# Uhlenbeck Compactness and Optimal Regularity in Lorentzian Geometry and for Yang-Mills Gauge Theories

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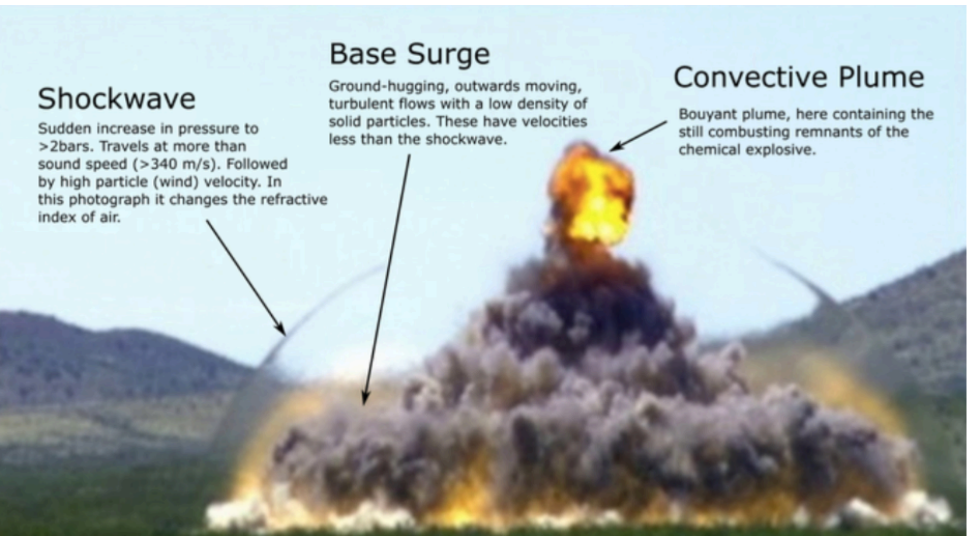
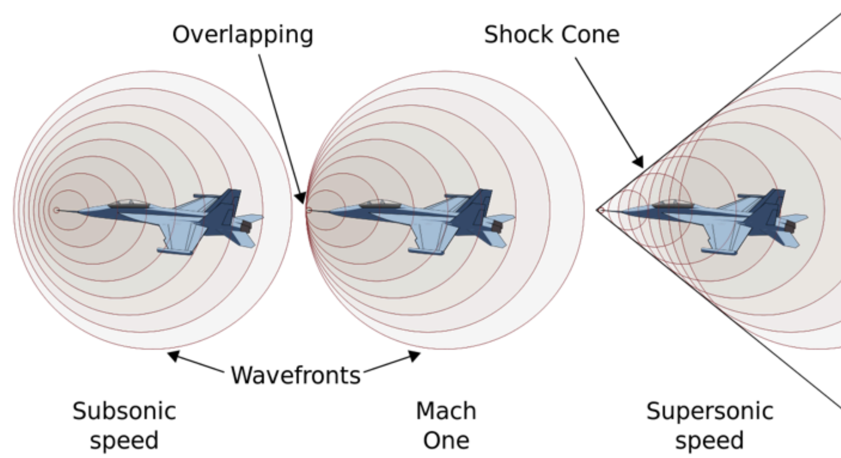
Collaborator: Blake Temple (University of California, Davis)

# A little background on Shock Waves:

- Shock waves (discontinuities) are typical phenomena in fluid dynamics:




- For the Einstein equations with a fluid source $T^{\mu\nu}$ (Cosmology; Stars)

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi T^{\mu\nu}$$
$$\nabla_\nu T^{\mu\nu} = 0$$

shock waves form when flow is compressive enough.

## Motivation from General Relativity:

- In coordinates where the Einstein equations are solvable, regularity issues often arise. (Singularity at Schwarzschild radius.)
- Shock wave solutions of Einstein eqn's are such [Israel, 1966]: Their metric tensors are only Lipschitz and appear singular.

**Can one remove these singularities in the metric tensor?**

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**Can one remove these singularities in the metric tensor?**

- Yes, … across a single shock surface. [Israel, 1966]
- Yes, … across two intersecting shock surfaces. [R., 2014]

By coordinate transformation to "optimal metric regularity".

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- The question remained open for general shock wave solutions.
  - E.g.: Glimm scheme based shock solutions of Einstein-Euler eqn's. [Groah & Temple (2004)]

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Can one remove these singularities in the metric tensor?

Corollary: [R. & Temple, Dec. 2019.]
These singularities are removable by a coordinate transformation.

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**Can one remove these singularities in the metric tensor?**

Thm 1: ("Optimal Regularity") [R. & Temple, Dec. 2019]
Any affine $L^\infty$ connection $\Gamma$ with $L^\infty$ Riemann curvature can be smoothed to $W^{1,p}$ (any $p < \infty$) by coordinate transformation.

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# Preview of results:

Thm 1: ("Optimal Regularity") [R. & Temple, Dec. 2019]

Any affine $L^\infty$ connection $\Gamma$ with $L^\infty$ Riemann curvature can be smoothed to $W^{1,p}$ (any $p < \infty$) by coordinate transformation.

Based on a novel system of elliptic PDE's

Thm 2: [R. & Temple, Dec. 2019.]

Uhlenbeck compactness in Lorentzian geometry (affine connections).

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Thm 3: [R. & Temple, May 2021]
The results of Thm’s 1 & 2 extend from tangent bundles to vector bundles, with compact and non-compact gauge groups. (Yang-Mills gauge theories of Particle Physics.)

# Optimal Regularity

## The setting:

- Connection components: $\Gamma \equiv \Gamma^k_{ij}$ $\quad (k, i, j = 1,\ldots,n)$

E.g.: $\Gamma^k_{ij} = g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ for a metric $g_{ij}$ .

- Their Riemann curvature: $\mathrm{Riem}(\Gamma) = \mathrm{Curl}(\Gamma) + [\Gamma, \Gamma]$

Both defined on an open & bounded set $\Omega \subset \mathbb{R}^n$.

---

The problem of optimal regularity is local.

- The set $\Omega \subset \mathbb{R}^n$ represents a chart $(x, U)$ on a manifold, $\Omega = x(U)$.

# Optimal regularity and coordinate transformations:

$$\boxed{\begin{array}{c} \Gamma \in C^1 \\ \big\downarrow \frac{\partial}{\partial y} \\ \mathrm{Riem}(\Gamma) \in C^0 \end{array}}$$

$$\mathrm{Riem}(\Gamma) \sim \mathrm{Curl}(\Gamma)$$

# Optimal regularity and coordinate transformations:

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makes this possible

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"Optimal Regularity"

$$\boxed{\begin{array}{c} \Gamma \in C^0 \\ \Big\downarrow \frac{\partial}{\partial x} \\ \mathrm{Riem}(\Gamma) \in C^0 \end{array}}$$

"Non-optimal Regularity"

# Optimal regularity and coordinate transformations:

$$\frac{\partial x}{\partial y} \in C^1$$

$$y \to x$$

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"Optimal Regularity"

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$$\Gamma \to \Gamma + \partial\left(\frac{\partial x}{\partial y}\right)$$

$$\mathrm{Riem}(\Gamma) \to \frac{\partial x}{\partial y} \cdot \mathrm{Riem}(\Gamma)$$

# Optimal regularity and coordinate transformations:

$$\begin{tikzcd}[column sep=huge]
\boxed{\begin{matrix}\Gamma \in C^1 \\ \big\downarrow \frac{\partial}{\partial y} \\ \mathrm{Riem}(\Gamma) \in C^0\end{matrix}} \arrow[r, bend left, "\frac{\partial x}{\partial y} \in C^1", "y \to x"'] & \boxed{\begin{matrix}\Gamma \in C^0 \\ \big\downarrow \frac{\partial}{\partial x} \\ \mathrm{Riem}(\Gamma) \in C^0\end{matrix}} \arrow[l, bend left, "y \leftarrow x\,?"]
\end{tikzcd}$$

"Optimal Regularity"

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$\boxed{\underline{\text{Question:}}\ \exists\, x \to y\,?}$

# Optimal regularity and coordinate transformations:

Typical, when solving Einstein equations.

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"Optimal Regularity" $\qquad y \leftarrow x\,?$ $\qquad$ "Non-optimal Regularity"

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"Optimal Regularity" "Non-optimal Regularity"

$y \leftarrow x\,?$

**Question:** $\exists\, x \to y\,?$

E.g.: Shock wave solutions of Einstein-Euler eqn's have non-optimal regularity.

# Optimal regularity and coordinate transformations:

$\mathrm{Riem}(\Gamma) \sim \text{fluid} \in L^\infty$, required for shock discontinuities.

$$\Gamma \in W^{1,\infty} \xrightarrow{\frac{\partial}{\partial y}} \mathrm{Riem}(\Gamma) \in L^\infty$$

"Optimal Regularity"

$y \leftarrow x\,?$

$$\Gamma \in L^\infty \xrightarrow{\frac{\partial}{\partial x}} \mathrm{Riem}(\Gamma) \in L^\infty$$

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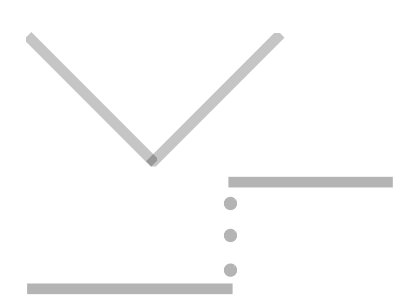
Function spaces:

$W^{1,\infty} = C^{0,1} = \text{Lipschitz continuous}$

$L^\infty = \text{bounded, but discontinuous}$



# Our optimal regularity result:

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Thm 1: $\exists\, x \to y\,!$

Thm 1 is based on a novel system of elliptic PDE's.

➡ Elliptic regularity theory requires spaces $W^{1,p}$, $p < \infty$.

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Thm 1: (R. & Temple, Dec. 2019.)
Let $\Gamma \in L^\infty$ with $\mathrm{Riem}(\Gamma) \in L^\infty$ in $x$-coordinates. Let $p \in (n, \infty)$.
Then there exists a coordinate transformation $x \to y$ with Jacobian $J \in W^{1,2p}$, such that $\Gamma \in W^{1,p}$(optimal regularity) in y-coordinates.

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Corollary: (First application to General Relativity)
Singularities in Lorentzian metrics of GR shock wave solutions are removable, because $\Gamma$ in $y$-coord's is Hölder continuous.

➡ Geodesic curves exist. (Particle trajectories)

➡ Locally inertial coordinates exist. (Newtonian limit)

Remark:

- Higher regularities $(m \geq 1, p > n)$: $\Gamma, \mathrm{Riem}(\Gamma) \in W^{m,p} \longrightarrow \Gamma \in W^{m+1,p}$
- Proof of Thm 1 is based on the *Regularity Transformation (RT-)equations*, a novel system of PDE's, elliptic regardless of metric (signature).

# A glimpse at the RT-equations

The "*Regularity Transformation (RT-)equations*":

$$
\begin{cases}
\Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A), \\
\Delta J = \delta(J\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \\
d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \\
\delta\vec{A} = v,
\end{cases}
$$

- $\Gamma$ denotes components of non-optimal connection (in $x$ coord's), which we want to smooth to optimal regularity.

- <u>Unknowns:</u> $(J,\tilde{\Gamma},A)$ are matrix-valued differential forms.
  - $J$ is Jacobian of coord. transformation to optimal regularity.
  - $\tilde{\Gamma}$ is a tensor related to connection of optimal regularity.
  - $A$ is an auxiliary field required to induce integrability for $J$.

The "*Regularity Transformation (RT-)equations*":

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- The RT-equations are elliptic PDE's on spacetime:
  - $\Delta$ is the Euclidean Laplacian in $\mathbb{R}^n$ (in x-coordinates).
  - $d$ exterior derivative; $\delta$ co-derivative
  - $d\delta + \delta d = \Delta$ implies $\overrightarrow{A}$-equations are elliptic.
- The RT-equations are based on a "Cartan Calculus" for matrix valued differential forms, w.r.t. the Euclidean metric in x-coordinates.

$$\begin{cases} \Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A), \\ \Delta J = \delta(J\Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A, \\ d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big), \\ \delta\vec{A} = v, \end{cases}$$

with $d\vec{J} = 0$ on $\partial\Omega$ (boundary data)

<ins>Thm 0</ins>: [R. & Temple, Dec. 2019]

Assume $\Gamma$, $\mathrm{Riem}(\Gamma) \in L^\infty$ in x-coordinates.

If $(J, \tilde{\Gamma}, A) \in W^{1,2p} \times W^{1,p} \times L^{2p}$ solves the RT-eqn (with $J^{-1} \in W^{1,2p}$), then $J$ is the Jacobian of a coordinate transformation $x \to y$ such that $\Gamma \in W^{1,p}$ in y-coordinates. (Equivalence holds)

- Thm 0 + Existence Theory $\Longrightarrow$ Thm 1 (optimal regularity).
- <ins>Equivalence:</ins> RT-eqn's are derived from the connection transformation law alone, via the *Riemann-flat condition* [R. & Temple 2017].
- Metric signature plays no role for optimal regularity!

# Optimal regularity and "harmonic coordinates":

- Riemannian geometry: [DeTurck & Kazdan, '81]

  In harmonic coordinates:

  Laplace-Beltrami operator, elliptic

  $$\mathrm{Ric}(\Gamma) \sim \Delta_g g$$

  ➡ Optimal regularity by elliptic regularity theory.

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➡ Optimal regularity by elliptic regularity theory.

- Lorentzian geometry: (Problematic!)

In harmonic coordinates:

D’Alembert operator, hyperbolic

$$\boxed{\mathrm{Ric}(\Gamma) \sim \Box_g g}$$

➡ No elliptic regularity theory can be applied…

Partial results were obtained in modified coord’s. [Anderson 2002]

# Uhlenbeck Compactness

**Thm 1:** (R. & Temple, Dec. 2019) ("Optimal Regularity")

Let $p \in (n, \infty)$. Assume that in x-coordinates

$$\|\Gamma\|_{L^\infty} + \|\mathrm{Riem}(\Gamma)\|_{L^\infty} \ \leq \ M$$

Then there exists coordinates y such that the transformed connection has optimal regularity, $\Gamma_y \in W^{1,p}$, and satisfies

$$\boxed{\|\Gamma_y\|_{W^{1,p}} \leq C(M)}$$

where $C(M) > 0$ depends only on $\Omega, n, p$ and $M > 0$.

Norms are taken component-wise in fixed $x$-coordinates.

**E.g.:** $$\|\Gamma\|_{L^p} \equiv \sum_{k,i,j} \|\Gamma^k_{ij}\|_{L^p} = \sum_{k,i,j} \Big( \int_\Omega |\Gamma^k_{ij}|^p dx \Big)^{\frac{1}{p}}$$

$$\|\Gamma\|_{W^{1,p}} \equiv \|\Gamma\|_{L^p} + \|D\Gamma\|_{L^p}$$

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**Thm 2:** (R. & Temple, Dec. 2019) ("Uhlenbeck compactness")

Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of $L^\infty$ connections in x-coordinates.

Assume: $\|\Gamma_i\|_{L^\infty} + \|\mathrm{Riem}(\Gamma_i)\|_{L^\infty} \;\leq\; M.$

Then for each $\Gamma_i$ there exists coordinates $y_i$ such that the transformed connection has optimal regularity, $\Gamma_{y_i} \in W^{1,p}$, and

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Thus, we have compactness: A subsequence $\Gamma_{y_j}$ converges weakly in $W^{1,p}$ and strongly in $L^p$.

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- Thm 2 only requires uniform bound on $\mathrm{Riem}(\Gamma_i)$, not all derivatives.
- The convergence is regular enough to pass limits through products!

Corollary: Let $(g_i)_{i\in\mathbb{N}}$ in $C^{0,1}$ with uniform curvature-type bound.
Assume $g_i \longrightarrow g$ weakly in $W^{1,p}$ and $\mathrm{Ric}(g_i) \longrightarrow 0$ weakly in $L^p$.
Then, $\mathrm{Ric}(g) = 0$, i.e. $g$ solves the vacuum Einstein equations.

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Thm 2 extends Uhlenbeck compactness to Lorentzian geometry!

Uhlenbeck Compactness in Riemannian Geometry:

[K. Uhlenbeck, '82] (Abel Prize 2019, Steele Prize 2007)

- Assumes only a uniform curvature bound, but $\Gamma_i \in W^{1,p}$.
- Applies to vector bundles over Riemannian manifolds.
  - Thm 2 applies to tangent bundles of arbitrary manifolds, including Lorentzian manifolds.

# Uhlenbeck Compactness and Optimal Regularity for Yang-Mills Gauge Theories

# The setting:

- Vector bundle: $V\mathscr{M} = \mathbb{R}^N \times \Omega$ (trivialisation taken w.l.o.g.)
- Gauge group: $SO(r,s) \equiv \left\{ U \in \mathbb{R}^{N\times N} \middle|\, U^T \eta U = \eta \ \& \ \det(U) = 1 \right\}$

  for $r + s = N$ and $\eta \equiv \mathrm{diag}(\underbrace{-1,\ldots,-1}_{r},\underbrace{1,\ldots,1}_{s})$.
- Connection on $V\mathscr{M}$: $\mathbf{A} : \Omega \longrightarrow so(r,s)$

  for $so(r,s) \equiv \left\{ X^T\eta + \eta X = 0 \ \& \ \mathrm{tr}(X) = 0 \right\}$, Lie algebra of $SO(r,s)$.

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Remark:

- We discard (w.l.o.g.) the affine connection on the base manifold and work with Euclidean metric as auxiliary Riemannian structure.
  Covariant derivative $D \equiv \partial + \cancel{\Gamma} + \mathbf{A}$.
- Method extends to $U(r,s)$ and $SU(r,s)$, as well as more general Lie groups (work in progress).

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- Under a gauge transformation $U : \Omega \longrightarrow SO(r,s)$,

  a connection transforms by $\mathbf{A}_{\mathsf{a}} = U^{-1}dU + U^{-1}\mathbf{A}_{\mathsf{b}}U$,

  where $\mathsf{b} = U \cdot \mathsf{a}$, (change of basis a to b; "gauge"=basis).

# Statement of Results:

Assumption: $\mathbf{A}_{\mathsf{a}} \in L^p(\Omega)$ & $d\mathbf{A}_{\mathsf{a}} \in L^p(\Omega)$, $(p > n)$. (non-optimal)

Thm 1: [R. & Temple, 2021]

There exists a gauge transformation $U \in W^{1,p}(\Omega, SO(r,s))$

to a gauge (basis) $\mathsf{b} = U \cdot \mathsf{a}$ such that $\mathbf{A}_{\mathsf{b}} \in W^{1,\frac{p}{2}}(\Omega)$ with

$$\|\mathbf{A}_{\mathsf{b}}\|_{W^{1,\frac{p}{2}}} + \|U\|_{W^{1,p}} \leq C(M).$$

where $M > 0$ is such that $\|\mathbf{A}_{\mathsf{a}}\|_{L^p} + \|d\mathbf{A}_{\mathsf{a}}\|_{L^p} < M$

Remark:

Thm 1 extends the optimal regularity result of DeTurck & Kazdan ['81] from Riemannian metrics to connections on vector bundles (over Lorentzian manifolds).

Thm 1: [R. & Temple, 2021]

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$$\|\mathbf{A}_{\mathsf{b}}\|_{W^{1,\frac{p}{2}}} + \|U\|_{W^{1,p}} \leq C(M).$$

By extra derivative, applying the Banach-Alaoglu Thm.

Thm 2: ("Uhlenbeck compactness") [R. & Temple, 2021]

Assume a sequence of connections $\mathbf{A}_i$ in a (fixed) gauge "a" satisfies $\|\mathbf{A}_i\|_{L^p} + \|d\mathbf{A}_i\|_{L^p} < M$. Then there exist gauge transformations $U_i$ in $SO(r,s)$ to gauges $\mathsf{b}_i = U_i \cdot \mathsf{a}$ such that

$$\|\mathbf{A}_{\mathsf{b}_i}\|_{W^{1,\frac{p}{2}}} + \|U_i\|_{W^{1,p}} \leq C(M).$$

Thus, a subsequence of $\mathbf{A}_{\mathsf{b}_i}$ converges weakly in $W^{1,\frac{p}{2}}$ to some $\mathbf{A}_{\mathsf{b}}$, (where b is the weak $W^{1,p}$-limit of $\mathsf{b}_i = U_i \cdot \mathsf{a}$).

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<u>Remarks:</u>

- Thm 2 extends Uhlenbeck's theorem ['82] from compact to non-compact groups $SO(r,s)$ and Lorentzian geometry. (Assuming a uniform bound on $\|\mathbf{A}_i\|_{L^p}$ instead of $\mathbf{A}_i \in W^{1,p}$ without a bound…)
- Thm's 1 & 2 are based on the "RT-equations associated to vector bundles".

# The RT-equations associated to vector bundles:

$$\Delta\tilde{\mathbf{A}} = \delta d\mathbf{A} - \delta(dU^{-1} \wedge dU) \qquad (1)$$

$$\Delta U = U\delta\mathbf{A} - (U^T\eta)^{-1}\langle dU^T; \eta dU\rangle \qquad (2)$$

$\delta$ co-derivative

$d$ exterior derivative

Matrix-valued inner product

Unknowns: $U$ in $SO(r,s)$ (regularising gauge transformation)

$\tilde{\mathbf{A}} \equiv U^{-1}\mathbf{A}_{\mathsf{b}}U$ ($\mathbf{A}_{\mathsf{b}}$ is connection of optimal regularity)

Remarks:

- The RT-equations are elliptic. ($\Delta$ is Euclidean Laplacian in $\mathbb{R}^n$.)
- The equations for $U$ and $\tilde{\mathbf{A}}$ are already decoupled.
- Solving RT-equation (2) with Dirichlet data $U \in SO(r,s)$ on $\partial\Omega$ yields regularising gauge transformation $U \in W^{1,p}(\Omega, SO(r,s))$.
- RT-equation (1) gives regularity boost. (No need to solve!)

# Derivation of the RT-equations:

- Start with connection transformation law to optimal regularity

(in $\mathrm{b} = U \cdot \mathrm{a}$):

$$\mathbf{A}_{\mathsf{a}} = U^{-1} dU + U^{-1} \mathbf{A}_{\mathsf{b}} U$$

Assume:
$\mathbf{A}_{\mathsf{a}} \in L^p$ & $d\mathbf{A}_{\mathsf{a}} \in L^p$
(non-optimal)

Assume:
$\mathbf{A}_{\mathsf{b}} \in W^{1,\frac{p}{2}}$
(optimal)

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- Start with connection transformation law to optimal regularity

(in $\mathrm{b} = U \cdot \mathrm{a}$):

$$\mathbf{A}_{\mathsf{a}} = U^{-1}dU + U^{-1}\mathbf{A}_{\mathsf{b}}U$$

$\mathbf{A} \equiv \mathbf{A}_{\mathsf{a}}$ $\qquad$ $\tilde{\mathbf{A}} \equiv U^{-1}\mathbf{A}_{\mathsf{b}}U$

$$\mathbf{A} = U^{-1}dU + \tilde{\mathbf{A}}$$

Take exterior derivative $d$!

$$d\tilde{\mathbf{A}} = d\mathbf{A} - dU^{-1} \wedge dU$$

Take co-derivative $\delta$!

$$\Delta U = U\,(\delta\mathbf{A} - \delta\tilde{\mathbf{A}}) + \langle dU; \mathbf{A} - \tilde{\mathbf{A}}\rangle$$

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- Introduce the matrix function $\alpha \in L^p(\Omega)$ by

$$\delta\tilde{\mathbf{A}} = U^{-1}\alpha$$

and <u>treat</u> $\alpha$ as a free parameter.

# Derivation of the RT-equations:

- Start with connection transformation law to optimal regularity

(in b $= U \cdot$ a): $$\mathbf{A}_{\mathsf{a}} = U^{-1}dU + U^{-1}\mathbf{A}_{\mathsf{b}}U$$

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$$\mathbf{A} = U^{-1}dU + \tilde{\mathbf{A}}$$

Take exterior derivative $d$!

$$d\tilde{\mathbf{A}} = d\mathbf{A} - dU^{-1} \wedge dU$$

Take co-derivative $\delta$!

Add d of $\delta\tilde{\mathbf{A}} = U^{-1}\alpha$ !

Use $\Delta = d\delta + \delta d$!

$$\Delta\tilde{\mathbf{A}} = \delta d\mathbf{A} - \delta(dU^{-1} \wedge dU) + d(U^{-1}\alpha)$$

Take co-derivative $\delta$!

$$\Delta U = U\left(\delta\mathbf{A} - \delta\tilde{\mathbf{A}}\right) + \langle dU; \mathbf{A} - \tilde{\mathbf{A}}\rangle$$

Substitute $\delta\tilde{\mathbf{A}} = U^{-1}\alpha$ !

$$\Delta U = U\delta\mathbf{A} + \langle dU; \mathbf{A} - \tilde{\mathbf{A}}\rangle - \alpha$$

- Next: Use $U \in SO(r,s)$ to determine $\alpha$!

$\Delta\tilde{\mathbf{A}} = \delta d\mathbf{A} - \delta(dU^{-1} \wedge dU) + d(U^{-1}\alpha)$ $\Delta U = U\delta\mathbf{A} + \langle dU; \mathbf{A} - \tilde{\mathbf{A}}\rangle - \alpha$

- Next: Use $U \in SO(r,s)$ to determine $\alpha$! (Omit $\det(U) = 1$!)

Define: $w \equiv U^T \eta U - \eta$

Observe: $U \in SO(r,s) \quad \Longleftrightarrow \quad w = 0 \quad \Longleftrightarrow \quad \begin{cases} \Delta w = 0 \\ w\,|_{\partial\Omega} = 0 \end{cases}$

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By Leibniz rule: $\Delta w = (\Delta U)^T \eta U + 2\langle dU^T; \eta dU\rangle + U^T \eta \Delta U$

Substitute red equation

$$\Longrightarrow \quad \alpha \equiv (U^T\eta)^{-1}\langle dU^T; \eta dU\rangle + \langle dU; \mathbf{A} - \tilde{\mathbf{A}}\rangle$$

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$$\Longrightarrow \quad \alpha \equiv (U^T\eta)^{-1}\langle dU^T; \eta dU\rangle + \langle dU; \mathbf{A} - \tilde{\mathbf{A}}\rangle$$

Cancellation: $(\delta\mathbf{A})^T \cdot U^T\eta U + U^T \eta U \cdot \delta\mathbf{A} = 0$

by interplay of Lie algebra with Lie group

This is crucial for regularity to close!

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$$\Longrightarrow \quad \alpha \equiv (U^T\eta)^{-1}\langle dU^T; \eta dU\rangle + \langle dU; \mathbf{A} - \tilde{\mathbf{A}}\rangle$$

- Substitution of $\alpha$ into blue and red equations yields RT-equations:

$$\Delta\tilde{\mathbf{A}} = \delta d\mathbf{A} - \delta(dU^{-1} \wedge dU) \qquad (1)$$

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- Why is $U$ a gauge transformation to optimal regularity:

Lemma: Let $U \in W^{1,p}(\Omega, SO(r,s))$ solve (2).
Then $\tilde{\mathbf{A}}' \equiv \mathbf{A} - U^{-1}dU$ solves (1) and $\tilde{\mathbf{A}}' \in W^{1,\frac{p}{2}}$.

Proof:

- That $\tilde{\mathbf{A}}'$ solves (1) follows by direct computation, substituting (2).
- $\tilde{\mathbf{A}}' \in W^{1,\frac{p}{2}}$ follows from (1) using Hölder inequality. ■

➡ $\mathbf{A}_{\mathrm{b}} \equiv U\tilde{\mathbf{A}}'U^{-1}$ is the connection in gauge $\mathrm{b} = U \cdot \mathrm{a}$
and $\mathbf{A}_{\mathrm{b}} \in W^{1,\frac{p}{2}}$ (optimal regularity)

➡ Thus, $U$ transformation to optimal regularity!

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- Why is $U$ in $SO(r,s)$?

$$\Delta U = U\delta\mathbf{A} - (U^T\eta)^{-1}\langle dU^T; \eta dU\rangle \quad (2)$$

- Why is $U$ in $SO(r,s)$:

- Define $w \equiv U^T\eta U - \eta$, where $U$ is a solution of (2).

  Recall: $U \in SO(r,s) \iff w = 0 \iff \begin{cases} \Delta w = 0 \\ w\,|_{\partial\Omega} = 0 \end{cases}$

- For $U \in SO(N)$:

  Computing $\Delta w$, substitution of (2) yields $\Delta w = 0$. Thus $U \in SO(N)$.

$$\boxed{\Delta U = U\delta\mathbf{A} - (U^T\eta)^{-1}\langle dU^T; \eta dU\rangle} \quad (2)$$

- Why is $U$ in $SO(r,s)$:

- Define $w \equiv U^T\eta U - \eta,$ where $U$ is a solution of (2).

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- For $U \in SO(N)$:

  Computing $\Delta w$, substitution of (2) yields $\Delta w = 0$. Thus $U \in SO(N)$.

- For $U \in SO(r,s)$:

  Computing $\Delta w$, by substituting (2) and using $\delta\mathbf{A} \in so(r,s)$, yields

$$\boxed{\Delta w = \delta\mathbf{A}^T \cdot w + w \cdot \delta\mathbf{A},}$$

  and Fredholm alternative allows for non-zero solutions. Problem!

$$\Delta U = U\delta\mathbf{A} - (U^T\eta)^{-1}\langle dU^T; \eta dU\rangle \quad (2)$$

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  $$\Delta w = \delta\mathbf{A}^T \cdot w + w \cdot \delta\mathbf{A},$$

  and Fredholm alternative allows for non-zero solutions. Problem!

  Resolution: (By a spectral perturbation argument)

  - Write $w$-eqn as $w = K(w)$ for the compact operator $K(w) \equiv \Delta^{-1}\big(\delta\mathbf{A}^T \cdot w + w \cdot \delta\mathbf{A}\big)$.
  - Solving (2) for $\mathbf{A}/\lambda$ in place of $\mathbf{A}$ for $\lambda \in (0,1]$ gives eigenvalue problem $K(w) = \lambda w$, (with $w$ defined in terms of new solution).
  - Since compact operators have a countable spectrum, $w = 0$ must hold for almost every $\lambda \in (0,1]$.
  - By continuous dependence in $\lambda$ of solutions constructed (Thm 3), it follows that $w = 0$ for original eigenvalue problem ($\lambda = 1$). $\blacksquare$

$$\Delta U = U\delta\mathbf{A} - (U^T\eta)^{-1}\langle dU^T;\eta dU\rangle \quad (2)$$

## Existence Theory:

**Thm 3:** [R. & Temple, 2021]

Let $\mathbf{A}_\mathrm{a} \in L^p(\Omega)$ & $d\mathbf{A}_\mathrm{a} \in L^p(\Omega)$, $(p > n)$. Then, locally, there exists a solution $U \in W^{1,p}(\Omega, SO(r,s))$ of (2), with Dirichlet data $U \in SO(r,s)$ on $\partial\Omega$, such that $\|U\|_{W^{1,p}} \leq C(M)$.

Initial bound $M > 0$: $\|\mathbf{A}_\mathrm{a}\|_{L^p} + \|d\mathbf{A}_\mathrm{a}\|_{L^p} < M$

**Proof:**

- Iteration via Poisson equations with $W^{-1,p}$-sources. (Linearisation)
- Requires $\epsilon$-rescaling of equations by domain restriction. $(U \equiv I + \epsilon v)$
- Elliptic estimates (for scalar PDE's) together with source estimates yield $W^{1,p}$-convergence to solution. ∎

# Conclusion:

The RT-equations establish:

- Optimal regularity,   (independent of metric signature).
- GR-shock waves are non-singular.
- Uhlenbeck compactness in Lorentzian geometry.
- Uhlenbeck compactness for compact & non-compact Lie groups.

# Outlook:

- Extension to lower regularities $\Gamma, d\Gamma \in L^p$?
  - (Include singularities of Schwarzschild type?)
- Applications of Uhlenbeck compactness in Lorentzian geometry?
  - Zero viscosity limits?
  - Assumptions of Cosmology?
  - Extending "classical" applications of Uhlenbeck compactness?
  - Cauchy Problem for Einstein/Yang-Mills equations?
  - …

1. M.R., "Spacetime is locally inertial at points of general relativistic shock wave interaction…", Adv. Theor. Math. Phys. 21.6 (2017), 1525-1611. arXiv:1409.5060
2. M.R. & B. Temple, "Shock Wave Interactions and the Riemann-flat condition: The Geometry behind…", Arch. Rat. Mech. Anal. 235 (2020), 31 pages. [arXiv:1610.02390]
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7. M.R. & B. Temple, "Uhlenbeck compactness and optimal regularity for Yang-Mills theory with Lorentzian geo. and non-compact Lie groups", (2021), 32 pages. [arXiv:2105.10765]

# Thank you very much for your attention!


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