# Null shells: general matching of two spacetimes across null hypersurfaces

Miguel Manzano (joint work with Marc Mars)

Fundamental Physics and Mathematics Institute
University of Salamanca

*miguelmanzano06@usal.es*
*marc@usal.es*

February 1, 2021

1. Motivation
2. Introduction: geometric objects
3. General matching of two spacetimes across their null boundaries
4. Step function, diffeomorphism between the sets of null generators of $\Omega^{\pm}$
   - Totally geodesic null boundaries
5. Tensors $\boldsymbol{Y}^{\pm}$, energy-momentum tensor of the shell
6. Comparison with explicit cut-and-paste constructions
   - Plane-fronted impulsive wave
7. Summary and future work
8. Backup slides
   - Null spherically fronted wave

## Thin shells (or surface layers): null case

- Idealized objects
- Description of "sufficiently" narrow concentrations of matter or energy
- "Sufficiently" meaning: they can be considered to be located on a hypersurface
- Causal character of the hypersurface: null, timelike, spacelike or mixed
- Null shells, useful for modelling
  (a) Infinitesimally thin concentrations of massless particles
  (b) Impulsive waves
- Null shells possess their own gravity and hence affect the spacetime geometry
- Construction of a null shell
  (a) Consider two spacetime regions (one at each side of the null shell)
  (b) Perform a matching according to the corresponding theory

# Motivation

## Null thin shells

(a) "Cut-and-paste" method [Penrose, 1968]

(b) Matching theory [Darmois, 1927]

## "Cut-and-paste"

- Metric with a Dirac delta distribution with support on the shell
- Singular metric: standard tensor distributional calculus is not sufficient
- Spacetime $(\mathcal{M}, g)$, embedded null hypersurface $\Omega \subset \mathcal{M}$
  - (a) Cut-and-paste procedure uses lightlike coordinates adapted to $\Omega$
  - (b) $\Omega$ is removed by a cut, which leaves two separated manifolds $\left(\mathcal{M}^{\pm}, g^{\pm}\right)$
  - (c) Paste: boundary identification with a jump on the coordinates
- Jump is responsible for the appearance of the Dirac delta
- [Penrose, 1968, 1972]: impulsive plane-fronted, spherically-fronted waves in Minkowski
- [Podolskỳ et al., 2017] : pp-waves with additional gyratonic terms

## Null thin shells

(a) "Cut-and-paste" method [Penrose, 1968]

(b) Matching theory [Darmois, 1927]

## Matching theory of Darmois

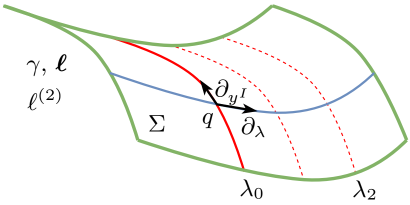
- Two spacetimes $\left(\mathcal{M}^{\pm}, g^{\pm}\right)$ with respective differentiable boundaries $\Omega^{\pm}$
- Construction of a new spacetime:
  - (a) Identification of the boundary points
  - (b) Identification of the full tangent spaces defined on $\Omega^{\pm}$
- $\left(\mathcal{M}^{\pm}, g^{\pm}\right)$ must satisfy the preliminary junction conditions [Clarke-Dray, 1987]
- $\Omega^{\pm}$ must be isometric with respect to their induced metrics
- Resulting spacetime:
  - (a) It verifies the well-known Israel equations [Israel, 1966]
  - (b) Thin layer with material content given by the jump on the extrinsic curvatures

$\nexists$ any systematic analysis of the connection between the cut-and-paste constructions and the matching theory of Darmois

## Objectives

- Detailed analysis of the Darmois matching theory across null hypersurfaces
- General conditions that allow for the matching in terms of
  - (a) The geometry of the ambient spaces
  - (b) The identification of boundaries
- Understand how the cut-and-paste construction fits into the matching framework

# Introduction: geometric objects



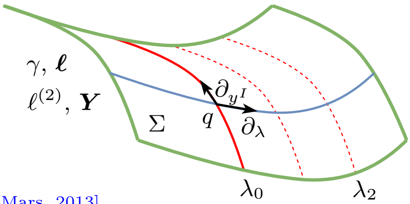

## $n-$dimensional manifold $\Sigma$

- $\gamma$: $2-$symmetric covariant tensor
- $\boldsymbol{\ell}$: $1-$form
- $\ell^{(2)}$: scalar function
- Coordinate system: $\{\lambda, y^A\}$

## Ambient tensor $\mathcal{A}|_p \in T_p\Sigma \times \mathbb{R}$

$$\mathcal{A}|_p\left((W,a),(Z,b)\right) := \gamma|_p(W,Z) + a\,\boldsymbol{\ell}|_p(Z) + b\,\boldsymbol{\ell}|_p(W) + ab\ell^{(2)}|_p, \\ W,Z \in T_p\Sigma, \qquad a,b \in \mathbb{R}$$

# Introduction: geometric objects




[Mars, 2013]

## (Metric) hypersurface data (MHD)

- $\{\Sigma, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ **metric hypersurface data**

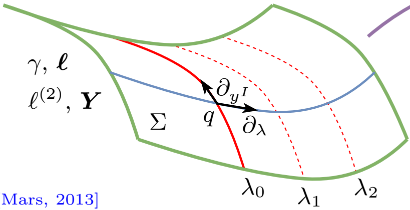
  $\mathcal{A}$ is non-degenerate everywhere in $\Sigma$

- $\{\Sigma, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \boldsymbol{Y}\}$ **hypersurface data**

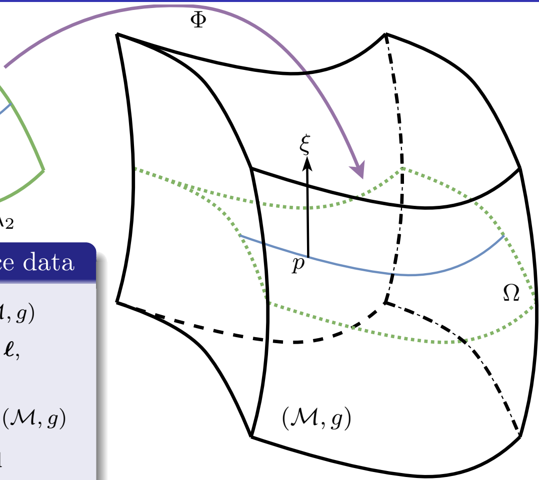
  $\{\Sigma, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ metric hypersurface data
  $\boldsymbol{Y}$ symmetric 2−covariant tensor field along $\Sigma$

## Ambient metric $\mathcal{A}|_p \in T_p\Sigma \times \mathbb{R}$

$$\mathcal{A}|_p \left((W,a),(Z,b)\right) := \gamma|_p (W,Z) + a\, \boldsymbol{\ell}|_p (Z) + b\, \boldsymbol{\ell}|_p (W) + ab\ell^{(2)}|_p, \quad W, Z \in T_p\Sigma, \qquad a, b \in \mathbb{R}$$




[Mars, 2013]



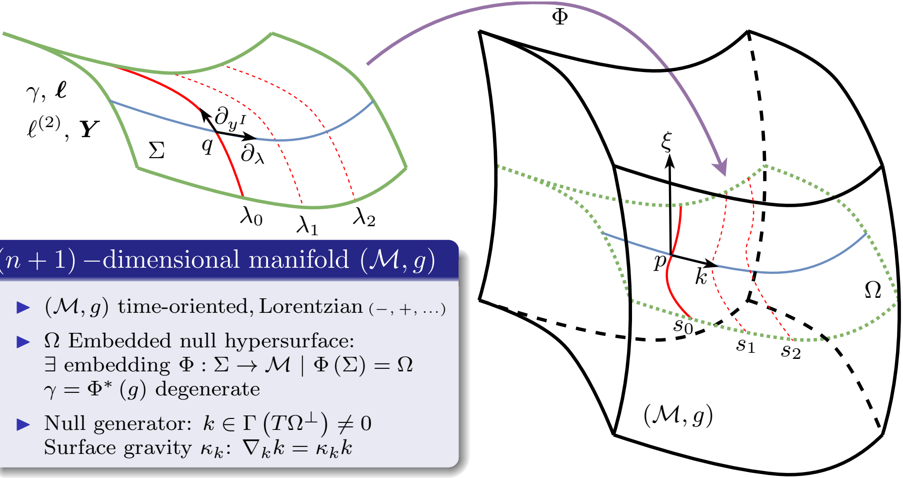

## Embedded (metric) hypersurface data

- $\{\Sigma, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ **embedded** in $(\mathcal{M}, g)$

$$\Phi^*(g) = \gamma,\ \Phi^*(g(\xi, \cdot)) = \boldsymbol{\ell},$$
$$\Phi^*(g(\xi, \xi)) = \ell^{(2)}$$

- $\{\Sigma, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \boldsymbol{Y}\}$ **embedded** in $(\mathcal{M}, g)$

$$\{\Sigma, \gamma, \boldsymbol{\ell}, \ell^{(2)}\} \text{ embedded}$$
$$\tfrac{1}{2}\Phi^*(\mathscr{L}_\xi g) = \boldsymbol{Y}$$




## $(n+1)$ −dimensional manifold $(\mathcal{M}, g)$

- $(\mathcal{M}, g)$ time-oriented, Lorentzian $(-,+,\dots)$
- $\Omega$ Embedded null hypersurface:
  $\exists$ embedding $\Phi : \Sigma \to \mathcal{M} \mid \Phi(\Sigma) = \Omega$
  $\gamma = \Phi^*(g)$ degenerate
- Null generator: $k \in \Gamma\left(T\Omega^\perp\right) \neq 0$
  Surface gravity $\kappa_k$: $\nabla_k k = \kappa_k k$

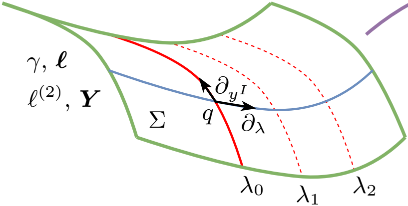
## Spacelike section, tangent plane and foliation of $\Omega$

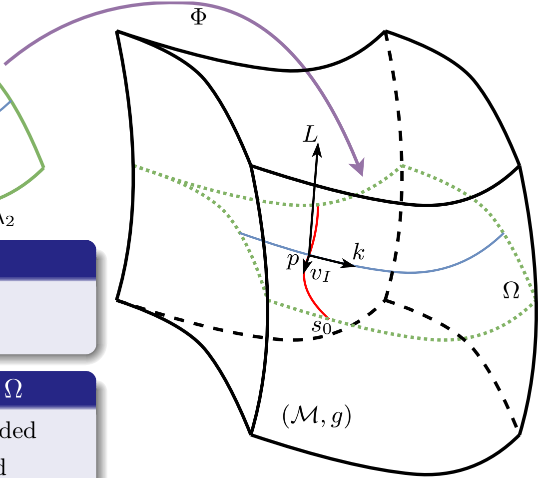
**Assumption**: existence of a function $s : \Omega \to \mathbb{R}$ such that $k(s) \neq 0$ in $\Omega$

Spacelike section: $S_{s_0} := \{p \in \Omega \mid s(p) = s_0,\ s_0 \in \mathbb{R}\}$

Tangent plane: $T_p S_{s(p)} := \{X \in T_p\Omega \mid X(s) = 0\}$

Family of $\{S_s\}$ define a foliation of $\Omega$ given by the subsets of $s = \text{const}$




### $\{k, v_I\}$: basis of $\Gamma(T\Omega)$

$$k(s) = 1, \quad v_I|_p \in T_p S_{s(p)},$$
$$[k, v_I] = [v_I, v_J] = 0$$

### $L$: transverse vector field along $\Omega$

- $\Omega$ boundary of $(\mathcal{M}, g)$: two-sided
- It always admits a vector field $L \notin T_p\Omega, \forall p \in \Omega$
- $\Omega$ null: $L$ can always be taken to be null everywhere

Definition of $n$ scalar functions on $\Omega$:

$$\varphi(p) := -\left.\langle L, k\rangle_g\right|_p \neq 0,$$
$$\psi_I(p) := -\left.\langle L, v_I\rangle_g\right|_p$$

### Tensors on $\Omega$

Given $p \in \Omega$, $Z, W \in T_p S_{s(p)}$ and a basis $\{v_I|_p\}$ of $T_p S_{s(p)}$

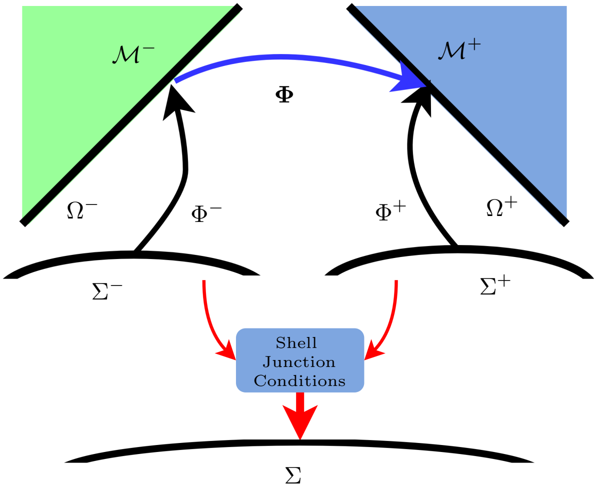
- Induced metric h of $S_{s(p)}$: $h(Z, W) := \langle Z, W \rangle_g$, $h$ components $h_{IJ}$
- $2^{\text{nd}}$ fundamental form $\boldsymbol{\chi}^k$ of $S_{s(p)}$ with respect to $k$: $\boldsymbol{\chi}^k(Z, W) := \langle \nabla_Z k, W \rangle_g$
- Tensors $\boldsymbol{\Theta}^L$ and $\boldsymbol{\sigma}_L$:

$$\boldsymbol{\Theta}^L(Z, W)|_p := \langle \nabla_Z L, W \rangle_g|_p \qquad \boldsymbol{\sigma}_L(Z)|_p := (1/\varphi)\langle \nabla_Z k, L \rangle_g|_p$$

- $\boldsymbol{\Theta}^L$ is the second fundamental form with respect to $L$ when $L$ is orthogonal to the section
- $\boldsymbol{\sigma}_L(Z)$ generalization of the torsion one-form

- Otherwise ($\psi_I = -\langle L, v_I \rangle_g \neq 0$), $\boldsymbol{\Theta}^L$ not a second fundamental form of $S_{s(p)}$ nor symmetric in general
- In many cases the most convenient choice of $L$ (e.g. to simplify computations) does not verify $\psi_I = 0$

# General matching of two spacetimes across their null boundaries



# General matching of two spacetimes across their null boundaries

## Matching problem

- Key objects to study the problem: $\mathbf{\Phi} := \Phi^{+} \circ (\Phi^{-})^{-1}$, $\xi^{\pm}$
  - $\mathbf{\Phi}$: points identification between $\Omega^{\pm}$, hence between $T_p\Omega^{-}$, $T_{\mathbf{\Phi}(p)}\Omega^{+}$
  - $\xi^{\pm}$: identification of the full tangent spaces $T_p\mathcal{M}^{-}$, $T_{\mathbf{\Phi}(p)}\mathcal{M}^{+}$
- $\Phi^{\pm}$, $\xi^{\pm}$ can be chosen freely on one side, say $\left(\mathcal{M}^{-}, g^{-}\right)$
- Matching problem requires determining $\Phi^{+}$, $\xi^{+}$ such that MHD$^{\pm}$ agree

## Choice of $\{\Phi^{-}, \xi^{-}\}$, construction of $\{\Phi^{+}, \xi^{+}\}$

- Basis of $T_{\Phi^{\pm}(q)}\Omega^{\pm}$: $\quad \{e_1^{\pm}|_{\Phi^{\pm}(q)} = \mathrm{d}\Phi^{\pm}|_q\,(\partial_\lambda)\,, \quad e_A^{\pm}|_{\Phi^{\pm}(q)} = \mathrm{d}\Phi^{\pm}|_q\left(\partial_{y^A}\right)\}$
- Use this to define the MHD$^{-}$ so that $\Phi^{-}$ is the identity map from $\Sigma$ to $\Omega^{-}$:

$$e_1^{-} = k^{-}, \quad e_I^{-} = v_I^{-}, \quad \xi^{-} = L^{-}$$

- First consequence: $e_1^{+}$ **null generator**. In terms of $\{k^{+}, v_I^{+}\}$, $L^{+}$:

$$e_1^{+} = \zeta k^{+}, \quad e_I^{+} = a_I k^{+} + b_I^J v_J^{+}, \quad \xi^{+} = \frac{1}{A}L^{+} + Bk^{+} + C^K v_K^{+}$$

# General matching of two spacetimes across their null boundaries

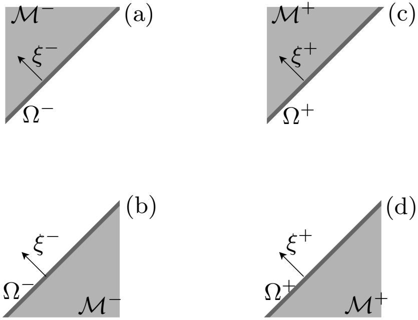
## Shell junction conditions (SJC)

$$h^{-}_{IJ}|_p = b^L_I b^K_J h^{+}_{LK}|_{\boldsymbol{\Phi}(p)}, \qquad \varphi^{-}|_p = \frac{\zeta\varphi^{+}}{A}\bigg|_{\boldsymbol{\Phi}(p)} \implies e^{+}_1 = (\varphi^{-}/\varphi^{+})Ak^{+},$$

$$-\psi^{-}_I|_p = -\frac{1}{A}\left(a_I\varphi^{+} + b^J_I\psi^{+}_J\right) + C^K b^J_I h^{+}_{JK}\bigg|_{\boldsymbol{\Phi}(p)},$$

$$0 = 2B\varphi^{+} + 2C^J\psi^{+}_J - AC^I C^J h^{+}_{IJ}\bigg|_{\boldsymbol{\Phi}(p)}$$

- ▷ $\psi^{\pm}_I$, $h^{\pm}_{IJ}$, $\varphi^{\pm}$ known
- ▷ $B$, $C^K$ uniquely solved in terms of $A$, $a_I$, $b^K_I$
- ▷ Matching depends on $A$, $a_I$, $b^K_I$




## Extra condition: riggings identification

- SJC: $\xi^{\pm}$ null and future
- Given $\{\Phi^{\pm}, \xi^{-}\}$, if $\exists$ solution for $\xi^{+}$ it is unique [Mars-Senovilla-Vera, 2007]
- Extra condition: $A > 0$

## Step function, diffeomorphism between the sets of null generators of $\Omega^{\pm}$

- The analysis of the shell matching conditions yields

$$\frac{\partial H\left(\lambda, y^A\right)}{\partial \lambda} = (\varphi^- / \varphi^+) A > 0, \qquad \frac{\partial H\left(\lambda, y^A\right)}{\partial y^I} = a_I, \qquad b_I^K = \frac{\partial h^K\left(y^A\right)}{\partial y^I}$$

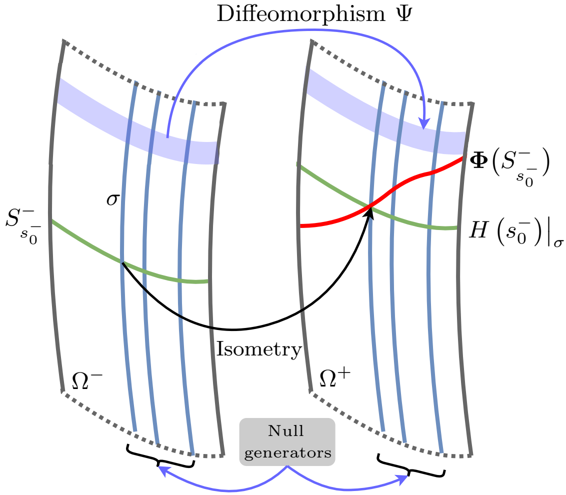
$$e_1^+ = \partial_\lambda H k^+, \qquad e_I^+ = \partial_{y^I} H k^+ + \partial_{y^I} h^J v_J^+$$

- Given $\{S^{\pm}_{s^{\pm}}\}$ defined by $s^{\pm}$, consequences of $k^{\pm}\left(s^{\pm}\right) = 1$, $v_I^{\pm}\left(s^{\pm}\right) = 0$ are

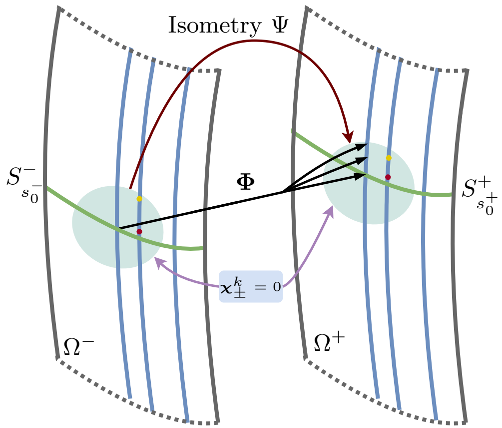
$$\left.\begin{array}{l} s^- \circ \Phi^- = \lambda + \text{const.} \\ s^+ \circ \Phi^+ = H + \text{const.} \end{array}\right\} \text{ on } \Sigma \quad \Longrightarrow \quad H(\lambda, y^A) \textbf{ step function}$$

- Immediate connection with cut-and-paste constructions
  - Initial works by Penrose: cutting out Minkowski across a null hyperplane and reattaching the two regions after shifting the null coordinate of one of the regions
  - This shift is recovered and generalized in the matching formalism

# Step function, diffeomorphism between the sets of null generators of $\Omega^{\pm}$



# Totally geodesic null boundaries




**Recall**: $e_1^- = k^-, \quad e_1^+ = (\partial_\lambda H) k^+,$

$$h^-_{IJ} = b^L_I b^K_J h^+_{LK},$$

$$k^\pm (h^\pm_{IJ}) = 2\boldsymbol{\chi}^{k^\pm}_\pm (v^\pm_I, v^\pm_J),$$

$$e_1^-(h^-_{IJ}) = \frac{\partial h^-_{IJ}}{\partial \lambda} = 2\boldsymbol{\chi}^{k^-}_- (v^-_I, v^-_J),$$

$$e_1^+(h^+_{IJ}) = \frac{\partial h^+_{IJ}}{\partial \lambda} = 2(\partial_\lambda H)\boldsymbol{\chi}^{k^+}_+ (v^+_I, v^+_J),$$

$$\frac{\partial h^-_{IJ}}{\partial \lambda} = \frac{\partial (b^A_I b^C_J)}{\partial \lambda} h^+_{AC} + b^A_I b^C_J \frac{\partial h^+_{AC}}{\partial \lambda} = b^A_I b^C_J \frac{\partial h^+_{AC}}{\partial \lambda}$$

$$\boldsymbol{\chi}^{k^-}_- (v^-_I, v^-_J) = (\partial_\lambda H) b^A_I b^C_J \boldsymbol{\chi}^{k^+}_+ (v^+_A, v^+_C)$$

# Tensors $\boldsymbol{Y}^{\pm}$, energy-momentum tensor of the shell

- Definitions: $\boldsymbol{Y}^{\pm} := \frac{1}{2}\Phi^{\pm *}\left(\mathcal{L}_{\xi^{\pm}} g^{\pm}\right), \quad \xi^{+} = \frac{1}{A}\left(L^{+} + X^{a} e_{a}^{+}\right), \quad \widehat{\rho}_{I} := b_{I}^{J} \rho_{J}$
- When the SJC are satisfied, the geometry of the shell is determined by the jump of the transverse tensors $\boldsymbol{Y}^{\pm}$, i.e. $\boldsymbol{V} := \left[\boldsymbol{Y}^{+} - \boldsymbol{Y}^{-}\right]$
- Energy-momentum tensor of the shell, denoted by $\tau$, is defined in the null case as
$$\tau^{11} := -\left(n^{1}\right)^{2} \gamma^{IJ} V_{IJ}, \qquad \tau^{1I} := \left(n^{1}\right)^{2} \gamma^{IJ} V_{1J}, \qquad \tau^{IJ} := -\left(n^{1}\right)^{2} \gamma^{IJ} V_{11}$$
- Its components can be interpreted as [e.g. Poisson, 2004]
$$\tau^{11} = \rho, \qquad \tau^{1I} = j^{I}, \qquad \tau^{IJ} = \gamma^{IJ} p$$
- Matching formalism provides the explicit form of $\boldsymbol{Y}^{\pm}$, $\tau$ in terms of $H$, $\Psi$ and the geometrical objects introduced before
- Imposing $b_{I}^{J} = \delta_{I}^{J}$, $X^{a} = 0$, $H\left(\lambda, y^{A}\right) = \lambda$ and $+ \longrightarrow -$ transforms $\boldsymbol{Y}^{+}$ into $\boldsymbol{Y}^{-}$

# Tensors $\boldsymbol{Y}^{\pm}$, energy-momentum tensor of the shell

$$Y^+_{11} = \varphi^- \left( \kappa^+_{k^+} \partial_\lambda H + \frac{\partial_\lambda \partial_\lambda H}{\partial_\lambda H} - \frac{\partial_\lambda \varphi^-}{\varphi^-} \right),$$

$$Y^+_{1J} = \varphi^- \left( \kappa^+_{k^+} \nabla^{\parallel}_J H - \boldsymbol{\sigma}^+_{L^+}\left(\hat{v}^+_J\right) + \frac{\partial_\lambda \partial_{y^J} H}{\partial_\lambda H} - \frac{X^L \boldsymbol{\chi}^{k^-}_- \left(v^-_J, v^-_L\right)}{\varphi^+ \partial_\lambda H} - \frac{\nabla^{\parallel}_J \varphi^-}{2\varphi^-} - \frac{\partial_\lambda \psi^-_J}{2\varphi^-} \right),$$

$$Y^+_{IJ} = \varphi^- \left( \frac{\kappa^+_{k^+} \nabla^{\parallel}_I H \ \nabla^{\parallel}_J H}{\partial_\lambda H} - \frac{\nabla^{\parallel}_{(I} H \ \partial_\lambda \hat{\psi}^+_{J)}}{\varphi^+ \left(\partial_\lambda H\right)^2} - \frac{2\nabla^{\parallel}_{(I} H \ \boldsymbol{\sigma}^+_{L^+}\left(\hat{v}^+_{J)}\right)}{\partial_\lambda H} + \frac{\boldsymbol{\Theta}^{L^+}_+ \left(\hat{v}^+_{(I}, \hat{v}^+_{J)}\right)}{\varphi^+ \partial_\lambda H} \right.$$
$$\left. + \frac{X^1 \boldsymbol{\chi}^{k^-}_- \left(v^-_I, v^-_J\right)}{\varphi^+ \partial_\lambda H} + \frac{\nabla^{\parallel}_I \nabla^{\parallel}_J H}{\partial_\lambda H} + \frac{\nabla^{\parallel}_{(I} \hat{\psi}^+_{J)}}{\varphi^+ \partial_\lambda H} - \frac{\nabla^{\parallel}_{(I} \psi^-_{J)}}{\varphi^-} \right)$$

$$\tau^{11} = -\frac{\gamma^{IJ}}{\varphi^-} \left( \frac{\kappa^+_{k^+} \nabla^{\parallel}_I H \ \nabla^{\parallel}_J H}{\partial_\lambda H} - \frac{\nabla^{\parallel}_I H \ \partial_\lambda \hat{\psi}^+_J}{\varphi^+ \left(\partial_\lambda H\right)^2} - \frac{2\nabla^{\parallel}_I H \ \boldsymbol{\sigma}^+_{L^+}\left(\hat{v}^+_J\right)}{\partial_\lambda H} + \frac{\boldsymbol{\Theta}^{L^+}_+ \left(\hat{v}^+_I, \hat{v}^+_J\right)}{\varphi^+ \partial_\lambda H} \right.$$
$$\left. + \frac{X^1 \boldsymbol{\chi}^{k^-}_- \left(v^-_I, v^-_J\right)}{\varphi^+ \partial_\lambda H} + \frac{\nabla^{\parallel}_I \nabla^{\parallel}_J H}{\partial_\lambda H} + \frac{\nabla^{\parallel}_I \hat{\psi}^+_J}{\varphi^+ \partial_\lambda H} - \frac{\nabla^{\parallel}_I \psi^-_J}{\varphi^-} - \frac{\boldsymbol{\Theta}^{L^-}_- \left(v^-_I, v^-_J\right)}{\varphi^-} \right),$$

$$\tau^{1I} = \frac{\gamma^{IJ}}{\varphi^-} \left( \kappa^+_{k^+} \nabla^{\parallel}_J H + \frac{\partial_\lambda \partial_{y^J} H}{\partial_\lambda H} - \frac{X^L \boldsymbol{\chi}^{k^-}_- \left(v^-_J, v^-_L\right)}{\varphi^+ \partial_\lambda H} - \left( \boldsymbol{\sigma}^+_{L^+}\left(\hat{v}^+_J\right) - \boldsymbol{\sigma}^-_{L^-}\left(v^-_J\right) \right) \right),$$

$$\tau^{IJ} = -\frac{\gamma^{IJ}}{\varphi^-} \left( \kappa^+_{k^+} \partial_\lambda H - \kappa^-_{k^-} + \frac{\partial_\lambda \partial_\lambda H}{\partial_\lambda H} \right)$$

# Comparison with explicit cut-and-paste constructions

## Motivation

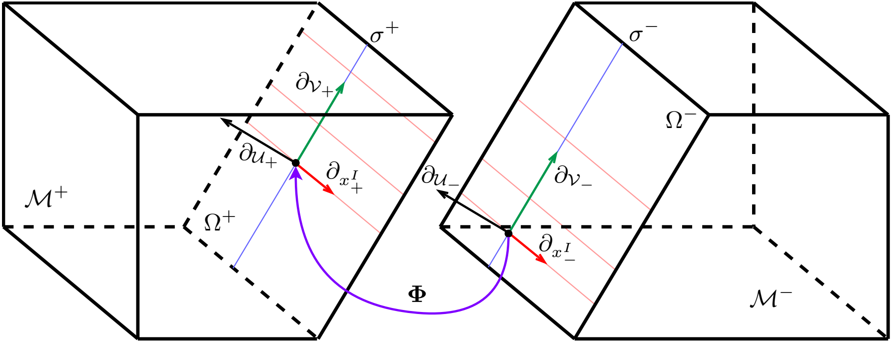
- This formalism allows for the analysis of multiple sorts of matchings at once
- Task easier and more systematic than with the cut and paste formalism
- Several matchings studied with cut-and-paste:
  - (a) Plane-fronted impulsive wave [Penrose, 1968]
  - (b) Robinson-Trautman null spherically fronted wave [Penrose, 1972]
  - (c) Generalizations of (a): non-zero $\Lambda$, pp-waves with gyratons [Podolský et al., 2017]
- They can be recovered and extended in the present formalism

## Plane-fronted impulsive wave

- Plane-fronted wave: $\mathrm{d}s^2 = -2\left(\mathrm{d}\mathcal{V} + \Psi\left(\mathcal{U}, x^A\right)\mathrm{d}\mathcal{U}\right)\mathrm{d}\mathcal{U} + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2$
- Impulsive limit: $\mathrm{d}s^2 = -2\left(\mathrm{d}\mathcal{V} + \delta\left(\mathcal{U}\right)\mathcal{H}\left(x^A\right)\mathrm{d}\mathcal{U}\right)\mathrm{d}\mathcal{U} + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2$
- $\exists$ coordinate transformation yielding a $C^0$-form [Podolský-Griffiths, 1999] [Podolský-Svarc-Steinbauer-Sämann, 2017] and proving $\mathcal{V}$ to be discontinuous on $\mathcal{U} = 0$
- Penrose's jump: $\mathcal{V}_+|_{\mathcal{U}_+=0} = \mathcal{V}_- + \mathcal{H}\left(x_-^A\right)|_{\mathcal{U}_-=0}$, obtained from transformation

# Matching of two Minkowski regions: plane case

- $(\mathcal{M}^{\pm}, g^{\pm})$: $\mathcal{U} \gtrless 0$ Minkowski regions, $\mathrm{d}s^2_{\pm} = -2\mathrm{d}\mathcal{V}_{\pm}\mathrm{d}\mathcal{U}_{\pm} + \delta_{AB} dx^A_{\pm} dx^B_{\pm}$, $s^{\pm} = \mathcal{V}_{\pm}$
- Penrose's jump $\mathcal{V}_+|_{\mathcal{U}_+=0} = \mathcal{V}_- + \mathcal{H}\left(x^A_-\right)|_{\mathcal{U}_-=0} \iff H\left(\lambda, y^A\right) = \lambda + \mathcal{H}\left(y^A\right)$




## Tensors $\boldsymbol{Y}^{\pm}$ and energy-momentum tensor of the shell

$$Y^-_{ab} = 0, \qquad Y^+_{11} = \frac{\partial_\lambda \partial_\lambda H}{\partial_\lambda H}, \qquad Y^+_{1I} = \frac{\partial_\lambda \partial_{y^I} H}{\partial_\lambda H}, \qquad Y^+_{IJ} = \frac{\partial_{y^I} \partial_{y^J} H}{\partial_\lambda H}$$

$$\tau^{11} = -\frac{\delta^{IJ} \partial_{y^I} \partial_{y^J} H}{\partial_\lambda H} = \rho, \quad \tau^{1I} = \frac{\delta^{IJ} \partial_\lambda \partial_{y^J} H}{\partial_\lambda H} = j^I, \quad \tau^{IJ} = -\frac{\delta^{IJ} \partial_\lambda \partial_\lambda H}{\partial_\lambda H} = \delta^{IJ} p$$

# Matching of two Minkowski regions: plane case

## No-shell case: $\boldsymbol{V} = 0$

$$H\left(\lambda, y^A\right) = a\lambda + c_J y^J + d,\ a > 0 \quad \Longleftrightarrow \quad \mathcal{V}_+ = a\mathcal{V}_- + c_J x_-^J + d$$

## Non-zero energy case: $\tau^{1J} = \tau^{IJ} = 0 \Longleftrightarrow \tau^{11}\left(y^A\right) \neq 0$

$$H\left(\lambda, y^A\right) = a\lambda + \mathcal{H}\left(y^A\right), \qquad \delta^{IJ}\frac{\partial^2\mathcal{H}}{\partial y^I \partial y^J} = -a\rho\left(y^A\right)$$

## General shell in Minkowski: $\tau^{ab} \neq 0$, $p = -\partial_\lambda\left(\ln\left(\partial_\lambda H\right)\right)$

$$H\left(\lambda, y^A\right) = \alpha\left(y^A\right)\int \exp\left(-\int p\left(\lambda, y^A\right) \mathrm{d}\lambda\right)\mathrm{d}\lambda + \mathcal{H}\left(y^A\right), \quad \alpha\left(y^A\right) > 0$$

- $$\begin{cases} e_1^- = k^-, \\ e_1^+ = (\partial_\lambda H)k^+ \\ \nabla^{\pm}_{k^{\pm}} k^{\pm} = 0 \end{cases} \Longrightarrow \begin{cases} e_1^-\left(s^-\right) = 1, & \nabla^-_{e_1^-} e_1^-\left(s^-\right) = 0 \\ e_1^+\left(s^+\right) = \partial_\lambda H, & \nabla^+_{e_1^+} e_1^+\left(s^+\right) = \partial_\lambda\partial_\lambda H \end{cases}$$
- $\exists$ *self-compression (resp. self-stretching) whenever the acceleration measured by $\lambda$ is strictly negative (resp. positive)*, effect ruled by $p\left(\lambda, y^A\right)$
- $p\left(\lambda, y^A\right) > 0$ *pushes points towards* $\downdownarrows H\left(\lambda, y^A\right)$ *(or* $\downdownarrows s^+$*) and vice versa*

## $\lambda$−dependent pressure: $p(\lambda) = -\frac{\mu''}{\mu'}$

- $\mu(\lambda)$ is any regular function with $\mu'(\lambda) > 0\ \forall \lambda$
- Step function: $H\left(\lambda, y^A\right) = \alpha\left(y^A\right)\mu(\lambda) + \mathcal{H}\left(y^A\right)$, conditions $\alpha(y^A) > 0$, $\lim_{\lambda\to\pm\infty}\mu(\lambda) = \pm\infty$
- Behaviour of $\tau^{ab}$ ruled by $\mu(\lambda)$, $\alpha\left(y^A\right), \mathcal{H}\left(y^A\right)$

$$\mu(\lambda) := (a+1)\lambda - \sqrt{a}\nu(\lambda)$$
$$\nu(\lambda) := \sqrt{(a+2)\lambda^2 + b^2},\ a > 0 \text{ and } b \text{ real constants}$$

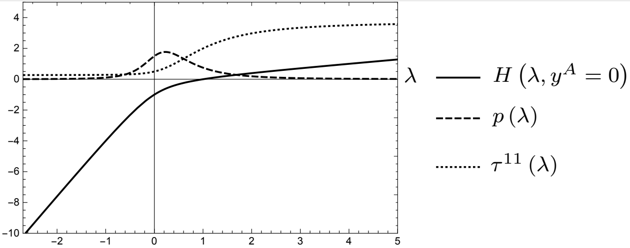



Figure: Matching of regions $\mathcal{U} \lessgtr 0$ of Minkowski: plot of $p(\lambda)$, $H\left(\lambda, y^A\right)$, $\tau^{11}(\lambda)$ along the null generator $\{y^A = 0\}$ for $a = 1$, $b = 1$, $\alpha(y^A) = 1$ and $\mathcal{H}(y^A) = \frac{1}{2(n-1)}\delta_{IJ}y^I y^J$.

# Summary and future work

## Summary

- Necessary and sufficient conditions that allow for the matching
- Matching depends on
  - (a) Diffeomorphism between the set of null generators in each boundary
  - (b) Step function: shift of points along the null generators
- Generically $\exists$ at most one possible matching
- Totally geodesic boundaries: multiple (even infinite) matchings are possible
- The expression for the energy-momentum tensor of a general null shell
- The most general null shell generated by matching two Minkowski regions
  - (a) Non-zero energy
  - (b) Non-zero energy flux
  - (c) Non-zero pressure: "self-compression/self-stretching"
- Penrose's cut-and-paste constructions and matching formalism connected

## Future work: cut-and-paste constructions

- Understand the 4-dimensional cases
- Find generalizations to arbitrary dimensions
  - (a) Null spherically fronted wave
  - (b) Impulsive gravitational waves with non-zero cosmological constant $\Lambda$
  - (c) Impulsive waves with gyratons

## Null spherically fronted wave

- [Penrose, 1972] : limiting case of Robinson-Trautman null spherically fronted wave
- Transformation $u = \mathcal{U}$, $v = \mathcal{V} + \eta\,\bar{\eta}\,\mathcal{U}$, $\zeta = \mathcal{U}\,\eta$ turns $\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + 2\mathrm{d}\zeta\mathrm{d}\bar{\zeta}$ into

$$\mathrm{d}s^2 = -2\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V} + 2\mathcal{U}^2\mathrm{d}\eta\mathrm{d}\bar{\eta}$$

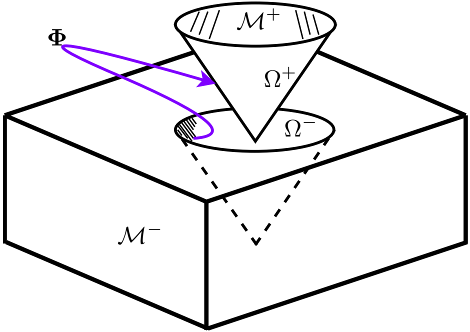
- Penrose cuts the spacetime along the null cone defined by $\mathcal{V} = 0$
- Paste is only possible if the identification between $\Omega^{\pm} \equiv \mathcal{V}_{\pm} = 0$ is given by

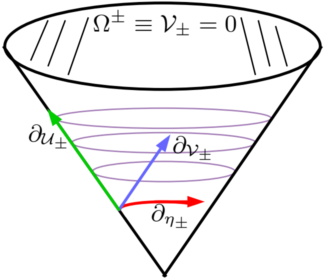
$$\mathcal{U}_+ = \mathcal{U}_- \left|\frac{\mathrm{d}h\left(\eta\right)}{\mathrm{d}\eta}\right|^{-1}, \qquad \mathcal{V}_+ = \mathcal{V}_- = 0, \qquad \eta_+ = h\left(\eta_-\right), \qquad \bar{\eta}_+ = \bar{h}\left(\bar{\eta}_-\right),$$

$h\left(\eta\right)$ is an holomorphic function

# Matching of two Minkowski regions: spherical case

- $(\mathcal{M}^{\pm}, g^{\pm})$: $\mathcal{V} \gtrless 0$ regions, $\mathrm{d}s_{\pm}^2 = -2\mathrm{d}\mathcal{U}_{\pm}\mathrm{d}\mathcal{V}_{\pm} + 2\mathcal{U}_{\pm}^2 \mathrm{d}\eta_{\pm}\mathrm{d}\bar{\eta}_{\pm}$, $s^{\pm} = \mathcal{U}_{\pm}$
- Notation: $\eta_- = y = y^2 = \overline{y^3}$, $\eta_+ = h(y, \bar{y}) = h^2(y, \bar{y}) = \overline{h^3(y, \bar{y})}$
- Penrose's jump: $\mathcal{U}_+ = \mathcal{U}_- \left| \dfrac{\mathrm{d}h(\eta)}{\mathrm{d}\eta} \right|^{-1}$, $\mathcal{V}_+ = \mathcal{V}_- = 0$, $\eta_+ = h(\eta_-)$




- Isometry condition:

$$h = h(y), \qquad \lambda^2 = H^2(\lambda, y, \bar{y}) \left| \frac{dh}{dy} \right|^2 \implies H(\lambda, y, \bar{y}) = \lambda \left| \frac{\mathrm{d}h}{\mathrm{d}y} \right|^{-1}$$

# Matching of two Minkowski regions: spherical case

## Tensors $\boldsymbol{Y}^{\pm}$ and energy-momentum tensor of the shell

- Transverse tensor $\boldsymbol{Y}^{-} = 0 \quad \Longrightarrow \quad \boldsymbol{V} = \boldsymbol{Y}^{+}$
- Only non-vanishing component: $V_{II} = \lambda \dfrac{\partial_{y^I} \partial_{y^I} \mathcal{F}}{\mathcal{F}}, \qquad \mathcal{F}(y, \bar{y}) = \left| \frac{dh}{dy} \right|^{-1}$
- Energy-momentum tensor: $\tau^{ab} = 0$
- Only $\boldsymbol{V} \neq 0$, shell of purely gravitational nature
- The most general matching gives rise to an impulsive gravitational wave

## Consistency check

- Absence of shell, i.e. $\boldsymbol{V} = 0$
- Under these conditions, $h(y)$ defines a Möbius transformation, i.e.

$$h(y) = \frac{\alpha y + \beta}{\gamma y + \delta}, \quad \alpha, \beta, \gamma, \delta \text{ complex constants}, \quad \alpha\delta - \beta\gamma = 1$$

- Möbius transf. can be absorbed by a Lorentz transf. in $\left(\mathcal{M}^{+}, g^{+}\right)$
- Matching recovers the global spacetime