α-Kähler quantization

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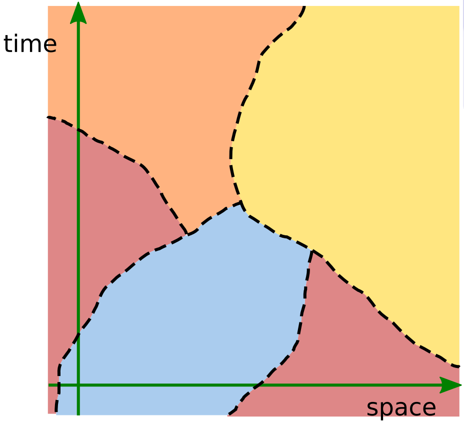
Overview

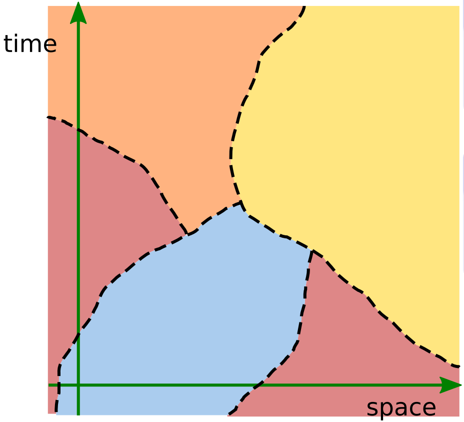
Spacetime locality



Classical Field Theory

Glue solutions in spacetime regions at boundaries.

Spacetime locality



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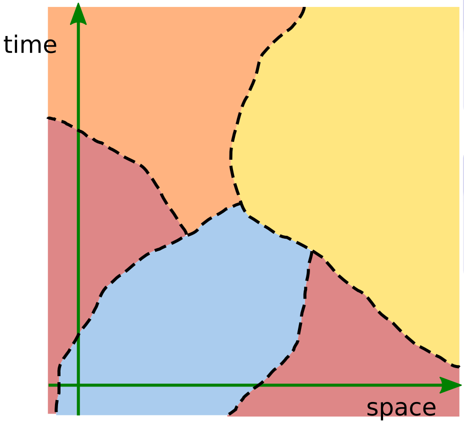
Glue solutions in spacetime regions at boundaries.

Quantum Field Theory

TQFT/GBQFT: Glue amplitudes in spacetime regions at boundaries.

Spacetime locality



Classical Field Theory

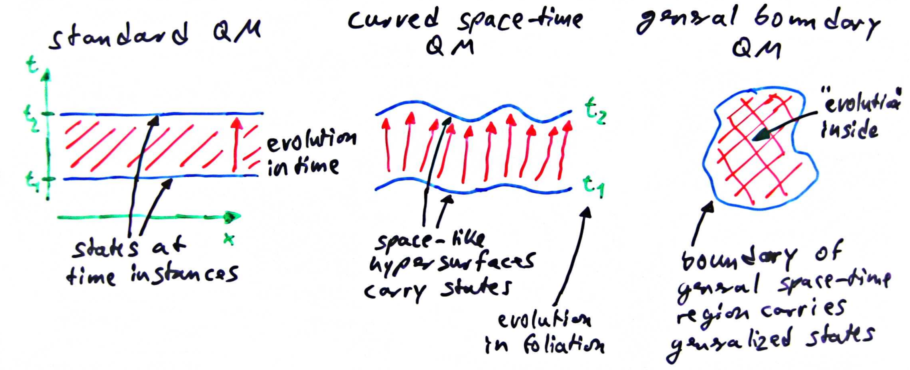
Glue solutions in spacetime regions at boundaries.

Quantum Field Theory

TQFT/GBQFT: Glue amplitudes in spacetime regions at boundaries.

- in non-relativistic (Q)M, there is only “temporal locality”
- spacetime locality might apply to **quantum gravity** as well

Amplitudes in spacetime regions



Lagrangian field theory

Formulate field theory in terms of first order Lagrangian density $\Lambda(\varphi, \partial\varphi, x)$. For a spacetime region M the **action** of a field ϕ is

$$S_M(\phi) := \int_M \Lambda(\phi(\cdot), \partial\phi(\cdot), \cdot).$$

Classical solutions in M are extremal points of this action. These are obtained by setting to zero the first variation of the action,

$$(\mathrm{d}S_M)_\phi(X) = \int_M X^a \left(\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu \frac{\delta\Lambda}{\delta\, \partial_\mu\varphi^a}\right)(\phi) + \int_{\partial M} X^a \partial_\mu \lrcorner \frac{\delta\Lambda}{\delta\partial_\mu\varphi^a}(\phi)$$

under the condition that the infinitesimal field X vanishes on ∂M. This yields the **Euler-Lagrange equations**,

$$\left(\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu \frac{\delta\Lambda}{\delta\, \partial_\mu\varphi^a}\right)(\phi) = 0.$$

The symplectic form

The boundary term can be defined for an arbitrary hypersurface Σ.

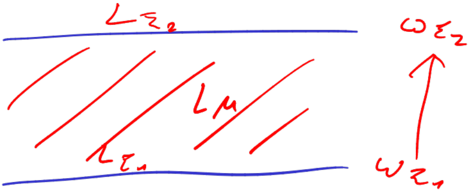
$$(\theta_\Sigma)_\phi(X) = -\int_\Sigma X^a \partial_\mu \lrcorner \frac{\delta\Lambda}{\delta\partial_\mu\varphi^a}(\phi)$$

This 1-form is called the **symplectic potential**. Its exterior derivative is the **symplectic 2-form**,

$$(\omega_\Sigma)_\phi(X,Y) = (\mathrm{d}\theta_\Sigma)_\phi(X,Y) = -\frac{1}{2}\int_\Sigma \Bigg((X^b Y^a - Y^b X^a)\,\partial_\mu \lrcorner \frac{\delta^2\Lambda}{\delta\varphi^b \delta\,\partial_\mu\varphi^a}(\phi) + (Y^a \partial_\nu X^b - X^a \partial_\nu Y^b)\,\partial_\mu \lrcorner \frac{\delta^2\Lambda}{\delta\,\partial_\nu\varphi^b \delta\,\partial_\mu\varphi^a}(\phi) \Bigg).$$

We denote the space of solutions in M by L_M and the space of germs of solutions on a hypersurface Σ by L_Σ.

Conservation of the symplectic form



globally hyp. s.t.
spacelike h.s.

$L_M \to L_{\Sigma_1} \times L_{\bar{\Sigma}_2}$

$\omega_{\Sigma_1} + \omega_{\bar{\Sigma}_2}$
$= \omega_{\Sigma_1} - \omega_{\Sigma_2}$

conservation

$\Leftrightarrow \quad L_M \subseteq L_{\Sigma_1} \times L_{\bar{\Sigma}_2}$

Lagrangian subspace

Lagrangian submanifolds

Let M be a region and $\phi \in L_{\partial M}$. Then ϕ may or may not be induced from a solution in M. If ϕ arises from a solution in M and X, Y arise from infinitesimal solutions in M, then,

$$(\omega_{\partial M})_\phi(X, Y) = (\mathrm{d}\theta_{\partial M})_\phi(X, Y) = -(\mathrm{dd}S_M)_\phi(X, Y) = 0.$$

This means, L_M induces an **isotropic** submanifold of $L_{\partial M}$.

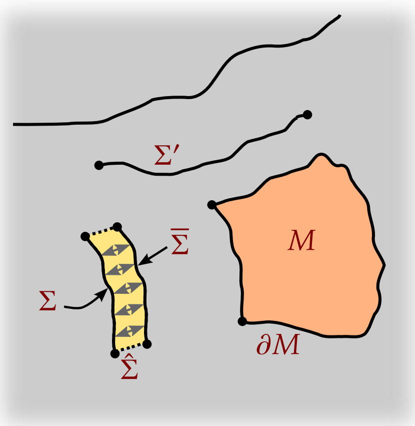
It is natural to require that the symplectic form is **non-degenerate**. We are then led to the converse statement: If given X we have $(\omega_{\partial M})_\phi(X, Y) = 0$ for all induced Y, then X itself must be induced. This means, L_M induces a **coisotropic** submanifold of $L_{\partial M}$.

L_M induces a **Lagrangian** submanifold of $L_{\partial M}$.

[Kijowski, Tulczyjew 1979]

Geometric setting – manifolds

Fix dimension d. Manifolds are **oriented** and may carry additional structure: differentiable, metric, complex, etc.



region M

d-manifold with boundary.

hypersurface Σ

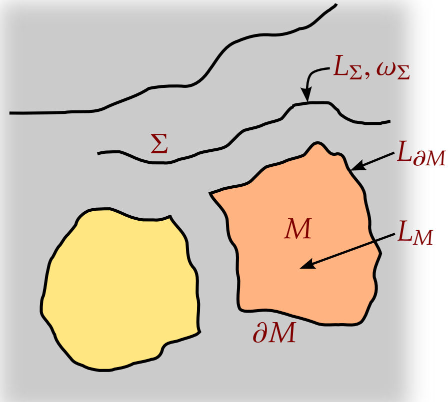
$d-1$-manifold with boundary, with germ of d-manifold.

slice region $\hat{\Sigma}$

$d-1$-manifold with boundary, with germ of d-manifold, interpreted as "infinitely thin" region.

Axiomatic classical field theory

[RO 2010]



per hypersurface Σ :

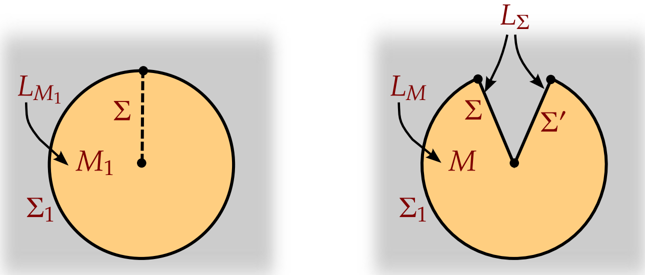
The **space of germs of solutions** near Σ. This is a **symplectic manifold** $(L_\Sigma, \omega_\Sigma)$.

per region M :

The **space of solutions** in M. Forgetting the interior yields a map $L_M \to L_{\partial M}$. Under this map L_M is a **Lagrangian submanifold** $L_M \subseteq L_{\partial M}$.

Axiomatic classical field theory

There are additional axioms related to gluing etc. like this one:



$$L_{M_1} \hookrightarrow L_M \rightrightarrows L_\Sigma$$

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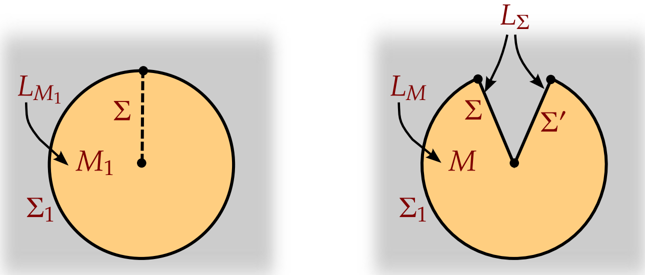


$$L_{M_1} \hookrightarrow L_M \rightrightarrows L_\Sigma$$

BUT, this does not work for many **non-compact regions**.

Canonical quantization in curved spacetime (review)

- L space of germs of solutions of the equations of motion (a real vector space). $L^{\mathbb{C}}$ complexification.
- $\omega : L \times L \to \mathbb{R}$ symplectic form – a bilinear antisymmetric form
- Define sesquilinear form $\boxed{(\phi, \phi') := 4\mathrm{i}\omega(\overline{\phi}, \phi')}$
- A **quantization** is determined by a complete set of modes $\{u_k\}_{k\in I}$:

$$(u_k, u_l) = \delta_{k,l}, \quad (\overline{u}_k, \overline{u}_l) = -\delta_{k,l}, \quad (u_k, \overline{u}_l) = 0, \quad \forall k, l \in I.$$

- $L^+ \subseteq L^{\mathbb{C}}$, $L^- \subseteq L^{\mathbb{C}}$ subspaces generated by the modes u_k, $\overline{u}_k$. Have $L^{\mathbb{C}} = L^+ \oplus L^-$ and $L^- = \overline{L^+}$.
- postulate corresponding creation and annihilation operators:

$$[a_k, a_l] = 0, \quad [a_k^\dagger, a_l^\dagger] = 0, \quad [a_k, a_l^\dagger] = \delta_{k,l}.$$

Positive-definite Lagrangian Subspaces

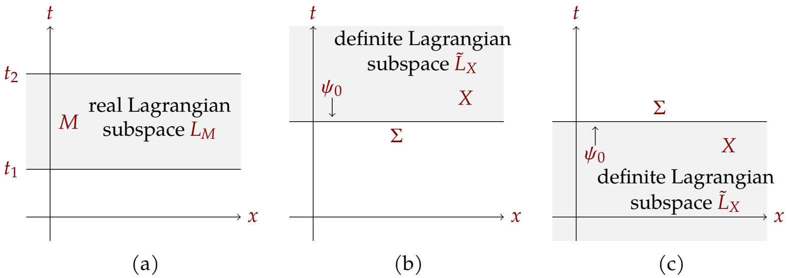
Alternative characterization:

Choice of vacuum corresponds to choice of **positive-definite Lagrangian subspace** L^+:

1. $L^+ \subseteq L^{\mathbb{C}}$ is Lagrangian subspace
2. L^+ is positive-definite with respect to $(\cdot,\cdot)$

$\{u_k\}_{k\in I}$ is just an ON-basis of L^+.

Extended classical axioms



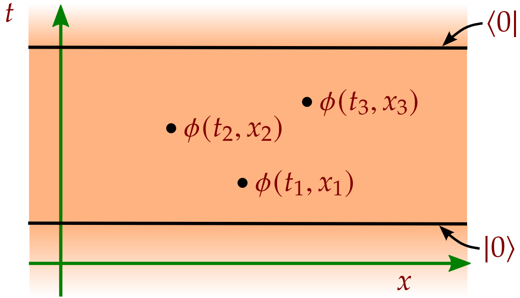
The classical axioms extend to general non-compact regions.

- For **compact** regions $L_M \subseteq L_{\partial M}^{\mathbb{C}}$ is the complexification of a **real Lagrangian subspace**.
- For **non-compact** regions $L_M \subseteq L_{\partial M}^{\mathbb{C}}$ is a general **complex Lagrangian subspace**.

The path integral and observables

To construct (asymptotic) amplitudes for **interacting theories** in QFT we need to insert **observables** into the path integral.

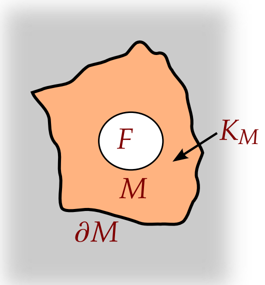
A typical example are n-point functions.



$$\langle 0|\mathbf{T}\phi(t_1,x_1)\phi(t_2,x_2)\phi(t_3,x_3)|0\rangle = \int \mathcal{D}\phi\, \phi(t_1,x_1)\phi(t_2,x_2)\phi(t_3,x_3)e^{\mathrm{i}S(\phi)}$$

Observables

In relativistic field theory observables need to be defined on configuration space.



per region M :

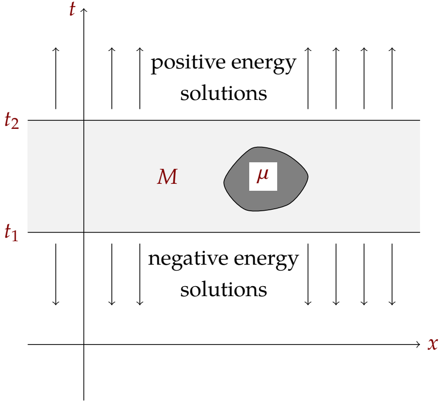
Assign the **configuration space** K_M in M. Have $L_M \subseteq K_M$. Also, a space C_M of **classical observables** given by maps $K_M \to \mathbb{R}$.

per region M with label F:

Assign the **classical observable** $F \in C_M$.

Evaluating the path integral with observables

Insert a source μ into the path integral of a transition amplitude.
$M = [t_1, t_2] \times \mathbb{R}^3$. $X = (-\infty, t_1] \times \mathbb{R}^3 \cup [t_2, \infty) \times \mathbb{R}^3$.



- Exterior vacuum is given by Lagrangian subspace $L_X^{\mathbb{C}} \subseteq L_{\partial X} = L_{\overline{\partial M}}$.
- Here, $L_X^{\mathbb{C}} = L_1^+ \oplus L_2^+ = L_1^- \oplus L_2^+$.
- Solve inhomogeneous equations $(\Box + m^2)\eta = \mu$ such that $\eta \in L_X^{\mathbb{C}}$. (Klein-Gordon example)

$$\int \mathcal{D}\phi\, e^{\mathrm{i}(S(\phi)+\int \mathrm{d}^4\, \phi\mu)} = \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^4\, \eta\mu\right)$$

Evaluating the path integral with observables

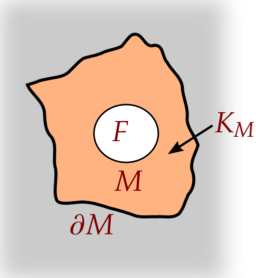
Write source μ as linear observable $D(\phi) := \int \mathrm{d}^4\, \phi(x)\mu(x)$.
Write exponentiated observable $F = \exp(\mathrm{i}D)$.

$$\rho_M^F(W) = \langle 0|\mathbf{T}F(\phi)|0\rangle = \int \mathcal{D}\phi\, F(\phi)e^{\mathrm{i}S(\phi)} = \int \mathcal{D}\phi\, e^{\mathrm{i}(S+D)(\phi)}$$
$$= \int \mathcal{D}\phi\, e^{\mathrm{i}(S(\phi)+\int \mathrm{d}^4\, \phi\mu)} = \exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^4\, \eta\mu\right) = \exp\left(\frac{\mathrm{i}}{2}D(\eta)\right)$$

This works for arbitrary **linear observables** D.

Evaluating the path integral with observables

- Suppose D is a **linear observable** $D : K_M \to \mathbb{R}$.
- Suppose F is the associated **Weyl observable**: $F = \exp(\mathrm{i}D)$
- Replace the quadratic action S by $S + D$. Let A_M^D denote the space of solutions of the new equations of motions.



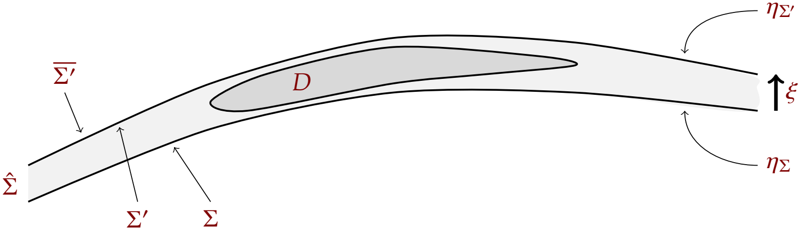
Let X denote the **exterior region**.
$L_X^{\mathbb{C}} \subseteq L_{\partial X}^{\mathbb{C}} = L_{\overline{\partial M}}^{\mathbb{C}}$ is Lagrangian.
$\eta \in \left(A_M^D \oplus \mathrm{i}L_M\right) \cap L_X^{\mathbb{C}}$ is (generically) unique.

$$D(\phi) = 2\omega_{\partial M}(\eta, \phi) \quad \text{for} \quad \phi \in L_M^{\mathbb{C}} \quad (1)$$

$$\boxed{\rho_M^F(W) = \exp\left(\frac{\mathrm{i}}{2} D(\eta)\right)} \qquad (2)$$

Slice observables

Σ a **hypersurface**, $\hat{\Sigma}$ associated **slice region**. $\partial\hat{\Sigma} = \Sigma \cup \overline{\Sigma'}$.
Set $K_{\hat{\Sigma}} = L_{\partial\hat{\Sigma}} = L_\Sigma \oplus L_{\overline{\Sigma'}}$.
Slice observable $D : K_{\hat{\Sigma}} \to \mathbb{C}$ induced by $D' : L_\Sigma \to \mathbb{C}$ via
$D(\phi, \phi') = D'(\frac{1}{2}(\phi + \phi'))$.



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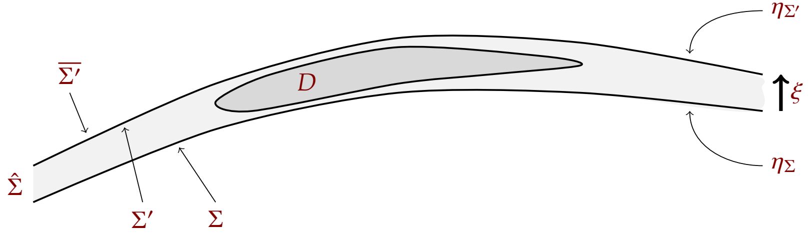
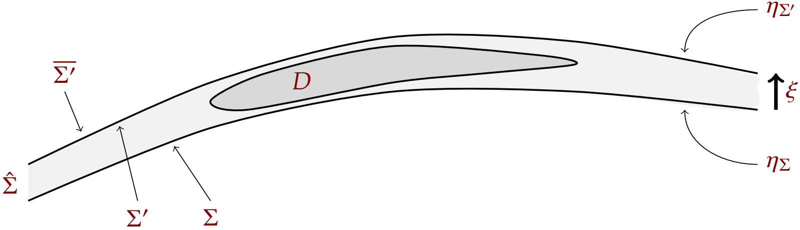


If D is **linear**, $\exists \xi \in L_\Sigma^{\mathbb{C}}$ s.t. $D'(\phi) = 2\omega_\Sigma(\phi, \xi)$. Then $D(\phi, \phi) = 2\omega_\Sigma(\phi, \xi)$.
Let $\eta \in A_M^D$. From (1) we conclude $D(\phi, \phi) = 2\omega_\Sigma(\eta_\Sigma - \eta_{\Sigma'}, \phi)$.

$$\boxed{\eta_{\Sigma'} - \eta_\Sigma = \xi}$$

Vacuum correlator of Weyl slice observable



Let D be linear with $D'(\phi) = 2\omega_\Sigma(\phi, \xi)$. Impose vacuum boundary conditions via Lagrangian subspace $L_X \subseteq L_{\partial X} = L_{\overline{\partial\hat{\Sigma}}} = L_\Sigma^+ \oplus L_\Sigma^-$.
Suppose factorization $L_X = L_\Sigma^+ \oplus L_\Sigma^-$ with $L_\Sigma^+ \subseteq L_\Sigma^{\mathbb{C}}$ and $L_\Sigma^- \subseteq L_{\Sigma'}^{\mathbb{C}}$.

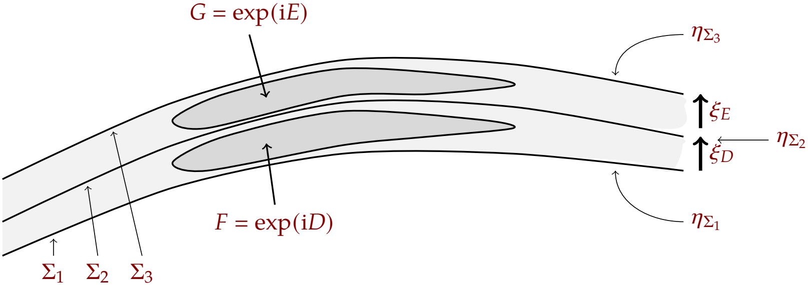
Then, $\eta_\Sigma \in L_\Sigma^+$ and $\eta_{\Sigma'} \in L_\Sigma^-$ while $\eta_{\Sigma'} - \eta_\Sigma = \xi$.

$$\boxed{\eta_\Sigma = -\xi^+ \qquad \eta_{\Sigma'} = \xi^-}$$

$$\rho_{\hat{\Sigma}}^F(W) = \exp\left(\frac{\mathrm{i}}{2} D'\left(\frac{1}{2}(\eta_\Sigma + \eta_{\Sigma'})\right)\right) = \exp\left(-\mathrm{i}\omega_{\overline{\Sigma}}(\xi^-, \xi^+)\right)$$

Composition of slice observables

Compose **linear slice observables** D and E and the corresponding **Weyl observables** F and G.



Compare $\rho_{\hat{\Sigma}}^{G \bullet F} = \rho_{\hat{\Sigma}}^{G} \diamond \rho_{\hat{\Sigma}}^{F}$ to $\rho_{\hat{\Sigma}}^{G \cdot F}$ via (2).

$$\exp\left(\frac{\mathrm{i}}{2}(E+D)(\eta)\right) = \exp\left(\frac{\mathrm{i}}{2}\left(E'\left(\frac{1}{2}(\eta_{\Sigma_2}+\eta_{\Sigma_3})\right) + D'\left(\frac{1}{2}(\eta_{\Sigma_1}+\eta_{\Sigma_2})\right)\right)\right)$$
$$= \exp\left(\mathrm{i}\omega_\Sigma(\xi_D,\xi_E)\right) \exp\left(\frac{\mathrm{i}}{2}(E'+D')\left(\frac{1}{2}(\eta_{\Sigma_1}+\eta_{\Sigma_3})\right)\right).$$

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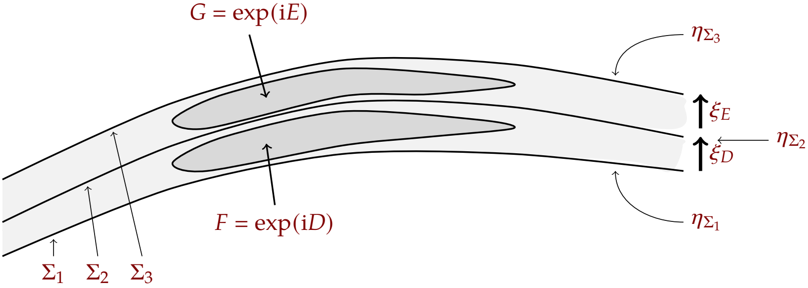

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$$\rho_{\hat{\Sigma}}^{G\bullet F} = \exp\left(\mathrm{i}\omega_\Sigma(\xi_D, \xi_E)\right) \rho_{\hat{\Sigma}}^{G\cdot F}$$

Quantum algebra of slice observables.

$$G \star F = \exp\left(\mathrm{i}\omega_\Sigma(\xi_D, \xi_E)\right) G \cdot F$$

GNS construction (I)

The vacuum correlation function defines a **linear functional on slice observables** $v_\Sigma(F) = \rho^F_{\hat{\Sigma}}(W)$.

The slice observables form a **$*$-algebra** with $F^*(\phi) = \overline{F(\overline{\phi})}$.
If the polarization is **Kähler**, we have a positive-definite Lagrangian subspace. In particular, $L^-_\Sigma = \overline{L^+_\Sigma}$. Moreover, for $\xi \in L_\Sigma$ with $\xi \neq 0$,

$$4\mathrm{i}\omega_{\overline{\Sigma}}(\xi^-, \xi^+) = 4\mathrm{i}\omega_{\overline{\Sigma}}(\overline{\xi^+}, \xi^+) = (\xi^+, \xi^+)_{\overline{\Sigma}} > 0$$

If F is a **Weyl observable** determined by $\xi \in L_\Sigma$,

$$v_\Sigma(F) = \exp\left(-\frac{1}{4}(\xi^+, \xi^+)_{\overline{\Sigma}}\right)$$

This implies that v_Σ is a **positive $*$-functional**.

GNS construction (II)

Construct the Hilbert space $\mathcal{H}_\Sigma$ from the quantum $*$-algebra $\mathcal{A}$ on L_Σ via the **GNS construction**.

Define on L_Σ the **hermitian sesquilinear form**.

$$[G, F] := v_\Sigma(G^* \star F)$$

Then $\mathcal{I} := \{A \in \mathcal{A} : [A, A] = 0\}$ is a **left ideal** and the **quotient** $\mathcal{A}/\mathcal{I}$ is a positive-definite inner product space. **Completion** yields the desired Hilbert space.

Here, $\mathcal{I}$ is generated by the relation $F \sim \mathbf{1}$ for F a Weyl observable determined by $\xi \in L_\Sigma^+$.

Under the quotient Weyl observables become **coherent states**.

α-Kähler quantization

$L_\Sigma^\pm \subseteq L_\Sigma^\mathbb{C}$ complementary Lagrangian subspaces. $L_\Sigma^\mathbb{C} = L_\Sigma^+ \oplus L_\Sigma^-$.

What if this is not a Kähler polarization?

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$L_\Sigma^\pm \subseteq L_\Sigma^\mathbb{C}$ complementary Lagrangian subspaces. $L_\Sigma^\mathbb{C} = L_\Sigma^+ \oplus L_\Sigma^-$.

What if this is not a Kähler polarization?

Positive-definite real structure α

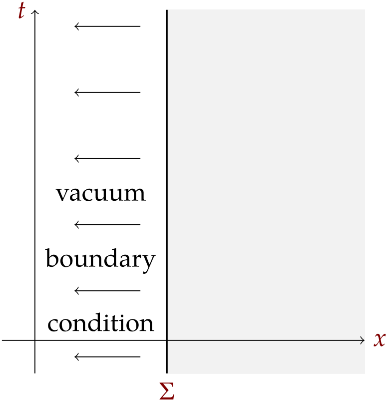
- $\alpha : L_\Sigma^\mathbb{C} \to L_\Sigma^\mathbb{C}$ anti-linear involution.
- $\alpha(L_\Sigma^\pm) = \alpha(L_\Sigma^\mp)$
- $\omega_\Sigma(\alpha(\phi), \alpha(\eta)) = \overline{\omega_\Sigma(\phi, \eta)}$
- $(\phi, \eta)_\Sigma^\alpha := 4\mathrm{i}\omega_\Sigma(\alpha(\phi), \eta)$ positive-definite on L_Σ^+.

Define new $*$-structure on $\mathcal{A}$: $F^\alpha(\phi) = \overline{F(\alpha(\phi))}$. This yields the twisted quantum $*$-algebra $\mathcal{A}^\alpha$. v_Σ is again a positive $*$-functional on $\mathcal{A}^\alpha$. We can apply the GNS construction to obtain the Hilbert space $\mathcal{H}_\Sigma^\alpha$.

Application: Evanescent particles

[D. Colosi, RO: arXiv:2104.12321]

Klein-Gordon theory in Minkowski space, Σ timelike hyperplane



- $L_\Sigma = L_\Sigma^{\mathrm{p}} \oplus L_\Sigma^{\mathrm{e}}$
- L_Σ^{p}: **propagating waves** (as on spacelike hypersurface)
- L_Σ^{e}: **evanescent waves**
- $L_\Sigma^{\mathrm{p},+} \subseteq L_\Sigma^{\mathrm{p}}$ is **positive-definite** Lagrangian subspace
- $L_\Sigma^{\mathrm{e},+} \subseteq L_\Sigma^{\mathrm{e}}$ is **real** Lagrangian subspace

This can be quantized via α-Kähler quantization. α arises from a **reflection**. (Compare **reflection-positivity** in Euclidean QFT.)