let: $\mathcal{F}$ (possibly non-compact) smooth manifold

$\mathcal{L} \in C^0(\mathcal{F}\times\mathcal{F}, \mathbb{R}^+_0)$

$$S = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \;\; \mathcal{L}(x,y)$$

volume constraint $\quad \rho(\mathcal{F}) = \text{const}$

weak EL equs*: $\quad \nabla_{\underline{u}}\, \ell|_M = 0 \quad \forall \underline{u} \in \mathfrak{J}^{\text{test}}$

$$\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) - s$$

<u>Remarks</u>: causal action principle

$\mathcal{L} \longrightarrow \mathcal{L}_\kappa \quad , \quad \mathcal{F} \longrightarrow \mathcal{F}^{\text{reg}}_c$

restrict attention to the finite-dimensional setting

<u>1st step</u>: assume that $\mathcal{L}$ is smooth

$$\mathfrak{J}^{\text{test}} = C^\infty(M,\mathbb{R}) \oplus \Gamma(M, T\mathcal{F})$$
$$\underline{u} = (\;a\;,\;u\;)$$

where $a \in C^\infty(M,\mathbb{R})$ is defined by:

$\exists\; \tilde{a} \in C^\infty(\mathcal{F},\mathbb{R})$ with $\tilde{a}|_M = a$.

ansatz: $\quad \tilde{\rho}_\tau = (F_\tau)_*(f_\tau\, \rho) \quad ; \quad \tau \in [0,\delta)$

where $\quad f_\tau \in C^\infty(M, \mathbb{R}^+), \quad F_\tau \in C^\infty(M,\mathcal{F})$

and smooth in τ

assume: $\quad \nabla_{\underline{v}} \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) = \int_{\mathcal{F}} \nabla_{1,\underline{v}} \mathcal{L}(x,y)\, d\rho(y)$

$\forall \underline{v} \in \mathfrak{J}^{\text{test}}$

Assume: $\tilde{\rho}_\tau$ satisfies the weak EL-eqns $\forall \tau$

$$\tilde{M}_\tau := \underline{F_\tau(M)} \quad \text{varied spacetime}$$

$$\tilde{x} \in \tilde{M}_\tau \qquad \nabla_{\underline{\tilde{u}}}\, \tilde{\ell}(\tilde{x}) = 0 \qquad \forall\, \underline{\tilde{u}} \in \mathfrak{J}^{\text{test}}_\tau$$

$$\tilde{\ell}(\tilde{x}) := \int_{\mathcal{F}} \mathcal{L}(\tilde{x}, y)\, d\tilde{\rho}_\tau - s$$

$$\tilde{x} = F_\tau(x) \qquad\qquad \forall\, \underline{u} \in \mathfrak{J}^{\text{test}}$$

$$\nabla_{\underline{u}(x)}\Big(\int_{\mathcal{F}} \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\, f_\tau(y)\, d\rho(y) - s \Big) = 0$$

$$\big(\tilde{u}(F_\tau(x)) = DF_\tau|_x\, u(x) \big)$$

Multiply by $f_\tau(x)$

$$0 = \nabla_{\underline{u}} \Big(\int_{\mathcal{F}} f_\tau(x)\, \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\, f_\tau(y)\, d\rho(y) - f_\tau(x)\, s \Big)$$

$$\underbrace{- \Big(\int_{\mathcal{F}} D_u f_\tau(x)\, \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\, f_\tau(y)\, d\rho(y) - D_u f_\tau(x)\, s \Big)}_{= D_u f_\tau(x)\; \tilde{\ell}(\tilde{x}) = 0}$$

Differentiate w.r.t. τ at $\tau = 0$.

$$0 = \nabla_{\underline{u}} \Big(\int_{\mathcal{F}} \frac{d}{d\tau}\Big(f_\tau(x)\, \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\, f_\tau(y) \Big)\Big|_{\tau=0}\, d\rho(y)$$

$$- \frac{d}{d\tau} f_\tau(x)\, s \Big)$$

introduce $\underline{v} = \frac{d}{d\tau}\left(f_\tau, F_\tau\right)\Big|_{\tau=0} \in \mathcal{J}$

$$\Rightarrow \quad 0 = \nabla_{\underline{u}}\left(\int \left(\nabla_{1,\underline{v}} + \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y)\, d\rho(y) - \nabla_{\underline{v}}\, s\right)$$

linearized field eqns in the smooth setting

$$\equiv \langle \underline{u}, \Delta \underline{v} \rangle(x).$$

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non-smooth setting

$\mathcal{J}^{\text{test}} \subset \mathcal{J}$

Convention: ∇ act only on the Lagrangian

$$\langle \underline{u}, \Delta \underline{v} \rangle\Big|_M = 0 \qquad \forall\, \underline{u} \in \mathcal{J}^{\text{test}}$$

solution $\underline{v} \in \mathcal{J}^{\text{lin}} \subset \mathcal{J}^{\text{vary}}$

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* In the newer literature, the weak EL eqns are also referred to as the restricted EL eqns. This avoids potential confusion with the notion of weak solutions of these equations.