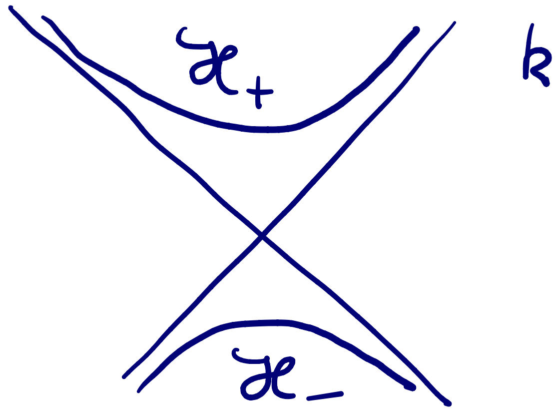
$\mathcal{D} = i \partial\!\!\!/$

$(\mathcal{H}_m , (.|.)_m)$ solution space



$\mathcal{H}_m = \mathcal{H}_+ \oplus \mathcal{H}_-$

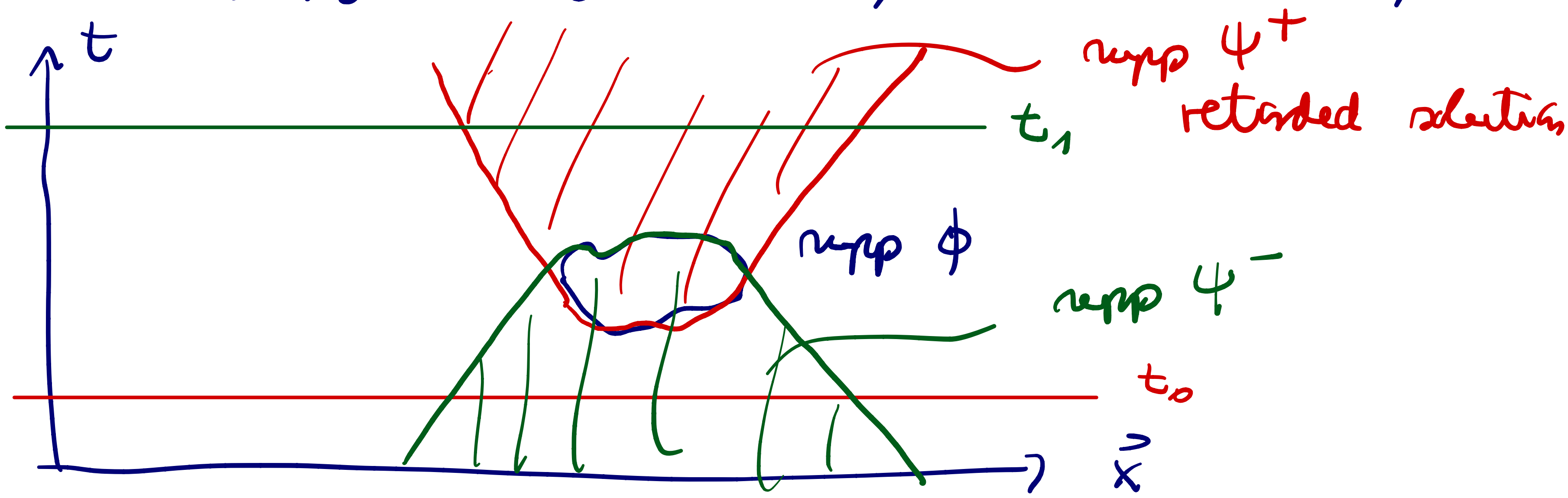
$\mathcal{H} = \mathcal{H}_-$ for vacuum ....

$P(x,y)$ unregularized kernel of the fermionic projector

$$= \int \frac{d^4k}{(2\pi)^4} \, \delta(k^2 - m^2) \, \Theta(-k^0) \, (k\!\!\!/ + m) \, e^{-ik(y-x)}$$

more abstractly

let $\phi_1, \phi_2 \in C_0^\infty(\mathcal{M}, S\mathcal{M})$ (no solutions!)



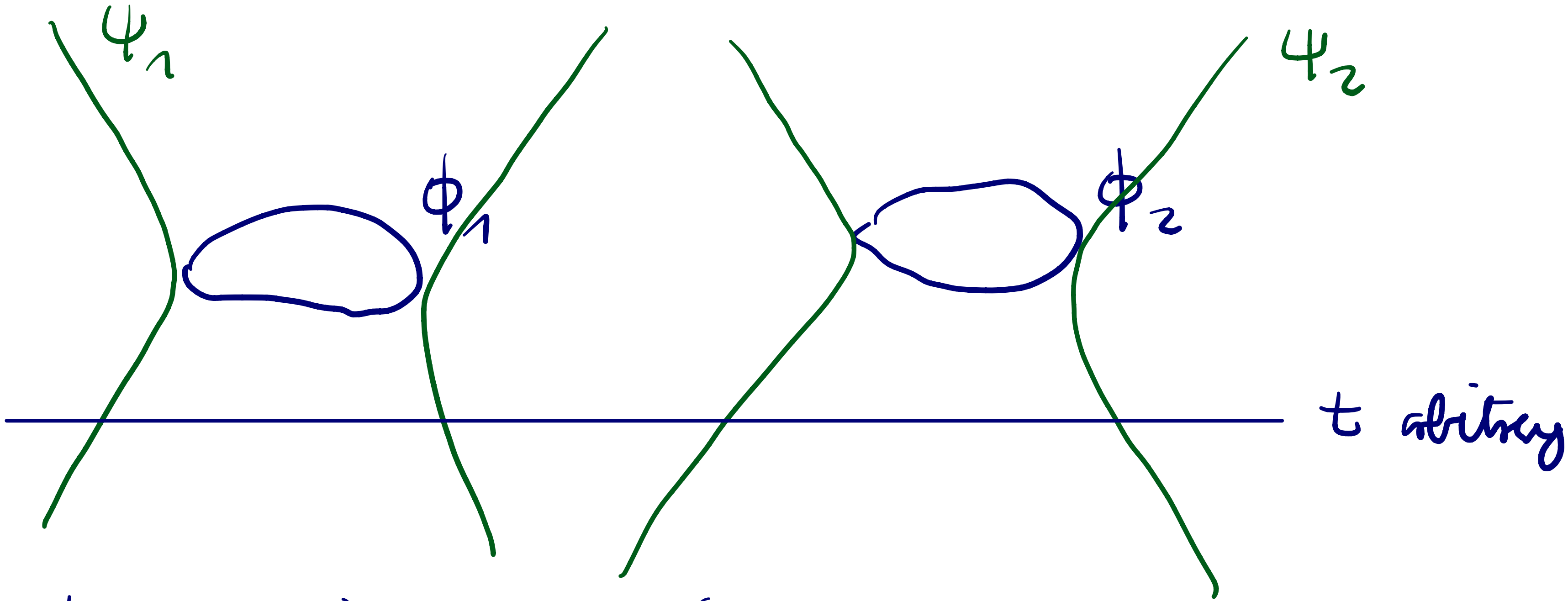
$$\begin{cases} (\mathcal{D} - m)\, \psi^\pm = \phi & \text{inhomogeneous Dirac equation} \\ \psi^+|_{t_0} = 0 & \psi^-|_{t_1} = 0 \end{cases}$$

This Cauchy problem has a unique global solution, smooth and finite propagation speed (linear symmetric hyperbolic system)

$\psi := \psi^+ - \psi^-$

$$(\mathcal{D} - m)\, \psi = (\mathcal{D} - m)\psi^+ - (\mathcal{D} - m)\psi^-$$

$$= \phi - \phi = 0$$



$$(\Psi_1|\Psi_2)_t = (\Psi_1|\Psi_2)_m$$

let $\Pi_{\mathcal{H}}$: $\mathcal{H}_m \longrightarrow \mathcal{H} = \mathcal{H}_-$

orthogonal projection

$$(\Psi_1 | \Pi_{\mathcal{H}} \Psi_2)$$

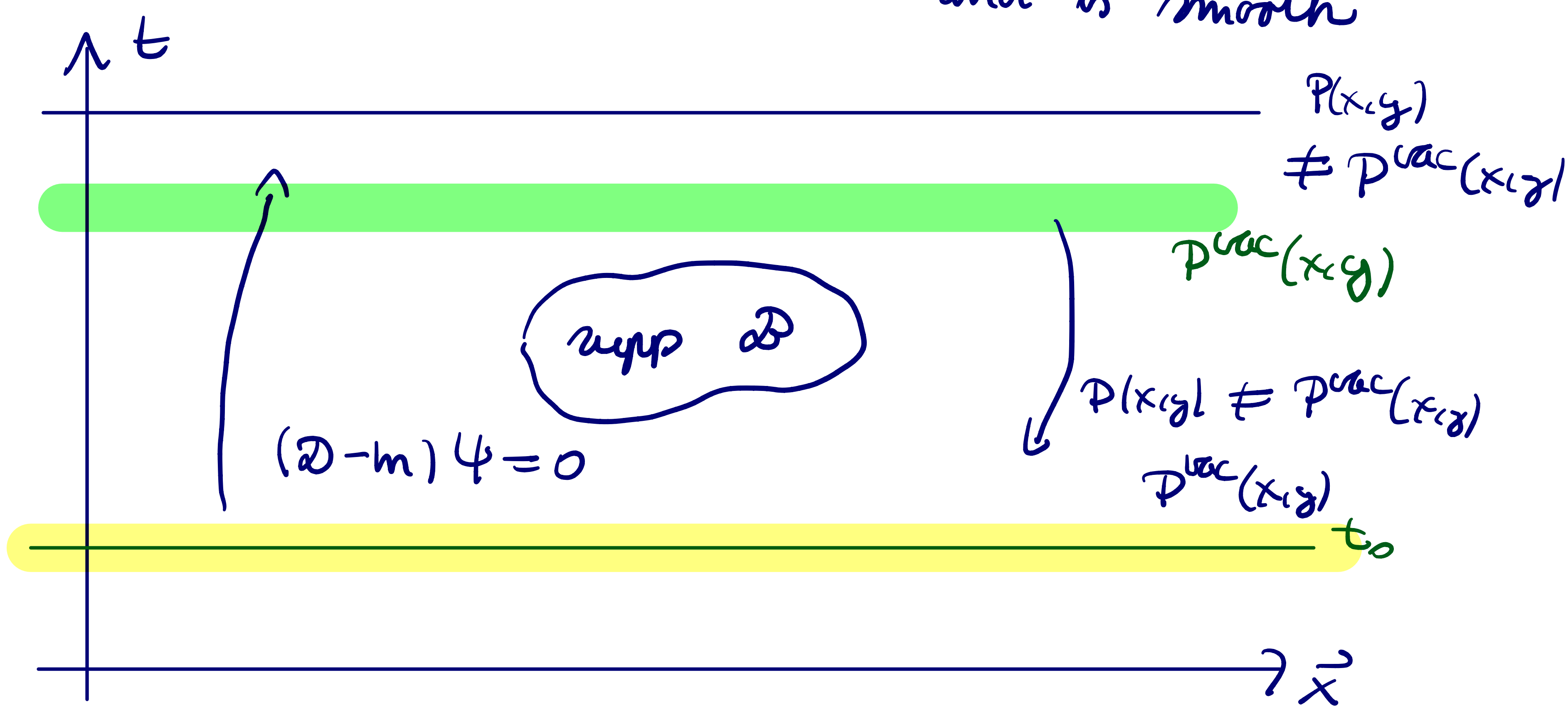
$$=: \langle \phi_1 | P \phi_2 \rangle$$

$$= \int d^4x \int d^4y \, \langle \phi_1(x) \,|\, P(x,y)\, \phi_2(y) \rangle$$

This defines P and $P(x,y)$ distributional kernel external potential $\mathcal{B}(x)$; for example $\mathcal{B}(x) = \not{A}(x)$

$$\mathcal{B}(x)^* = \mathcal{B}(x) \quad ; \quad \mathcal{D} = i\not{\partial} + \mathcal{B}$$

assume for simplicity that $\mathcal{B}$ has compact support and is smooth



<u>Question</u>: Is there still a canonical splitting

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \ ?$$

<u>static situation</u>: $\mathcal{B}(t,\vec{x}) = \mathcal{B}(\vec{x})$ static & C^∞_0

separation ansatz

$$\psi(t,\vec{x}) = e^{-i\omega t}\, \check{\psi}(\vec{x})$$

separation constant $\omega \in \mathbb{R}$

The sign of ω gives a splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

<u>time-dependent situation</u>

no time separation possible

the sign of the frequency is not well-defined

$\Rightarrow$ there is no obvious decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Answer: Yes, there is a canonical decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

$\mathcal{H}_+, \mathcal{H}_-$ generalize the subspaces of positive and negative frequency

perturbatively: causal perturbation expansion for $P^{sea}(x,y)$

$(.|.)_m$

$$<\psi|\phi> = \int_M \prec \psi(x) | \phi(x) \succ d^4x$$

$<\psi_m|\phi_m>$ is ill-defined because the time integral diverges

$I = (m_a, m_b)$ $\qquad m_a, m_b > 0$

$\in$

m

Consider a family $(\psi_m)_{m \in I}$

integrate over mass

$$(\mathfrak{p}\psi)(x) := \int_I \psi_m \, dm$$

no longer a solution

typically decays for large times due to destructive interference of oscillatory behavior in m

$(\psi_m)_{m \in I}, (\phi_m)_{m \in I}$

global

$$<\mathfrak{p}\psi | \mathfrak{p}\phi> = \int_I (\psi_m | S_m \phi_m) \, dm$$

exists provided that the strong mass oscillation property holds

$(S_m)_{m \in I}$ family of symmetric bounded operators

$S_m \in L(\mathcal{H}_m)$ fermionic signature operator fix m.

$$S_m = \int_{\sigma(S_m)} \lambda \, dE_\lambda$$

$$\chi_-(S_m) := \int_{-\infty}^{0} dE_\lambda \quad : \mathcal{H}_m \circlearrowleft$$

projection

choose $\mathcal{H} = \chi_-(S_m)(\mathcal{H}_m) \subset \mathcal{H}_m$ closed

$$\langle \phi_1 | P \phi_2 \rangle := (\psi_1 \mid \pi_{\mathcal{H}} \psi_2)_m$$

defines $P(x,y)$ canonically

$\hookleftarrow$ unregularized kernel of the fermionic projector