Let $\mathcal{F}$ be (possibly non-compact) manifold.

$\mathcal{L} \in C^0(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0)$

$$S = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \; \mathcal{L}(x,y)$$

Let ρ be a minimizer ($\rho(\mathcal{F}) = \infty$ is possible)

assume that ρ is locally finite

Let $x \in M$, U an open neighborhood of x

$\rho(U) \neq 0$ and finite

Consider the variation

$$\tilde{\rho}_\tau = \chi_{M \setminus U} \, \rho + (1-\tau) \, \chi_U \, \rho + \tau \, \rho(U) \, \delta_{x_0} \quad \text{for } \tau \in [0,1)$$

where $x_0 \in \mathcal{F}$.

This variation preserves the total volume in the sense

$|\tilde{\rho}_\tau - \rho| < \infty$, $\quad (\tilde{\rho}_\tau - \rho)(\mathcal{F}) = 0$

$\mathcal{S}(\tilde{\rho}_\tau) - \mathcal{S}(\rho) = \ldots =$ well-defined and finite expression

$$0 \leq \frac{d}{d\tau} \left(\mathcal{S}(\tilde{\rho}_\tau) - \mathcal{S}(\rho) \right) \Big|_{\tau=0}$$

$$= \rho(U) \; \ell(x_0) - \int_U \ell(x) \, d\rho(x) \qquad \forall \, U, \, x_0$$

where

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x,y) \, d\rho(y) - s$$

$$\Rightarrow \quad \ell(x_0) \geq \frac{1}{\rho(\mathcal{U})} \int_{\mathcal{U}} \ell(x)\, d\rho(x)$$

$$\Rightarrow \quad \ell|_{\mathcal{M}} \equiv \inf_{\mathcal{F}} \ell = 0 .$$

for suitable ρ

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weak EL eqns *

$$\Rightarrow \quad \nabla_{\underline{u}} \ell|_{\mathcal{M}} = 0 \qquad \forall\ \underline{u} \in \mathfrak{J}^{\text{test}} \subset \mathfrak{J}$$

$$\left(\nabla_{\underline{u}} \ell \right)(x) := a(x)\, \ell(x) + D_u \ell(x)$$

$$\underline{u} = (a, u) , \qquad a \in C^\infty(\mathcal{M}, \mathbb{R})$$

$$u \in \Gamma(T\mathcal{F}, \mathcal{M})$$

* In the newer literature, the weak EL eqns are also referred to as the <u>restricted</u> EL eqns. This avoids potential confusion with the notion of weak solutions of these equations.