$\mathbb{R}^n$

multi-index $\alpha = (p_1, \dots, p_n) \in \mathbb{N}_0^n$

$\partial^\alpha := \left(\frac{\partial}{\partial x^1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{p_n}$

$x^\alpha := (x^1)^{p_1} \cdots (x^n)^{p_n}$

$|\alpha| := p_1 + \dots + p_n$ order of α

$$\|f\|_{p,q} := \max_{\substack{\alpha,\, |\alpha| \le p \\ \beta,\, |\beta| \le q}} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|$$

Schwartz norms

$\mathcal{S}(\mathbb{R}^n) := \{ f \in C^\infty(\mathbb{R}^n) \mid \|f\|_{p,q} < \infty$ for all $p, q \in \mathbb{N}_0 \}$

Schwartz space

Thus f and all its partial derivatives have rapid decay.

topology on $\mathcal{S}(\mathbb{R}^n)$

Def: $f_n \longrightarrow f$ in $\mathcal{S}(\mathbb{R}^n)$ if $\|f - f_n\|_{p,q} \longrightarrow 0 \ \forall p, q$

$\mathcal{S}'(\mathbb{R}^n)$ tempered distributions

$:= \mathcal{S}(\mathbb{R}^n)^* = \{ T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ linear and continuous $\}$

where continuous means that $\exists p, q$ and $c > 0$ s.t.

$$|Tf| \le c \, \|f\|_{p,q} \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

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distributions can be understood as "generalized functions":

let $g \in L^\infty(\mathbb{R}^n)$

$$T_g f := \int_{\mathbb{R}^n} g(x) f(x) \, d^n x$$

$$|T_g f| \leq \int_{\mathbb{R}^n} \underbrace{|g(x)|}_{\leq \|g\|_{L^\infty}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}} \cdot \underbrace{(1+|x|^2)^{\frac{n+1}{2}} |f(x)| d^n x}_{\leq c \|f\|_{n+1,0}}$$

$$\leq c \, \|g\|_{L^\infty} \, \|f\|_{n+1,0}$$

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derivatives of distributions

note that $\forall g \in (C^\infty \cap L^\infty)(\mathbb{R}^n)$

$$T_{\partial_i g}(f) = \int_{\mathbb{R}^n} (\partial_i g(x)) \, f(x) \, d^n x$$

$$= \int_{\mathbb{R}^n} g(x) \, (-\partial_i f(x)) \, d^n x$$

$$= T_g(-\partial_i f)$$

<u>Def</u>: $(\partial^\alpha T)(f) := T(\underbrace{(-1)^{|\alpha|} \partial^\alpha f}_{\in \mathcal{S}})$

Fourier transform

$$\begin{cases} (\mathcal{F} f)(p) = \int_{\mathbb{R}^n} f(x) \, e^{ipx} \, d^n x \\ (\mathcal{F}^{-1} \hat{f})(x) = \int_{\mathbb{R}} \hat{f}(p) \, e^{-ipx} \, \frac{d^n p}{(2\pi)^n} \end{cases}$$

$\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ linear and continuous

$\mathcal{F}^{-1} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{F}^{-1} = \mathbb{1}_{\mathcal{S}(\mathbb{R}^n)}$

---

let $g \in \mathcal{S}(\mathbb{R}^n)$

$$T_{\mathcal{F}g} f = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g(x) \, e^{ipx} \, d^n x \right) f(p) \, d^n p$$

$$= \int_{\mathbb{R}^n} d^n x \; g(x) \underbrace{\left(\int_{\mathbb{R}^n} f(p)\, e^{ipx}\, d^n p \right)}_{=: (\mathcal{F} f)(x)}$$

$$= T_g(\mathcal{F} f)$$

<u>Def</u>: $(\mathcal{F}T)(f) := T(\mathcal{F}f)$

Fourier transform of tempered distributions

<u>Remarks</u>: (1) for $f, g \in L^2(\mathbb{R}^n)$, $\mathcal{F}f, \mathcal{F}^{-1}f \in L^2(\mathbb{R}^n)$

$$\langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2(\mathbb{R}^n)} = \langle f, g \rangle_{L^2(\mathbb{R}^n)} \cdot \frac{1}{(2\pi)^n} \; ?$$

Plancherel's theorem

(2) $C_0^\infty(\mathbb{R}^n)$, $\|f\|_p := \max_{\alpha \,|\, |\alpha| \le p} \sup_{\mathbb{R}^n} |\partial^\alpha f|$

$\mathcal{D}(\mathbb{R}^n) := \left(C_0^\infty(\mathbb{R}^n) \right)^*$

distributions