Def: Let E be a set. A mapping
$$d : E \times E \longrightarrow \mathbb{R}$$
is a <u>metric</u> if

(i) $d(x,y) \geq 0 \quad \forall x,y \in E$ and
$$d(x,y) = 0 \iff x = y$$

(ii) $d(x,y) = d(y,x) \quad \forall x,y \in E$

(iii) triangle inequality
$$d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in E.$$

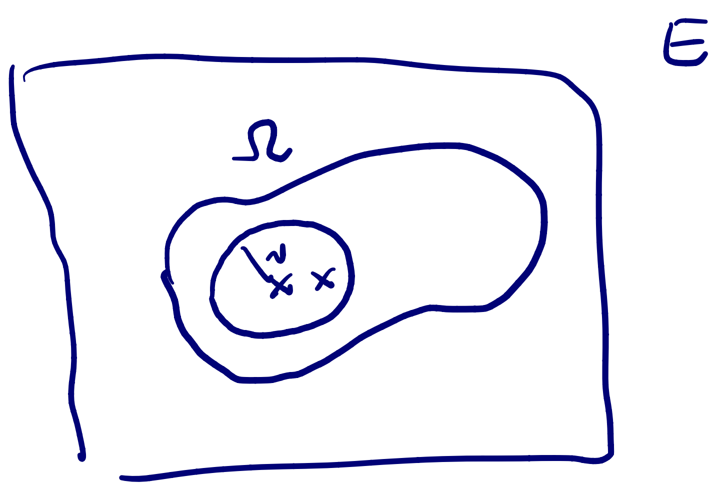
(E,d) is a <u>metric space</u>

<u>Example</u> $E = \mathbb{R}^3$ and
$$d(x,y) = \|x-y\| = \left(\sum_{\alpha=1}^{3} |x^\alpha - y^\alpha|^2\right)^{\frac{1}{2}}$$

<u>Def:</u> $B_r(x) := \{ y \in E \mid d(x,y) < r \} \quad r \geq 0$

open ball of radius r centred at x.

A subset $\Omega \subset E$ is <u>open</u> if
$$\forall x \in \Omega \ \exists r > 0 \text{ s.t. } B_r(x) \subset \Omega$$



$$\mathcal{O} := \{ \Omega \subset E \text{ open} \} \subset \mathcal{P}(E) \quad \text{topology}$$

<u>Lemma</u>: (i) $\emptyset \in \mathcal{O}$, $E \in \mathcal{O}$

(ii) $\Omega_1, \dots, \Omega_n \in \mathcal{O} \Longrightarrow \Omega_1 \cap \dots \cap \Omega_n \in \mathcal{O}$

(iii) $(\Omega_i)_{i \in J}$ and $\Omega_i \in \mathcal{O} \Longrightarrow \bigcup_{i \in J} \Omega_i$ open

↳ general index set

<u>Def</u>: Let E be a set.

Let $\mathcal{O} \subset \mathcal{P}(E)$ be a family of subsets of E with the above properties (i) – (iii).

$\mathcal{O}$ is the <u>topology</u>

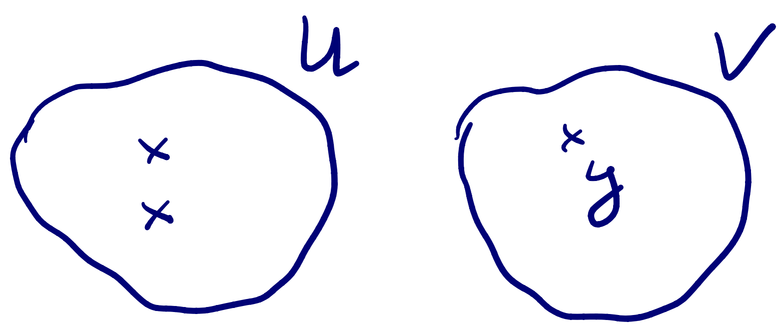
The sets in $\mathcal{O}$ are called <u>open</u>

$(E, \mathcal{O})$ is a <u>topological space</u>

<u>Def</u>: $(E, \mathcal{O})$ is <u>Hausdorff</u> if

$\forall x, y \in E$, $x \neq y$, $\exists\, U, V \in \mathcal{O}$ with

$U \cap V = \emptyset$ and $x \in U$, $y \in V$



<u>Def</u>: $A \subset E$ is <u>closed</u> if $\complement A = E \setminus A$ is open

$\overline{A} := \bigcap \{ B \supset A \mid B \text{ closed} \}$ closure

$\overline{A} \supset A$

$A \subset E$ is <u>compact</u> if every open covering of A has a finite subcovering.

<u>Def</u>: Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be two topological spaces

A mapping $f : E \to F$ is continuous if the preimage of every open set is open.