smooth setting, ρ a critical measure, ...

Noether-like theorem:

$\Phi_\tau : M \to \mathcal{F}$, $\tau \in (-\delta, \delta)$

symmetry $\mathcal{L}(\Phi_\tau(x), y) = \mathcal{L}(x, \Phi_{-\tau}(y))$

Let $\Omega \subset M$ be compact

$$\frac{d}{d\tau} \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right) \Big|_{\tau=0} = 0$$

$$v(x) = \frac{d}{d\tau} \Phi_\tau(x) \Big|_{\tau=0} \in \Gamma(M, T\mathcal{F})$$

symmetry $(D_{1,v} + D_{2,v})\, \mathcal{L}(x,y) = 0$

$$\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, (D_{1,v} - D_{2,v})\, \mathcal{L}(x,y) = 0,$$

corresponding to J_k^Ω (not J_{k+1}^Ω)!

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general class of conservation laws for $k=1$.

$\tilde{\rho}_s = (F_s)_* (f_s \rho)$ family of critical measures

$$\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, (\partial_{1,s} - \partial_{2,s}) \left(f_s(x)\, \mathcal{L}(F_s(x), F_s(y))\, f_s(y) \right) \Big|_{s=0}$$

$$= s \int_\Omega \partial_s f_s(x)\, d\rho(x) \Big|_{s=0}$$

$\underline{v} = \frac{d}{ds}(f_s, F_s)\Big|_{s=0}$ is a solution of the linearized field eqns $\langle \underline{u}, \Delta \underline{v} \rangle(x) = 0$

$$\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(x)\, (\nabla_{1,\underline{v}} - \nabla_{2,\underline{v}})\, \mathcal{L}(x,y)$$

$$= \int_\Omega \nabla_{\underline{v}}\, s\, d\rho(x)$$

$\underline{\text{conserved one-form}}$

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alternative proof:

$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(x) \underbrace{\left(\nabla_{1,\underline{v}} - \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y)}_{\text{anti-symmetric}}$$

$$= \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(x) \left(\nabla_{1,\underline{v}} - \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y)$$

$$+ \underbrace{\int_\Omega d\rho(x) \int_{\Omega} d\rho(x) \left(\nabla_{1,\underline{v}} - \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y)}_{=0}$$

$$= \int_\Omega d\rho(x) \int_{M} d\rho(x) \left(\nabla_{1,\underline{v}} - \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y)$$

$$= \int_\Omega \nabla_{\underline{v}}\, \mathfrak{s} \; d\rho(x)$$

↑ linearized field eqns

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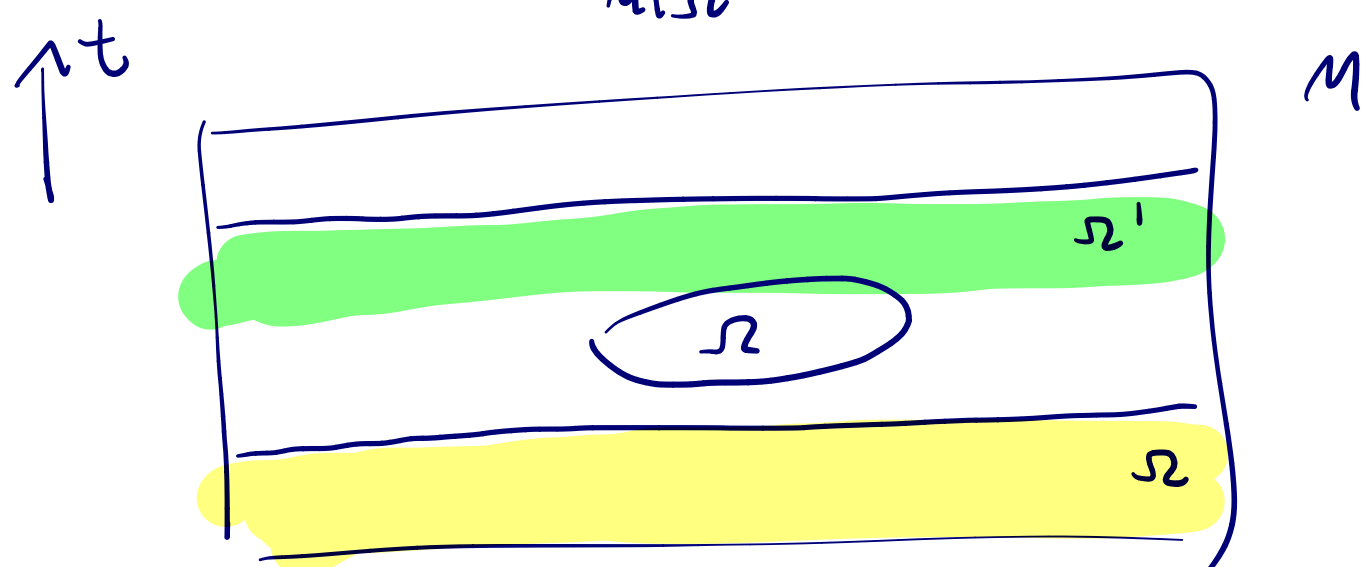
back to causal fermion systems, $\dim \mathscr{H} < \infty$

$\mathcal{C}(x) = i\,[A, x]$, $\quad A$ symmetric operator on $\mathscr{H}$

$\underline{v} = (0, \mathcal{C})$ $\quad$ commutator jet

is a solution of the linearized field eqns without scalar component

$$\Rightarrow \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \left(D_{1,\mathcal{C}} - D_{2,\mathcal{C}}\right) \mathcal{L}(x,y) = 0$$



$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \; (D_{1,\varphi} - D_{2,\varphi}) \,\mathcal{L}(x,y)$$

$$= \int_{\Omega'} d\rho(x) \int_{M\setminus\Omega'} d\rho(y) \; (D_{1,\varphi} - D_{2,\varphi} | \mathcal{L}(x,y))$$

Consider the case $A = |u\rangle\langle u|$, $u \in \mathcal{H}$

$$\langle u|u\rangle^\Omega_\rho := \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \; \big(D_{1,\varphi(A)} - D_{2,\varphi(A)} \big) \mathcal{L}(x,y)$$

Polarization gives

$\langle u|v\rangle^\Omega_\rho : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ — commutator inner product, sesquilinear form

can be expressed in terms of physical wave functions ψ^u and ψ^v

generalized to wave functions $\psi, \phi \in C^0(M, SM)$

Back to causal variational principles

assume that M has a smooth manifold structure

$\underline{v} = (\operatorname{div} v, v)$ inner solution, $v \in \Gamma(M, TM)$

conserved one-form

$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \; \big(\nabla_{1,\underline{v}} - \nabla_{2,\underline{v}} \big) \mathcal{L}(x,y)$$

$$\overset{\text{Gauß}}{=} \int_{\partial\Omega} d\mu(v,x) \int_{M\setminus\Omega} d\rho(y) \; \mathcal{L}(x,y)$$

$$- \int_\Omega d\rho(x) \Big(- \int_{\partial\Omega} d\mu(v,y) \Big) \mathcal{L}(x,y)$$

$$= \int_{\partial\Omega} d\mu(v,x) \Big(\int_M \mathcal{L}(x,y)\, d\rho(y) \Big)$$

$$= \alpha \cdot \int_{\partial\Omega} d\mu(v, x)$$

explain $d\mu$

$$d\mu = h(x)\, d^k x$$

volume form

$$\omega = h(x)\, \epsilon_{i_1 \cdots i_k}\, dx^{i_1} \cdots dx^{i_k} \quad k\text{-form}$$

$$\omega \llcorner v = h(x)\, \epsilon_{i_1 \cdots i_k}\, v^{i_1}\, dx^{i_2} \wedge \cdots \wedge dx^{i_k} \quad (k-1)\text{-form}$$

$d\mu(v, x)$ is the corresponding volume measure