$\mathcal{F}$ smooth manifold

$\rho$ solve EL eqns

$$\nabla_{\underline{u}} \left( \int_M \left(\nabla_{1,\underline{v}} + \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y)\, d\rho(y) - \nabla_{\underline{v}}\, \mathfrak{s} \right) = 0 \qquad \forall\, \underline{u} \in \mathfrak{J}^{\text{test}}$$

1) unitary invariance for causal fermion systems

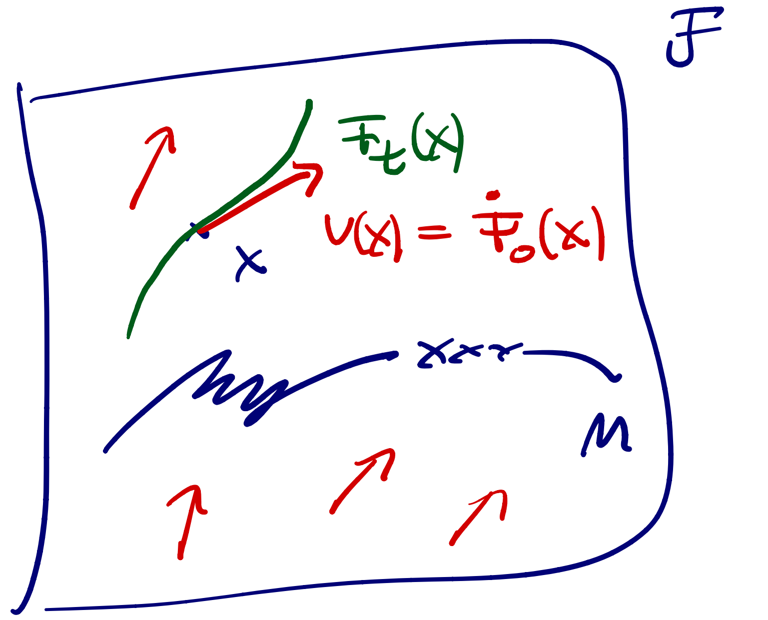
$\dim \mathcal{H} < \infty$, $\qquad \mathcal{F} = \mathcal{F}^{\text{reg}}$

let $A \in \mathrm{L}(\mathcal{H})$ symmetric

$\mathcal{U}_t = e^{itA}$ family of unitary transformations

let $x \in \mathcal{F}$

$$F_t(x) := \mathcal{U}_t\, x\, \mathcal{U}_t^{-1}$$




$$v(x) := \frac{d}{dt} F_t(x) \Big|_{t=0}$$
$$= i\,[A, x]$$

commutator vector field

$$\mathcal{C}_A(x) = i\,[A, x]$$

$v(x) := \mathcal{C}_A(x)$ with $x \in M$ is in $\Gamma(M, T\mathcal{F})$

$\underline{v} := (0, v)$ is the commutator jet

The Lagrangian is unitarily invariant

$$\mathcal{L}\big(F_t(x), F_t(y)\big) = \mathcal{L}(x,y)$$

<u>Lemma</u>: For any symmetric $A \in \mathrm{L}(\mathcal{H})$, the corresponding commutator jet satisfies the linearized field eqns.

**Proof:** $\frac{d}{dt} \mathcal{L}(F_t(x), F_t(y)) \big|_{t=0}$

$$= \left(\nabla_{1,\underline{v}} + \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y) = 0$$

$\nabla_{\underline{v}}\, s = 0$ (because $\underline{v}$ has no scalar component)

$$\Rightarrow \left( \int_M \left(\nabla_{1,\underline{v}} + \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y)\, d\rho(y) - \nabla_{\underline{v}}\, s \right) = 0$$

$$\forall x \in \mathcal{F}$$

$$\Rightarrow \nabla_{\underline{u}} \left( \int_M \left(\nabla_{1,\underline{v}} + \nabla_{2,\underline{v}}\right) \mathcal{L}(x,y)\, d\rho(y) - \nabla_{\underline{v}}\, s \right) = 0$$

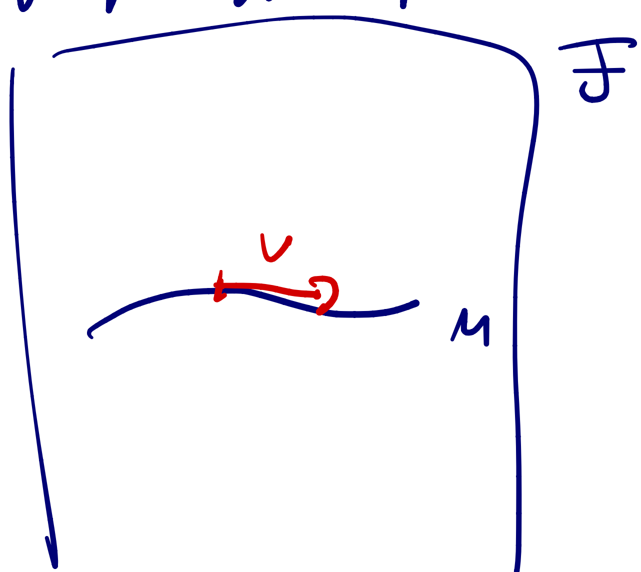
Note: The term when $\nabla_{\underline{u}}$ acts on $\underline{v}$ vanishes because of the EL equations. $\square$

## 2) Diffeomorphism invariance of $M$

Let $\Phi_\tau : M \to M$ family of diffeomorphisms

$v(x) := \frac{d}{d\tau} \Phi_\tau(x) \big|_{\tau=0}$

infinitesimal generator vector field on $M$




$(\Phi_\tau)_* \rho =: \tilde{\rho}_\tau$

the change of the measure is described infinitesimally by the divergence of $v$.

**Def:** Spacetime $M := \operatorname{supp} \rho$ has a <u>smooth manifold structure</u> if

(i) $M$ is a $k$-dim. smooth submanifold of $\mathcal{F}$

(iii) In a chart $(X^i, \mathcal{U})$ of $M$, the measure $\rho$ can be written as

$$d\rho = h(x)\, d^k x$$

with $h$ a smooth positive function.

Let $v \in \Gamma(M, TM)$ be a vector field on $M$
( $\in \Gamma(M, J\mathcal{F})$ and $v(x) \in T_x M \subset T_x \mathcal{F}$ )

The divergence of $v$ is defined by

$$\operatorname{div} v = \frac{1}{h} \partial_i ( h\, v^i(x) )$$

is independent of the choice of charts. This can be seen in analogy to the Koszul formula

$$\operatorname{div} V = \frac{1}{\sqrt{\det g}} \partial_i ( \sqrt{\det g}\, V^i )$$

in Riemannian geometry where $d\mu_M = \sqrt{\det g}\, d^k x$.

The jet $\underline{v} = (\operatorname{div} v, v)$ is an <u>inner solution</u>

<u>Lemma</u>: $\underline{v} = (\operatorname{div} v, v)$ is a solution of the linearized field eqns if $v \in \Gamma_0(M, TM)$ has compact support and $\mathcal{L}$ is smooth

<u>Proof</u>: Let $f \in C^\infty(M, \mathbb{R})$.

$$\int_M D_v f \, d\rho = \int_M \underbrace{D_v f(x)}_{= v^i \partial_i f} h(x)\, d^k x$$

$$= - \int_M f(x) \underbrace{\frac{1}{h} \partial_i ( h v^i )}_{\operatorname{div} v} \underbrace{h\, d^k x}_{d\rho} = - \int f \operatorname{div} v \, d\rho$$

$$\Rightarrow \int_M \nabla_{\underline{v}} f \, d\rho = 0$$

$$\langle \underline{u}, \Delta \underline{v} \rangle = \nabla_{\underline{u}} \left( \int_M \left(\nabla_{1,\underline{v}} + \cancel{\nabla_{2,\underline{v}}}\right) \mathcal{L}(x,y)\, d\rho(y) - \nabla_{\underline{v}}\, s \right)$$

$$= \nabla_{\underline{u}} \nabla_{\underline{v}}\, \ell(x)$$

$$= \nabla_{\underline{v}} \nabla_{\underline{u}}\, \ell(x) \longleftarrow \text{both derivatives act on } \ell$$

$$= \nabla_{\underline{v}} \left( \nabla_{\underline{u}(x)}\, \ell(x) \right) - \nabla_{D_v u}\, \ell(x)$$

acts on both $\ell$ and $\underline{u}(x)$

$\nabla_{\underline{u}}\, \ell(x) = 0 \quad \forall x \in M$ due to EL eqns

Since $v$ is tangential to $M$,

$\nabla_{\underline{v}} \left( \nabla_{\underline{u}}\, \ell \right)$ exists on $M$ and vanishes

$\nabla_{D_v u}\, \ell = 0$ due to EL eqn. $\square$

generalizations:

- $v$ has no compact support, but decays suitably at infinity.
- $\mathcal{L}$ not smooth: careful with $\mathfrak{J}^{\text{test}}, \ldots$