# Minimal symmetry breaking of the cosmological principle and dominating dust

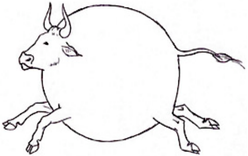
Galliano Valent, André Tilquin, Thomas Schücker

*To the memory of Vaughan Jones*

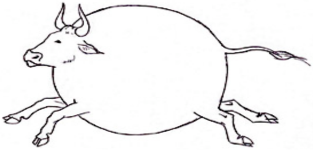
Symmetries reduce the complexity of systems and may lead to useful approximations. Observed deviations from the symmetric approximation may destroy all symmetries or only some.

**Ex** 2*d***:** On our planet, the paradigm of the **spherical cow** is useful:

**‘geographic’ deviations**: Mount Everest
$8.8\,\mathrm{km}\cdot 2\pi/(40\,000\,\mathrm{km}) \;\approx\; 1.4\cdot 10^{-3}$.



**‘geometric’ deviations**: (equatorial – polar) radius = 21.3 km,
$21.3\,\mathrm{km}\cdot 2\pi/(40\,000\,\mathrm{km}) \;\approx\; 3.3\cdot 10^{-3}$.



**Example** 3*d***:** In our universe, the spherical cow is called cosmological principle and it is useful:
**geographic deviations**: well established in the **C**osmic **M**icrowave **B**ackground (CMB), at $\approx 10^{-5}$;
**geometric deviations**, modelled by axial Bianchi I universes,

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - a(t)^2\,[\mathrm{d}x^2 + \mathrm{d}y^2] - c(t)^2\,\mathrm{d}z^2,$$

are signaled in CMB data by Cea [2014] at $\approx 10^{-2}$ ($1\sigma$),

and in the Lemaître-Hubble diagram (740 type 1a supernovae) by Tilquin, Schücker & Valent [2014] at $\approx 10^{-2}$ ($1\sigma$).

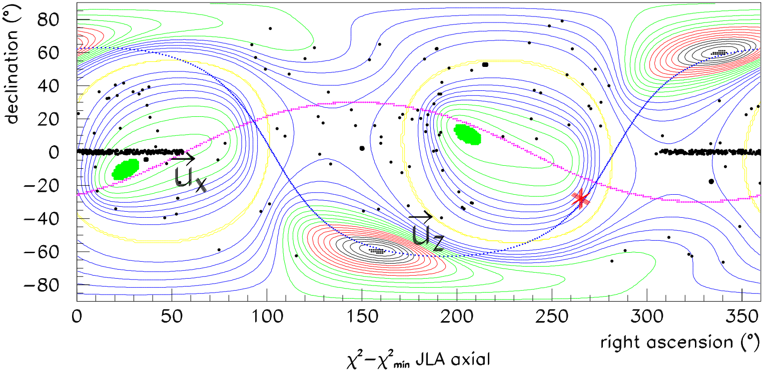



Figure: Confidence level contours of privileged directions in arbitrary color codes for axial Bianchi I universes. Black points represent 740 supernova positions. Note the accumulation of supernovae in the equatorial plane of the Earth. The blue line is the galactic plane and the purple line is the plane transverse to the main privileged direction (gray speck). The red star is the direction towards our galactic center.

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One year of Large Synoptic Survey Telescope data should yield 50 000 type 1a supernovae (starting ~~2022~~ after Oct. 2023) and a precision better than

$$6 \cdot 10^{-4}.$$

**Theorem**: The isometry group of a $d$-dimensional space or spacetime (with $d \geq 2$) is a Lie group of dimension $n \leq d\,(d+1)/2$.

**Examples in $d = 2$:**
The sphere has the $n = d\,(d+1)/2 = 3$ dimensional isometry group $O(3)$, it is “maximally symmetric”;
oblate and prolate axial ellipsoids (pumpkin and rugby ball) have the $n = 1$ dimensional isometry group $O(2)$;
generic ellipsoids have a discrete isometry group only, $n = 0$.

**Theorem** (Guido Fubini 1903): The isometry group of a $d$-dimensional space, $d \leq 3$, cannot be of dimension $n = d\,(d+1)/2 - 1$.

In $d = 2$ dimensions the cylinder is a counter example, because it has two isometries. At the level of Killing vectors, Fubini’s theorem was shown by Bianchi (1898) and is also true in two dimensions. Indeed the cylinder has three independent Killing vectors.

If we say: axial ellipsoids realize **minimal symmetry breakings** of the sphere, we mean:

(1) These ellipsoids can be infinitesimally close to the sphere.

(2) They have the highest possible number of symmetries:

$n = d\,(d+1)/2 - 2 = 1.$

**Examples in $d = 1 + 3$:**

In relativistic cosmology, we are tempted to start with a maximally symmetric space-time: anti de Sitter spaces, Minkowski space or de Sitter spaces. However none of them admits dynamics and we must be more modest: $d = 3$.

**The cosmological principle** postulates maximally symmetric spaces of simultaneity: 3-spheres, $\mathbb{R}^3$ and pseudo 3-spheres with the $n = 6$ dimensional isometry groups: $O(4)$, $O(3) \ltimes \mathbb{R}^3$, $O(3,1)$.

Adding time as an orthogonal $\mathbb{R}$ to these 3-spaces of simultaneity, one obtains the 'Robertson-Walker' universes.

**Definition:** A **minimal symmetry breaking** of the cosmological principle is

(1) a smooth family of **deformations** of a maximally symmetric 3-space,

(2) such that the isometry group of all deformations has maximal dimension, $n = d\,(d+1)/2 - 2 = 4$ according to Fubini.

**Example:** axial Bianchi I universes, 2 scale factors;
3 translations + 1 rotation (around the $z$ axis) = 4 symmetries.

**Counter-example:** generic Bianchi I universes, 3 scalefactors:
3 translations + no rotation = **three** symmetries.

# Three symmetries: Bianchi universes

There is only one 1-dimensional Lie algebra: the Abelian one.

There are two 2-dimensional Lie algebras: the Abelian one and the (solvable) one of $2 \times 2$ triangular matrices with vanishing trace.

In 1898 Luigi Bianchi classifies all 3-dimensional, real Lie algebras. His list starts with seven 3-dimensional Lie algebras: Bianchi I, II, III, IV, V, VIII and IX. Bianchi I is Abelian (3 translations), Bianchi II is the Heisenberg algebra. Bianchi III is the direct sum of the 1-dimensional and the solvable 2-dimensional Lie algebras. Bianchi IX is $so(3)$. In addition he finds two uncountable families of 3-dimensional Lie algebras, each indexed by a real parameter $h$: Bianchi $\text{VI}_h$, $h \neq 0$, $h \neq 1$ and $\text{VII}_h$, $h \geq 0$. ($h \neq$ Planck's constant!)

Bianchi also shows that all of these Lie algebras can be represented as **infinitesimal** isometries ('Killing vectors') on 3-spaces. Adding time as an orthogonal $\mathbb{R}$, one obtains the Bianchi universes.

# Four symmetries: axial Bianchi universes

Bianchi classifies all 4-dimensional Lie algebras that occur as infinitesimal isometries on 3-spaces.

He finds that all of these 4-dimensional Lie algebras contain 3-dimensional Lie sub-algebras. (In fact, any real or complex 4-dimensional Lie algebra has a 3-dimensional ideal.) Therefore these 3-spaces give rise to universes, that are special types of Bianchi universes. Let us call them **‘axial’** Bianchi universes.

# Minimal symmetry breaking of the cosmological principle

Which of the Bianchi universes qualify as minimal symmetry breaking of the cosmological principle?

Condition (1) ('Infinitesimally close to maximal symmetry')

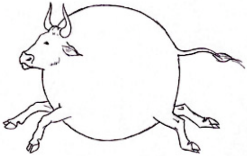
eliminates the Bianchi II, III, IV, $\mathrm{VI}_h$, and Bianchi VIII universes.

Condition (2) ('Four symmetries')

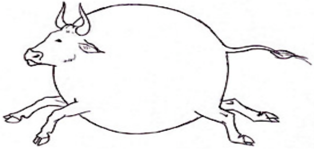
eliminates the axial Bianchi $\mathrm{VII}_h$ universes,
because they are isomorphic
to the axial Bianchi I universe for $h = 0$ and
to the axial Bianchi V universe for $h \neq 0$.

We remain with the axial Bianchi I, V and IX universes, whose 3-spaces of simultaneity are smooth deformations of Eucidean $\mathbb{R}^3$, of the pseudo spheres and of the spheres.

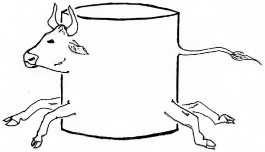
# In pictures



Robertson-Walker + CMB + ...



axial Bianchi I, V, IX + CMB + ...



Bianchi II, III, IV, VI$_h$, VIII + CMB + ...

# Axial Bianchi IX universes with dust

The metric tensor contains two scale factors $a(t)$ and $c(t)$:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{4}a^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{4}(a^2\cos^2 x + c^2 \sin^2 x) & -\frac{1}{4}c^2 \sin x \\ 0 & 0 & -\frac{1}{4}c^2 \sin x & -\frac{1}{4}c^2 \end{pmatrix}.$$

The 3+1 Killing vectors

$$\cos y\,\partial_x + \tan x\ \sin y\,\partial_y - \frac{\sin y}{\cos x}\,\partial_z,\ \partial_y,$$

$$\sin y\,\partial_x - \tan x\ \cos y\,\partial_y + \frac{\cos y}{\cos x}\,\partial_z;\ \partial_z,$$

generate the isometry group $O(3) \times O(2)$.

Consider any geodesic with affine parameter $q$ and write $\dot{} := \mathrm{d}/\mathrm{d}q$. According to Emmy Noether, we have 4 conserved quantities $A$, $B$, $\tilde{A}$ and $C$ with:

$$
\begin{aligned}
\dot{x} &= \pm \frac{1}{a^2}\left[A^2 + \tilde{A}^2 - \left(\frac{B \sin x - C}{\cos x}\right)^2\right]^{1/2}, \\
\dot{y} &= \frac{B - C \sin x}{a^2 \cos^2 x}, \\
\dot{z} &= \frac{C}{c^2} - \sin x \, \frac{B - C \sin x}{a^2 \cos^2 x}, \\
A \cos y - \tilde{A} \sin y &= -\frac{B \sin x - C}{\cos x}.
\end{aligned}
$$

Any dust test-particle with vanishing initial 3-velocity, $A = B = C = 0$, will remain co-moving.

To compute the Lemaître-Hubble diagram we solve the Einstein equations with cosmological constant and co-moving dust with energy-momentum tensor $T_{\mu\nu} = \mathrm{diag}(\rho(t), 0, 0, 0)$.
Using $\mathrm{d}/\mathrm{d}t =:'$ and the Hubble parameters $H := a'/a$, $H_c := c'/c$, they read:

$$\begin{aligned}
3H^2 + 2H(H_c - H) + \frac{1}{a^2}\left(4 - \frac{c^2}{a^2}\right) &= \Lambda + 8\pi G\rho, \quad (tt) \\
H' + H_c' + H^2 + H_c^2 + HH_c + \frac{c^2}{a^4} &= \Lambda, \quad (xx) \\
2\,H' + 3\,H^2 + \frac{1}{a^2}\left(4 - 3\,\frac{c^2}{a^2}\right) &= \Lambda, \quad (zz)
\end{aligned}$$

and solve them perturbatively to first order in a small deviation $\eta$ from maximal symmetry defined by

$$c(t) =: a(t)\,[1 + \eta(t)].$$

To illustrate the cosmological principle here is a story set in the Black Forest, on a farm in Oberwolfach, home to Traudi, her owner's preferred cow. Traudi is ill and the veterinarian is helpless. Sparing no effort, the farmer calls on a physician, more expensive, but as helpless as his colleague. The farmer has a nephew with a PhD in biology and asks him to see Traudi, again without success. Finally, when he sees a theoretical physicist on his way to a conference, the farmer driven to dispair asks him for help. The physicist sits down next to Traudi, pulls out his note pad and starts calculating. During hours the farmer watches the physicist's intense concentration from a respectful distance and feels a timid ray of optimism. He pulls closer, caresses Traudi between the horns and asks: 'is there hope?' 'Indeed there is', replies the physicist with unconcealed pride, 'I just solved the case of the spherical cow.'