$$\begin{cases} (\eta_t)_{t \in J} & \text{local foliation by space layers} \\ \text{hyperbolicity conditions} & \\ (\underline{v}, \underline{v})^t \geq \frac{1}{C^2} \int_{\mathcal{U}} \left(\|\underline{v}(x)\|_x^2 + |\Delta_2[\underline{v}, \underline{v}]| \right) d g_t & \forall \underline{v} \in \mathcal{J}_{\mathcal{U}} \end{cases}$$

Under these assumptions, we call

$$L := \bigcup_{t \in J} \operatorname{supp} \Theta_t \quad \subset \mathcal{U}$$

a lens-shaped region.

<u>energy estimates</u>:

$$\|\underline{v}\|_{L^2(L)} \leq \Gamma \, \|\Delta \underline{v}\|_{L^2(L)} \qquad \forall \underline{v} \in \mathcal{J}_{\mathcal{U}} \text{ with } \|v\|^t = 0$$

$$\|\underline{v}\|^t := \sqrt{(\underline{v}, \underline{v})^t}$$

$$L^2(L) := L^2(L, \underbrace{(\eta_{t_1} - \eta_{t_0})}_{\eta_J} \, d g)$$

<u>uniqueness of strong solutions</u>

$$\begin{cases} \Delta \underline{v} = \underline{w} \\ \|\underline{v}\|^{t_0} = 0 \end{cases} \qquad \left(\langle \underline{u}, \Delta \underline{v} \rangle(x) = \langle \underline{u}, \underline{w} \rangle \quad \forall \underline{u} \in \mathcal{J}^{\text{test}} \right)$$

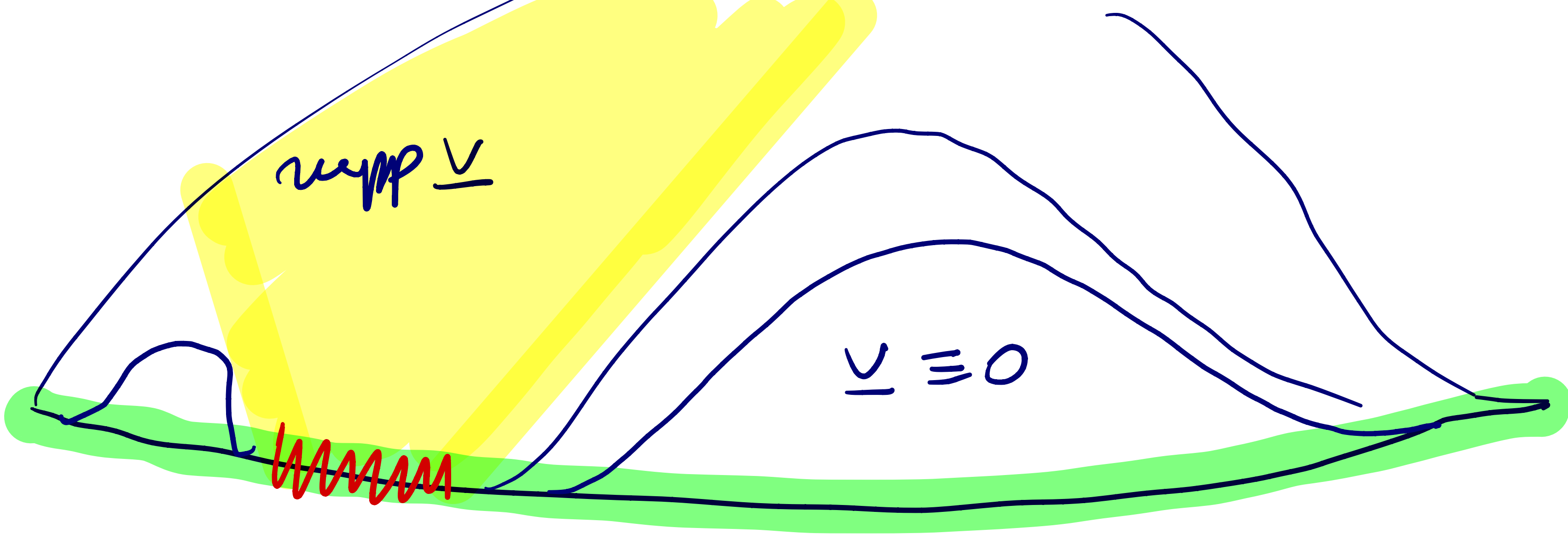
$\underline{v} \in \mathcal{J}_{\mathcal{U}}$

Let $\underline{v}_1, \underline{v}_2$ be two such solutions.

Then $\underline{v} := \underline{v}_1 - \underline{v}_2$ is a homogeneous eqn:

$$\begin{cases} \Delta \underline{v} = 0 \\ \|v\|^{t_0} = 0 \end{cases}$$

$$\overset{\text{energy estimates}}{\Longrightarrow} \quad \|v\|_{L^2(\mathcal{U})} = 0 \implies v|_L \equiv 0$$



Existence. K.O. Friedrichs

We need weak solutions — linear symmetric hyperbolic systems

idea "integrate by parts"

$$\langle \underline{u}, \Delta \underline{v} \rangle(x) = \langle \underline{u}, \underline{w} \rangle(x)$$

$$\Rightarrow \int_L \langle \underline{u}, \Delta \underline{v} \rangle(x)\, \eta_L(x)\, d\rho(x) = \int_L \langle \underline{u}, \underline{w} \rangle(x)\, \eta_L(x)\, d\rho(x)$$

$$= \int_L \langle \Delta \underline{u}, \underline{v} \rangle(x)\, \eta_L(x)\, d\rho(x) + \dots$$

<u>Lemma</u> (Green's formula) $\forall\, \underline{u}, \underline{v} \in \mathcal{J}_u$

$$\langle \underline{u}, \Delta \underline{v} \rangle_{L^2(L)} - \langle \Delta \underline{u}, \underline{v} \rangle_{L^2(L)}$$

$$= \sigma^{t_1}(\underline{u}, \underline{v}) - \sigma^{t_0}(\underline{u}, \underline{v}),$$

where σ^t is the softened symplectic form,

$$\sigma^t(\underline{u}, \underline{v}) := \int_{\mathcal{U}} d\rho(x) \int_{\mathcal{U}} d\rho(y)\; \eta_t(x)\, (1 - \eta_t(y))$$

$$\times \left(\nabla_{1,\underline{u}} \nabla_{2,\underline{v}} - \nabla_{1,\underline{v}} \nabla_{2,\underline{u}} \right) \mathcal{L}(x,y).$$

Proof:

$$\langle \underline{u}, \Delta \underline{v} \rangle_{L^2(L)} - \langle \Delta \underline{u}, \underline{v} \rangle_{L^2(L)}$$

$$= \int_U \left(\langle \underline{u}, \Delta \underline{v} \rangle(x) - \langle \Delta \underline{u}, \underline{v} \rangle(x) \right) \eta_s(x) \, d\rho(x)$$

$$= \int_U \eta_s(x) \, d\rho(x) \Big(\nabla_{\underline{u}} \Big(\int_U \big(\cancel{\nabla_{1,\underline{v}}} + \nabla_{2,\underline{v}} \big) \mathcal{L}(x,y) - \cancel{\nabla_{\underline{v}} \mathcal{D}} \Big) - \nabla_{\underline{v}} \Big(\int_U \big(\cancel{\nabla_{1,\underline{u}}} + \nabla_{2,\underline{u}} \big) \mathcal{L}(x,y) - \cancel{\nabla_{\underline{u}} \mathcal{D}} \Big) \Big)$$

$$= \int_U d\rho(x) \, \underbrace{\eta_s(x)}_{= \eta_{t_1}(x) - \eta_{t_0}(x)} \int_U \left(\nabla_{1,\underline{u}} \nabla_{2,\underline{v}} - \nabla_{1,\underline{v}} \nabla_{2,\underline{u}} \right) \mathcal{L}(x,y) \, d\rho(y)$$

$$= \int_U d\rho(x) \int_U d\rho(y) \; \eta_t(x) \left(\nabla_{1,\underline{u}} \nabla_{2,\underline{v}} - \nabla_{1,\underline{v}} \nabla_{2,\underline{u}} \right) \mathcal{L}(x,y) \Big|_{t=t_0}^{t=t_1}$$

vanishes in the integral by symmetry

$$= \int_U d\rho(x) \int_U d\rho(y) \Big(\underbrace{\eta_t(x) - \eta_t(x)\eta_t(y)}_{\eta_t(x)\,(1 - \eta_t(y))} \Big) \left(\nabla_{1,\underline{u}} \nabla_{2,\underline{v}} - \nabla_{1,\underline{v}} \nabla_{2,\underline{u}} \right) \mathcal{L}(x,y) \Big|_{t=t_0}^{t=t_1}$$

$$= \sigma^t(\underline{u}, \underline{v}) \Big|_{t=t_0}^{t=t_1} .$$

$\square$

derive weak equations:

assume $\quad \Delta \underline{v} = \underline{w} \quad$ strong solution

$$\Rightarrow \quad \langle \underline{u}, \Delta \underline{v} \rangle_{L^2(L)} = \langle \underline{u}, \underline{w} \rangle_{L^2(L)}$$

$\|$ Green's formula

$$\langle \Delta \underline{u}, \underline{v} \rangle - \sigma^{t_1}(\underline{u}, \underline{v}) + \sigma^{t_0}(\underline{u}, \underline{v})$$

the weak equation should include the condition that the initial data vanishes

In order to get rid of the symplectic form at t_1, we test in

$$\overline{\mathcal{J}_u} = \{ \underline{u} \in \mathcal{J}_u \mid (1-\eta_{t_1})\underline{u} = 0, \ \|\underline{u}\|^{t_1} = 0, \ \sigma^{t_1}(\underline{u}, \cdot) = 0 \}$$

$$\Rightarrow \quad \langle \underline{u}, \underline{w} \rangle_{L^2(L)} = \langle \Delta \underline{u}, \underline{v} \rangle_{L^2(L)} - \sigma^{t_0}(\underline{u}, \underline{v}) \quad \forall\, \underline{u} \in \overline{\mathcal{J}_u}$$

If $\underline{v}$ vanishes initially, $\underline{v} \in \underline{\mathcal{J}_u}$, then

$$\langle \Delta \underline{u}, \underline{v} \rangle_{L^2(L)} = \langle \underline{u}, \underline{w} \rangle_{L^2(L)} \quad \forall\, \underline{u} \in \overline{\mathcal{J}_u} \qquad (*)$$

This is the weak formulation of the Cauchy problem.

<u>Thm</u>: Assume: L is lens-shaped regions.
Then $\forall\, \underline{w} \in L^2(L)$ there is $\underline{v} \in L^2(L)$ which satisfies the weak Cauchy problem $(*)$.
Moreover,

$$\|\underline{v}\|_{L^2(L)} \leq \Gamma \, \|\underline{w}\|_{L^2(L)}$$

Proof: On $\overline{\mathcal{J}_u}$ one has an analogous energy estimate
$$\| \underline{u} \|_{L^2(L)} \leq \Gamma \, \| \Delta \underline{u} \|_{L^2(L)} \quad \forall \underline{u} \in \overline{\mathcal{J}_u}.$$
Introduce the bilinear form
$$\langle \cdot | \cdot \rangle : \overline{\mathcal{J}_u} \times \overline{\mathcal{J}_u} \to \mathbb{R},$$
$$(\underline{u}, \underline{v}) \mapsto \langle \Delta \underline{u}, \Delta \underline{v} \rangle_{L^2(L)}$$
This is positive definite because
$$\langle \underline{u} | u \rangle = \| \Delta u \|^2_{L^2(L)} \geq \frac{1}{\Gamma} \| \underline{u} \|^2_{L^2(L)}$$
Taking the completion gives a Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$, with norm $|||\cdot|||$
Consider the functional $\langle \underset{\substack{\cap \\ L^2(L)}}{\underline{w}}, \cdot \rangle_{L^2(L)}$.
The estimate
$$\left| \langle \underline{w}, \underline{u} \rangle_{L^2(L)} \right| \leq \| \underline{w} \|_{L^2(L)} \| \underline{u} \|_{L^2(L)}$$
$$\leq \| w \|_{L^2(L)} \, \Gamma \, \underbrace{\| \Delta \underline{u} \|_{L^2(L)}}_{= |||\underline{u}|||}$$
shows that functional is bounded on $\mathcal{H}$.
Thus the Fréchet-Riesz theorem yields $\underline{V} \in \mathcal{H}$ s.t.
$$\langle \underline{w}, \underline{u} \rangle_{L^2(L)} = \langle \underline{V} | \underline{u} \rangle$$
$$= \langle \underbrace{\Delta \underline{V}}_{=: \underline{v}}, \Delta \underline{u} \rangle_{L^2(L)}$$
Thus $\underline{v} := \Delta \underline{V}$ is the desired solution.
Note that $\Delta : \mathcal{H} \to L^2(L)$ is well-defined by continuity $\square$