$(\mathcal{H}, (\cdot|\cdot))$ $\quad (\psi|\phi) = \int_Y (\psi_m|\phi_m)_m \, dm$

$\mathcal{H}^\infty \subset \mathcal{H}$ domain

$T: \mathcal{H} \to \mathcal{H}$, $\quad (T\psi)_m = m\psi_m$

$p: \mathcal{H}^\infty \to C^\infty_{sc}(\mathcal{M}, S\mathcal{M})$; $\quad (p\psi)(x) := \int_Y \psi_m(x)\, dm$

weak mass oscillation property: $\forall \psi \in \mathcal{H}^\infty \; \exists \, c(\psi)$ s.t.

(i) $|\langle p\psi | p\phi \rangle| \leq c(\psi) \, \|\phi\| \quad \forall \phi \in \mathcal{H}^\infty$

(ii) $\langle pT\psi | p\phi\rangle = \langle p\psi | pT\phi\rangle \quad \forall \phi, \psi \in \mathcal{H}^\infty$

(i) $\Longrightarrow$ $\forall \psi \in \mathcal{H}^\infty$,

$\langle p\psi | p\,.\rangle : \mathcal{H}^\infty \longrightarrow \mathbb{C}$ is bounded

$\Rightarrow$ can be extended uniquely to $\mathcal{H}$

Fréchet-Riesz: $\exists! \, u \in \mathcal{H}$ s.t.

$(u|\phi) = \langle p\psi | p\phi\rangle \quad \forall \phi \in \mathcal{H}^\infty$

The mapping $\psi \mapsto u$ is linear.

$S: \mathcal{H}^\infty \longrightarrow \mathcal{H}$, $\quad S\psi = u$.

$\Rightarrow (S\psi|\phi) = \langle p\psi | p\phi\rangle \quad \forall \psi, \phi \in \mathcal{H}^\infty$

Do we get an operator $S_m : \mathcal{H}_m \longrightarrow \mathcal{H}_m$?

(ii) $\Longrightarrow$ $ST\psi = TS\psi \quad \forall \psi \in \mathcal{H}^\infty$

$\overset{?}{\Longrightarrow} (S\psi)_m = S_m \psi_m$

problem: S is a densely defined, symmetric operator with $\mathcal{D}(S) = \mathcal{H}^\infty$

moreover: $[S, T] = 0$ on $\mathcal{H}^\infty$

task: construct self-adjoint extension of S
show that spectral measures of S and T commute

$\rightsquigarrow$ $S_m\, dm$ can be constructed

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strong mean oscillation property

$$|\langle p\psi | p\phi\rangle| \le c \int_J \|\psi_m\|_m \|\phi_m\|_m \, dm \quad \forall \psi, \phi \in \mathcal{H}^\infty$$

<u>Thm</u>: The following statements are equivalent:

(i) The strong mean oscillation property holds.

(ii) $\exists\, c > 0$ s.t. $\forall \psi, \phi \in \mathcal{H}^\infty$,

$$|\langle p\psi | p\phi\rangle| \le c\, \|\psi\|\, \|\phi\|$$

$$\langle pT\psi | p\phi\rangle = \langle p\psi | pT\phi\rangle$$

(iii) $\exists\, (S_m)_{m\in J}$, $S_m \in L(\mathcal{H}_m)$ uniformly bounded,

$$\sup_{m\in J} \|S_m\| < \infty$$

s.t.

$$\langle p\psi | p\phi\rangle = \int_J (\psi_m | S_m \phi_m)_m \, dm.$$

Proof:

(i) $\Rightarrow$ (ii)

$$|\langle p\psi | p\phi\rangle| \le c \int_J \|\phi_m\|_m \, \|\psi_m\|_m \, dm$$

Cauchy-Schwarz $\le c \left(\int_J \|\phi_m\|_m^2 \, dm\right)^{\frac{1}{2}} \left(\int_J \|\psi_m\|_m^2 \, dm\right)^{\frac{1}{2}}$

def. of $(\cdot|\cdot)$ $\le c \, \|\phi\| \, \|\psi\|$.

Next, given N we subdivide $J = (m_L, m_R)$ $(m_L, m_R > 0)$

$$m_\ell := \frac{\ell}{N}(m_R - m_L) + m_L$$

$$m_0 = m_L < m_1 < \dots < m_L = m_R$$

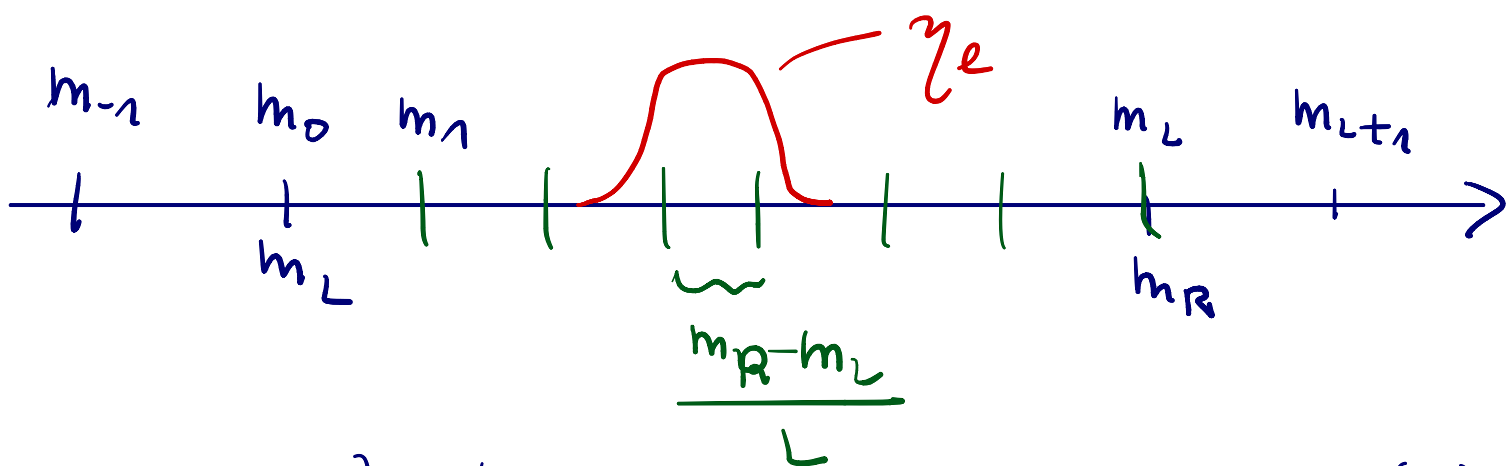
subintervals $(m_{\ell-1}, m_\ell)$, $\ell = 1, \dots, N$

Choose $\eta_1, \dots, \eta_L \in C_0^\infty(\mathbb{R})$, $\eta_\ell \geq 0$

$$\sum_{\ell=1}^{L} \eta_\ell|_J = 1|_J \qquad \text{(partition of unity)}$$

$$\operatorname{supp} \eta_\ell \subset (m_{\ell-2}, m_{\ell+1})$$

$$m_{-1} = m_L - 1, \qquad m_{L+1} = m_R + 1$$



introduce $\eta(T)$ by $(\eta(T)\psi)_m = \eta(m)\psi_m$

$$\langle \rho T\psi | \rho\phi\rangle - \langle \rho\psi | \rho T\phi\rangle$$

$$= \sum_{\ell,\ell'=1}^{N} \Big(\langle \rho T \eta_\ell(T)\psi | \rho\, \eta_{\ell'}(T)\, \phi\rangle - \langle \rho\, \eta_\ell(T)\psi | \rho\, T\, \eta_{\ell'}(T)\, \phi\rangle \Big)$$

$$= \sum_{\ell,\ell'=1}^{N} \Big(\langle \rho (T-m_\ell)\eta_\ell(T)\psi | \rho\, \eta_{\ell'}(T)\, \phi\rangle - \langle \rho\, \eta_\ell(T)\psi | \rho (T-m_\ell)\eta_{\ell'}(T)\, \phi\rangle \Big)$$

Take absolute value and use strong mean oscillation property. This gives

$$|\langle \rho T\psi | \rho\phi\rangle - \langle \rho\psi | \rho T\phi\rangle|$$

$$\leq c \sum_{\ell,\ell'}^{N} \int_Y \Big(|m-m_\ell|\, \eta_\ell(m)\ \eta_{\ell'}(m)\ \|\psi_m\|\ \|\phi_m\| + \big(\eta_\ell(m)\ |m-m_\ell|\ \eta_{\ell'}(m)\ \|\psi_m\|\ \|\phi_m\|\big)\Big)$$

$$\leq 2c \sum_{\ell,\ell'} \int_Y \underbrace{|m-m_\ell|}_{\leq 2\frac{|Y|}{N}}\ \underbrace{\eta_\ell(m)\ \ \eta_{\ell'}(m)}_{=0 \text{ unless } |\ell-\ell'|\leq 1}\ \|\psi_m\|\ \|\phi_m\|$$

$$\leq 4c\ \frac{|Y|}{N}\ 2 \int_Y \|\psi_m\|_m\ \|\phi_m\|_m\ dm$$

Since N is arbitrary, we conclude that the left side is zero.

(iii) $\Rightarrow$ (i):

$$|\langle \rho\psi | \rho\phi\rangle| = \left| \int_Y (\psi_m | S_m\, \phi_m)_m\ dm \right|$$

$$\leq 1 \int_Y \|\psi_m\|_m \underbrace{\|S_m\|}_{< c} \|\phi_m\| \, dm$$

$$\leq c \int_Y \|\psi_m\|_m \|\phi_m\|_m \, dm$$

(ii) $\Rightarrow$ (iii)

The Fréchet-Riesz implies that $\exists !\, S \in L(\mathcal{H})$ s.t.

$$\langle \rho\psi | \rho\phi \rangle = (\psi | S\phi) \quad \forall \psi, \phi \in \mathcal{H}^\infty$$

$$\leq c \|\psi\| \|\phi\|$$

S is symmetric and $\|S\| \leq c$.

Moreover, S and T commute on $\mathcal{H}$

The spectral theorem for commuting selfadjoint operators gives us a spectral measure F on $\sigma(S) \times J$ such that

$$S^p T^q = \int_{\sigma(S) \times J} \nu^p m^q \, dF_{\nu, m} \quad \forall p, q \in \mathbb{N}$$

Let $\psi, \phi \in \mathcal{H}^\infty$, we define

$$\mu_{\psi,\phi}(\Omega) := \int_{\sigma(S) \times \Omega} \nu \, d(\psi | F_{\nu,m} \phi)$$

Then $\mu_{\psi,\phi}(J) = (\psi | S\phi)$ and

$$\mu_{\psi,\phi}(\Omega) = \int_{\sigma(S) \times J} \nu \, d(\chi_\Omega(T)\psi | F_{\nu,m} \chi_\Omega(T)\phi)$$

$$= (\chi_\Omega(T)\psi | S \chi_\Omega(T)\phi)$$

$$\Rightarrow |\mu_{\psi,\phi}(\Omega)| \leq c\, \|\chi_\Omega(T)\psi\|\, \|\chi_\Omega(T)\phi\|$$

def of (.l.)

$$\leq c \left(\int_\Omega \|\psi_m\|_m^2\, dm \int_\Omega \|\phi_m\|_m^2\, dm \right)^{\frac{1}{2}}$$

$$\leq c\, |\Omega| \underbrace{\sup_{m\in J} \|\psi_m\| \sup_{m\in J} \|\phi_m\|}_{<\infty}$$

$\Rightarrow \mu_{\psi,\phi}$ is absolutely continuous w.r.t. dm

The Radon-Nikodym theorem gives $f_{\psi,\phi} \in L^1(J)$ s.t.

$$\mu_{\psi,\phi}(\Omega) = \int_\Omega f_{\psi,\phi}(m)\, dm$$

Moreover, for any $m \in J$

$$|f_{\psi,\phi}(m)| \leq c\, \|\psi_m\|_m\, \|\phi_m\|_m$$

Thus $f_{\cdot,\cdot}(m)$ is a bounded sesquilinear form on $\mathcal{H}_m \times \mathcal{H}_m$.

Again Fréchet-Riesz gives $S_m \in L(\mathcal{H}_m)$ with $\|S_m\| \leq c$

$$f_{\psi,\phi}(m) = (\psi_m | S_m \phi_m)$$

$$\Rightarrow \langle P\psi | P\phi \rangle = (\psi | S\phi) = \int_{\sigma(S)\times J} s\, d(\psi | F_{s,m}\phi)$$

$$= \int_J d\mu_{\psi,\phi} = \int_J f_{\psi,\phi}(m)\, dm$$

$$= \int_J (\psi_m | S_m \phi_m)_m\, dm. \qquad \square$$

<u>Corollary</u>: strong mean oscillation property

$\Longrightarrow$ weak mean oscillation property.

Dependence on $\mathcal{J}$

Given m,

$\mathcal{J} = (m_L, m_R) \ni m$

$\tilde{\mathcal{J}} = (\tilde{m}_L, \tilde{m}_R) \ni m$

Is $S_m = \tilde{S}_m$?

<u>Prop</u>: The $(S_m)_{m \in \mathcal{J}}$ can be chosen s.t. $\forall \psi, \phi \in \mathcal{H}^\infty$

$(\psi_m | S_m \phi_m)_m$ is continuous in m.

Choosing $(S_m)_{m \in \mathcal{J}}$ in this way, the S_m do not depend on the choice of $\mathcal{J} \ni m$.