Let $\mathcal{F}$ be a noncompact manifold,

ρ a regular Borel measure with $\rho(\mathcal{F}) = \infty$,

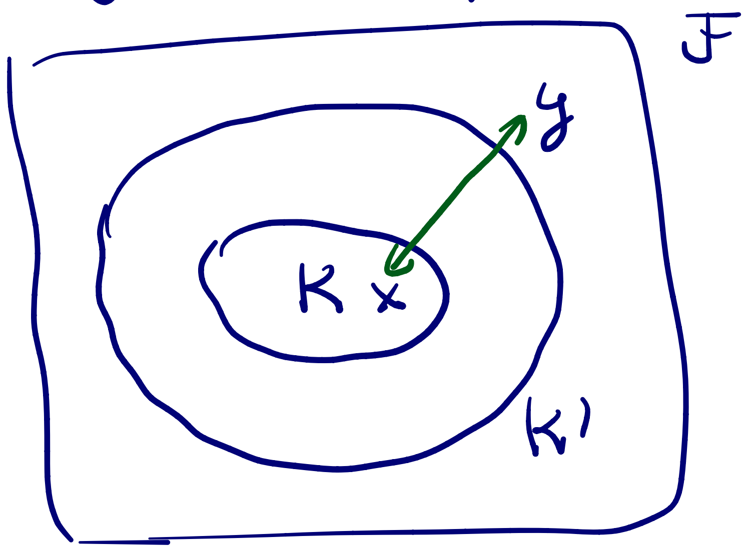
but locally finite (i.e. $\rho(K) < \infty \;\; \forall K \subset \mathcal{F}$ compact)

$\mathcal{L} \in C^0(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0)$ (always $\mathcal{L}(x,y) = \mathcal{L}(y,x)$)

has compact range, i.e.

$\forall K \subset \mathcal{F}$ compact $\exists K' \subset \mathcal{F}$ compact s.t.

$\mathcal{L}(x,y) = 0$ if $x \in K$ and $x' \notin K'$



(also assume $\mathcal{L}(x,x) > 0 \quad \forall x \in \mathcal{F}$)

$$S = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \; \mathcal{L}(x,y) \quad \text{typically} = \infty$$

volume constraint $\rho(\mathcal{F}) = \text{const} = \infty$

Definition variations of finite volume

Let $\tilde{\rho}$ be regular Borel measure locally finite and

$|\rho - \tilde{\rho}| < \infty$ and $(\rho - \tilde{\rho})(\mathcal{F}) = 0$

or simpler: compact variations

assume $\exists K \subset \mathcal{F}$ with

$\rho|_{\mathcal{F} \setminus K} = \tilde{\rho}|_{\mathcal{F} \setminus K}$ and $\rho(K) = \tilde{\rho}(K)$

$$S(\tilde{\rho}) - S(\rho) \qquad\qquad \mathcal{F} = K \cup (\mathcal{F}\setminus K)$$

$$\underset{\text{formally}}{=} \cancel{\int_{\mathcal{F}\setminus K} d\tilde{\rho}(x) \int_{\mathcal{F}\setminus K} d\tilde{\rho}(y)}\; \mathcal{L}(x,y) + 2\int_K d\tilde{\rho}(x) \int_{\mathcal{F}\setminus K} d\tilde{\rho}(y)\; \mathcal{L}(x,y)$$

$$+ \int_K d\tilde{\rho}(x) \int_K d\tilde{\rho}(y)\; \mathcal{L}(x,y)$$

$$- \cancel{\int_{\mathcal{F}\setminus K} d\rho(x) \int_{\mathcal{F}\setminus K} d\rho(y)}\; \mathcal{L}(x,y) - 2\int_K d\rho(x) \int_{\mathcal{F}\setminus K} d\rho(y)\; \mathcal{L}(x,y)$$

$$\underbrace{- \int_K d\rho(x) \int_K d\rho(y)\; \mathcal{L}(x,y)}_{\text{well-defined and finite}}$$

<u>Def</u>: $S(\tilde{\rho}) - S(\rho)$ is defined by $\nearrow$

ρ is a <u>minimizer</u> if $\quad S(\tilde{\rho}) - S(\rho) \geq 0$

$\forall \tilde{\rho}$ compact variations.

<u>EL-eqns</u>:

$$\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) - s$$

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0$$

Let $K_1 \subset K_2 \subset \dots \subset \mathcal{F}$, K_j compact

and $\bigcup_{j=1}^{\infty} K_j = \mathcal{F}$ exhaustion by compact sets

On K_j consider

$$\mathcal{S}_{K_j} = \int_{K_j} d\rho(x) \int_{K_j} d\rho(y)\, \mathcal{L}(x,y)$$

Let ρ_j be a minimizer, $M_j := \operatorname{supp} \rho_j \subset K_j$

$\uparrow$ normalized regular Borel measure on K_j

EL eqns: $\ell_j(x) := \int_{K_j} \mathcal{L}(x,y)\, d\rho_j(y) - \mathfrak{s}_j$

$$\ell_j|_{M_j} \equiv \inf_{K_j} \ell_j = 0 \,, \quad \mathfrak{s}_j > 0$$

Rescale: $\tilde{\rho}_j = \frac{1}{\mathfrak{s}_j} \rho_j$

$$\tilde{\ell}_j := \int_{K_j} \mathcal{L}(x,y)\, d\tilde{\rho}_j(y) - 1$$

$$\tilde{\ell}_j|_{M_j} = \inf_{K_j} \tilde{\ell}_j = 0$$

vague convergence $\tilde{\rho}_j \longrightarrow \rho$

means: $\forall K \subset \mathcal{F}$ compact $\tilde{\rho}_j|_K \longrightarrow \rho|_K$

abstract result: $\exists$ subsequence $\tilde{\rho}_{j_\ell} \longrightarrow \rho$ vaguely

Lemma: $\forall K \subset \mathcal{F}$ $\exists$ C_K s.t.

$$\tilde{\rho}_j(K) \leq C_K \quad \forall j$$

Proof: $\mathcal{L}(x,x) > 0$ $\forall x \in K$ by assumption.

Cover K by $U_1, \dots, U_L$ s.t.

$$\mathcal{L}(x,y) \geq \delta \qquad \forall x,y \in U_e$$

for some $\delta > 0$. Then for sufficiently large j, $\forall x$

$$1 = \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\tilde{\rho}_j(y) \geq \int_{U_e} \mathcal{L}(x,y)\, d\tilde{\rho}_j(y) \qquad (x \in U_e)$$

extended by zeros to $\mathcal{F}$

$$\geq \delta\, \tilde{\rho}_j(U_e)$$

Hence $\tilde{\rho}_j(U_e) \leq \frac{1}{\delta}$.

Since x is arbitrary, this holds for all U_e.

$$\Rightarrow \quad \tilde{\rho}_j(K) \leq \frac{L}{\delta}. \qquad \square$$

Let $x \in \mathcal{F}$. Then $x \in K_j$ for large j

EL eqns:
$$\int_{K'} \mathcal{L}(x,y)\, \tilde{\rho}_j(y) \geq 1$$

extend by zero to all of $\mathcal{F}$

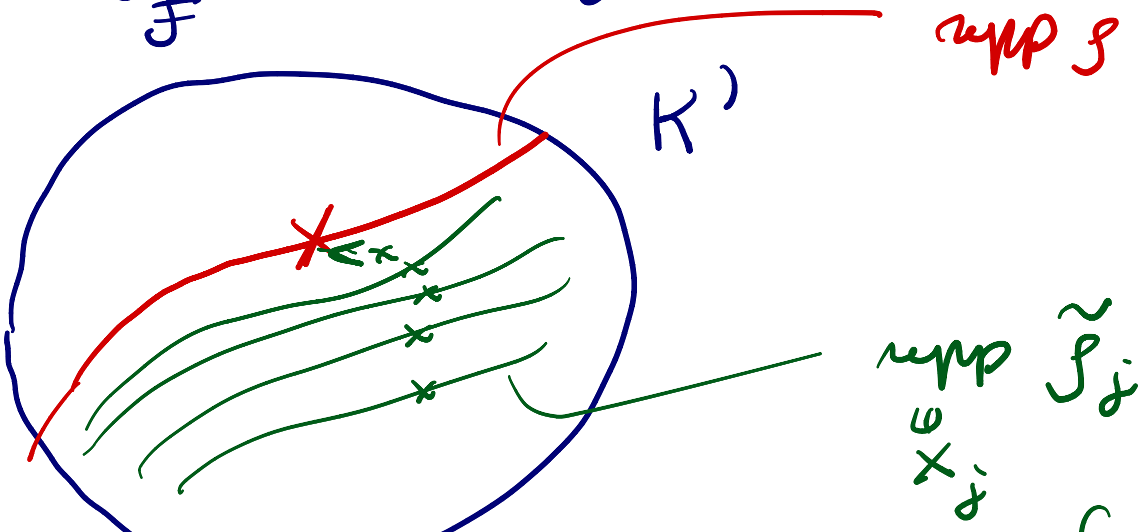
$\mathcal{L}$ has compact range

$$\Rightarrow \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) \geq 1$$

This shows that ρ is nontrivial (in a quantified way)

Remaining task: Show that

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) = 1 \qquad \forall x \in \operatorname{supp} \rho$$



Then $\displaystyle\int_{K'} \mathcal{L}(x_j, y)\, d\tilde{\rho}_j(y) = 1$

For any $x \in \operatorname{supp} \rho \cap K'$

$\exists \;\; x_j \in \operatorname{supp} \tilde{\rho}_j \cap K'$ and $x_j \longrightarrow x$

$$\int_{K'} \mathcal{L}(x_j, y)\, d\tilde{\rho}_j(y) = 1$$

$\downarrow$ use that ℓ_j are equicontinuous.

$$\int_{K} \mathcal{L}(x,y)\, d\rho(y)$$