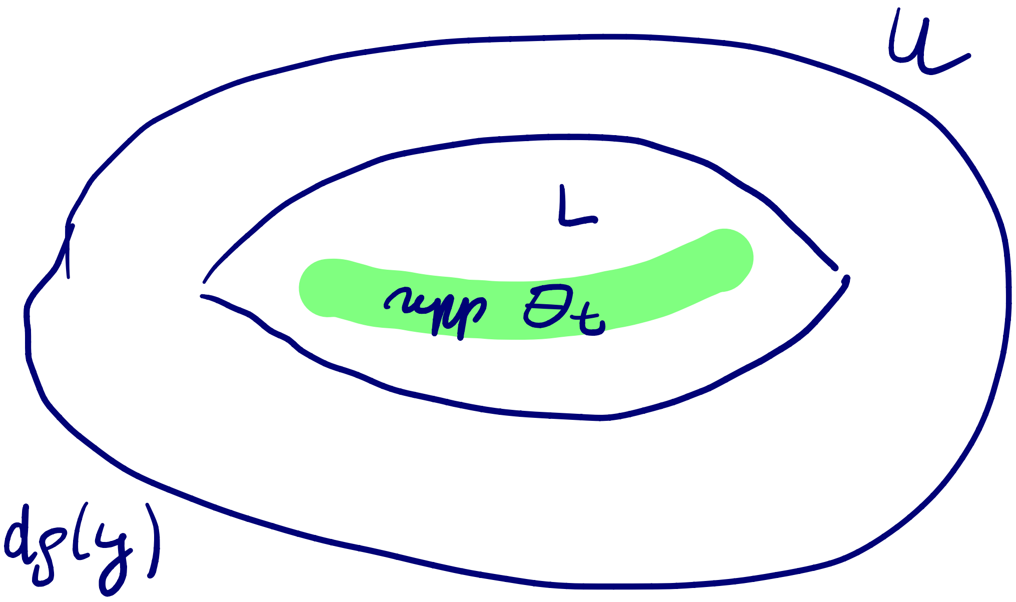
$(\eta_t)_{t \in J}$ local foliation by surface layers

$\theta_t := \partial_t \eta_t \geq 0$ compact support

$d\rho_t := \theta_t \, d\rho$

softened surface layer inner product



$$(\underline{u}, \underline{v})^t := \int_U d\rho(x) \int_U d\rho(y)$$

$$\times \; \eta_t(x)\,\big(1-\eta_t(y)\big)\,\big(\nabla_{1,\underline{u}} \nabla_{1,\underline{v}} - \nabla_{2,\underline{u}} \nabla_{2,\underline{v}}\big)\,\mathcal{L}(x,y)$$

$$E(t) := (\underline{v}, \underline{v})^t$$

$x \leftrightarrow y$, gives minus sign

$$\frac{d}{dt} E(t) = \int_U d\rho(x) \int_U d\rho(y)$$

$$\times \Big(\theta_t(x)\,\big(1-\eta_t(y)\big) + \eta_t(x)\,\big(-\theta_t(y)\big) \Big)$$

$$\times \big(\nabla_{1,\underline{v}}^2 - \nabla_{2,\underline{v}}^2\big)\,\mathcal{L}(x,y)$$

$$= \int_U d\rho(x) \int_U d\rho(y)\; \theta_t(x)\,\big(1 - \cancel{\eta_t(y)} + \cancel{\eta_t(y)}\big)\big(\nabla_{1,\underline{v}}^2 - \nabla_{2,\underline{v}}^2\big)\,\mathcal{L}(x,y)$$

$$= \int_U d\rho_t(x) \int_U \big(\nabla_{1,\underline{v}}^2 - \nabla_{2,\underline{v}}^2\big)\,\mathcal{L}(x,y)\; d\rho(y)$$

$$\langle \underline{v}, \Delta \underline{v} \rangle(x) = \nabla_{\underline{v}} \Big(\int_U \big(\nabla_{1,\underline{v}} + \nabla_{2,\underline{v}}\big)\,\mathcal{L}(x,y)\, d\rho(y) - \underbrace{\nabla_{\underline{v}}\, s}_{= b(x)\, s} \Big)$$

because $\nabla \mathcal{L}(x,y) = 0$ $\forall x \in L,\; y \in M \setminus U$

let $x \in L$

$\underline{v} = (b, v)$

$$= \int_u \left(\nabla_{1,\underline{v}}^2 + \nabla_{1,\underline{v}} \nabla_{2,v} \right) \mathcal{L}(x,y)\, d\rho(y)$$
$$- \sigma\, b(x)^2$$

$$0 = \int_u \langle \underline{v}, \Delta \underline{v} \rangle \, d\rho_t + \sigma \int_u b^2 \, d\rho_t \qquad | \times 2$$
$$- \int_u d\rho_t(x) \int_u \left(\nabla_{1,\underline{v}}^2 + \nabla_{1,\underline{v}} \nabla_{2,\underline{v}} \right) \mathcal{L}(x,y)\, d\rho(y).$$

$$\frac{d}{dt} E(t) = 2 \int_u \langle \underline{v}, \Delta \underline{v} \rangle \, d\rho_t + \sigma \int_u b^2 \, d\rho_t$$
$$- \int_u d\rho_t(x) \underbrace{\int_u \left(\nabla_{1,\underline{v}} + \nabla_{2,\underline{v}} \right)^2 \mathcal{L}(x,y)\, d\rho(y)}_{=: \Delta_2[\underline{v},\underline{v}](x)}$$

<u>Lemma</u> (energy identity)

$$\frac{d}{dt} (\underline{v}, \underline{v})^t = 2 \int_u \langle \underline{v}, \Delta \underline{v} \rangle \, d\rho_t + \sigma \int b^2 \, d\rho_t$$
$$- \int_u \Delta_2[\underline{v}, \underline{v}] \, d\rho_t$$

<u>Def</u> (hyperbolicity condition)

$\exists\, C > 0 \quad \forall\, t \in J$

$$(\underline{v}, \underline{v})^t \geq \frac{1}{C^2} \int_u \left(\|\underline{v}(x)\|_x^2 + |\Delta_2[\underline{v}, \underline{v}]| \right) d\rho_t(x)$$

$$\forall\, v \in \mathcal{J}_u \leftarrow \text{suitable jet space}$$

where $\|\underline{v}(x)\|$ is defined as follows: (<u>not</u> dense in L^2_{loc}.)

- choose a Riemannian metric g on $\mathcal{F}$
- $\langle \underline{v}(x), \underline{v}(x) \rangle_x := g(v(x), v(x)) + b(x)^2$

$-\quad \|v(x)\| := \sqrt{\langle \underline{v}(x), v(x)\rangle}$

$$\frac{d}{dt}(\underline{v},\underline{v})^t \leq 2\int_{\mathcal{U}} \langle \underline{v}, \Delta \underline{v}\rangle \, d\rho_t + \left(\rho C^2 + C^2\right) \quad (\underline{v},\underline{v})^t \, d\rho_t$$

introduce $\|v\|^t := (\underline{v},\underline{v})^t \qquad c = \frac{\rho C^2}{2} + \frac{C^2}{2}$

$$\frac{d}{dt}(\underline{v},\underline{v})^t = 2\,\|v\|^t \frac{d}{dt}\|v\|^t$$

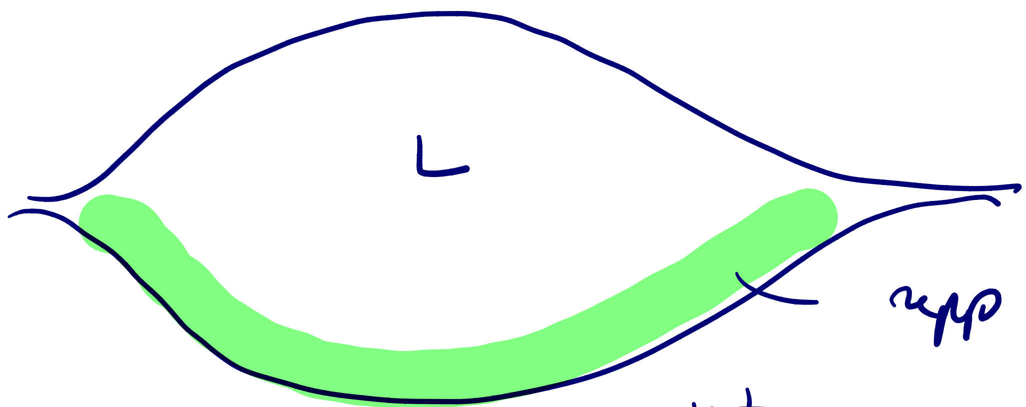
$$2\,\|v\|^t \frac{d}{dt}\|v\|^t \leq 2\,\underbrace{\|\underline{v}\|_{L^2(\mathcal{U},d\rho_t)}}_{\leq C\|v\|^t} \|\Delta \underline{v}\|_{L^2(\mathcal{U},d\rho_t)} + 2c\,\|\underline{v}\|^t\,\|\underline{v}\|^t$$

We then obtain the following estimate:

<u>Lemma</u> If the hyperbolicity assumption hold, $\exists\, C, c$

$$\frac{d}{dt}\|v\|^t \leq C\,\|\Delta \underline{v}\|_{L^2(\mathcal{U},d\rho_t)} + c\,\|\underline{v}\|^t \qquad \forall \underline{v} \in \mathcal{J}_{\mathcal{U}}$$

$\mathcal{J} = [t_0, t_1]$



assume that $\|v\|^{t_0} = 0$

$L^2(L)$; $\langle \underline{u}, \underline{v}\rangle_{L^2(L)} := \int_{\mathcal{U}} \langle \underline{u}(x), \underline{v}(x)\rangle_x \, \eta_{\mathcal{J}}(x)\, d\rho(x)$

$$\eta_{\mathcal{J}}(x) := \eta_{t_1}(x) - \eta_{t_0}(x)$$

$$\eta_{\mathcal{J}}(x) = \int_{t_0}^{t_1} \underbrace{\partial_t \eta_t(x)}_{\theta_t} \, dt$$

Thus

$$\langle \underline{u}, \underline{v} \rangle = \int_{t_0}^{t_1} dt \underbrace{\int_{\mathcal{U}} \langle \underline{u}(x), \underline{v}(x) \rangle_x \overbrace{\theta_t \, d\rho(x)}^{d\rho_t(x)}}_{\langle \underline{u}, \underline{v} \rangle_{L^2(\mathcal{U}, d\rho_t)}}$$

Prop: Assume that hyperbolicity conditions hold. Then $\exists \Gamma$ s.t.

$$\| \underline{v} \|_{L^2(L)} \leq \Gamma \, \| \Delta \underline{v} \|_{L^2(L)} \qquad \forall \underline{v} \in \mathcal{J}_{\mathcal{U}} \text{ with } \| \underline{v} \|^{t_0} = 0$$

Proof: $\frac{d}{dt}\left(e^{-2ct} (\underline{v}, \underline{v})^t \right) \leq 2c e^{-2ct} \| v \|^t \, \| \Delta v \|_{L^2(\mathcal{U}, d\rho_t)}$

Integrate on both sides from t_0 to t

$$\Rightarrow e^{-2ct} (\underline{v}, \underline{v})^t \leq 2c \int_{t_0}^{t} e^{-2ct'} \| \underline{v} \|^{t'} \, \| \Delta \underline{v} \|_{L^2(\mathcal{U}, d\rho_{t'})} dt'$$

$$\Rightarrow (\underline{v}, \underline{v})^t \leq 2C \int_{t_0}^{t} \underbrace{e^{2c(t-t')}}_{\leq e^{2c(t_1 - t_0)}} \| \underline{v} \|^{t'} \, \| \Delta \underline{v} \|_{L^2(\mathcal{U}, d\rho_t)} dt'$$

$$\leq 2C e^{2c(t_1 - t_0)} \left(\int_{t_0}^{t} (\| \underline{v} \|^{t'})^2 dt' \right)^{\frac{1}{2}} \times \left(\underbrace{\int_{t_0}^{t} \int_{\mathcal{U}} \langle \Delta \underline{v}, \Delta \underline{v} \rangle_x \, d\rho_t(x) \, dt'}_{\| \Delta v \|_{L^2(L)}} \right)^{\frac{1}{2}}$$

$$\Rightarrow (\underline{v}, \underline{v})^t \leq 2C\, e^{C(t_1 - t_0)} \, \|\Delta \underline{v}\|_{L^2(L)} \times \left(\int_{t_0}^{t_1} (\underline{v}, \underline{v})^{t'} \, dt' \right)^{\frac{1}{2}}.$$

$$\int_{t_0}^{t_1} (\underline{v}, \underline{v})^t \, dt \leq 2C\, (t_1 - t_0)\, e^{C(t_1 - t_0)} \, \|\Delta \underline{v}\|_{L^2(L)} \times \left(\int_{t_0}^{t_1} (\underline{v}, \underline{v})^{t'} \, dt' \right)^{\frac{1}{2}}$$

$$\Rightarrow \left(\int_{t_0}^{t} (\underline{v}, \underline{v})^t \, dt \right)^{\frac{1}{2}} \leq 2C\, (t_1 - t_0)\, e^{C(t_1 - t_0)} \, \|\Delta \underline{v}\|_{L^2(L)}$$

Using the hyperbolicity condition once again, we obtain

$$\|\underline{v}\|_{L^2(L)} = \left(\int_{t_0}^{t_1} \|\underline{v}\|^2_{L^2(U, dg_t)} \, dt \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_{t_0}^{t_1} (\underline{v}, \underline{v})^t \, dt \right)^{\frac{1}{2}}$$

$$\leq \underbrace{2C^2 (t_1 - t_0)\, e^{C(t_1 - t_0)}}_{\Gamma} \, \|\Delta \underline{v}\|_{L^2(L)}$$

$\square$