$$\mathcal{D} = i\not{\partial} + \mathcal{B}$$

$(\psi_m \mid m \in J$

$$\left\langle \int_J \psi_m \, dm \,\Big|\, \int_J \psi'_{m'} \, dm' \right\rangle$$

$$= \int_J (\psi_m \mid S_m \psi'_m)_m \, dm$$

$S_m \in L(\mathcal{H}_m)$ symmetric , $\chi_-(S_m)$

$$(\mathcal{D} - m)\psi = \phi \in C_0^\infty(\mathcal{M}, S\mathcal{M})$$

$$\begin{cases} s_m^\vee : & C_0^\infty(\mathcal{M}, S\mathcal{M}) \longrightarrow C_{sc}^\infty(\mathcal{M}, S\mathcal{M}) \quad \text{advanced} \\ s_m^\wedge : & \text{—— " ——} \quad \text{retarded} \end{cases}$$

$s_m^\wedge \phi = \psi_+$, $s_m^\vee \phi = \psi_-$ Green's operator

$$k_m = \frac{1}{2\pi i}(s_m^\vee - s_m) : C_0^\infty(\mathcal{M}, S\mathcal{M}) \longrightarrow C_{sc}^\infty(\mathcal{M}, S\mathcal{M}) \cap \mathcal{H}_m$$

causal fundamental solution

$$P(\phi, \phi') := c\,(k_m \phi \mid \chi_-(S_m)\, k_m \phi')_m$$

unregularized kernel of the fermionic projector

Consider

$\psi_m = k_m \phi$, $\psi'_m = k_m \phi'$

$$\int_J dm \int_J dm' \, \langle k_m \phi \mid k_{m'} \phi' \rangle$$

$$= \int_J (k_m \phi \mid S_m \, k_m \phi)_m \, dm$$

$$\langle k_m \phi \mid k_{m'} \phi' \rangle \underset{\uparrow}{=} \langle \phi \mid k_m k'_m \phi' \rangle$$

formally

$$\underbrace{k_m k_{m'}}_{\text{product of integral operators in spacetime}} = \delta(m-m') \, (\ldots)$$

$$(\ldots) = S_m k_m$$

$$(k_m k_{m'})(x,y) = \int_M k_m(x,z) \, k_{m'}(z,y) \, d^4 z$$

spectral calculus in spacetime

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related concept: spectral calculus for $\mathcal{D}$

$$\mathcal{D}\psi = m\psi$$

goal: $\mathcal{D} = \int m \, dE_m$

cannot be realized mathematically because $\mathcal{D}$ is a symmetric operator on on the <u>indefinite</u> inner product space $\mathcal{K}$ $\langle . | . \rangle$

$\mathcal{D} = i \not{\partial}$

work in Fourier space k

$dE_m = p_m \, dm$ where

$$p_m(k) = (\not{k} + m) \, \delta(k^2 - m^2)$$

$$E_u E_v = E_{u \cap v}$$

$$\Rightarrow p_m p_{m'} = \delta(m-m') \, p_m$$

$$p_m(k) \, p_{m'}(k) = (\not{k}+m)(\not{k}+m') \, \delta(k^2-m^2) \, \delta(k^2-m'^2)$$

$$= (\not{k}+m)(\not{k}+m') \underbrace{\delta(m^2-m'^2)}_{\frac{1}{2|m|}\left(\delta(m-m') + \cancel{\delta(m+m')}\right)} \delta(k^2-m^2)$$

assume that $m, m' > 0$

$$= \frac{1}{2m} \, \delta(m-m') \, \big(\underbrace{k^2}_{= m^2} + \underbrace{(m+m')}_{2m} \not{k} + \underbrace{mm'}_{m^2} \big) \, \delta(k^2-m^2)$$

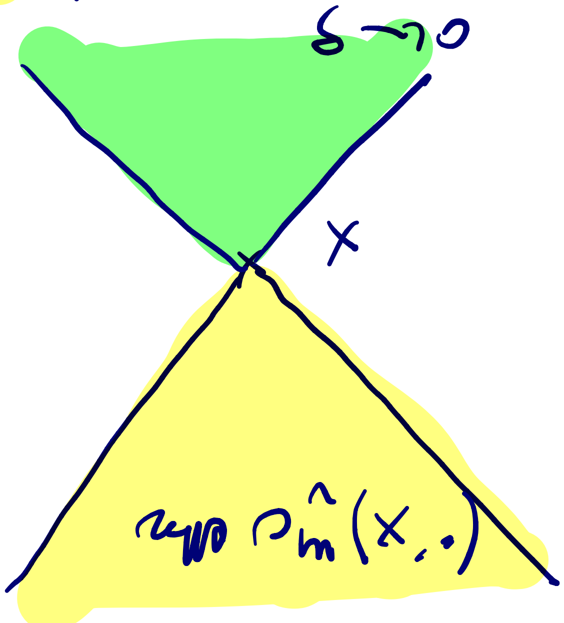
$$= \frac{1}{\cancel{2m}} \delta(m-m') \, \cancel{2m} \, \underbrace{(\slashed{k}+m)\,\delta(k^2-m^2)}_{p_m}$$

$$= \delta(m-m') \, p_m \, .$$

$$s_m(x,y) = \int \frac{d^4k}{(2\pi)^4} \, s_m(k) \, e^{-ik(x-y)}$$

$$s_m^{\vee}(k) = \lim_{\delta \searrow 0} \frac{\slashed{k}+m}{k^2-m^2-i\delta k^0}$$

$$s_m^{\wedge}(k) = \lim_{\delta \searrow 0} \frac{\slashed{k}+m}{k^2-m^2+i\delta k^0}$$



$$\lim_{\varepsilon \searrow 0} \frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i \, \delta(x)$$

$$k_m = \frac{1}{2\pi i}\left(s_m^{\vee} - s_m^{\wedge}\right)$$

$$= (\slashed{k}+m)\,\delta(k^2-m^2) \; \overset{\substack{S_m \\ \parallel}}{\epsilon(k^0)}$$

$$k_m \, k_{m'} = \delta(m-m') \, p_m$$

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perturbative treatment $\quad \mathcal{D} = i\slashed{\partial} + \mathcal{B}$

$\tilde{s}_m^{\vee}$, $\tilde{s}_m^{\wedge}$ standard perturbation expansion

$$\tilde{k}_m := \frac{1}{2\pi i}\left(\tilde{s}_m^{\vee} - \tilde{s}_m^{\wedge}\right)$$

$\tilde{k}_m \, \tilde{k}_{m'}$ gives $\tilde{S}_m$

use contour integrals involving the resolvent R_λ

gives unique explicit perturbation expansion for $\tilde{P}(x,y)$

$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\tilde{s}_m = \mathbb{1}$$

$$(i\partial\!\!\!/ - m + \mathcal{B})\,\tilde{s}_m = \mathbb{1} \qquad \Big| \cdot s_m$$

$$(\mathbb{1} + s_m \mathcal{B})\,\tilde{s}_m = s_m$$

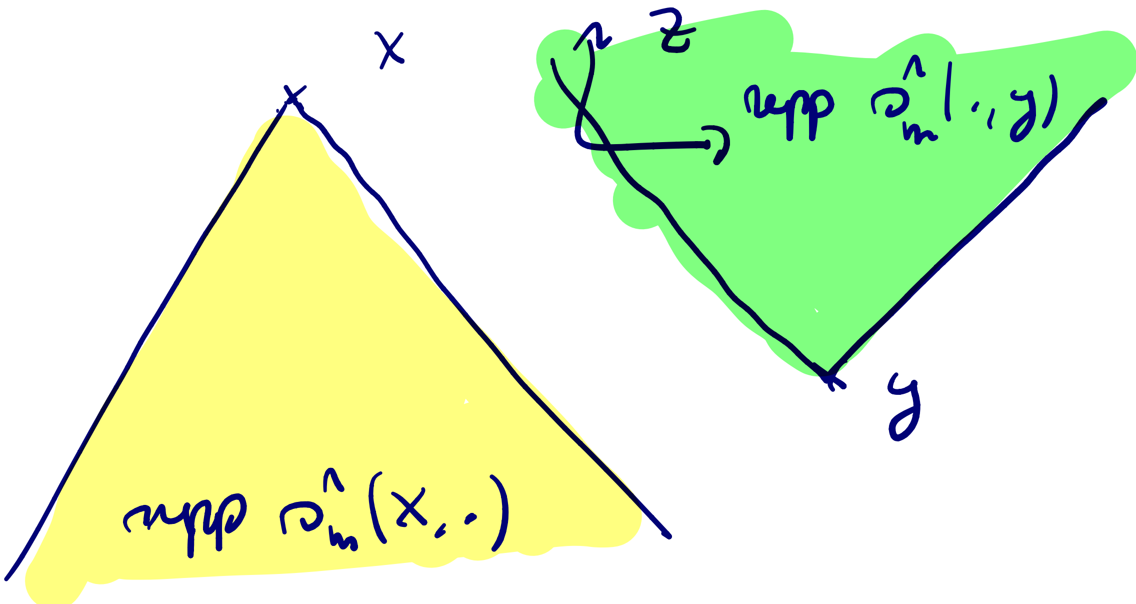
$$(\mathbb{1} + s_m \mathcal{B})^{-1} = \sum_{p=0}^{\infty} (-s_m \mathcal{B})^p$$

$$\tilde{s}_m^{\wedge} = \sum_{p=0}^{\infty} (-s_m^{\wedge} \mathcal{B})^p\, s_m^{\wedge}$$

This ensures causal support properties,

1st order $(-s_m^{\wedge} \mathcal{B}\, s_m^{\wedge})(x,y)$

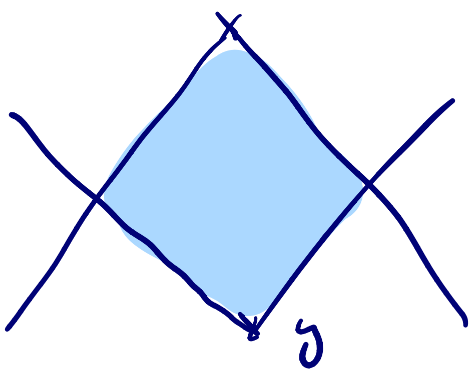
$$= -\int d^4z\; s_m^{\wedge}(x,z)\, \mathcal{B}(z)\, s_m^{\wedge}(z,y)$$



$\Longrightarrow$ $(s_m \mathcal{B}\, s_m)(x,.)$

is supported in the

past light cone



$\tilde{k}_m$

good notation: leave out m and $\delta(m-m')$
write multiplication with $\cdot$

$$\begin{cases} p \cdot p = p \\ k \cdot k = p \\ p \cdot k = k \cdot p = k \end{cases}$$

k has eigenvalues ± 1

$$R_\lambda := (k-\lambda)^{-1}$$

$$= \frac{1}{1-\lambda}\,\frac{p+k}{2} + \frac{1}{-1-\lambda}\,\frac{p-k}{2} + \frac{1}{-\lambda}(\mathbb{1}-p)$$

$$\tilde{k}_m = k_m \underbrace{- s_m \mathcal{D} k_m - k_m \mathcal{D} s_m + \dots}$$

$s_m := \frac{1}{2}(s^\vee_m + s^\wedge_m)$ symmetric Green's operator

$$= k_m + \Delta k_m$$

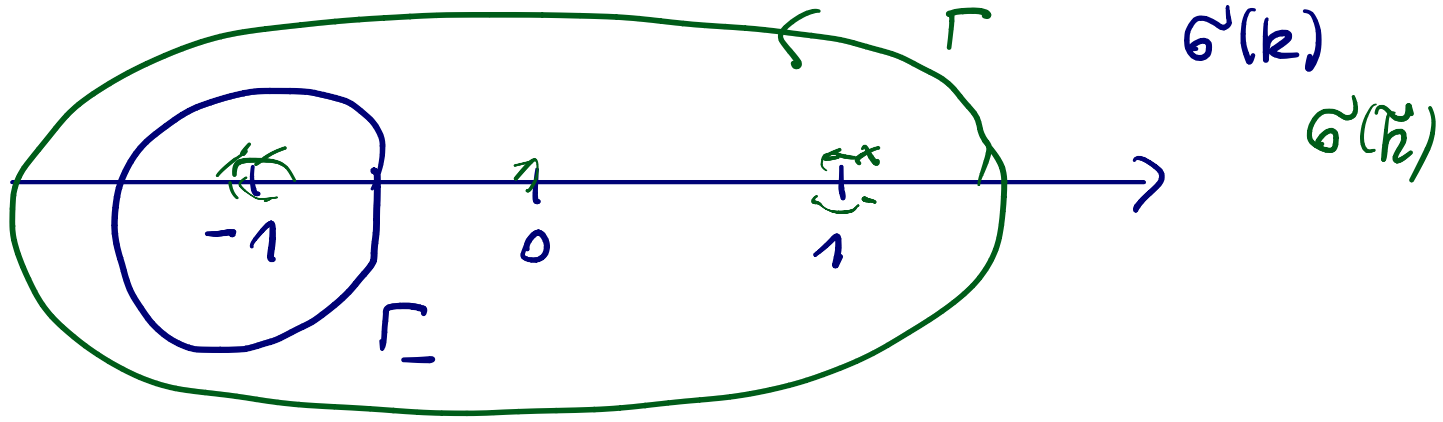
$$\tilde{k} = k + \Delta k$$

$$\tilde{R}_\lambda = (\tilde{k}-\lambda)^{-1} = (k-\lambda+\Delta k)^{-1}$$

$$= \big((k-\lambda)\cdot(\mathbb{1} + R_\lambda \Delta k)\big)^{-1}$$

$$= (\mathbb{1} + R_\lambda \Delta k)^{-1} \cdot R_\lambda$$

$$= \sum_{p=0}^{\infty} (-R_\lambda \Delta k)^p \cdot R_\lambda$$



$$f(\tilde{k}) := -\frac{1}{2\pi i}\int_\Gamma f(\lambda)\, \tilde{R}_\lambda \, d\lambda$$ f holomorphic

resolvent identity $\tilde{R}_\lambda \cdot \tilde{R}_{\lambda'} = \frac{1}{\lambda-\lambda'}\left(\tilde{R}_\lambda - \tilde{R}_{\lambda'}\right)$

$$f(\tilde{k}) \cdot g(\tilde{k}) = (fg)(\tilde{k})$$ functional calculus

$$\tilde{k} = -\frac{1}{2\pi i}\int_\Gamma \lambda \, \tilde{R}_\lambda \, d\lambda$$

$$\tilde{P} := -\frac{1}{2\pi i}\int_{\Gamma_-} -\lambda \, \tilde{R}_\lambda \, d\lambda = \chi_- \tilde{k}$$

causal: is based on perturbation expansion for

explicit formulas to every order $\tilde{s}^\vee_m$, $\tilde{s}^\wedge_m$

normalization

spatial normalization

$$\int \tilde{P}(x,z)\, \gamma^0\, \tilde{P}(z,y)\, d^3y = \frac{1}{2\pi}\tilde{P}(x,y)$$

$$\int \tilde{k}_m(x,z)\, \gamma^0\, \tilde{k}_m(z,y)\, d^3y = \frac{1}{2\pi}\tilde{k}_m(x,y)$$

$$P(\phi,\phi') := c\,(\tilde{k}_m \phi \mid \chi_-(S_m)\tilde{k}_m \phi')_m$$

$$= \frac{c}{2\pi}\langle \tilde{\phi} \mid \underbrace{\boxed{\chi_-(\tilde{S}_m)\tilde{k}_m}}_{=\tilde{P}} \phi' \rangle$$