$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\tilde{\Psi} = 0$$

$$\tilde{P} = P - s_m \mathcal{B} P - P \mathcal{B} s_m + \ldots$$

$$P(k) = (k\!\!\!/ + m)\,\delta(k^2 - m^2)\,\Theta(-k^0)$$

$$s_m = \frac{1}{2}\left(s_m^\vee + s_m^\wedge\right)$$

How does $\tilde{P}(x,y)$ look like in position space

Hadamard form

$$P(x,y) = \lim_{\varepsilon \searrow 0} i\partial\!\!\!/_x \left(\frac{U(x,y)}{\sigma_\varepsilon(x,y)} + V(x,y)\,\log \sigma_\varepsilon(x,y) + W(x,y) \right),$$

singular on light cone

$$\xi = y - x$$

$$\sigma_\varepsilon(x,y) := \xi^2 - i\varepsilon\,\xi^0$$

light-cone expansion: – Minkowski space
- systematic method for computing the Hadamard expansion to every order in perturbation theory
- also compute the smooth contributions

1992 – 1998 " 2002 $A_{xy} = P(x,y)\,P(y,x)$

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overview:
- $\tilde{s}_m^\vee, \tilde{s}_m^\wedge$ light-cone expansion of advanced and retarded Green's operators order by order in perturbation expansion
- $\tilde{k}_m := \frac{1}{2\pi i}\left(\tilde{s}_m^\vee - \tilde{s}_m^\wedge\right)$
- residual argument: light-cone expansion of $\tilde{k}_m$

$\xrightarrow{\text{computational procedure}}$ light-cone expansion for $\tilde{P}$

- resum perturbation expansion to get non-perturbative formulas
- smooth contributions need to be analyzed.

Def: A distribution $A(x,y)$ on $M \times M$ is of the order $\mathcal{O}((y-x)^{2p})$ on the light cone if ($p \in \mathbb{Z}$) $(y-x)^{-2p} A(x,y)$ is a regular distribution.

An expansion of the form

$$A(x,y) = \sum_{j=g}^{\infty} A^{[j]}(x,y)$$

if the $A^{[j]}(x,y)$ are of the order $\mathcal{O}((y-x)^{2j})$ and

$$A(x,y) - \sum_{j=g}^{p} A^{[j]}(x,y) \text{ is of the order } \mathcal{O}((y-x)^{2p+2}).$$

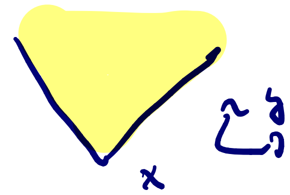
analogy to Taylor expansion of a smooth function
the Taylor polynomials approximate the function,
but Taylor series does not need to converge
however: $(y-x)^2$ vanishes on the light cone
nonlocal expansion.

no external potential

$$s^{\vee}_m(x,y) = (i\partial\!\!\!/ + m)\, S^{\wedge}_{m^2}(x,y) \quad ; \quad a = m^2$$

$$S^{\vee}_a(k) = \lim_{\delta \to 0} \frac{1}{k^2 - a - i\delta k^0} \qquad \text{Klein-Gordon Green's operator}$$

Compute Fourier transform



$$S_a^\vee(x,y) = -\frac{1}{2\pi}\,\delta\big((y-x)^2\big)\,\Theta(y^0-x^0) + \frac{a}{4\pi}\,\frac{J_1\big(\sqrt{a(y-x)^2}\big)}{\sqrt{a(y-x)^2}}\,\Theta\big((y-x)^2\big)\,\Theta(y^0-x^0)$$

Expand in powers of a

$$S_a^\vee(x,y) = -\frac{1}{2\pi}\,\delta\big((y-x)^2\big)\,\Theta(y^0-x^0) + a\sum_{p=0}^{\infty} c_p \big(a\,(y-x)^2\big)^p\,\Theta\big((y-x)^2\big)\,\Theta(y^0-x^0)$$

expanding in powers of a, one gets the desired light-cone expansion.

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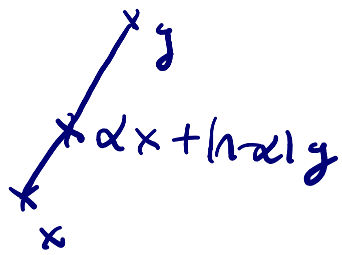
external potential $\mathcal{B}$

$$-s_m^\vee\,\mathcal{B}\,s_m^\vee$$

scalar potential

pull out Dirac matrices: $\quad S_a^\vee\, V\, S_a^\vee$

define $\quad S^{(l)}(x,y) := \left(\frac{d}{da}\right)^{l} S_a^\vee(x,y)\Big|_{a=0}$



<u>Lemma</u>: For any $V \in C^\infty(M,\mathbb{R})$ and $l, r \geq 0$

$$\big(S^{(l)}\,V\,S^{(r)}\big)(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\int_0^1 \alpha^{l}\,(1-\alpha)^{r}\,(\alpha-\alpha^2)^n\,(\Box^n V)\Big|_{\alpha y+(1-\alpha)x}\,d\alpha \times S^{(l+r+n+1)}(x,y),$$

$$S_a^{\vee}(p) = \lim_{\delta \searrow 0} \frac{1}{p^2 - a - i\delta p^0}$$ leave out $\vee$ $\lim_{\delta\searrow 0}$

$$\frac{\partial}{\partial p^k} S_a = -\left(\frac{d}{da} S_a\right) 2 p_k$$

Expand in powers of a

$$\frac{\partial}{\partial p^k} S^{(\ell)}(p) = -2 p_k \; S^{(\ell+1)}(p)$$ assume $\ell \geq 1$

$$\frac{\partial}{\partial x^k} S^{(\ell)}(x,y) = \int \frac{d^4p}{(2\pi)^4} S^{(\ell)}(p) \, (-i p_k) \, e^{-ip(x-y)}$$

$$= \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \frac{\partial}{\partial p^k} S^{(\ell-1)}(p) \; e^{-ip(x-y)}$$

$$= -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} S^{(\ell-1)}(p) \frac{\partial}{\partial p^k} e^{-ip(x-y)}$$

$$= \frac{1}{2} (y-x)_k \; S^{(\ell-1)}(x,y) \quad , \quad \ell \geq 1,$$

$$\Rightarrow -\Box \, S^{(\ell)}(x,y)$$

$$= -\frac{1}{2} \frac{\partial}{\partial x^k}\left((y-x)^k \; S^{(\ell-1)}(x,y)\right)$$

$$= 2 \, S^{(\ell-1)}(x,y) + \frac{1}{4} (y-x)^2 \; S^{(\ell-2)}(x,y), \quad \ell \geq 2 \qquad (1)$$

alternatively:

$$(-\Box_x - a) \, S_a(x,y) = \delta^4(x-y)$$

expanding in powers of a gives

$$-\Box \, S^{(\ell)}(x,y) = \delta_{\ell,0} \, \delta^4(x-y) + \ell \, S^{(\ell-1)}(x,y), \quad \ell \geq 0 \qquad (2)$$

Combining (1) and (2) gives

$$(y-x)^2 \; S^{(\ell)}(x,y) = -4\ell \; S^{(\ell+1)}(x,y) \; ; \quad \ell \geq 0$$

Finally,

$$\frac{\partial}{\partial x^k} S^{(\ell)}(x,y) = -\frac{\partial}{\partial y^k} S^{(\ell)}(x,y) \; , \quad \ell \geq 0$$

Computation rules:

$$S^{(p)} = \left(\frac{d}{da}\right)^p S_a^v \Big|_{p=0} \qquad (1)$$

$$\frac{\partial}{\partial p_R} S^{(l)}(p) = -2p_R \, S^{(l+1)}(p) \qquad , l \geq 0 \qquad (2)$$

$$\frac{\partial}{\partial x^R} S^{(l)}(x,y) = -\frac{\partial}{\partial y^R} S^{(l)}(x,y) \qquad , l \geq 0 \qquad (3)$$

$$= \frac{1}{2} (y-x)_R \, S^{(l-1)}(x,y) \qquad , l \geq 1 \qquad (4)$$

$$-\Box \, S^{(l)}(x,y) = \delta_{l,0} \, \delta^4(x,y) + l \, S^{(l-1)}(x,y) \qquad , l \geq 1 \qquad (5)$$

$$(y-x)^2 S^{(l)}(x,y) = -4l \, S^{(l+1)}(x,y) \qquad , l \geq 0 \qquad (6)$$

Lemma

$$(S^{(\ell)}\, V\, S^{(r)})(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^\ell (1-\alpha)^r (\alpha-\alpha^2)^n \,\Box^n V\big|_{\alpha y+(1-\alpha)x}\, S^{(n+\ell+r+1)}(x,y)$$

Proof strategy: show that

$$\left.\begin{aligned} &\Box(\text{lhs}-\text{rhs}) = 0 \\ &(\text{lhs}-\text{rhs})\big|_{y^0\in(t-\delta,\,t+\delta)} = 0 \end{aligned}\right\}$$

next, using causality

$$\xRightarrow[\text{uniqueness of solutions of Cauchy problem}]{} \quad \text{lhs}-\text{rhs} \equiv 0$$

$$-\Box_x (S^{(\ell)} V S^{(r)})(x,y) \overset{(5)}{=} \delta_{\ell,0}\, V(x)\, S^{(r)}(x,y) + \ell\, (S^{(\ell-1)} V S^{(r)})(x,y)$$

$$-\Box_x \int_0^1 \alpha^\ell (1-\alpha)^r (\alpha-\alpha^2)^n \,\Box^n V\big|_{\alpha y+(1-\alpha)x}\, S^{(n+\ell+r+1)}(x,y)$$

$$= -\int_0^1 \alpha^\ell (1-\alpha)^{r+2} (\alpha-\alpha^2)^n \,\Box^{n+1} V\big|_{\alpha y+(1-\alpha)x}\, S^{(n+\ell+r+1)}(x,y)$$

$$\overset{(4)}{-}\cancel{2}\int_0^1 \alpha^\ell (1-\alpha)^{r+1} (\alpha-\alpha^2)^n \,\partial_k \Box^n V\big|_{\alpha y+(1-\alpha)x} \times \frac{1}{\cancel{2}}(y-x)^k\, S^{(n+\ell+r)}(x,y)$$

$$\overset{(5)}{+}\int_0^1 \alpha^\ell (1-\alpha)^r (\alpha-\alpha^2)^n \,\Box^n V\big|_{\alpha y+(1-\alpha)x}\, (n+\ell+r+1)\, S^{(n+\ell+r)}(x,y)$$

The green expression can be written as a derivative w.r.t. α

$$(y-x)^k \, \partial_k \, \Box^n V \big|_{\alpha y + (1-\alpha)x}$$

$$= \frac{d}{d\alpha} \Box^n V \big|_{\alpha y + (1-\alpha)x} .$$

Then we integrate by parts. We thus obtain

$$\int_0^1 \alpha^\ell (1-\alpha)^{r+1} (\alpha-\alpha^2)^n \, \partial_k \Box^n V \big|_{\alpha y+(1-\alpha)x} (y-x)^k \, d\alpha$$

$$= \int_0^1 \alpha^\ell (1-\alpha)^{r+1} \underbrace{(\alpha-\alpha^2)^n}_{= \alpha^n (1-\alpha)^n} \frac{d}{d\alpha} \Box^n V \big|_{\alpha y+(1-\alpha)x} \, d\alpha$$

$$= -\delta_{\ell,0} \, \delta_{n,0} \, V(x)$$

$$- \int_0^1 (n+\ell) \, \alpha^\ell (1-\alpha)^{r+2} (\alpha-\alpha^2)^{n-1} \, \Box^n V \big|_{\alpha y+(1-\alpha)x}$$

$$+ \int_0^1 (n+r+1) \, \alpha^\ell (1-\alpha)^r (\alpha-\alpha^2)^n \, \Box^n V \big|_{\alpha y+(1-\alpha)x}$$

$$= -\delta_{n,0} \, \delta_{\ell,0} \, V(x) \qquad (1-\alpha)^2 = (1-\alpha) - \underbrace{\alpha(1-\alpha)}_{= (\alpha-\alpha^2)}$$

$$- n \int_0^1 \alpha^\ell (1-\alpha)^{r+2} (\alpha-\alpha^2)^{n-1} \, \Box^n V \big|_{\alpha y+(1-\alpha)x}$$

$$+ (n+\ell+r+1) \int_0^1 \alpha^\ell (1-\alpha)^r (\alpha-\alpha^2)^n \, \Box^n V \big|_{\alpha y+(1-\alpha)x}$$

$$- \ell \int_0^1 \alpha^{\ell-1} (1-\alpha)^r (\alpha-\alpha^2)^n \, \Box^n V \big|_{\alpha y+(1-\alpha)x} .$$

We thus obtain

$$-\mathbb{D}_x \int_0^1 \alpha^\ell (1-\alpha)^r (\alpha-\alpha^2)^n \, \Box^n V\big|_{\alpha y+(1-\alpha)x} \, S^{(n+\ell+r+1)}(x,y)$$

$$= \delta_{n,0}\, \delta_{\ell,0}\, V(x)\, S^{(r)}(x,y)$$

$$+ \ell \int_0^1 \alpha^{\ell-1} (1-\alpha)^r (\alpha-\alpha^2)^n \, \Box^n V\big|_{\alpha y+(1-\alpha)x} \, S^{(n+\ell+r)}(x,y)$$

$$- \int_0^1 \alpha^\ell (1-\alpha)^{r+2} (\alpha-\alpha^2)^n \, \Box^{n+1} V\big|_{\alpha y+(1-\alpha)x} \, S^{(n+\ell+r+1)}(x,y)$$

$n \to n+1$

$$+ n \int_0^1 \alpha^\ell (1-\alpha)^{r+2} (\alpha-\alpha^2)^{n-1} \, \Box^n V\big|_{\alpha y+(1-\alpha)x} \, S^{(n+\ell+r)}(x,y).$$

Dividing by $n!$ and summing over n, we obtain

$$-\mathbb{D}_x \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 \alpha^\ell (1-\alpha)^r (\alpha-\alpha^2)^n \, \Box^n V\big|_{\alpha y+(1-\alpha)x} \, S^{(n+\ell+r+1)}(x,y)$$

$$= \delta_{\ell,0}\, V(x)\, S^{(r)}(x,y)$$

$$+ \ell \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 \alpha^{\ell-1} (1-\alpha)^r (\alpha-\alpha^2)^n \, \Box^n V\big|_{\alpha y+(1-\alpha)x} \, S^{(n+\ell+r)}(x,y)$$

$\ell=0$:
$$\begin{cases} \mathbb{D}_x (\text{lhs} - \text{rhs}) = 0 \\ (\text{lhs} - \text{rhs})\big|_{x^0 > y^0+1} = 0 \end{cases}$$

$$\Longrightarrow \quad \text{lhs} \equiv \text{rhs}$$

$\ell-1 \Longrightarrow \ell$: Using the induction hypothesis,

$$\mathbb{D}_x (\text{lhs} - \text{rhs}) = 0$$

just as before $\text{lhs} \equiv \text{rhs}$. $\Box$