$ with its canonical distance is a metric space, we can define a notion of Hausdorff distance $\hat{d}_H$ for its closed subsets and a related topology.
The distribution of cones is continuous (locally Lipschitz) on U if the map $q \mapsto \hat{C}_q$ is continuous (resp. locally Lipschitz).

Definitions: cone structures

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Example

A time oriented Lorentzian manifold (M, g) has an associated canonical cone structure given by the distribution of causal cones

$$C_x = \{y \in T_x M \backslash \{0\} : g(y, y) \leq 0, \; y \text{ future directed}\}.$$

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Proposition

Let (M, g) be a time oriented Lorentzian manifold. If g is continuous (locally Lipschitz) then $x \mapsto C_x$ is continuous (resp. locally Lipschitz).

Closed cone structure

A *closed cone structure* (M, C) is a cone structure which is a closed subbundle of the slit tangent bundle $TM\backslash 0$.

Equivalently, a closed cone structure is an upper semi-continuous cone structure, namely for every $p \in M$ and for every open neighborhood $N(\hat{C}_p)$ of $\hat{C}_p$ we can find a neighborhood $N(p)$ of p such that $\forall q \in N(p), \hat{C}_q \subset N(\hat{C}_p)$.

Definitions: proper cone structures

Proper cone

A *proper cone* is a closed sharp convex cone with non-empty interior.

Physically *sharpness* means: the speed of light is finite in every direction.

Proper cone structure

A *proper cone structure* is a closed cone structure in which additionally the cone bundle is proper, in the sense that $(\mathrm{Int}_{TM}\, C)_x \neq \emptyset$ for every x.

Here the *interior* is with respect to the bundle topology. Equivalently, it is proper if it contains a C^0 distribution of proper cones.

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Given two cones C_p and C'_p we write $C_p \le C'_p$ if $C_p \subset C'_p$, and $C_p < C'_p$ if $C_p \subset (\mathrm{Int}C')_p$. They correspond to opening partially or strictly the cones.

Definitions: causal and chronological relations

Causality theory concerns the study of the global qualitative properties of absolutely continuous solutions to the differential inclusion

$$\dot{x}(t) \in C_{x(t)}\,,$$

where $x \colon I \to M$ is called a parametrized *continuous causal curve*.
For every subset U of M we define the causal relation

$$J(U) = \{(p, q) \in U \times U \colon p = q \text{ or there is a continuous causal curve contained in } U \text{ from } p \text{ to } q\}.$$

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For a proper cone structure a timelike curve is a piecewise C^1 solution to

$$\dot{x}(t) \in (\mathrm{Int}C)_{x(t)}\,,$$

and the chronological relation is

$$I(U) = \{(p,q) \in U \times U \colon \text{ there is a timelike curve contained in } U \text{ from } p \text{ to } q\}.$$

Some properties

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It is interesting to explore the properties of the relation $\mathring{J} := \mathrm{Int} J$ which will be used to define the notion of geodesic.

Proposition

The relation $\mathring{J}$ is open, transitive and contained in J. Moreover, in a proper cone structure $I \subset \mathring{J}$, $\overline{\mathring{J}} = \bar{J}$ and $\partial \mathring{J} = \partial J$.

Local causality properties do not differ

The closed cone structures have local non-imprisoning properties

Proposition

Let (M, C) be a closed cone structure. For every $x \in M$ we can find a coordinate open neighborhood $U \ni x$, and a flat Minkowski metric g on U such that at every $y \in U$, $C_y \subset (\mathrm{Int} C^g)_y$. Furthermore, for every Riemannian metric h there is a constant $\delta_h(U)$ such that all continuous causal curves in $\bar{U}$ have h-arc length smaller than δ.

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and local globally hyperbolic neighborhoods exist.
A consequence of the Hopf-Rinow theorem is

Corollary

Let (M, C) be a closed cone structure and let h be a complete Riemannian metric. A continuous causal curve $x\colon [0, a) \to M$ is future inextendible iff its h-arc length is infinite.

Existence results from differential inclusion theory

Theorem

(Zaremba, Marchaud) Let (M, C) *be a closed cone structure. Every point* $p \in M$ *is the starting point of an inextendible continuous causal curve. Every continuous causal curve can be made inextendible through extension.*

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Under stronger regularity conditions it can be improved as follows

Theorem

Let (M, C) be a locally Lipschitz closed cone structure. For every $x_0 \in M$ and $y_0 \in C_{x_0}$, there is a C^1 causal curve passing through x_0 with velocity y_0. If the cone structure is proper and y_0 is timelike the curve can be found timelike.

Identity $\bar{I} = \bar{J}$ in the locally Lipschitz case

It has been observed by Chrusciel and Grant that in C^0 Lorentzian geometry there can be *causal bubbles* $J^+(p)\backslash\overline{I^+(p)} \neq \emptyset$, and that they are absent in the locally Lipschitz theory.

The same is true for general cone structures, due to the so called *relaxation theory*.

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We recall a key, somehow little known result by Filippov

Theorem

Let U be an open subset of $\mathbb{R}^n$, and let $x \mapsto \hat{C}_x \subset \mathbb{R}^n$ be a Lipschitz multivalued map defined on U with non-empty compact convex values. Let $\sigma\colon [0,a] \to U$, be a solution of $\dot{x} \in \hat{C}_{x(t)}$ with initial condition $\sigma(0) = p \in U$. For any $\epsilon > 0$ there exists a C^1 solution $\gamma\colon [0,a] \to U$ to $\dot{x} \in \hat{C}_{x(t)}$ with initial condition $\gamma(0) = p$, such that $\|\gamma - \sigma\| \leq \epsilon$.

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Starting from this result the idea is to push in the timelike direction the curve γ so as to get a C^1 timelike approximation to σ. Since every continuous causal curve is approximated by a timelike curve we have

Theorem

Let (M, C) be a locally Lipschitz proper cone structure and let h be a Riemannian metric. Every point admits an open neighborhood U with the following property. Every h-arc length parametrized continuous causal curve in U with starting point $p \in U$ can be uniformly approximated by a C^1 timelike solution with the same starting point, thus $\overline{I^+(p, U)} = J^+(p, U)$.

We arrive at a classical result of causality theory.

Theorem

Let (M, C) be a locally Lipschitz proper cone structure. Let γ be a continuous causal curve obtained joining a continuous causal curve η and a timelike curve σ (or with order exchanged). Then γ can be deformed in an arbitrarily small neighborhood $O \supset \gamma$ to give a timelike curve $\bar\gamma$ connecting the same endpoints of γ. In particular, $J \circ I \cup I \circ J \subset I$, $\bar J = \bar I$, $\partial J = \partial I$, $I = \mathring{J}$. For every subset S, $\overline{J^+(S)} = \overline{I^+(S)}$, $\partial J^+(S) = \partial I^+(S)$, $I^+(S) = \mathrm{Int}(J^+(S))$, and time dually.

Closed cone structures preserve the validity of limit curve theorems.

Theorem

Let (M, C) and (M, C_k), $k \geq 1$, be closed cone structures, $C_{k+1} \leq C_k$, $C = \cap_k C_k$, and let h be a Riemannian metric on M. If the continuous C_k-causal curves $x_k \colon I_k \to M$ parametrized with respect to h-length converge h-uniformly on compact subsets to $x \colon I \to M$ then x is a continuous C-causal curve.

Lemma (Limit curve lemma)

Let (M, C) and (M, C_n), $n \geq 1$, be closed cone structures, $C_{n+1} \leq C_n$, $C = \cap_n C_n$. Let $x_n : (-\infty, +\infty) \to M$, be a sequence of inextendible continuous causal curves parametrized with respect to h-arc length, and suppose that $p \in M$ is an accumulation point of the sequence $x_n(0)$. There is a inextendible continuous causal curve $x \colon (-\infty, +\infty) \to M$, such that $x(0) = p$ and a subsequence x_k which converges h-uniformly on compact subsets to x.

How to improve from “locally Lipschitz” to “upper semi-continuous”.

The main idea is to prove some result for locally Lipschitz proper cone structure $C_k > C$ and then get the result applying the limit curve theorem for $C_k \to C$. This is possible because

Theorem

Let (M, C) be a closed cone structure. Then the family of locally Lipschitz proper cone structures C' such that $C < C'$ is non-empty. Moreover, for every locally Lipschitz proper structure $\tilde{C} > C$ we can find a countable subfamily of locally Lipschitz proper cone structures $\{C_k\}$ such that $C < C_{k+1} < C_k < \tilde{C}$, $C = \cap_k C_k$.

Definition

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An example of result proved with the said strategy. Here the horisoms is $E^+(p,U) = J^+(p,U)\backslash \mathrm{Int} J^+(p,U)$.

Theorem

Let (M,C) be a closed cone structure. Every point in M has an arbitrarily small coordinate neighborhood U with the following property. The relation $J(U)$ is closed and for every $p \in U$ and $q \in E^+(p,U)\backslash\{p\}$ there is a future lightlike geodesic joining p and q entirely contained in $E^+(p,U)$ (and time dually). Moreover, if (M,C) is locally Lipschitz every continuous causal curves connecting p to q is a lightlike geodesic contained in $E^+(p,U)$.

Lightlike geodesics might branch, be non unique or might not extend as inextendible lightlike geodesics. However, they have lightlike tangents wherever they are differentiable.

Theorem

Let (M, C) be a closed cone structure. The following properties are equivalent:

(i) *Stable causality,*

(ii) *Existence of a smooth temporal function,*

(iii) *Existence of a time function,*

(iv) *K-causality.*

Moreover, in this case $J_S = K = T_1 = T_2$ where

$$T_1 = \{(p, q) \colon t(p) \le t(q) \text{ for every time function } t\},$$
$$T_2 = \{(p, q) \colon t(p) \le t(q) \text{ for every smooth temporal function } t\}.$$

Equivalence between (i) and (ii) was also obtained by Fathi and Siconolfi in the C^0 case and by Bernard and Suhr in the upper semi-continuous case. Our proof is different, and shall explain some key steps later. Equivalence between (i),(iii) and (iv) is new.

Global hyperbolicity: definitions

Definition

A *causal diamond* is a set of the form $J^+(p) \cap J^-(q)$ for $p, q \in M$. The convex hull of a set S is $J^+(S) \cap J^-(S)$.

The first definition is imported from mathematical relativity, while the second definition is new (joint work with R. Hounnonkpe).

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Definition

A closed cone structure (M, C) is *globally hyperbolic* if the following equivalent conditions hold

(α) Non-imprisonment and the causal convex hull operation preserves boundedness.

(β) Causality and the causal convex hull operation preserves compactness.

Let h be a Riemannian metric. A function t is h-steep if for every $v \in C$, $\mathrm{d}t(v) \geq \|v\|_h$.

Theorem

Let (M, C) be a closed cone structure and let h be a complete Riemannian metric. Then the next conditions are equivalent:

(i) *global hyperbolicity,*

(ii) *existence of a Cauchy time function,*

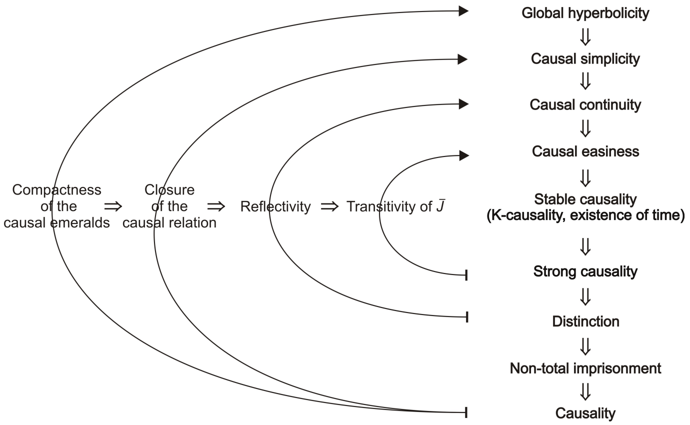
(iii) *existence of a smooth h-steep Cauchy temporal function,*

(iv) *existence of a (stable) Cauchy hypersurface.*

Finally, under global hyperbolicity M is smoothly diffeomorphic to a product $\mathbb{R} \times S$ where the projection to $\mathbb{R}$ is a smooth h-steep Cauchy temporal function (the fibers of the smooth projection to S are not necessarily causal), and every stable Cauchy hypersurface is smoothly diffeomorphic to S.
Additionally, for a proper cone structure all Cauchy hypersurfaces are diffeomorphic to S and the fibers of the smooth projection to S are smooth timelike curves.

The equivalence between (i) and (iii) and the splitting was previously obtained by Bernard and Suhr with a different proof.

Causal ladder and transverse ladder



The causal ladder and the transverse ladder for closed cone structures. The arrows crossing a property use it in the implication.

Length of vectors (metricity)? It is causality on $M \times \mathbb{R}$.

Let (M, C) be a closed cone structure, and let $\mathscr{F}: C \to [0, +\infty)$ be a concave positive homogeneous function. In general relativity it would be

$$\mathscr{F}(v) = \sqrt{-g(v, v)}.$$

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Let us introduce the cone structure on $M^\times = M \times \mathbb{R}$ defined at $P = (p, r)$ by

$$C_P^\times = \{(y, z) \colon y \in C_p, \ |z| \le \mathscr{F}(y)\}.$$

It is indeed easy to check that this is a non-empty convex sharp cone. We say that the cone structure $(M^\times, C^\times)$ is a *Lorentz-Finsler space* $(M, \mathscr{F})$.

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Definition

$(M, \mathscr{F})$ is a closed (proper) Lorentz-Finsler space iff $(M^\times, C^\times)$ is a closed (resp. proper) cone structure. We say that $(M, \mathscr{F})$ is locally Lipschitz (or C^0) if $C^\times$ is locally Lipschitz (resp. C^0).

Causal geodesics are by definition just projections of $C^\times$-lightlike geodesics on $(M^\times, C^\times)$.

The next result proves that our approach to the regularity of Lorentz-Finsler spaces is compatible with the natural definitions coming from Lorentzian geometry.

Theorem

For a time oriented Lorentzian manifold (M, g) the metric g is continuous (locally Lipschitz) iff $C^{\times}$ is continuous (resp. locally Lipschitz).

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An interesting large class of closed Lorentz-Finsler spaces is selected by the next theorem (that explains the terminology *Lorentz-Finsler*)

Theorem

Let $C \subset TM\backslash 0$ be a proper cone structure and let $\mathscr{F}: C \to [0, +\infty)$ be a positive homogeneous C^0 function, such that $\mathscr{F}^{-1}(0) = \partial C$. Suppose that $\mathscr{L} = -\mathscr{F}^2/2$ is $C^1(C) \cap C^2(\mathrm{Int}C)$, that it has Lorentzian vertical Hessian $\mathrm{d}^2_y \mathscr{L}$, and that $\mathrm{d}_y \mathscr{L} \neq 0$ on ∂C. Then $\mathscr{F}$ is concave, and $(M, \mathscr{F})$ is a locally Lipschitz proper Lorentz-Finsler space (hence both C and $C^\times$ are locally Lipschitz).

Theorem

(Upper semi-continuity of the length functional)
Let $(M, \mathscr{F})$ *and* $(M, \mathscr{F}_n)$ *be closed Lorentz-Finsler spaces. Let* $x_n \colon [a_n, b_n] \to M$, *be continuous* C_n*-causal curves, parametrized with respect to h-arc length, which converge uniformly on compact subsets to* $x \colon [a, b] \to M$, $a_n \to a$, $b_n \to b$, *where for every* n, $C_{n+1} \le C_n$, $\mathscr{F}_{n+1} \le \mathscr{F}_n|_{C_{n+1}}$, *and* $C = \cap_n C_n$, $\lim_{n\to\infty} \mathscr{F}_n = \mathscr{F}$. *Then* x *is a continuous* C*-causal curve and*

$$\limsup_n \ell_n(x_n) \le \ell(x).$$

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$$\limsup_n \ell_n(x_n) \le \ell(x).$$

Theorem

(Generalization of the Avez-Seifert theorem)
Let $(M, \mathscr{F})$ *be a globally hyperbolic closed Lorentz-Finsler space, then* ℓ *is maximized, namely for every* $(p, q) \in J$ *we can find a continuous causal curve* $x \colon [0, 1] \to M$, $p = x(0)$, $q = x(1)$*, such that* $\ell(x) = d(p, q)$*.*

Notice that it holds under upper semi-continuity of both C and $\mathscr{F}$.

Example of singularity theorem: Penrose's theorem

Definition

A future trapped set is a non-empty set S such that $E^+(S)$ is compact.

The new key result, which allows us to improve the differentiability assumption on the cone structure from 'locally Lipschitz and proper' to 'upper semi-continuous' is

Theorem (Stability of compact trapped sets)

Let (M, C) be a stably causal closed cone structure. Let S be a compact set such that $E^+(S)$ is compact. Then there is a locally Lipschitz proper cone structure $\tilde{C} > C$ such that for every locally Lipschitz proper cone structure $C < \hat{C} < \tilde{C}$, $\hat{E}^+(S)$ is compact.

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Theorem (Improved Penrose's singularity theorem)

Let (M, C) be a globally hyperbolic closed cone structure admitting a non-compact stable Cauchy hypersurface. Then there are no compact future trapped sets.

Part IV: The distance formula for Lorentz-Finsler structures

We have already shown that the time functions represent topology and order. Can they be used to represent distance?
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Case study: Gelfand's approach to phase space in (quantum) mechanics

The classical phase space is the spectrum of the commutative algebra of observables $\mathscr{F}$, where an element of $\mathscr{F}$ is a continuous function $f(q,p)$ on phase space. Phase space is reconstructed from $\mathscr{F}$ identifying each point (p,q) with a homomorphism through the evaluation map

$$h_{(p,q)} : \mathscr{F} \to \mathbb{R}(\text{or } \mathbb{C}), \qquad f \mapsto f(p,q)$$

So phase space is recovered from the homomorphisms.

Can a similar approach be followed for spacetime? Is spacetime a sort of spectrum for some functional space?

In order to follow this program it is necessary to prove that the next ingredients are recovered

- topology,
- causal order,
- metric

The last item may be too ambitious (for instance I expect it to be difficult to characterize functional spaces which lead to pseudo-Riemannian geometry instead of pseudo-Finsler geometry). We replace it with

- (Lorentzian) distance

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- (Lorentzian) distance

This is basically Connes' program for the unification of fundamental forces applied to the spacetime manifold.

The Riemannian version

Let (M, g) be a complete Riemannian manifold. The distance is defined by

$$d(p, q) = \inf\{L(\gamma) : \gamma \text{ path from } p \text{ to } q\}$$

but it can also be written

Riemannian distance formula

$$d(p, q) = \sup\{|f(q) - f(p)| : f \in C^1(M) \text{ such that } |\nabla f| \leq 1\}.$$

This formula has an algebraic formulation in the context of Connes spectral triples (points are replaced by states, functions by operators, $|\nabla f| \leq 1$ by the condition $\|[D, f]\| \leq 1$ where D is the Dirac operator).

Observation

Connes did not investigate how to recover spacetime with its Lorentzian signature. He worked in Riemannian geometry instead, and in subsequent work focused on the interior space (Standard Model).

$\mathscr{F}$-Steep temporal function

Parfionov and Zapatrin (2000) introduced the concept of $\mathscr{F}$-steep temporal function.

Definition ($\mathscr{F}$-Steep temporal function)

On the closed Lorentz-Finsler space $(M, \mathscr{F})$, a function $t\colon M \to \mathbb{R}$ is $\mathscr{F}$-steep temporal if it is C^1 and such that for every $v \in C$, $\mathrm{d}t(v) \geq \mathscr{F}(v)$.

Steep temporal functions are expected to encode both the causal and the metrical information on the Lorentz-Finsler space.

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Steep temporal functions are expected to encode both the causal and the metrical information on the Lorentz-Finsler space.
They asked to generalize Connes' (Riemannian) distance formula

$$d(p,q) = \sup_{f \in C^1(M)} \{|f(p) - f(q)| : |f'| \leq 1\}$$

to the physical Lorentzian case. The generalization is non-trivial since spacetime has order.

They proposed to prove Eq. (1) below, the Lorentzian generalizations being fundamental for Connes' noncommutative program on the unification of forces (see Connes 1985 book). There were partial answers from Moretti (2003), Franco (2010), Rennie and Whale (2016), who however could not prove that the representing function could be taken C^1 as proposed by Parfionov and Zapatrin.

Theorem

Let $(M, \mathscr{F})$ be a globally hyperbolic closed Lorentz-Finsler space and let $\mathscr{S}$ be the family of smooth Cauchy $\mathscr{F}$-steep temporal functions. The family $\mathscr{S}$ is non-empty and it represents

(a) *the causal order J, namely $(p,q) \in J \Leftrightarrow f(p) \le f(q)$, $\forall f \in \mathscr{S}$;*

(b) *the manifold topology, namely for every open set $O \ni p$ we can find $f, h \in \mathscr{S}$ in such a way that $p \in \{q \colon f(q) > 0\} \cap \{q \colon h(q) < 0\} \subset O$;*

(c) *the distance, in the sense that the distance formula holds true: for every $p, q \in M$, with $a^+ = \max\{a, 0\}$,*

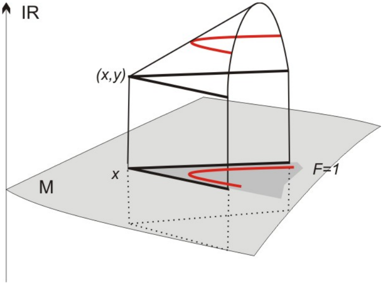
$$d(p,q) = \inf\big\{[f(q) - f(p)]^+ : \ f \in \mathscr{S}\big\}. \tag{1}$$

We have also versions for causally continuous and stably causal spacetimes.

Lorentz-Finsler translated into causal structure

(M, F) is equivalent to a cone structure in $M^\times = M \times \mathbb{R}$

$$C^{\downarrow}_{(x,y)} = \{(v, z) \in T_x M \times \mathbb{R} : z \leq F(v)\}.$$



A *closed Lorentz-Finsler space* is one for which this cone structure is closed (which is equivalent to $x \mapsto C_x$ and $x \mapsto F_x$ upper semi-continuous).

The stable distance

In stably causal spacetimes let $D\colon M \times M \to [0, +\infty]$ be a function called *stable distance*: for $p, q \in M$

$$D(p,q) = \inf{}_{F' > F} d'(p,q),$$

It is upper semi-continuous and satisfies the reverse triangle inequality: $(p,q) \in J_S$ and $(q,r) \in J_S$ implies

$$D(p,q) + D(q,r) \leq D(p,r).$$

We have $D = d$ under global hyperbolicity.

Definition

A closed Lorentz-Finsler space (M, F) is *stable* if it is stably causal and $D < +\infty$.

$D < +\infty$ means that d remains finite under small perturbations of the indicatrix.

Theorem

Let (M, F) be a closed Lorentz-Finsler space and let $\mathscr{S}$ be the family of smooth F-steep time functions. The Lorentz-Finsler space (M, F) is stable if and only if $\mathscr{S}$ is non-empty. In this case $\mathscr{S}$ represents

(a) *the order J_S, namely $(p, q) \in J_S \Leftrightarrow f(p) \le f(q)$, $\forall f \in \mathscr{S}$;*

(b) *the manifold topology, namely for every open set $O \ni p$ we can find $f, h \in \mathscr{S}$ in such a way that $p \in \{q \colon f(q) > 0\} \cap \{q \colon h(q) < 0\} \subset O$;*

(c) *the stable distance, in the sense that the distance formula holds true: for every $p, q \in M$*

$$D(p, q) = \inf\big\{[f(q) - f(p)]^+ \colon\ f \in \mathscr{S}\big\}.$$

We can now mathematically justify a derivation by Eckstein and Franco (2012) thus arriving at (see also my joint work with Canarutto)

Theorem

If (M, g) is a n-dimensional spin Lorentzian manifold which is stably causal such that the Lorentzian distance d is continuous and finite, and if we define:

- The algebra $A = C^1(M, R)$ with pointwise multiplication,
- The Hilbert space $H = L^2(M, S)$ of square integrable sections of the spinor bundle S over M (using a positive definite inner product on the spinor bundle),
- The Dirac operator $D = -ie^\mu_a \gamma^a \nabla^S_\mu$ associated with the spin connection ∇^S,
- The fundamental symmetry $J = i\gamma^0$ where γ^0 is the first flat gamma matrix,
- The chirality operator $\chi = \pm i^{\frac{n}{2}+1}\gamma^0 \cdots \gamma^{n-1}$,

then for all $p, q \in M$, if n is even:

$$d(p,q) = \inf_{f\in A}\big\{[f(q) - f(p)]^+ : \forall \varphi \in H, \langle \varphi, J([D,f] + i\chi)\varphi\rangle \leq 0\big\}.$$

The existence of a steep time function allows one to import Nash embedding theorem to the Lorentzian framework

Theorem (Müller and Sanchez 2011)

Let (M, g) be a C^3 Lorentzian manifold. The next assertion are equivalent

(i) (M, g) admits and isometric embedding in Minkowski spacetime.

(ii) (M, g) is admits a steep time function.

The problem is moved to that of proving the existence of steep time function. So by our existence theorem we get

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(ii) (M, g) is admits a steep time function.

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Theorem

A C^3 Lorentzian *spacetime is stable iff it is isometrically embeddable in Minkowski spacetime.*

Conclusions

- We showed that the theory of topological ordered spaces provides a natural framework for studying rough spacetime geometry while giving meaning to causality. This approach suggests that closed relations are more useful, at the fundamental level, than open relations.
- We developed causality theory for non-round cone structure showing that most results are preserved under upper semi-continuity of the cone distribution.
- We showed the usefulness of non-round cone structures for the proof of the existence of steep temporal functions and hence for the proof of the Lorentzian generalization of Connes' distance formula.

Thank you for the attention.