<u>Def</u>: Let $(\mathcal{H}, \langle.|.\rangle_{\mathcal{H}})$ be a complex Hilbert space.
For $p, q \in \mathbb{N}_0$, let
$\mathcal{F} := \{ A \in L(\mathcal{H})$ symmetric, finite rank,
has at most p positive,
at most q negative eigenvalues $\}$
Let ρ be a measure on $\mathcal{F}$.
$(\mathcal{H}, \mathcal{F}, \rho)$ is a topological fermion system.

$$\begin{cases} p = q: & \text{causal fermion system} \\ p = 0: & \text{Riemannian fermion system.} \end{cases}$$

Riemannian fermion system
$x, y \in \mathcal{F}$, x, y are negative semidefinite
$$xy = (-x)(-y) = \sqrt{-x}\,\sqrt{-x}\,(-y)$$
$$\simeq \sqrt{-x}\,(-y)\,\sqrt{-x} \leftarrow \text{symmetric, positive semidefinite}$$
$\Rightarrow$ eigenvalues are real and non-negative
then causal structure is trivial

Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a topological fermion system
$M := \operatorname{supp} \rho$ base space $\subset \mathcal{F}$
$S_x := x(\mathcal{H}) \subset \mathcal{H}$ of dimension $\leq p+q$
spin space, $\prec .|. \succ_x := -\langle . | x . \rangle_{\mathcal{H}}$

<u>Def</u>: system is <u>regular</u> if
$\operatorname{rank} x = p+q \quad \forall x \in M$

<u>Def</u>: <u>associated vector bundle</u>

$$SM := \bigcup_{x \in M} (x, S_x)$$

$$\pi : (x, S_x) \mapsto x$$

$$SM \longrightarrow M \quad \text{bundle projection}$$

This is a vector bundle with fibre $S_x \simeq \mathbb{C}^{p+q}$ and structure group $U(q,p)$

↑ isometries of S_x

<u>Thm</u>: For any topological vector bundle $\mathcal{B} \to M$ where the fibre is endowed with an inner product of signature (p,q) and structure group the isometries of this inner product

Then $\exists$ topological fermion system such that its associated vector bundle is $\mathcal{B}$.

If M is compact, then $\dim \mathcal{H} < \infty$

— " — non-compact, then $\mathcal{H}$ is separable.

<u>Def</u>: $\mathrm{Symm}(S_x)$ linear symmetric operators on S_x
↑ w.r.t. $\langle . | . \rangle$

A subspace $K \subset \mathrm{Symm}(S_x)$ is a <u>Clifford subspace</u> if

(i) $\forall u, v \in K$,
$$\{u, v\} := uv + vu \sim \mathbb{1}_{S_x}$$

(ii) The bilinear form $\langle . , . \rangle : K \times K \to \mathbb{R}$ defined by
$$\frac{1}{2}\{u, v\} = \langle u, v \rangle \mathbb{1}_{S_x}$$
is non-degenerate and has signature (r, s)

for causal fermion systems: Clifford extension
$K \ni s_x$ Euclidean sign operator

note: – Clifford subspaces are not unique

– a-priori no connection to the geometry of M in a neighborhood of x

Clifford section $\mathcal{C}l(M)$: $x \longmapsto K$ continuous, $K \subset \mathcal{C}l_x$

connection to spinor bundle:

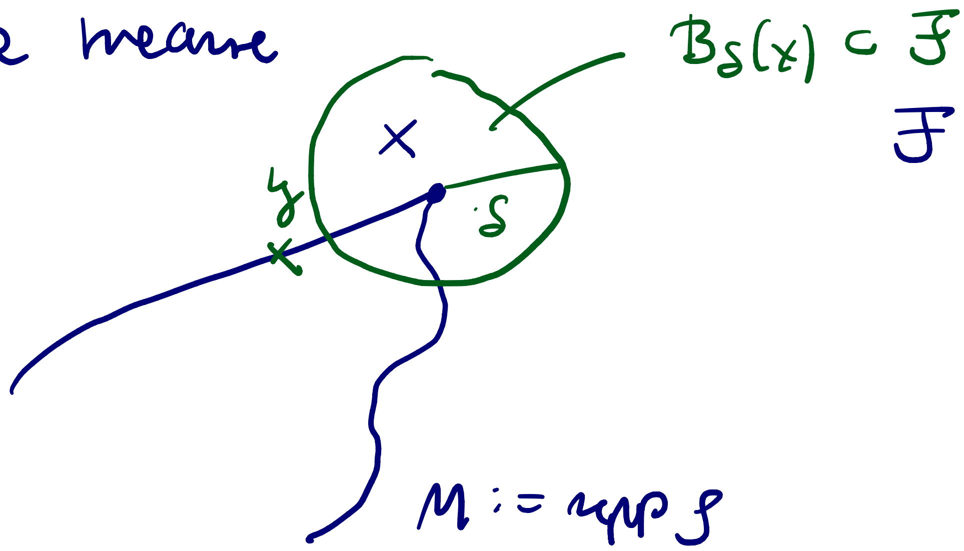
– assume that M is a smooth manifold

– assume mapping $\underline{T_x M} \longrightarrow \mathcal{C}l_x$ continuous

spin structure

Is there a canonical choice of $\mathcal{C}l(M)$ and of the spin structure?

tangent cone measure



$\operatorname{Symm}(S_x)$: Clifford subspaces

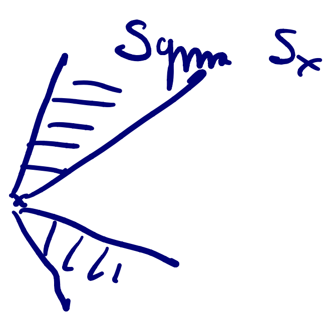
consider $A : M \longrightarrow \operatorname{Symm} S_x$ with $A(x) = 0$

for example $y \longmapsto \pi_x (y-x)\, X \,|_{S_x}$

introduce measure μ_x on $\operatorname{Symm}(S_x)$ by

$$\mu_x(\Omega) := \rho\big(A^{-1}(\Omega)\big)$$

Def: $\mathbb{R}^+ A$, $A \in \operatorname{Symm} S_x$ is a conical set



$$\mu^*_{\mathrm{con}}(A) = \lim_{\delta \to 0} \frac{1}{\rho(B_\delta(x))} \rho\left(A^{-1}(A) \cap B_\delta(x) \right)$$

this defines an outer measure ...

extend to measure μ_{con} on conical subsets of $\mathrm{Sym}(S_x)$

The tangent cone measure can be used to construct distinguished Clifford subspaces

$$\underset{\mathrm{Sym}(S_x)}{L(\mathcal{U})} \overset{}{} \quad \text{with } \mathcal{U} \subset \mathrm{Sym}(S_x)$$

$$L(\mathcal{U}) = \int_{S_1(0) \subset \mathrm{Sym}(S_x)} \mathrm{Tr}_{\mathrm{Sym}(S_x)} \left(\Pi_{\mathcal{U}} \, \Pi_{\langle e \rangle} \right) d\mu_x(\mathbb{R}^+ e)$$

minimize $\mathcal{U}$ over all Clifford subspaces

gives tangential Clifford section.