Let $\mathcal{F}$ be a smooth compact manifold,

$\mathcal{L} \in C^0(\mathcal{F}\times\mathcal{F}, \mathbb{R}^+_0)$

$\mathcal{D}_1 = \{ \rho$ regular Borel measure, $\rho(\mathcal{F}) = 1 \}$.

minimize $\quad S = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y)\, \mathcal{L}(x,y)$

under variations in $\mathcal{D}_1$.

<u>Thm</u>: $\exists\, \rho \in \mathcal{D}_1$ with

$$S(\rho) = \inf_{\rho' \in \mathcal{D}_1} S(\rho') =: s .$$

general strategy: direct method

Let $(\rho_n)_{n\in\mathbb{N}}$ be a minimizing sequence, i.e. $\rho_n \in \mathcal{D}_1$ and

$$S(\rho_n) \longrightarrow s .$$

goal: a subsequence $(\rho_{n_\ell})_{\ell\in\mathbb{N}}$ converges,

$$\rho_{n_\ell} \longrightarrow \rho$$

and

$$S(\rho_{n_\ell}) \longrightarrow S(\rho)$$

$$\left.\begin{aligned} E &= C^0(\mathcal{F}, \mathbb{R}) \\ \|f\| &= \sup_{x\in\mathcal{F}} |f(x)| \end{aligned}\right\} \text{ Banach space } (E, \|.\|)$$

$E^* = \{ \phi : E \to \mathbb{R}$ linear and bounded $\}$

bounded means: $\exists\, c$ s.t. $|\phi(u)| \leq c\,\|u\| \;\forall u \in E$

E^* is again a Banach space with norm

$$\|\phi\| := \sup_{u\in E,\, \|u\|=1} |\phi(u)| .$$

$$\begin{tikzcd} E & E^* & E^{**} \arrow[from=1-1, to=1-3, bend right] \end{tikzcd}$$

$\phi_n \longrightarrow \phi$ in E^* if $\|\phi_n - \phi\| \longrightarrow 0$

$\phi_n \rightharpoonup \phi$ in E^* if $\psi(\phi_n - \phi) \longrightarrow 0 \ \forall \psi \in E^{**}$

$\Rightarrow \phi_n \overset{*}{\rightharpoonup} \phi$ in E^* if $(\phi_n - \phi)(\underset{E}{\underset{\cap}{u}}) \longrightarrow 0 \ \forall u \in E$

weak-*-convergence

<u>Thm</u> (Banach-Alaoglu)

The closed unit ball in E^* is weak-*-compact, i.e. $\forall \ \phi_n \in E^*$ $\exists$ subsequence ϕ_{n_ℓ} and $\phi \in E^*$ with

$$\phi_{n_\ell} \overset{*}{\rightharpoonup} \phi .$$

In our setting, E is separable, in which case the Banach-Alaoglu thm can be proved with a diagonal sequence argument.

Let $\rho \in \mathcal{D}_1$, define $\phi \in E^*$ by

$$\phi(f) = \int_{\mathcal{F}} f(x) \, d\rho(x)$$

$$|\phi(f)| \leq \int_{\mathcal{F}} |f(x)| \, d\rho(x) \leq \|f\| \, \overbrace{\rho(\mathcal{F})}^{=1}$$

then $\|\phi\| \leq 1$.

Moreover, ϕ is <u>positive</u> in the sense that

$$\phi(f) \geq 0 \quad \text{if } f \text{ is non-negative}$$

Thm (Riesz representation theorem)

For every $\phi \in E^*$ positive, there is a regular Borel measure ρ with

$$\phi(f) = \int_{\mathcal{F}} f(x)\, d\rho(x) \qquad \forall f \in E = C^0(\mathcal{F}, \mathbb{R})$$

Let ρ_n be the minimizing sequence

Let $\phi_n \in E^*$ be the corresponding linear functionals,

$$\phi_n(f) := \int_{\mathcal{F}} f(x)\, d\rho_n(x).$$

Then $\|\phi_n\| \leq 1$.

By the Banach-Alaoglu theorem, $\exists$ subsequence

ϕ_{n_ℓ} s.t. $\qquad \phi_{n_\ell} \xrightarrow{*} \phi \in E^*$.

i.e. $\qquad \phi_{n_\ell}(f) \longrightarrow \phi(f) \;\; \forall f \in E$

ϕ is positive because for any $f \geq 0$,

$$\phi(f) = \lim_{\ell\to\infty} \underbrace{\phi_{n_\ell}(f)}_{\geq 0} \geq 0$$

The Riesz representation theorem gives a Borel measure ρ s.t.

$$\phi(f) = \int_{\mathcal{F}} f(x)\, d\rho(x) \qquad \forall f \in E.$$

We conclude that $\forall f \in E$,

$$\int_{\mathcal{F}} f(x)\, d\rho_{n_\ell}(x) \longrightarrow \int_{\mathcal{F}} f(x)\, d\rho(x)$$

Choosing $f \equiv 1$ as a constant function,

$$1 = \rho_{h_e}(\mathcal{F}) \longrightarrow \rho(\mathcal{F})$$

$\Rightarrow$ ρ is normalized, $\rho \in \mathcal{D}_1$.

The action converges because

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho_{h_e}(y) \longrightarrow \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y),$$

by continuity in y.

The resulting function is continuous in x. Thus

$$S(\rho_{h_e}) \longrightarrow S(\rho).$$

This concludes the proof.

<u>Remark</u>: Replacing $\mathcal{F}$ by compact topological space and $\mathcal{L}$ by a lower semi-continuous function on $\mathcal{F} \times \mathcal{F}$, the above existence result again holds, using a version of Fatou's lemma for measures on $\mathcal{F} \times \mathcal{F}$.