Let $\mathcal{F}$ be a smooth manifold

$$\mathcal{L} \in C^\infty(\mathcal{F}\times\mathcal{F}, \mathbb{R}^+_0)$$

Let ρ be a critical measure,

$$D_{\underline{u}}\, \ell\big|_M = 0 \qquad \forall u \in \mathfrak{J}^{test} = \mathfrak{J}$$

$$\ell(x) := \int_M \mathcal{L}(x,y)\, d\rho(y) - s$$

<u>symmetry</u> of the Lagrangian

$$\Phi_\tau : \mathcal{F} \to \mathcal{F}$$

with
$$\mathcal{L}(\Phi_\tau(x), \underbrace{\Phi_\tau(y)}_{=\tilde{y}}) = \mathcal{L}(x, \underset{\Phi_{-\tau}(\tilde{y})}{\overset{}{y}}) \quad \forall x,y \in \mathcal{F}$$

often sensible assumption: one-parameter group

$$\Phi_0 = \mathbb{1}_{\mathcal{F}}$$

$$\Phi_\tau\, \Phi_{\tau'} = \Phi_{\tau+\tau'}$$

$$\Phi_{-\tau} = \Phi_\tau^{-1}$$

$$\mathcal{L}(\Phi_\tau(x), \tilde{y}) = \mathcal{L}(x, \Phi_{-\tau}(\tilde{y}))$$

<u>Def</u>: A variation $\Phi : (-\tau_{max}, \tau_{max}) \times M \to \mathcal{F}$ is a <u>symmetry</u> of the Lagrangian if

$$\mathcal{L}(x, \Phi_\tau(y)) = \mathcal{L}(\Phi_{-\tau}(x), y) \quad \forall x,y \in M$$

(note: no group property).

<u>Def</u>: A variation Φ of the above form is continuously differentiable if

$$\ell \circ \Phi : (-\tau_{max}, \tau_{max}) \times M \to \mathbb{R}$$

is continuous on $(-\tau_{max}, \tau_{max}) \times M \to \mathbb{R}$
and is continuously differentiable in τ.

Moreover, we assume that ρ is locally finite, i.e.

$$\rho(K) < \infty \qquad \forall\, K \subset M \text{ compact}.$$

<u>Thm</u> (Noether-like thm)

Under the above assumptions on ρ and $\mathcal{L}$, let Φ_τ a continuously differentiable symmetry of the Lagrangian.

Then for any compact $\Omega \subset M$,

$$\frac{d}{d\tau} \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \Big(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \Big)\Big|_{\tau=0} = 0$$

<u>Proof</u>:

$$0 = \int_\Omega d\rho(x) \int_\Omega d\rho(y) \Big(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_{-\tau}(y)) \Big)$$

$$= \int_\Omega d\rho(x) \int_\Omega d\rho(y) \Big(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \Big)$$

$$= \int_\Omega d\rho(x) \int_M d\rho(y)\; \chi_\Omega(y) \Big(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \Big)$$

$$\chi_\Omega(y) = (1 - \chi_{M\setminus\Omega}(y))$$

$$= \int_\Omega d\rho(x) \Big(\ell(\Phi_\tau(x)) + \cancel{\nu} - \ell(\Phi_{-\tau}(x)) - \cancel{\nu} \Big)$$

$$- \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \Big(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \Big).$$

We thus obtain

$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right)$$

$$= \int_\Omega \left(\ell(\Phi_\tau(x)) - \ell(\Phi_{-\tau}(x)) \right) d\rho(x)$$

Since $\ell(\Phi_{\pm\tau}(x))$ is continuously differentiable, the right side is differentiable at $\tau=0$, and the derivative vanishes in view of the weak EL eqns.

Thus the left side is also differentiable at $\tau=0$, and its derivative vanishes. $\square$

Assume that the τ-derivative and the integrals can be interchanged in the Noether-like theorem. Then

$$0 = \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \; \frac{d}{d\tau} \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right)\Big|_{\tau=0}$$

$$\frac{d}{d\tau} \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right)\Big|_{\tau=0}$$

symmetry condition

$$= \frac{d}{d\tau} \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right)\Big|_{\tau=0}$$

$$= (D_{1,v} - D_{2,v}) \mathcal{L}(x,y) \quad , \quad v(x) = \frac{d}{d\tau} \Phi_\tau(x)\Big|_{\tau=0}$$

$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \; (D_{1,v} - D_{2,v}) \mathcal{L}(x,y) = 0 \qquad \forall \Omega \subset M \text{ compact}$$

examples:

$$\Phi_\tau(x) = U_\tau \, x \, U_\tau^{-1}$$

$$U_\tau = e^{i\tau A}, \quad A \in L(\mathcal{H}) \text{ symmetric}$$

unitary invariance of $\mathcal{L}$ gives rise to symmetry gives rise to current conservation.

$$v(x) = i\,[A, x] \quad \text{commutator vector field}$$