Quantization:
The Big Picture

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A brief history of "quantization" 1: Heisenberg c.s.

- Heisenberg (1925), *Quantum-mechanical re-interpretation of kinematical and mechanical relations*, contains essence of quantization: classical observable (function) $\rightsquigarrow$ matrix
- Born–Heisenberg–Jordan (1926), *On quantum mechanics II*, relates quantization to classical limit, in the light of CCR

$$pq - qp = \frac{h}{2\pi i} \cdot \mathbf{1} \tag{5}$$

'*We will later discuss the physical significance of this relation according to the correspondence principle*'

'*classical mechanics may be regarded as the limiting case of quantum mechanics when $\hbar$ tends to zero*'

'*one can see from eq. (5) that in the limit $h = 0$, the new theory would converge to the classical theory, as is physically required.*' (if only things were as simple as that . . .)

A brief history of "quantization" 2: Schrödinger

Schrödinger 1926 (2.5), *On the relation between the quantum mechanics of Heisenberg, Born, and Jordan, and mine*: '*to each function of the position- and momentum-co-ordinates there may be related a matrix in such a manner that these matrices, in every case, satisfy the formal calculating rules of Born and Heisenberg (among which I also reckon the so-called "quantum condition" or "commutation relation"* [i.e.

$$(p_l q_l - q_l p_l)^{ik} = K\delta_{ijk} \qquad (K = h/(2\pi\sqrt{-1})) \tag{11}$$

]), (...), it is understood that we could have also found relation (11) by taking the two matrices allied to q_l and p_l, viz.

$$q_l^{ik} = \int q_l \rho(x) u_i(x) u_k(x) dx;$$

$$p_l^{ik} = K \int \rho(x) u_i(x) \frac{\partial u_k(x)}{\partial q_l} dx$$

This is $q^j \mapsto \hat{q}_j = M(x^j)$ and $p_j \mapsto \hat{p}_j = -i\hbar\partial/\partial x^j$ quantization

A brief history of "quantization" 3: Dirac

Dirac (1925), *The fundamental equations of quantum mechanics*:

'*The difference between the Heisenberg product of two quantum quantities is equal to $ih/2\pi$ times their Poisson bracket expression*'

$$xy - yx = \frac{ih}{2\pi}\{x, y\} \quad \Rightarrow \quad q_k p_l - p_l q_k = \frac{ih}{2\pi}\{q_k, p_l\} = \frac{ih}{2\pi}\delta_{kl}$$

'*The strong analogy between the quantum P.B. [i.e. commutator $\times(2\pi i/h)$] and the classical P.B. leads us to make the assumption that the quantum P.B.'s, or at any rate the simpler ones of them, have the same values as the corresponding classical P.B.'s.*'

'*The correspondence between the quantum and classical theories lies not so much in the limiting agreement when $h \to 0$ as in the fact that the mathematical operations on the two theories obey in many cases the same laws.*' *within entirely different mathematical contexts!* Dirac (1930): '*classical mechanics may be regarded as the limiting case of quantum mechanics when $\hbar$ tends to zero.*'

Intermezzo: ‘The magic year 1927’ (Mackey)

- Hilbert, von Neumann, & Nordheim (1927), *On the foundations of quantum mechanics*
- von Neumann (1927a), *Mathematical foundation of QM*
- von Neumann (1927b), *Probabilistic development of QM*
- Peter & Weyl (1927), *The completeness of the primitive representations of a closed continuous group* (24/7/1926)
- (Wigner (1927ab): $\mathscr{S}_n$ and SO(3) symmetries of Hamiltonian)
- Weyl (1927), *QM and group theory* (13/10/1927)
- Weyl (1928), *Group Theory and Quantum Mechanics* (book)

Independently of QM, Hilbert’s former PhD student Hermann Weyl had begun to develop the theory of unitary group representations

Because of QM, Hilbert’s postdoc John von Neumann formalized Hilbert-school functional analysis into theory of Hilbert spaces

These lines crossed in 1927, leading vN and W to study unitary group representation theory on infinite-dimensional Hilbert spaces

A brief history of "quantization" 4: Weyl

Weyl (1927) distinguished two '*very similar questions*' in QM:

1. '*How to construct the Hermitian matrix that represents some quantity of a known physical system?*' ('*left open by JvN*')
2. '*Given this Hermitian form, what is their physical meaning?*' (Weyl regarded this problem as solved by von Neumann)

Weyl: group theory answers 1. CCR $[p,q] = -i\hbar \rightsquigarrow$ projective unitary representation of $\mathbb{R}^2 \cong$ unitary representation of $\mathrm{Heis}(\mathbb{R}^2)$:

- $p \rightsquigarrow$ unitary representation U of $\mathbb{R}$: $U(a) = \exp(ia\hat{p})$
- $q \rightsquigarrow$ unitary representation V of $\mathbb{R}$: $V(b) = \exp(ib\hat{q})$

This changes the CCR (involving unbounded operators) to

$$U(a)V(b) = e^{i\hbar ab}V(b)U(a)$$

Quantization formula for phase space functions $f(p,q)$ added in 2nd ed. (1931) of *Group Theory and Quantum Mechanics* (1928)

$$f(p,q) \mapsto \int\int da\, db\, \hat{f}(a,b)e^{ia\hat{p}+ib\hat{q}} = \int\int da\, db\, \hat{f}(a,b)e^{i\hbar ab/2}U(a)V(b)$$

A brief history of "quantization" 5: Stone–von Neumann

Weyl (1928, without proof), Stone (1930, with sketch of proof), von Neumann (1931, with complete proof), Stone–vN theorem:

Theorem

For fixed $\hbar \neq 0$, up to unitary equivalence there is a unique unitary irreducible representation (i.e. on Hilbert space) of the Weyl–CCR

$$U(a)V(b) = e^{i\hbar ab} V(b)U(a)$$

such that $a \mapsto U(a)$ and $b \mapsto V(b)$ are strongly continuous (equivalently via Stone's theorem: there are associated self-adjoint operators $\hat{p}$ and $\hat{q}$ such that $U(a) = \exp(ia\hat{p})$ and $V(b) = \exp(ib\hat{q})$)

C*-algebraically: uniqueness of regular irreducible representations of Weyl-CCR algebra $\mathscr{W}(M,\sigma)$ over finite-dimensional symplectic space (M,σ), but this algebra is very awkward ($=$ twisted group C*-algebra over $\mathbb{R}^2$ but with horrible discrete topology on $\mathbb{R}^2$) and should be replaced by Buchholz–Grundling C*-algebra

Later approaches to quantization

After first push (1925–1930) quantization theory branched off:

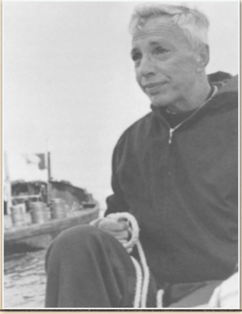
- Deformation quantization (emphasizing classical limit, downplaying symmetry): Groenewold (1946), Moyal (1949)
  - (a) Formal (= algebraic) deformation quantization: Berezin (1975), Flato–Lichnerowicz–Sternheimer (1976)
  - (b) Strict (= C*-algebraic) deformation quantization (Rieffel)

  Applicable to arbitrary phase spaces (= Poisson manifolds)
- Weyl's program (emphasizing symmetry, downplaying classical limit): Mackey (1957), induced group representations
  Applicable to arbitrary configuration spaces (notably: G/H)
- Geometric quantization (*idem dito*): Souriau (1969), Kostant (1970), constructs Hilbert spaces from line bundles etc.
  Applicable to homogeneous symplectic manifolds $M = G/H$
- Phase space (= Borel = POVM = coherent state) quantization, path integral quantization, stochastic ...

In remainder I will explain strict deformation quantization and Weyl's program and, though complementary, relate these

Hermann Weyl
(1885-1955)

George Mackey
(1916-2006)



"Hip" Groenewold
(1910-1996)

Marc Rieffel
(1937)

Mackey's approach to quantization (slightly reformulated)

Mackey breaks phase space symmetry between p and q! He exponentiates CCR $[p,q] = -i\hbar$ to system of imprimitivity:

- $p \rightsquigarrow$ unitary representation of $\mathbb{R}$ on $\mathscr{H}$, $U(a) = \exp(ia\hat{p})$
- $q \rightsquigarrow$ representation π of C*-algebra $C_0(\mathbb{R})$ on $\mathscr{H}$ (abstraction of Schrödinger's $\hat{q}\psi(x) = x\psi(x)$ to $\pi(f)\psi(x) = f(x)\psi(x)$)
- CCR $\rightsquigarrow$ covariance condition $U(a)\pi(f)U(a)^* = \pi(f \circ -a)$

Generalization: group G acts on (configuration) space Q

- Unitary representation U of G on Hilbert space $\mathscr{H}$
- Nondegenerate representation π of C*-algebra $C_0(Q)$ on $\mathscr{H}$
- Covariance condition $U(a)\pi(f)U(a)^* = \pi(f \circ a^{-1})$

Classification or irreps known if $Q = G/H$ with left G=action:

Theorem (Mackey's imprimitivity theorem)

Covariant irreps of $(G, C_0(G/H)) \leftrightarrow$ unitary irreps of H (up to ...)

Corollary: if $Q = G$ (e.g. $Q = G = \mathbb{R}^n$) then $\exists!$ 1 irrep ($H = \{e\}$)
(Mackey's PhD advisor) Stone–von Neumann uniqueness theorem

Mackey's approach: smooth setting

Mackey never mentioned Poisson brackets or explained which phase space he quantized (obvious choice T^*Q cannot be right)

- Mackey worked topologically. If G Lie group, Q manifold, action $G \circlearrowright Q$ smooth, and $U(G)$ regular (i.e. exponentiates Lie algebra $\mathfrak{g}$ representation à la Stone–Nelson), then system of imprimitivity may be defined via commutation relations:

$$[dU(A), dU(B)] = dU([A,B]) \qquad [\pi(f), \pi(g)] = 0$$

$$[dU(A), \pi(f)] = \pi(\xi_A f) \quad \xi_A f(q) := \frac{d}{dt} f\left(e^{-tA} q\right)_{t=0}$$

which for $G = Q = \mathbb{R}^n$ and $G \circlearrowright Q$ left action retrieves CCR:

$$[\hat{p}_j, \hat{p}_k] = 0 \qquad [\hat{q}^j, \hat{q}^k] = 0 \qquad [\hat{p}_j, \hat{q}_k] = -i\hbar\delta_{jk}$$

This is the key for a formulation in terms of Poisson brackets:

Mackey's quantization: Poisson brackets

Underlying phase space (= manifold M with Poisson bracket on $C^\infty(M)$) for Mackey turns out to be $M = \mathfrak{g}^* \times Q$ with P.B.

$$\{\hat{A}, \hat{B}\} = -\widehat{[A,B]} \qquad \{f,g\} = 0 \qquad \{\hat{A}, f\} = -\xi_A f$$

where, for $A \in \mathfrak{g}$, $\hat{A} \in C^\infty(\mathfrak{g}^*)$ is $\hat{A}(\theta) := \theta(A)$, and $f \in C^\infty(Q)$

With $Q_\hbar(A) := i\hbar dU(A)$ and $Q_\hbar(f) := \pi(f)$, the CCR become:

$$\frac{i}{\hbar}[Q_\hbar(\hat{A}), Q_\hbar(\hat{B})] = Q_\hbar(\{\hat{A}, \hat{B}\}); \quad \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] = Q_\hbar(\{f,g\}) = 0$$

$$\frac{i}{\hbar}[Q_\hbar(\hat{A}), Q_\hbar(f)] = Q_\hbar(\{\hat{A}, f\})$$

Mackey's quantization fits Dirac's "P.B. $\rightsquigarrow$ commutator" ideology

though underlying phase space $\mathfrak{g}^* \times Q$ is (often) not symplectic!

(Poisson manifolds M are foliated by symplectic leaves on which the P.B. is nondegenerate in that any point can dynamically move to any other point via Hamiltonian flow; this is not true for M)

Mackey's quantization: inequivalent possibilities

Special case $Q = G/H$: by imprimitivity theorem inequivalent irreducible representations of system of imprimitivity $(G, C_0(G/H))$ $\leftrightarrow$ unitary irreducible representations of H. Mackey's key example:

$$G = E(3) := SO(3) \ltimes \mathbb{R}^3 \qquad Q = \mathbb{R}^3 \cong G/SO(3)$$

so inequivalent irreducible representations $\leftrightarrow$ unitary irreps of H

$\leftrightarrow$ spin $j = 0, 1, 2 \ldots$ (if $SO(3) \rightsquigarrow SU(2)$ then $j \in \mathbb{N}/2$)

This was Mackey's explanation of spin which fitted (Pauli's) idea that spin is a purely QM phenomenon. This is wrong (Souriau)

- Quantum: irreps of system of imprimitivity $(G, C_0(Q))$
- Classical: symplectic leaves of Poisson manifold $M = \mathfrak{g}^* \times Q$

Symplectic leaves of $M = \mathfrak{g}^* \times (G/H) \leftrightarrow$ coadjoint orbits of $\mathfrak{h}^*$ (so M is symplectic iff $\mathfrak{h} = 0$ and then $M \cong T^*G$). For $G = E(3)$ and $Q = \mathbb{R}^3$ have $\mathfrak{h}^* \cong \mathbb{R}^3$ with Euler's P.B., whose symplectic leaves are spheres, so classical spin exists with continuous values $j \geq 0$

Groenewold–Rieffel: (strict) deformation quantization

Groenewold's (and Moyal's) idea: for phase space M, deform commutative algebra $C^\infty(M)$ "in the direction of the Poisson bracket" (Dirac) into some non-commutative algebra (Heisenberg)

Formalized by Rieffel (1989) in language of C*-algebras = "nice" algebras of bounded operators on Hilbert space, also defined abstractly, with ensuing representation theory on Hilbert spaces, like groups, which unify commutative and non-commutative worlds

- Continuous field of C*-algebras (Dixmier, 1962) is (*not necessarily locally trivial*) bundle whose fibers are C*-algebras
- Strict quantization of Poisson manifold M is continuous field of C*-algebras over $I \subseteq [0,1]$ with commutative C*-algebra $A_0 = C_0(M)$ and non-commutative C*-algebras $A_\hbar$ at $\hbar > 0$, plus quantization maps $Q_\hbar : A_0 \to A_\hbar$ satisfying the condition

$$\lim_{\hbar \to 0} \left\| \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\}) \right\|_{A_\hbar} = 0$$

Some examples of strict deformation quantization

- Non-commutative tori (Rieffel, 1989)
- Weyl quantization (Rieffel, 1994; Connes, 1994)
- Toeplitz quantization of compact Kähler manifolds (Bordemann– Meinrenken–Schlichenmaier, 1994)
- Coadjoint orbits of compact Lie groups (1998)
- Mackey quantization ("The big picture", 1998)
- Symmetric spaces (Bieliavsky, 2000)
- Symplectic manifolds (Natsume–Nest–Peter, 2003)
- ∞-dim. symplectic spaces (Binz–Honegger–Rieckers, 2004)
- Symplectic groupoids (Hawkins, 2008)
- Quantum field theory (Buchholz–Lechner-Summers, 2011)
- Philosophy of physics, classical limit (Feintzeig, 2017–2021)
- Buchholz–Grundling reolvent algebra (van Nuland, 2019)
- Quantum spin systems (van de Ven *et al.*, 2021–2021)
  N.B. $N = 1/\hbar$, then $\hbar \to 0$ = thermodynamic limit $N \to \infty$

Intermezzo: Lie groupoids and Lie algebroids

Groupoids (small categories with inverses) generalize sets, groups, group actions, equivalence relations: partial (associative) multiplication and local units (inverse defined everywhere)

1. group G: multiplication always defined, single unit
2. space M: multiplication $x \cdot y = x$ defined iff $y = x$
3. pair groupoid $M \times M$, $(u, v) \cdot (w, x) = (u, x)$ defined iff $v = w$
4. equivalence relation $R \subset M \times M$ (Grothendieck)
5. action groupoid $\Gamma = G \ltimes Q$ from group action $G \circlearrowright Q$: multiplication $(g, x) \cdot (h, y) = (gh, x)$ defined iff $y = gx$

Smooth (Lie) groupoids have associated "infinitesimal" objects: Lie algebroids E (vector bundles), e.g. $\mathfrak{g}$, M, TM, $E \subset TM$, $\mathfrak{g} \times Q$ (!)

Lie groupoid Γ has associated C*-algebra $C^*(\Gamma)$ (Connes)

Lie algebroid E has associated Poisson manifold E^* (Weinstein)

Combining Mackey with Groenewold–Rieffel

Given smooth group action $G \circlearrowright Q$ Mackey unwittingly quantized Poisson manifold $\mathfrak{g}^* \times Q$, which is the Poisson manifold associated to the Lie agebroid of the action Lie groupoid $G \ltimes Q$, suggesting:

- Strict deformation quantization of commutative C*-algebra $C_0(\mathfrak{g}^* \times Q)$ by non-commutative C*-algebras $A_\hbar = C^*(G \ltimes Q)$

This works well, and can be generalized to vast class of examples:

- Lie groupoid Γ defines both C*-algebra $C^*(\Gamma)$ and Lie algebroid $\mathrm{Lie}(\Gamma)$ with associated Poisson manifold $\mathrm{Lie}(\Gamma)^*$
- Continuous field of C*-algebras on $[0,1]$ with fibers $A_0 = C_0(\mathrm{Lie}(\Gamma)^*)$ and $A_\hbar = C^*(\Gamma)$ for $\hbar \in (0,1]$
- Quantization maps $Q_\hbar : A_0 \to A_\hbar$ generalizes Weyl's formula
- Axiom $\lim_{\hbar \to 0} \left\| \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f,g\}) \right\|_{C^*(\Gamma)} = 0$ holds

Proof is based on fact that C*-algebra $C^*(A)$ of continuous cross-sections of this field A is the C*-algebra of a Lie groupoid, namely the tangent groupoid of Γ (introduced by Connes in NCG)

Summary

- Quantization pioneers: Heisenberg (the very idea of quantization), Schrödinger (the first prescription for p and q), Dirac (Poisson bracket $\rightsquigarrow$ commutator), Weyl (group theory)
- First mathematical result: Stone–von Neumann theorem
- After first push (1925–1930) quantization theory branched off:

  Deformation quantization (emphasizing classical limit)

  Mackey–Weyl program (emphasizing symmetry and S-vN)
- Deformation quantization: deform commutative algebra of classical algebra of observables in direction of Poisson bracket so as to become non-commutative (Groenewold, Moyal)
- Unwittingly, Mackey quantized (non-symplectic) Poisson manifold $\mathfrak{g}^* \times Q$ and hence can be bracketed under Dirac
- Mackey quantization and a vast generalization of it to Lie groupoid C*-algebras (on quantum side, borrowed from NCG) and Lie algebroid Poisson manifolds (on classical side) falls under the scope of Rieffel's strict deformation quantization

Some literature

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