C^*-algebraic formulation of interacting quantum field theory applied to Fermions

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[1] joint work with Romeo Brunetti, Michael Dütsch and Klaus Fredenhagen, [arXiv:2103.05740]

Outline of the talk

Preliminaries

- A new idea for constructing local nets for interacting theories has been proposed in: Buchholz, D. and Fredenhagen, K., *A C^*-algebraic approach to interacting quantum field theories*, CMP 2020.

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- Main idea: theory described by an abstract C^*-algebra generated by a collection of unitaries, with a number of relations.
- These unitaries are interpreted as local S-matrices and are labelled by local functionals.
- Let M be a globally hyperbolic spacetime, $E \to M$ a vector bundle and $\mathcal{E} \doteq \Gamma(M, E)$, its space of smooth sections.

Functionals and Lagrangians

- The space of local functionals $\mathcal{F}_{\mathrm{loc}}$ is a subspace of the space of smooth compactly supported functionals on $\mathcal{E}$ consisting of those that can be written as $F(\phi) = \int\limits_M \alpha(j_x^k(\phi))d\mu(x)$, where α is a compactly-supported function on the jet bundle.

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- Here by the support of a functional F we mean:

$$\begin{aligned}\mathrm{supp}\ F = \{x \in M | \forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi_1, \varphi_2 \in \mathcal{E}, \mathrm{supp}\ \varphi_2 \subset U \\ \text{such that } F(\varphi_1 + \varphi_2) \neq F(\varphi_1)\}\end{aligned}$$

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- A generalized Lagrangian is a map $C^\infty_0(M) \equiv \mathcal{D} \ni f \mapsto L(f) \in \mathcal{F}_{\mathrm{loc}}$ with $\mathrm{supp}\, L(f) \subset \mathrm{supp}\, f$ and with $L(f+g+f') = L(f+g) - L(g) + L(g+f')$ if $\mathrm{supp}\, f \cap \mathrm{supp}\, f' = \varnothing$.

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- We restrict ourselves to generalized Lagrangians that lead to Green hyperbolic equations of motion.

Classical Dynamics

- Let L be a Lagrangian, $\varphi \in \mathcal{E}$. Define $\delta L : \mathcal{D} \times \mathcal{E} \to \mathbb{R}$ by

$$\delta L(\psi)[\varphi] \doteq L(f)[\varphi + \psi] - L(f)[\varphi]\,,$$

where $\varphi \in \mathcal{E}$, $\psi \in \mathcal{E}_c$ (compactly supported configuration) and $f \equiv 1$ on $\operatorname{supp}\psi$ (the map $\delta L(\psi)[\varphi]$ thus defined does not depend on the particular choice of f).

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Local S-matrices

Let F_1, F_2 be local functionals and let $F_1 \prec F_2$ denote the relation: $\operatorname{supp} F_1$ is not to the future of $\operatorname{supp} F_2$ (i.e. $\operatorname{supp} F_1$ does not intersect $J^+(\operatorname{supp} F_2)$). Local S-matrices are unitaries $\mathcal{S}(F)$, where $F \in \mathcal{F}_{\mathrm{loc}}$, required to satisfy the following relations:

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- S2 **Locality**: $\mathcal{S}$ satisfies the Hammerstein property, i.e. $F_1 \prec F_2$ implies that

$$\mathcal{S}(F_1 + F + F_2) = \mathcal{S}(F_1 + F)\mathcal{S}(F)^{-1}\mathcal{S}(F + F_2)\,,$$

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- **S3** **Schwinger-Dyson equation** For a fixed Lagrangian L, $\mathcal{S}(F)\mathcal{S}(\delta L(\varphi)) = \mathcal{S}(F^\varphi + \delta L(\varphi)) = \mathcal{S}(\delta L(\varphi))\mathcal{S}(F)$, where $F^\varphi(\psi) \doteq F(\varphi + \psi)$, $\varphi, \psi \in \mathcal{E}$.

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The C^*-algebra generated by above generators and relations is denoted by $\mathfrak{A}_L$.

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$$F(\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_n) = F_n(\mathrm{v}_1, \ldots, \mathrm{v}_n)\,, \quad F(1_{\Lambda V}) = F_0 \in \mathbb{R}\,.$$

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- The pointwise product of fermionic functionals is defined by

$$(F \cdot G)_n(\mathrm{v}_1, \ldots, \mathrm{v}_n) = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \sum_{k=0}^{n} \tfrac{1}{k!(n-k)!} F_k(\mathrm{v}_{\sigma(1)}, \ldots, \mathrm{v}_{\sigma(k)}) G_{n-k}(\mathrm{v}_{\sigma(k+1)}, \ldots, \mathrm{v}_{\sigma(n)})\ .$$

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- By derivative of a fermionic functional we always mean left derivative.

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- We thus obtain in a first step subalgebras $\mathfrak{A}_G$ of tensor products $G \otimes \mathfrak{A}$ of Grassmann algebras G with $\mathfrak{A}$ that are generated by even elements and the Grassmann algebra itself (understood as $G \otimes 1_{\mathfrak{A}}$).

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- The aim is to reconstruct the algebra $\mathfrak{A}$ from that family of subalgebras. To this end we equip this family of subalgebras with the following structure.

Category theory formulation

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- Let now $R : \mathfrak{Grass} \to \mathfrak{Alg}^{\mathbb{Z}_2}$ be the inclusion functor.

Covariant Grassmann multiplication algebra I

Definition

A covariant Grassmann multiplication algebra is a pair $(\mathfrak{G}, \iota)$ consisting of a functor $\mathfrak{G} : \mathfrak{Grass} \to \mathfrak{Alg}^{\mathbb{Z}_2}$ and a natural embedding $\iota : R \Rightarrow \mathfrak{G}$ i.e. a family $(\iota_G)_G$ of injective homomorphisms $\iota_G : G \to \mathfrak{G}G$ with

$$\iota_{G'} \circ \chi = \mathfrak{G}\chi \circ \iota_G\,, \qquad \text{for homomorphisms } \chi : G \to G'\,.$$

$$\begin{tikzcd} G \arrow[r, "\chi"] \arrow[d, "\iota_G"'] & G' \arrow[d, "\iota_{G'}"] \\ \mathfrak{G}G \arrow[r, "\mathfrak{G}\chi"'] & \mathfrak{G}G' \end{tikzcd}$$

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2. Let $\lambda_i \in \mathbb{R}$ and $\chi_i : G \to G'$, $i = 1, \dots, n$ be homomorphisms between Grassmann algebras with $\sum_{i=1}^{n} \lambda_i \chi_i = 0$. Then:
$$\sum_{i=1}^{n} \lambda_i\, \mathfrak{G}\chi_i = 0$$
.

Example

- Consider the functor $\mathfrak{G}^{\mathfrak{A}}$ with a graded unital algebra $\mathfrak{A}$ which maps each Grassmann algebras G to the tensor product $\mathfrak{G}^{\mathfrak{A}} G = G \otimes \mathfrak{A}$ with the product

$$(\eta_1 \otimes a_1) \cdot (\eta_2 \otimes a_2) \doteq (-1)^{\mathrm{dg}(\eta_2)\mathrm{dg}(a_1)} (\eta_1 \eta_2) \otimes (a_1 a_2)\,,$$

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- The morphisms $\chi : G \to G'$ are mapped to morphisms $\mathfrak{G}^{\mathfrak{A}}\chi : G \otimes \mathfrak{A} \to G' \otimes \mathfrak{A}$ by means of

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- In the following we simplify the notation by identifying $\iota_G(\eta)$ with η for $\eta \in G$ and $1_G \otimes a$ with a for $a \in \mathfrak{A}$, and similarly we write ηa for $\eta \otimes a \in G \otimes \mathfrak{A}$.

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- We identify ηF with the map $\omega \mapsto \eta F(\omega)$. The $\wedge$-symbol for the product in ΛV is usually omitted.
- The family $(F_G)_G$ is a natural transformation $\mathfrak{F} : \mathfrak{G}^{\Lambda V} \Longrightarrow \mathfrak{G}^{\mathbb{R}}$, i.e.:

$$\mathfrak{G}^{\mathbb{R}}\chi \circ F_G = F_{G'} \circ \mathfrak{G}^{\Lambda V}\chi\ .$$

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Application to fermionic functionals II

- F is already fixed if we know the maps F_G on all elements of the form $\exp \sum_{i \in I} \mathrm{v}^i \eta_i$ with odd elements $\eta_i \in G$, $\mathrm{v}^i \in \Lambda^1(V) = V$ and a finite index set $I \in \mathcal{P}_{\mathrm{finite}}(\mathbb{N})$, where $F_G(1_G) = F_0 1_G$.

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- In particular we can define shifts in the arguments as needed for the unitary Dyson-Schwinger equation.
- A shifted functional $F^{\vec{w}}$, with $\vec{w} = \sum_{j \in J} \vec{w}^j \theta_j$ with odd elements θ_j of some Grassmann algebra G' and $\vec{w}^j \in V$, $J \in \mathcal{P}_{\mathrm{finite}}(\mathbb{N})$, is defined as a family $(F_G^{\vec{w}})_G$ of G-module maps from $G \otimes \Lambda V$ to $G \otimes G'$.

Application to fermionic functionals III

- Explicitly:

$$F_G^{\vec{w}}\big(\exp\sum_{i\in I}\mathrm{v}^i\eta_i\big) = F_{G\otimes G'}\big(\exp\big(\sum_{i\in I}\mathrm{v}^i\eta_i + \sum_{j\in J}\vec{w}^j\theta_j\big)\big)$$
$$= \sum_{n\geq 0}\sum_{i_1<\ldots<i_n} F_n^{\vec{w}}(\mathrm{v}^{i_1},\ldots,\mathrm{v}^{i_n})\eta_{i_n}\cdots\eta_{i_1}\ ,$$

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- with alternating multilinear G'-valued maps $F_n^{\vec{w}}$ as components.

$$F_n^{\vec{w}}(\mathrm{v}^1,\ldots,\mathrm{v}^n)=\sum_{k\geq 0}\sum_{j_1<\ldots<j_k\in J} F_{k+n}(\mathrm{v}^1,\ldots,\mathrm{v}^n,\vec{w}^{j_1},\ldots,\vec{w}^{j_k})\,\theta_{j_k}\cdots\theta_{j_1}\ .$$

Universality

Reconstruction Theorem

Let $\mathfrak{G}$ be a covariant Grassmann multiplication algebra as defined above. Then there exists a graded unital algebra $\mathfrak{A}$ and a natural embedding

$$\sigma \equiv (\sigma_G)_G : \mathfrak{G} \Longrightarrow \mathfrak{G}^{\mathfrak{A}}$$

such that for any other graded unital algebra $\mathfrak{A}'$ with a natural embedding $\sigma' : \mathfrak{G} \Longrightarrow \mathfrak{G}^{\mathfrak{A}'}$ there exists a unique homomorphism $\tau : \mathfrak{A} \to \mathfrak{A}'$ with $\sigma'_G = (\mathrm{id} \otimes \tau) \circ \sigma_G$.

$$\begin{tikzcd}
 & \mathfrak{A}_G \doteq \mathfrak{G}G \arrow[dl, "\sigma_G"'] \arrow[dr, "\sigma'_G"] & \\
G \otimes \mathfrak{A} \arrow[rr, "\mathrm{id} \otimes \tau"'] & & G \otimes \mathfrak{A}'
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The algebra of Fermi fields

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- V is interpreted as the space of field configurations.
- We construct a covariant Grassmann multiplication algebra $\mathfrak{G} : \mathfrak{Grass} \to \mathfrak{Alg}^{\mathbb{Z}_2}$, by specifying the algebras $\mathfrak{A}_G \equiv \mathfrak{G}G$. These are generated by invertible elements $S_G(F)$ with $F \in G \otimes \mathcal{F}_{\mathrm{loc}}$ with the following properties and relations.

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- **(Quantization condition)** $S_G(\eta) = \iota_G(e^{i\eta})$ for $\eta \in G$.

The algebra of Fermi fields (properties and relations, continued)

- **(Causal factorization)**

$$S_G(F_1+F_2+F_3)=S_G(F_1+F_2)S_G(F_2)^{-1}S_G(F_2+F_3)$$

for even functionals F_1, F_2, F_3 with $\mathrm{supp}\, F_1 \cap J_-(\mathrm{supp}\, F_3)=\varnothing$ where J_- denotes the past of the region in the argument.

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- **(Dynamics)** Let $\vec{h}=\sum\limits_{i\in I}\eta_i\vec{h}^i$ with odd elements $\eta_i\in G$, $\vec{h}^i\in V_c$ and $I\in\mathcal{P}_{\text{finite}}(\mathbb{N})$.Then

$$S_G(F)=S_G(F^{\vec{h}}+\delta_{\vec{h}}L)$$

where

$$\delta_{\vec{h}}L=L(f)^{\vec{h}}-1_G\otimes L(f)$$

with $f\equiv 1$ on $\mathrm{supp}\,\vec{h}$ and 1_G denotes the unit of G.

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  which is characteristic for the *on-shell* algebra.
- Using the above relations one can, in particular, derive the CAR relations for the free Dirac field.
- As in the general case, we define $\mathfrak{A}$ as the inductive limit of this system with injections $\iota_n : \mathfrak{A}^n \to \mathfrak{A}$, where $\mathfrak{A}^n \subset \mathfrak{A}_{\Lambda\mathbb{R}^n}$ are defined in the course of the proof of the Reconstruction Theorem.

Involution

- To define involution, we set $\mathrm{v}^* = \mathrm{v}$ on the real vector space V and for linear maps on ΛV, we set:

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- The (universal) involution on the inductive limit of these spaces, denoted by $\mathfrak{A}$ is induced by
$$\iota_n(a)^* \doteq (-1)^{n(n-1)/2 + n(\mathrm{dg}(a)+n)} \iota_n(a^*) \ . \qquad (1)$$

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- The γ-matrices are then hermitian with respect to the sesquilinear form.

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- Smeared fields are defined as usual, that is, $\psi(s)[h] \doteq \langle s, h\rangle$, where $s \in V_c$ and $\overline{\psi}(s)[h] \doteq \langle h, s\rangle$.
- The Dirac Lagrangian $L = \overline{\psi} \wedge \not{D}\psi$ with the Dirac operator $\not{D} = i\gamma\partial - m$ associates to any compactly supported test function f a 2-form $L(f)$ on V, namely

$$L(f)[h_1, h_2] = \langle fh_1, \not{D}(fh_2)\rangle - \langle fh_2, \not{D}(fh_1)\rangle\ .$$

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- We extend the sesquilinear form $\langle\cdot,\cdot\rangle$ to a $G \otimes \mathbb{C}$-valued map $\langle\cdot,\cdot\rangle_G$ on $(G \otimes V_c) \times (G \otimes V_c)$ by

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- In particular we have: $\psi_G(\eta s) = \eta\psi_G(s)$, $\overline{\psi}_G(\eta s) = \eta\overline{\psi}_G(s)$.

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- This can be re-written as: $\delta_{\vec{h}} L_G = \psi_G(\not{D}\vec{h}) - \overline{\psi}_G(\not{D}\vec{h}) + \langle \vec{h}, \not{D}\vec{h} \rangle_G$.

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- The extended Lagrangian $L(f)_G$ is a quadratic form on even elements of $G \otimes V_c$.
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$$L(f)_G[e^h] = \frac{1}{2} L(f)_G[hh] = \frac{1}{2} \sum L(f)[h^i \wedge h^j] \eta_j \eta_i = \langle fh, \not{D} fh \rangle_G \,.$$

- The variation under a shift $\vec{h} = \sum_{i \in I} \vec{h}^i \theta_i$, with odd elements $\theta_i \in G$, $\vec{h}^i \in V_c$ is:

$$\delta_{\vec{h}} L_G[e^h] = \delta_{\vec{h}} L_G[1 + h] = \langle \vec{h}, \not{D} h \rangle_G + \langle h, \not{D} \vec{h} \rangle_G + \langle \vec{h}, \not{D} \vec{h} \rangle_G \,.$$

- This can be re-written as: $\delta_{\vec{h}} L_G = \psi_G(\not{D}\vec{h}) - \overline{\psi}_G(\not{D}\vec{h}) + \langle \vec{h}, \not{D}\vec{h} \rangle_G$.
- Let $s \in (G \otimes V_c)_{\text{even}}$ and let $\mathfrak{D}_G(s) \doteq \psi_G(s) - \overline{\psi}_G(s)$ be the smeared *classical* "doubled Dirac field" viewed as an element in $(G \otimes \mathcal{F}_{\text{loc}})_{\text{even}}$.

Quantum Dirac field I

Proposition

Let $s = \sum_{i=1}^{n} \eta_i s^i$ with $s^i \in V_c$ and η_i odd elements of G. The S-matrix S_G built with the doubled Dirac field has the expansion

$$S_G(\mathfrak{D}_G(s)) = 1_{\mathfrak{A}} + \sum_{k=1}^{n} \frac{i^k}{k!} \sum_{i_1 < \cdots < i_k} \eta_{i_k} \dots \eta_{i_1} B_k(s^{i_1} \wedge \cdots \wedge s^{i_k}) \qquad (2)$$

with $\mathbb{R}$-multilinear alternating maps $B_k : V_c^k \to \mathfrak{A}$, $k = 1, \dots, n$, (the time ordered products of the doubled Dirac field).

Quantum Dirac field II

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$$S_G(\mathfrak{D}_G(\eta s)^*)^{-1} = S_G(\mathfrak{D}_G(-\eta s))^{-1} = (1 - i\eta B_1(s))^{-1} = 1 + i\eta B_1(s)$$

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- It is an $\mathfrak{A}$-valued *antilinear* functional on V_c.

CAR

Theorem

The quantized Dirac field Ψ satisfies the canonical anticommutation rules over V_c:

$$\{\Psi(s^1)^*, \Psi(s^2)^*\} = \{\Psi(s^1), \Psi(s^2)\} = 0\,,\ \{\Psi(s^1), \Psi(s^2)^*\} = \langle s^2, i\not{S} s^1\rangle 1_{\mathfrak{A}}\,,$$

where

$$\not{S} = (i\gamma\partial + m)\Delta$$

with Δ the commutator function of the scalar theory.

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- The operator norm in the induced GNS representation is a C*-norm. We then equip the algebra with the maximal C*-norm.

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  $\Psi(f)^*\Psi(f)$ is a selfadjoint projection.
- Hence for every non-zero C*-seminorm

$$||\Psi(f)|| = ||f||_{V_c}$$

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Theorem

The maximal C*-seminorm on $\mathfrak{B}$ exists and is a C*-norm.

Thank you very much for your attention!