A product picture for QED

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Coulomb gauge QED

$$H_{\mathrm{QED}} = \int \frac{1}{2}\boldsymbol{\pi}^{\perp\,2} + \frac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{A})^2 + \psi^*\gamma^0\boldsymbol{\gamma}\cdot(-i\boldsymbol{\nabla} - \mathrm{e}\boldsymbol{A})\psi + m\psi^*\gamma^0\psi \, d^3x + V_{\mathrm{Coulomb}}$$

where

$$V_{\mathrm{Coulomb}} = \frac{1}{2}\int \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\, d^3x d^3y \; = \frac{1}{2}\int \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi \, d^3x$$

where

$$\rho(\boldsymbol{x}) = \mathrm{e}\psi(\boldsymbol{x})^*\psi(\boldsymbol{x}), \text{ and } \phi(\boldsymbol{x}) = \int \frac{\rho(\boldsymbol{y})}{4\pi\epsilon_0|\boldsymbol{x}-\boldsymbol{y}|} d^3y \text{ (and satisfies } \nabla^2\phi = -\rho)$$

Note that $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ (Coulomb gauge condition) and $\boldsymbol{\nabla}\cdot\boldsymbol{\pi}^{\perp} = 0$,

Interpretation: $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and $\boldsymbol{E} = -\boldsymbol{\pi}^{\perp} - \boldsymbol{\nabla}\phi$, i.e. $\boldsymbol{E}^{\text{trans}} = -\boldsymbol{\pi}^{\perp}$ and $\boldsymbol{E}^{\text{long}} = -\boldsymbol{\nabla}\phi$.

Here we use that given any vector field $\boldsymbol{F}(\boldsymbol{x})$, we have $\boldsymbol{F} = \boldsymbol{F}^{\text{trans}} + \boldsymbol{F}^{\text{long}}$, where $\boldsymbol{\nabla} \cdot \boldsymbol{F}^{\text{trans}} = 0$ and $\boldsymbol{F}^{\text{long}}$ arises as $\boldsymbol{\nabla}\Phi$ for some Φ.

Proof: Take $F_i^{\text{long}}(\boldsymbol{k}) = \frac{k_i k_j}{k^2} F_j(\boldsymbol{k})$, $F_i^{\text{trans}}(\boldsymbol{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) F_j(\boldsymbol{k})$.

Note $\boldsymbol{E}^{\text{long}} = -\boldsymbol{\nabla}\phi$ is equivalent to $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$ (So Gauss's law holds by definition!)

CCR and CAR

$$[A_i(\boldsymbol{x}), {\pi^{\perp}}_j(\boldsymbol{y})] = i\delta_{ij}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) + i\frac{\partial^2}{\partial x^i \partial x^j}\left(\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\right),$$

$$\{\psi(\boldsymbol{x}), \psi^*(\boldsymbol{y})\} = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$$

(And all other CR/AR zero.)

$\boldsymbol{A}$ and $\boldsymbol{\pi}^{\perp}$ are represented by defining

$$A_i(\boldsymbol{k}) = \left(\frac{a_i^{\rm trans}(\boldsymbol{k}) + a_i^{+\,\rm trans}(\boldsymbol{k})}{\sqrt{2}|k|^{1/2}}\right), \pi^{\perp}{}_i(\boldsymbol{k}) = -i|k|^{1/2}\left(\frac{a_i^{\rm trans}(\boldsymbol{k}) - a_i^{+\,\rm trans}(\boldsymbol{k})}{\sqrt{2}}\right)$$

The Dirac ψ and ψ^* will be represented with creation and annihilation operators too. We omit the details.

We can think of the representation Hilbert space as $\mathcal{F}(\mathcal{H}_{\rm one}^{\rm trans}) \otimes \mathcal{H}_{\rm Dirac}$ where $\mathcal{H}_{\rm one}^{\rm trans}$ consists of transverse elements in the one-particle Hilbert space $\mathcal{H}_{\rm one} = L^2(\mathbb{R}^3)^3$.

Alternatively, we can think of the representation space as the Coulomb gauge physical subspace CGPS of the augmented QED Hilbert space. $\mathcal{F}(\mathcal{H}_{\rm one}^{\rm trans}) \otimes \Omega^{\rm long} \otimes \mathcal{H}_{\rm Dirac} \subset \mathcal{F}(\mathcal{H}_{\rm one}^{\rm trans}) \otimes \mathcal{F}(\mathcal{H}_{\rm one}^{\rm long}) \otimes \mathcal{H}_{\rm Dirac} =$
$\mathcal{F}(\mathcal{H}_{\rm one}) \otimes \mathcal{H}_{\rm Dirac}$.
I.e. we admit the possibility of longitudinal photons but they are always in their vacuum state.
CRITIQUE: $V_{\rm Coulomb}$ quartic, nonlocal; Commutation Relations complicated, nonlocal; Pre-Faradayan! Longitudinal electric field doesn't have independent existence (/doesn't even exist!). (Capacitor example.)

CRITIQUE continued: It's not a product picture

A product picture would have

$H_{QED} = H_{electromag} \otimes I + I \otimes H_{Dirac} + H_{Int}$ on $\mathcal{H}_{electromag} \otimes \mathcal{H}_{Dirac}$.

But Coulomb gauge formulation is not a product picture in that sense.

Would like a product picture formulation as a warm-up problem for quantum gravity. My *matter-gravity entanglement hypothesis* (see e.g. BSK 1998) presupposes Quantum Gravity has a product picture and then equates the physical entropy of a closed system with its its matter-gravity entanglement entropy. This then offers an explanation e.g. for the black-hole information loss puzzle and the thermal atmosphere puzzle.

But because gravity is like a gauge theory and so its canonical formulation involves constraints, one might worry that it does not admit a product picture. QED serves as a simple analogy. In Coulomb gauge, Gauss's law is an analogous constraint. But we will see that it *does* admit a product picture.

A product picture

$$H_{\mathrm{QED}}^{\mathrm{PP}} = \int \frac{1}{2}\tilde{\boldsymbol{\pi}}^2 + \frac{1}{2}(\boldsymbol{\nabla}\times\hat{\boldsymbol{A}})^2 + \psi^*\gamma^0\boldsymbol{\gamma}\cdot(-i\boldsymbol{\nabla} - \mathrm{e}\hat{\boldsymbol{A}})\psi + m\psi^*\gamma^0\psi\, d^3x$$

Looks like H_{QED} except for tilde over the $\boldsymbol{\pi}$ and the hat over the A, and the fact that V_{Coulomb} is absent. Also now the CR/AR are the simpler:

$$[\hat{A}_i(\boldsymbol{x}), \tilde{\pi}_j(\boldsymbol{y})] = i\delta_{ij}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \quad \{\psi(\boldsymbol{x}), \psi^*(\boldsymbol{y})\} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$$

Representation (on the product picture physical subspace – see next slide):

$$\hat{A}_i(\boldsymbol{k}) = \left(\frac{a_i(\boldsymbol{k}) + a_i^+(\boldsymbol{k})}{\sqrt{2}|k|^{1/2}}\right), \quad \tilde{\boldsymbol{\pi}} = \boldsymbol{\pi}^\perp + \tilde{\boldsymbol{\pi}}^{\mathrm{long}}$$

where

$$\pi^\perp{}_i(\boldsymbol{k}) = -i|k|^{1/2}\left(\frac{a_i^{\mathrm{trans}}(\boldsymbol{k}) - a_i^{+\mathrm{trans}}(\boldsymbol{k})}{\sqrt{2}}\right)$$

and

$$\tilde{\pi}_i^{\mathrm{long}}(\boldsymbol{k}) = -\frac{2i|k|^{1/2}}{\sqrt{2}} a_i^{\mathrm{long}}(\boldsymbol{k}) \quad \text{Not self adjoint!}$$

... on the product picture physical subspace (PPPS) (of the same augmented QED Hilbert space) $= U$(Coulomb gauge physical subspace) where

$$U = \exp\left(i \int \hat{A}^i(\boldsymbol{x}) \partial_i \phi(\boldsymbol{x})\, d^3x\right)$$

Note: All states in PPPS (including the vacuum) are entangled between charged matter and longitudinal photons!

Intepretation: $\boldsymbol{B} = \boldsymbol{\nabla} \times \hat{\boldsymbol{A}}, \quad \boldsymbol{E} = -\tilde{\boldsymbol{\pi}}$

Theorem 1: $\tilde{\boldsymbol{\pi}}$ and $H^{\rm PP}_{\rm QED}$ are self-adjoint on the product picture physical subspace and map it to itself.

Theorem 2: $H^{\rm PP}_{\rm QED}$ on PPPS is (unitarily) equivalent to $H_{\rm QED}$ on CGPS.

Theorem 3: $\forall\, \boldsymbol{\Psi} \in$ PPPS, $\boldsymbol{\nabla} \cdot \boldsymbol{E}\boldsymbol{\Psi}$ $(= -\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\pi}}\boldsymbol{\Psi}) = \rho\boldsymbol{\Psi}$, i.e. Gauss's law holds as an operator equation on the PPPS. (Contrast Coulomb gauge, where it held by definition.)

Proofs of Theorems 3 and 1: They follow easily from $U\tilde{\boldsymbol{\pi}}U^{-1} = \tilde{\boldsymbol{\pi}} - \boldsymbol{\nabla}\phi$

Proof of Theorem 2: First define

$$\check{H}_{\mathrm{QED}} = H_{\mathrm{QED}} + \int \frac{1}{2}(\tilde{\boldsymbol{\pi}}^{\mathrm{long}})^2 + \tilde{\boldsymbol{\pi}} \cdot \boldsymbol{\nabla}\phi d^3x$$

and notice that, restricted to the CGPS, $\check{H}_{\mathrm{QED}} = H_{\mathrm{QED}}$.

Then on the full augmented QED Hilbert space, calculate $U\check{H}_{\mathrm{QED}}U^{-1}$ and you find that it equals $H^{\mathrm{PP}}_{\mathrm{QED}}$.

Ingredients include:

$$U\tilde{\boldsymbol{\pi}}U^{-1} = \tilde{\boldsymbol{\pi}} - \boldsymbol{\nabla}\phi, \quad U\psi U^{-1} = \breve{\psi} = e^{-\left(ie\frac{\partial_i}{\nabla^2}\hat{A}^i\right)}\psi$$

In fact we have the table ...

Quantity	Coulomb gauge	product picture
Hamiltonian	$\check{H}^{\mathrm{Dirac}}_{\mathrm{QED}}$ (or $H^{\mathrm{Dirac}}_{\mathrm{QED}}$)	$H^{\mathrm{PP,Dirac}}_{\mathrm{QED}}$
electric field	$\boldsymbol{E}_{\mathrm{C}} = -(\tilde{\boldsymbol{\pi}} + \boldsymbol{\nabla}\phi)$ (or $-(\boldsymbol{\pi}^{\perp} + \boldsymbol{\nabla}\phi)$)	$\boldsymbol{E}_{\mathrm{PP}} = -\tilde{\boldsymbol{\pi}}$
vector potential	$\hat{\boldsymbol{A}}$ (or $\boldsymbol{A}$)	$\hat{\boldsymbol{A}}$
Dirac field	ψ	$\breve{\psi} = e^{-\left(ie\frac{\partial_i}{\nabla^2}\hat{A}^i\right)}\psi$
adjoint Dirac field	ψ^*	$\breve{\psi}^* = e^{\left(ie\frac{\partial_i}{\nabla^2}\hat{A}^i\right)}\psi^*$
Dirac electrical potential	ϕ	ϕ
vacuum state	$\Omega^{\mathrm{trans}} \otimes \Omega^{\mathrm{long}} \otimes \Omega_{\mathrm{Dirac}}$ (or $\Omega^{\mathrm{trans}} \otimes \Omega_{\mathrm{Dirac}}$)	$\Omega^{\mathrm{trans}} \otimes U(\Omega^{\mathrm{long}} \otimes \Omega_{\mathrm{Dirac}})$ (entangled)

Previous attempts at temporal gauge quantization (cf. Löffelholz, Morchio, Strocchi 2003) fell foul of the

Contradictory Commutator Theorem: There can be no pair of 3-vector operators $\mathbf{A}$ and $\boldsymbol{\pi}$ on a Hilbert space H such that

(a) $\mathbf{A}$ and $\boldsymbol{\pi}$ satisfy the canonical commutation relations

$$[\mathrm{A}_i(\boldsymbol{x}), \pi_j(\boldsymbol{y})] = i\delta_{ij}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \; [\mathrm{A}_i(\boldsymbol{x}), \mathrm{A}_j(\boldsymbol{y})] = 0 = [\pi_i(\boldsymbol{x}), \pi_j(\boldsymbol{y})]$$

(b) $\mathbf{A}$ and $\boldsymbol{\pi}$ are each self-adjoint;
(c) For some vector $\boldsymbol{\Psi} \in \mathrm{H}$ ($\boldsymbol{\Psi} \neq 0$)

$$\boldsymbol{\nabla} \cdot \boldsymbol{\pi}\boldsymbol{\Psi} = -\rho\boldsymbol{\Psi}$$

for some operator-valued function of $\boldsymbol{x}$, ρ.
(d) ρ commutes with $\mathbf{A}$.

Proof: (a) easily implies that the quantity $\langle\boldsymbol{\Psi}|[\mathrm{A}_i(\boldsymbol{x}), \boldsymbol{\nabla} \cdot \boldsymbol{\pi}(\boldsymbol{y})]\boldsymbol{\Psi}\rangle$ is equal to $-i(\nabla_i\delta^{(3)})(\boldsymbol{x} - \boldsymbol{y})$, while (b), (c) and (d) imply that the same quantity is zero – a contradiction!

Maxwell-Schrödinger theory

There's a similar story with many non-relativistic particles interacting with the EM field. You have to treat the particles are finite radius balls, though, with charge density ρ.

$$H_{\mathrm{QED}}^{\mathrm{Schr}} = \int \frac{1}{2}\boldsymbol{\pi}^{\perp 2} + \frac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{A})^2 d^3x + \sum_{I=1}^{N} \frac{\left(\boldsymbol{p}_I - \int \boldsymbol{A}(\boldsymbol{x})\rho_I(\boldsymbol{x}-\boldsymbol{x}_I)\, d^3\boldsymbol{x}\right)^2}{2M_I} + V_{\mathrm{Coulomb}}^{\mathrm{Schr}}$$

where $$V_{\mathrm{Coulomb}}^{\mathrm{Schr}} = \frac{1}{2}\sum_{I=1}^{N}\sum_{J=1}^{N}\int\int \frac{\rho_I(\boldsymbol{x}-\boldsymbol{x}_I)\rho_J(\boldsymbol{y}-\boldsymbol{x}_J)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\, d^3x d^3y$$

on the appropriate CGPS gets replaced by

$$H_{\mathrm{QED}}^{\mathrm{PP\,Schr}} = \int \frac{1}{2}\tilde{\boldsymbol{\pi}}^2 + \frac{1}{2}(\boldsymbol{\nabla}\times\hat{\boldsymbol{A}})^2\, d^3x + \sum_{I=1}^{N} \frac{\left(\boldsymbol{p}_I - \int \hat{\boldsymbol{A}}(\boldsymbol{x})\rho_I(\boldsymbol{x}-\boldsymbol{x}_I)\, d^3\boldsymbol{x}\right)^2}{2M_I}$$

on the PPPS, all of whose states are again entangled between our charged balls and longitudinal photons.

And we again have

$$U\check{H}_{\rm QED}^{\rm Schr}U^{-1} = H_{\rm QED}^{\rm PP\,Schr},$$

and

$$U\boldsymbol{p}_I U^{-1} = \boldsymbol{p}_I - \int \hat{\boldsymbol{A}}^{\rm long}(\boldsymbol{x})\rho_I(\boldsymbol{x}-\boldsymbol{x}_I)\,d^3x.$$

etc.

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In the usual approximation where we neglect terms which lead to radiative corrections (and suppress the $\mathcal{F}(\mathcal{H}_{\rm one}^{\rm trans})\otimes$ in the Hilbert space) the Coulomb gauge story gives us the familiar Schrödinger equation. Note though that with extended balls, it is natural to include the (finite) self energies of the balls in $V_{\rm Coulomb}^{\rm Schr}$ and in E. Let us illustrate with the Hydrogen atom ...

The Hydrogen atom in the product picture

$$\left(\frac{\boldsymbol{p}_1^2}{2M_1} + \frac{\boldsymbol{p}_2^2}{2M_2} + V_{\mathrm{Coulomb}}^{\mathrm{Schr}}\right)\Omega^{\mathrm{long}} \otimes \Psi_{\mathrm{Schr}} \approx E\Omega^{\mathrm{long}} \otimes \Psi_{\mathrm{Schr}}.$$

In the product picture, this (on the CGPS) gets replaced by

$$\left(\frac{\boldsymbol{p}_1^2}{2M_1} + \frac{\boldsymbol{p}_2^2}{2M_2} + \frac{1}{2}\tilde{\boldsymbol{\pi}}^{\mathrm{long}^2}\right)U\Omega^{\mathrm{long}} \otimes \Psi_{\mathrm{Schr}} \approx EU\Omega^{\mathrm{long}} \otimes \Psi_{\mathrm{Schr}}.$$

(on the PPPS). The binding energy is now understood to be the decrease in the energy of the longitudinal part of the electric field when the balls are closer to one another. The (entangled) quantum state, $U\Omega^{\mathrm{long}} \otimes \Psi_{\mathrm{Schr}}$, schematically takes the form – when we think of $\mathcal{F}(\mathcal{H}_{\mathrm{one}}^{\mathrm{long}}) \otimes L^2(\mathbb{R}^6)$ as $L^2(\mathbb{R}^6, \mathcal{F}(\mathcal{H}_{\mathrm{one}}^{\mathrm{long}}))$ –

$$(\boldsymbol{x}_1, \boldsymbol{x}_2) \mapsto \Psi^{\mathrm{long}}(\boldsymbol{x}_1, \boldsymbol{x}_2)|\text{proton at } \boldsymbol{x}_1, \text{electron at } \boldsymbol{x}_2\rangle$$

where $\Psi^{\mathrm{long}}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ deserves to be called the coherent state of longitudinal photons which corresponds to the classical electric field due to the presence of the proton at $\boldsymbol{x}_1$ and the electron at $\boldsymbol{x}_2$. (Could say a lot more about that ... ... also about partial trace ... $\tilde{\boldsymbol{\pi}}$ versus $\hat{\boldsymbol{\pi}}$ ... more about Gauss's law and charged field commutator ....)

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