Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system,

$M := \operatorname{supp} \rho$ spacetime

$$S = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \quad \mathcal{L}(x,y) \qquad \text{causal action}$$

Constraints: $\rho(\mathcal{F})$ constant — volume constraint

$\int \operatorname{tr}(x)\, d\rho(x)$ constant — trace constraint

$$\mathcal{T} := \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y)\; |xy|^2 \leq C \qquad \text{boundedness constraint.}$$

ρ is a minimizer

(1) If $\dim \mathcal{H} < \infty$ and $\rho(\mathcal{F}) < \infty$

Minimize in the class of all regular Borel measures on $\mathcal{F}$.

(2) If $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) = \infty$

Assume that ρ is again regular Borel measure which is <u>locally finite</u> (i.e. $\rho(K) < \infty$ $\forall K \subset \mathcal{F}$ compact)

Let $K \subset M$ be compact.

Let $(\rho_\tau)_{\tau \in [0,\delta)}$ with $\rho_0 = \rho$

and $\rho_\tau |_{M \setminus K} = \rho |_{M \setminus K}$

volume constraint

$$0 = \rho_\tau(\mathcal{F}) - \rho(\mathcal{F})$$

$$= \cancel{\rho_\tau(M \setminus K)} + \rho_\tau(\mathcal{F} \setminus (M \setminus K))$$

$$- \cancel{\rho(M \setminus K)} - \rho(K)$$

formulate the volume constraint as

$$\rho_\tau(\mathcal{F} \setminus (M \setminus K)) = \rho(K) \qquad \forall \tau$$

ρ is a minimiser if for all such variations which satisfy the constraints,

$$\mathcal{S}(\rho_\tau) - \mathcal{S}(\rho) \geq 0 \qquad \forall \tau$$

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computational procedure: lagrange multipliers

$$\begin{aligned} S_{\kappa,\lambda,\nu} &= S + \kappa (J - C_1) \\ &\quad - \lambda \left(\int_{\mathcal{F}} t_f(x)\, d\rho(x) - C_2 \right) \\ &\quad - \nu \left(\rho(\mathcal{F}) - C_3 \right) \\ &= \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \left(\underbrace{\mathcal{L}(x,y) + \kappa\, |x-y|^2}_{= \mathcal{L}_\kappa} \right) \\ &\quad - \lambda \int_{\mathcal{F}} t_f(x)\, d\rho(x) - \nu\, \rho(\mathcal{F}) \\ &\quad - \kappa C_1 + \lambda C_2 + \nu C_3 \qquad \longleftarrow \text{irrelevant constant} \end{aligned}$$

specific class of variations chosen

$$\begin{cases} f_\tau \in C^0_0(M, \mathbb{R}^+) \\ F_\tau \in C^0(M, \mathcal{F}) \end{cases} \qquad F_\tau(x) \in L(x)$$

$$\rho_\tau := (F_\tau)_* (f_\tau\, \rho)$$

assume $\quad f_0 = 1_M, \qquad F_0(x) = x \quad \forall x \in M$

$$f_\tau|_{M \setminus K} = 1, \qquad F_\tau|_{M \setminus K} = \mathbb{1}$$

Thm: Under the above assumptions, for a minimizing measure ρ, the local trace is constant, i.e.

$$tr(x) = c \qquad \forall\, x \in M.$$

Proof: $\int_{\mathcal{F}} \phi(x) \, d\rho_\tau(x)$

$$= \int_{\mathcal{F}} \phi(\mathcal{F}_\tau(x)) \, f_\tau(x) \, d\rho(x)$$

def. of push-forward measure + approximation argument

Choose $\mathcal{F}_\tau(x) = \dfrac{x}{\sqrt{f_\tau(x)}}$

$$\int_{\mathcal{F}\setminus(M\setminus K)} \mathcal{L}(x,y) \, d\rho_\tau(y) = \int_K \mathcal{L}(x, \mathcal{F}_\tau(y)) \, f_\tau(y) \, d\rho(y)$$

$$= \int_K \mathcal{L}\Big(x, \frac{y}{\sqrt{f_\tau(y)}}\Big) \, f_\tau(y) \, d\rho(y)$$

$$= \int_K \mathcal{L}(x,y) \cancel{\frac{1}{(\sqrt{f_\tau(y)})^2}} \, \cancel{f_\tau(y)} \, d\rho(y)$$

$\mathcal{L}(x,.)$ is homogeneous of degree two

$$= \int_K \mathcal{L}(x,y) \, d\rho(y) \quad \text{is independent of } \tau$$

$\Rightarrow \mathcal{S}, \mathcal{J}$ are independent of τ

$$\mathcal{S}_{K,\lambda,\nu}(\rho_\tau) - \mathcal{S}_{K,\lambda,\nu}(\rho)$$

$$= -\lambda \int_K tr(\mathcal{F}_\tau(x)) \, f_\tau(x) \, d\rho(x) - \nu \int_K f_\tau(x) \, d\rho(x)$$

$$+ \lambda \int_K tr(x) \, d\rho(x) \qquad - \nu \, \rho(K)$$

$$= -\lambda \int_K \mathrm{tr}(x) \left(\sqrt{f_\tau(x)} - 1 \right) d\rho(x) - \nu \int \left(f_\tau(x) - 1 \right) d\rho(x).$$

Choose $\quad f_\tau(x) = 1 + \tau\, g(x)$

$$\Rightarrow \frac{d}{d\tau} S_{K,\lambda,\nu}(\rho_\tau)\Big|_{\tau=0} = 0$$

because ρ is a minimum

$$= -\frac{\lambda}{2} \int_K \mathrm{tr}(x)\, g(x)\, d\rho(x) - \nu \int g(x)\, d\rho(x)$$

$$= \int_K \left(-\frac{\lambda}{2} \mathrm{tr}(x) - \nu \right) g(x)\, d\rho(x)$$

Since g is arbitrary, it follows that

$$\mathrm{tr}(x) = -\frac{2\nu}{\lambda} \quad \text{on } \operatorname{supp} \rho. \qquad \square$$

In order to justify the Lagrange multiplier method, there are two methods:

(1) infinite-dimensional analysis on the Banach space $\quad \{ \mu \text{ signed regular Borel measure on } \mathcal{F} \mid |\mu| < \infty \}$

$$\|\mu\| := |\mu| \text{ total variation}$$

for the boundedness constraint, this seems to be the only method.

(2) for trace and volume constraints: arrange the constraints "by hand" with a rescaling procedure.

Assume $\rho(\mathcal{F}) = 1$

Let $\rho_\tau = (F_\tau)_* (f_\tau \rho)$ a variation

$$\int_{\mathcal{F}} d\rho_\tau = \int_{\mathcal{F}} f_\tau \, d\rho \neq 1 \text{ in general}$$

$$\int_{\mathcal{F}} \operatorname{tr}(x) \, d\rho_\tau(x) = \int_{\mathcal{F}} \operatorname{tr}(F_\tau(x)) \, f_\tau(x) \, d\rho(x)$$

$$\neq \int_{\mathcal{F}} \operatorname{tr}(x) \, d\rho(x) \text{ in general}$$

Introduce a new variation $(\tilde{f}_\tau, \tilde{F}_\tau)$ with

$$\tilde{f}_\tau(x) = \frac{1}{\int_{\mathcal{F}} f_\tau \, d\rho} f_\tau(x)$$

$$\tilde{F}_\tau(x) = \frac{\int \operatorname{tr}(y) \, d\rho(y)}{\int \operatorname{tr}(\tilde{F}_\tau(y)) f_\tau(y) \, d\rho(y)} F_\tau(x)$$

Then the volume and trace constraints are respected

$$0 = \frac{d}{d\tau} \left(\mathcal{S}(\tilde{\rho}_\tau) + k \, J(\tilde{\rho}_\tau) \right) \Big|_{\tau=0}$$

$$= \ldots = \frac{d}{d\tau} \left(\mathcal{S}(\rho_\tau) + k \, J(\rho_\tau) \right) \Big|_{\tau=0}$$

$$+ \text{(Lagrange multiplier terms)}.$$