$\tilde{\rho}_{s,t} = (F_{s,t})_* (f_{s,t}\, \rho)$ two-parameter family of critical measures

Let $\Omega \subset M$ compact.

$$\mathcal{J}_2^\Omega := \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, \big(\partial_{1,s} - \partial_{2,s}\big)\big(\partial_{1,t} + \partial_{2,t}\big) \times \Big(f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y) \Big)\Big|_{s=t=0}$$

$$\mathcal{J}_2^\Omega = S \int_\Omega \partial_s \partial_t f_{s,t}(x)\Big|_{s=t=0} d\rho(x)$$

$$\left. \begin{cases} \underline{u} := \partial_s \big(f_{s,t}, F_{s,t}\big)\big|_{s=t=0} \\ \underline{v} := \partial_t \big(f_{s,t}, F_{s,t}\big)\big|_{s=t=0} \end{cases} \right\}$$ solutions of the linearized field eqns

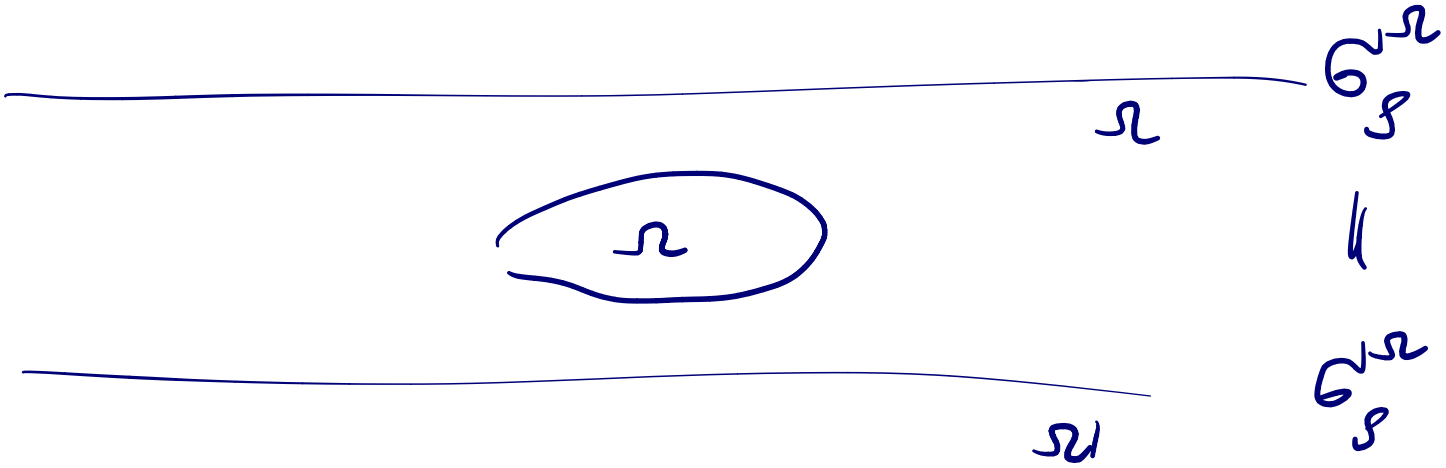
Anti-symmetrize in s and t

$$\sigma_\rho^\Omega(\underline{u}, \underline{v}) = \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, \big(\partial_{1,s}\partial_{2,t} - \partial_{2,s}\partial_{1,t}\big) \times \Big(f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y) \Big)\Big|_{s=t=0}$$

$$= \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, \big(\nabla_{1,\underline{u}} \nabla_{2,\underline{v}} - \nabla_{1,\underline{v}} \nabla_{2,\underline{u}}\big)\, \mathcal{L}(x,y)$$

symplectic form

$= 0$ conservation law



is anti-symmetric, but in general degenerate
(then there could be $\underline{v}$ with $\sigma(\underline{u}, \underline{v}) = 0 \quad \forall \underline{u}$)

$$\sigma_\rho^\Omega : \mathfrak{J}^{\text{lin}} \times \mathfrak{J}^{\text{lin}} \to \mathbb{R}$$

bilinear and antisymmetric, conserved.

One can also symmetrize

$$\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\partial_{1,s} \partial_{1,t} - \partial_{2,s} \partial_{2,t} \right) \times (\ldots) \Big|_{s=t=0}$$

$$= \underbrace{\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\nabla_{1,\underline{u}} \nabla_{1,\underline{v}} - \nabla_{2,\underline{u}} \nabla_{2,\underline{v}} \right) \mathcal{L}(x,y) + \ldots}_{(\underline{u}, \underline{v})_\rho^\Omega}$$

surface layer inner product

$$(. | .)_\rho^\Omega : \mathfrak{J}^{\text{lin}} \times \mathfrak{J}^{\text{lin}} \to \mathbb{R}$$

bilinear and symmetric, not conserved

in applications: nice positivity properties

is approximately conserved
useful for estimates,
as needed in the proof of existence of linearized solutions

$k > 2$: $\quad \gamma_\rho^\Omega \quad$ conserved one-form

$$\sigma_\rho^\Omega = d \gamma_\rho^\Omega$$