$\mathcal{D} = i\not{\partial}$

$(\mathcal{D} - m)\,\psi_m = 0$

$\mathcal{H}_m \cap C^\infty_{sc}(\mathcal{M}, S\mathcal{M})$

$\mathcal{J} = (m_L, m_R)$ with $m_L > 0$

$\psi = (\psi_m)_{m\in\mathcal{J}}$ family of solutions

$(\psi | \phi) = \int_{\mathcal{J}} (\psi_m | \phi_m)_m \, dm$

$\mathcal{H} \cap C^\infty_{sc,0}(\mathcal{M}\times\mathcal{J}, S\mathcal{M}) =: \mathcal{H}^\infty$

$\mathfrak{p} : \mathcal{H}^\infty \longrightarrow C^\infty_{sc}(\mathcal{M}, S\mathcal{M})$

$(\mathfrak{p}\psi)(x) := \int_{\mathcal{J}} \psi_m(x)\, dm$

<u>Thm</u>: $\langle \mathfrak{p}\psi | \mathfrak{p}\phi \rangle = \int_{\mathcal{J}} (\psi_m | S_m \phi_m)_m \, dm$

where $S_m \in L(\mathcal{H}_m)$ uniformly bounded.

$(\mathcal{D} - m)\,\psi_m = 0$

$\psi_m|_{t=0} = \psi_{m,0}(\vec{x})$

$$\begin{cases} \psi_m(x) = \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - m^2)\, \epsilon(k^0)\, (\not{k} + m)\, \phi_m(k)\, e^{-ikx} \\ \psi_{m,0}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \hat{\psi}_{m,0}(\vec{k})\, e^{i\vec{k}\vec{x}} \end{cases}$$

$$\psi_m|_{t=0} = \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - m^2)\, \epsilon(k^0)\, (\not{k} + m)\, \phi_m(k)\, e^{i\vec{k}\vec{x}}$$

$$\Rightarrow \hat{\psi}_{m,0}(\vec{k}) = \int \frac{dk^0}{2\pi}\, \delta(k^2 - m^2)\, \epsilon(k^0)\, (\not{k} + m)\, \phi_m(k)$$

$\omega := k^0$; $\delta(k^2 - m^2) = \delta(\omega^2 - |\vec{k}|^2 - m^2)$

$\omega(\vec{k}) := \sqrt{|\vec{k}|^2 + m^2}$

$$\Rightarrow \hat{\psi}_{m,0}(\vec{k}) = \frac{1}{2\pi}\,\frac{1}{2\omega(\vec{k})}\Big(\epsilon(\omega)\,(\omega\gamma^0 - \vec{k}\vec{\gamma} + m) \times \phi_m(\cancel{\omega},\vec{k})\Big)\Big|_{\omega=-\omega(\vec{k})}^{\omega=+\omega(\vec{k})}$$

w.l.o.g. using that $\not{k}+m$ has a two-dim. kernel

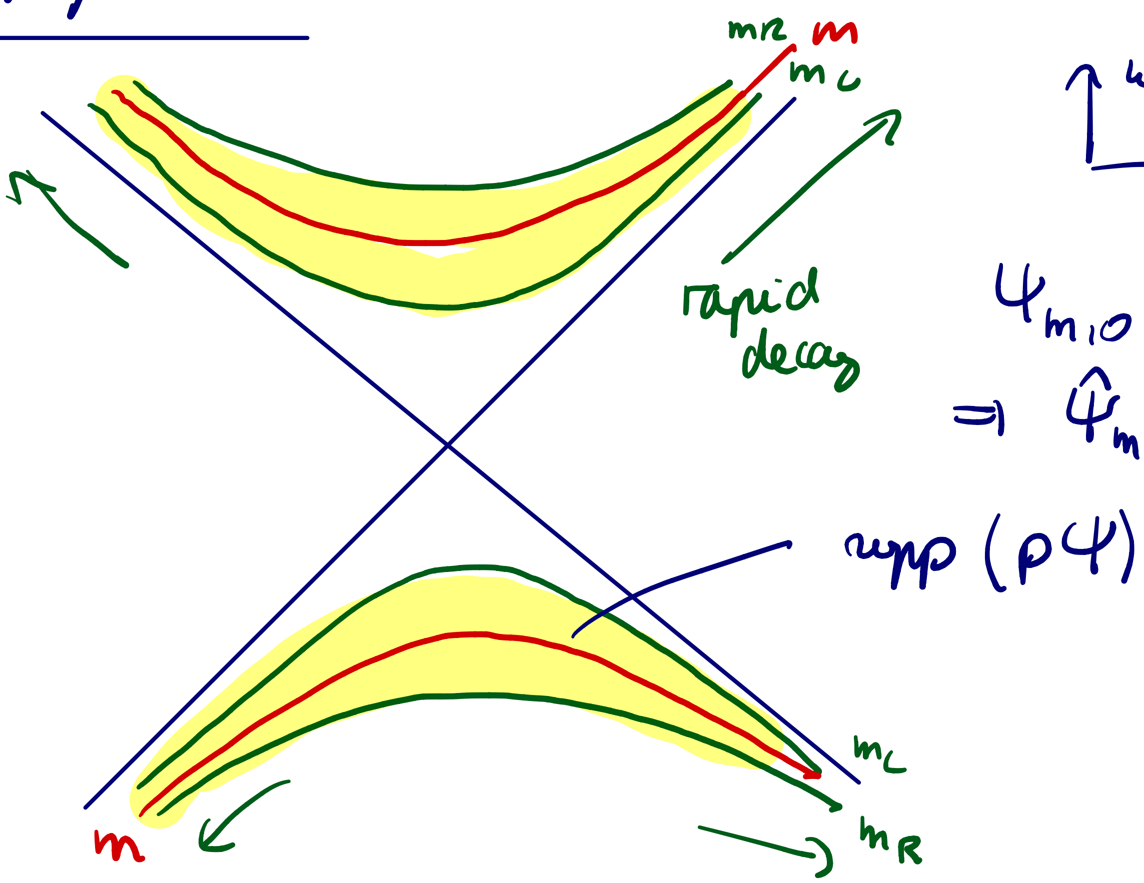
$$= \frac{1}{4\pi\,\omega(\vec{k})} \times 2\,\omega(\vec{k})\,\gamma^0\,\phi_m(\vec{k})$$

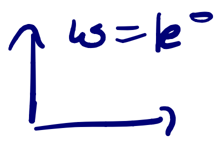
$$= \frac{1}{2\pi}\,\gamma^0\,\phi_m(\vec{k})$$

$$\Rightarrow \phi_m(\vec{k}) = 2\pi\,\gamma^0\,\hat{\psi}_{m,0}(\vec{k})$$

$$\psi_m(x) = \int \frac{d^4k}{(2\pi)^4}\,\psi_m(k)\,e^{-ikx}$$

$$\underline{\psi_m(k) = 2\pi\,\delta(k^2-m^2)\,\epsilon(k^0)\,(\not{k}+m)\,\gamma^0\,\hat{\psi}_{m,0}(\vec{k})}$$





$$\psi_{m,0} \in C_0^\infty(\mathbb{R}^3)$$

$$\Rightarrow \hat{\psi}_{m,0} \in \mathcal{S}(\mathbb{R}^3)$$

Plancherel:

$$\langle\psi|\phi\rangle = \int \frac{d^4k}{(2\pi)^4}\,\langle\psi(k)|\phi(k)\rangle$$

$\langle\psi_m|\psi_m\rangle$ diverges

Compute $P\psi$

$$\psi_m(k) = 2\pi\,\delta(k^2-m^2)\,\epsilon(k^0)\,(\cancel{k}+m)\gamma^0\,\hat{\psi}_{m,0}(\vec{k})$$

$$(P\psi)(k) = 2\pi \int_{m_L}^{m_R} dm \times \delta(k^2-m^2)\,\epsilon(k^0)\,(\cancel{k}+m)\gamma^0\,\hat{\psi}_{m,0}(\vec{k})$$

$$= 2\pi\,\frac{1}{2m}\,\epsilon(k^0)\,(\cancel{k}+m)\gamma^0\,\hat{\psi}_{m,0}(\vec{k})\Big|_{m=\sqrt{k^2}}$$

smooth, supported in region

$$m_L^2 \leq k^2 \leq m_R^2$$

Now $\langle P\psi | P\phi\rangle$ exists.

$\langle P\psi | P\phi\rangle$

$$= \int \frac{d^4k}{(2\pi)^4}\left(\frac{\pi}{m}\right)^2 \langle \underbrace{(\cancel{k}+m)}_{=2m}\gamma^0\,\hat{\psi}_{m,0}(\vec{k})\,|\,(\cancel{k}+m)\gamma^0\,\hat{\phi}_{m,0}(\vec{k})\rangle\Big|_{m=\sqrt{k^2}}$$

$$(\cancel{k}+m)^2 = k^2 + 2\cancel{k}m + m^2$$

$$= m^2 + 2\cancel{k}m + m^2$$

$$= 2m(\cancel{k}+m)$$

$$= \int \frac{d^4k}{4\pi^2}\,\frac{1}{2m}\,\langle \gamma^0\,\hat{\psi}_{m,0}(\vec{k})\,|\,(\cancel{k}+m)\gamma^0\,\hat{\phi}_{m,0}(\vec{k})\rangle\Big|_{m=\sqrt{k^2}}$$

$$\int \cdots dk^0 \rightsquigarrow \int \cdots dm$$

$$m^2 = (k^0)^2 - |\vec{k}|^2 \Longrightarrow m\,dm = |k^0|\,dk^0$$

$$= \frac{1}{4\pi^2}\int_J dm \int \frac{d^3k}{2|k^0|}\,\langle \gamma^0\,\hat{\psi}_{m,0}(\vec{k})\,|\,(\cancel{k}+m)\gamma^0\,\hat{\phi}_{m,0}(\vec{k})\rangle\Big|_{\substack{k^0 = \pm\sqrt{\vec{k}^2+m^2} \\ = \pm\,\omega(\vec{k})}}$$

$$\Rightarrow |\langle p\psi | p\phi\rangle|$$

$$\leq c \int_J dm \int \frac{d^3k}{(2\pi)^3} \|\hat\psi_{m,0}(\vec k)\| \, \|\hat\phi_{m,0}(\vec k)\|$$

Cauchy Schwarz

$$\leq c \int_J dm \left(\int \frac{d^3k}{(2\pi)^3} \|\hat\psi_{m,0}(\vec k)\|^2\right)^{\frac12} \left(\int \frac{d^3k}{(2\pi)^3} \|\hat\phi_{m,0}(\vec k)\|^2\right)^{\frac12}$$

Plancherel

$$= c \int_J \|\hat\psi_m\|_m \, \|\hat\phi_m\|_m \, dm.$$

Thus the strong mass oscillation property holds.

$$\langle p\psi | p\phi\rangle$$

$$= \frac{1}{4\pi^2} \int_J dm \int \frac{d^3k}{2|k^0|} \langle \gamma^0 \hat\psi_{m,0}(\vec k) | (\not k + m)\gamma^0 \hat\phi_{m,0}(\vec k)\rangle \Big|_{k^0 = \pm\omega(\vec k)}$$

$$\Pi_\pm(\vec k) := \frac{1}{2k^0} (\not k + m)\gamma^0 \Big|_{k^0 = \pm\omega(\vec k)}$$

orthogonal projection operators on $\mathbb{C}^4$

$$= \frac{1}{4\pi^2} \int_J dm \int d^3k \sum_\pm \left(\pm \langle \hat\psi_{m,0}(\vec k) | \gamma^0 \Pi_\pm \hat\phi_{m,0}(\vec k)\rangle\right)$$

$$= \frac{(2\pi)^3}{4\pi^2} \int_J dm \int \frac{d^3k}{(2\pi)^3} \langle \hat\psi_{m,0}(\vec k) | \gamma^0 (\Pi_+ - \Pi_-) \hat\phi_{m,0}(\vec k)\rangle$$

$$= 2\pi \int_J dm \, (\psi_m | (\Pi_+ - \Pi_-) \phi_m)_m$$

$$\Rightarrow S_m = 2\pi(\Pi_+ - \Pi_-).$$