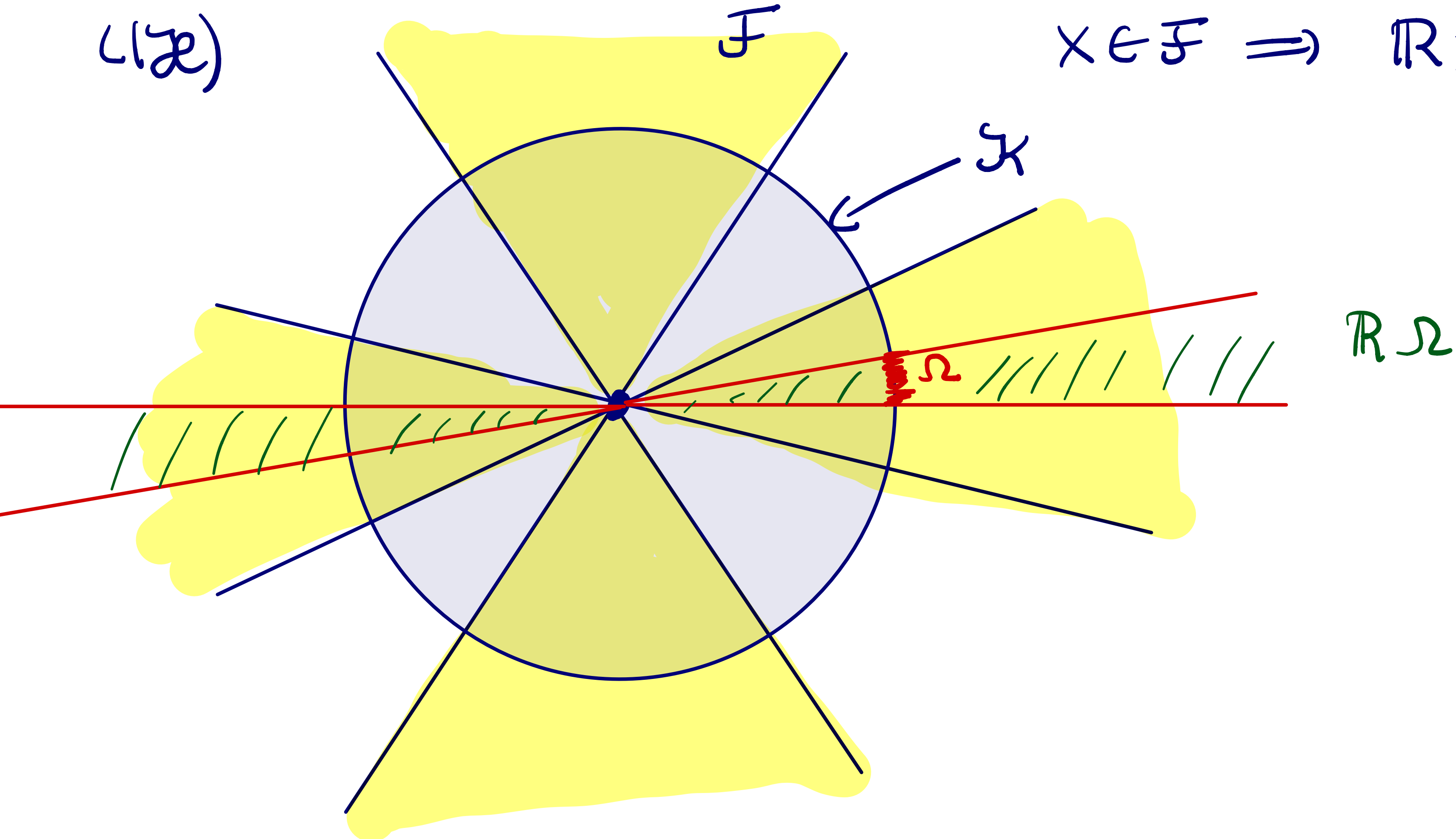
$\dim \mathcal{H} =: f < \infty$

given $n \in \mathbb{N}$, $\quad \mathcal{F} \subset L(\mathcal{H})$ closed subset

$$d(x,y) = \|x-y\|$$

$x \in \mathcal{F} \Longrightarrow \mathbb{R}x \subset \mathcal{F}$



$$\mathcal{P}_1 := \{ \rho \text{ regular Borel measure on } \mathcal{F},\ \rho(\mathcal{F}) = 1 \}$$

minimize

$$S = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \ \mathcal{L}(x,y)$$

under the constraints

$$\int_{\mathcal{F}} \operatorname{tr}(x) \, d\rho(x) = c \qquad \text{(TC)}$$

$$J = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \ |xy|^2 \leq \ell \qquad \text{(BC)}$$

Our goal is to prove the following theorem:

Thm: Given $c > 0$ and $\varepsilon > 0$, $\exists\, \rho \in \mathcal{D}_1$ s.t.

$$S(\rho) = \inf_{\substack{\rho' \in \mathcal{D}_1 \text{ and} \\ \text{satisfies (TC) and (BC)}}} \mathcal{S}(\rho')$$

and ρ also satisfies (TC) and (BC).

difficulties: $\mathcal{F}$ is not compact

make use of scaling behavior

$$X \longrightarrow \lambda X$$

$\mathcal{S}, \mathcal{J}$ homogeneous of degree 4 in λ

trace constraint is homogeneous of degree one.

Def $\mathcal{K} := \{ \rho \in \mathcal{F} \mid \|\rho\| = 1 \} \cup \{0\}$

is a compact topological space

$$m^{(0)}(\Omega) := \frac{1}{2}\rho(\mathbb{R}_+\Omega \setminus \{0\}) + \frac{1}{2}\rho(\mathbb{R}_-\Omega \setminus \{0\}) + \rho(\Omega \cap \{0\})$$

regular Borel measure on $\mathcal{K}$

$$m^{(1)}(\Omega) := \frac{1}{2}\int_{\mathbb{R}^+\Omega} \|x\|\, d\rho(x) - \frac{1}{2}\int_{\mathbb{R}^-\Omega} \|x\|\, d\rho(x)$$

difference of two regular Borel measures on $\mathcal{K}$

$$m^{(2)}(\Omega) := \frac{1}{2}\int_{\mathbb{R}^+\Omega} \|x\|^2\, d\rho(x) + \frac{1}{2}\int_{\mathbb{R}^-\Omega} \|x\|^2\, d\rho(x)$$

regular Borel measure on $\mathcal{K}$

these are the <u>moment measures</u>.

The causal action and the constraints can be expressed in terms of the moment measures

$$S(\rho) = \int_{\mathcal{K}} dm^{(2)}(p) \int_{\mathcal{K}} dm^{(2)}(q)\, \mathcal{L}(p,q)$$

$$J(\rho) = \int_{\mathcal{K}} dm^{(2)}(p) \int_{\mathcal{K}} dm^{(2)}(q) \quad |pq|^2 \le C$$

$$\int_{\mathcal{F}} \mathrm{tr}(x)\, d\rho(x) = \int_{\mathcal{K}} \mathrm{tr}(p)\; dm^{(1)}(q) = c$$

__Lemma__ There is $\varepsilon(f,n) > 0$ such that $\forall \rho \in \mathfrak{B}_1$ the corresponding moment measures satisfy for all Borel sets $\Omega \subset \mathcal{K}$ the inequalities

$$|m^{(1)}(\Omega)|^2 \le m^{(0)}(\Omega)\; m^{(2)}(\Omega)$$

$$m^{(0)}(\mathcal{K}) = 1$$

$$m^{(2)}(\mathcal{K}) \le \frac{\sqrt{J(\rho)}}{\varepsilon}.$$

$$:= \frac{1}{2} \int_{\mathbb{R}^+\Omega} \|x\|\; d\rho(x) + \frac{1}{2} \int_{\mathbb{R}^-\Omega} \|x\|\, d\rho(x).$$

__Proof__: $m^{(0)}(\mathcal{K}) = \rho(\mathcal{F}) = 1$

$$|2\, m^{(1)}(\Omega)| = \int_{\mathbb{R}\Omega} 1 \cdot \|x\|\; d\rho(x)$$

Schwarz

$$\le \left(\int_{\mathbb{R}\Omega} 1\; d\rho(x) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}\Omega} \|x\|^2\; d\rho(x) \right)^{\frac{1}{2}}$$

$$\le \rho(\mathbb{R}\Omega)^{\frac{1}{2}} \left(\int_{\mathbb{R}\Omega} \|x\|^2\; d\rho(x) \right)^{\frac{1}{2}}$$

$$\leq \left(2m^{(0)}(\Omega)\right)^{\frac{1}{2}} \quad \left(2m^{(2)}(\Omega)\right)^{\frac{1}{2}}$$

$$\phi : \mathcal{K}\times\mathcal{K} \longrightarrow \mathbb{R}, \qquad (p,q) \longmapsto |pq|$$

ϕ is continuous and

$$\phi(p,p) = |p^2| = \operatorname{tr}(p^2) \geq \|p\|^2 = 1 \quad \text{if } p \neq 0$$

Thus for any $r \in \mathcal{K}\setminus\{0\}$ $\exists$ $U(r)$ open neighborhood in $\mathcal{K}$ s.t.

$$\phi(p,q) \geq \frac{1}{2} \qquad \forall p,q \in U(r).$$

Since $\mathcal{K}\setminus\{0\}$ is compact, there are $r_1,\dots,r_N \in \mathcal{K}\setminus\{0\}$ s.t.

$$U(r_1) \cup \dots \cup U(r_N) \supset \mathcal{K}\setminus\{0\}.$$

Thus there is $i \in \{1,\dots,N\}$ with

$$m^{(2)}(U(r_i)) \geq \frac{m^{(2)}(\mathcal{K})}{N}.$$

$$\Rightarrow J(g) \geq \int_{U(r_i)} dm^{(2)}(p) \int_{U(r_i)} dm^{(2)}(q)\, \underbrace{|pq|^2}_{\geq \frac{1}{4}}$$

$$\geq \frac{1}{4}\, m^{(2)}(U(r_i))^2$$

$$\geq \frac{1}{4}\, \frac{m^{(2)}(\mathcal{K})^2}{N^2}.$$

Choosing $\varepsilon = \frac{1}{2N}$, we get the result. $\square$

Thus $m^{(0)}$ normalized measure

$m^{(2)}(\mathcal{K})$ a-priori bounded

$|m^{(2)}|(\mathcal{K})$ — " —.

Proof of the existence theorem:

Let (ρ_k) be a minimizing sequence.

Let $m_k^{(\ell)}$, $\ell = 0,1,2$ be the sequences of the corresponding moment measures

Using Banach-Alaoglu and Riesz representation thm

$$m_k^{(\ell)} \longrightarrow m^{(\ell)} \qquad \ell = 0,1,2$$

and $m^{(0)}$ normalized

and previous lemma also holds for the limit measures.

Remaining question: Is there $\rho \in \mathcal{D}_1$ which realizes the moment measures $m^{(\ell)}$?

$\ell = 1$ or 2

$$dm^{(\ell)} = \underbrace{f}_{\in L^1(\mathcal{K}, dm^{(0)})} dm^{(0)} + dm^{(\ell)}_{sing}$$

Radon-Nikodym decomposition

and $m^{(0)}$, $m^{(\ell)}_{sing}$ are mutually singular, i.e.

$$\mathcal{K} = E \mathbin{\dot\cup} F, \qquad E, F \text{ Borel sets}$$

$$m^{(0)}(\,.\,) = m^{(0)}(\,.\cap E)$$

$$m^{(\ell)}_{sing}(\,.\,) = m^{(\ell)}_{sing}(\,.\cap F).$$

The estimate

$$|m^{(1)}(\Omega)|^2 \leq m^{(0)}(\Omega)\, m^{(2)}(\Omega)$$

yields:

$$m^{(1)}_{sing} = 0$$

and $|f^{(1)}|^2 \leq f^{(2)}$ because

$$\left| \int_\Omega f^{(1)} \, dm^{(0)} \right|^2 \leq \left(\int_\Omega f^{(2)} \, dm^{(0)} \right) m^{(0)}(\Omega)$$

$$\approx \left| f^{(1)}(x) \right|^2 m^{(0)}(\Omega)^2 \qquad \approx f^{(2)}(x) \; m^{(0)}(\Omega)^2$$

$$\Rightarrow \quad f^{(1)} \in L^2(\mathcal{K}, dm^{(0)})$$

$$m = m^{(0)}, \qquad f = f^{(1)}$$

$$\begin{cases} dm^{(1)} = f \, dm \\ dm^{(2)} = |f|^2 \, dm + dn, \end{cases}$$

positive measure

where $dn := dm^{(2)}_{\text{sing}} + \underbrace{\left(f^{(2)} - |f|^2 \right) dm}_{\geq 0}$

Dropping n gives new moment measures

$$\begin{cases} d\tilde{m}^{(1)} = f \, dm \\ d\tilde{m}^{(2)} = |f|^2 \, dm \end{cases}$$

These new moment measures again satisfy the volume constraint and the trace constraint

The action and the boundedness constraint, however, could become smaller.

Thus the remaining task is to realize the new moment measures by $\rho \in \mathfrak{B}_1$.

This is accomplished in the following

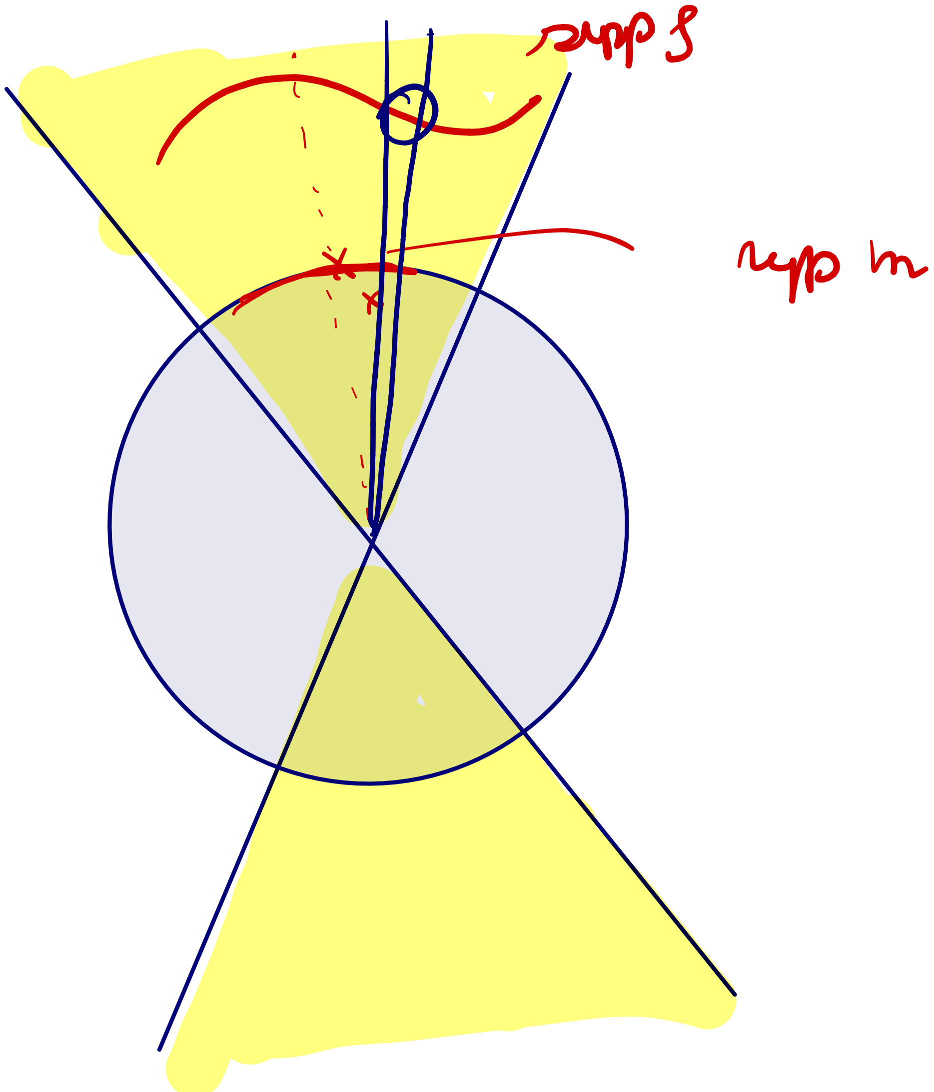
<u>Lemma</u> Given a normalized Borel measure m on $\mathcal{K}$ and $f \in L^2(\mathcal{K}, dm)$. Then there is a measure $\tilde{\rho} \in \mathcal{B}_1$ s.t. the corresponding moment measures have the form

$$\tilde{m}^{(0)} = m, \qquad \tilde{m}^{(1)} = f\, dm$$
$$\tilde{m}^{(2)} = |f|^2\, dm.$$

<u>Proof</u> $F : \mathcal{K} \to \mathcal{F},$

$$x \longmapsto f(x)\, x$$

$$\tilde{\rho} := F_* m$$



$\square$