$(\mathcal{H}, \mathcal{F}, \rho)$, $M := \operatorname{supp} \rho$

$x \in M$, $S_x := x(\mathcal{H}) \subset \mathcal{H}$

$\prec \psi | \phi \succ_x := -\langle \psi | x \, \phi \rangle_{\mathcal{H}}$

$u \in \mathcal{H}$, $\psi^u(x) := \pi_x u$

local correlation operator $F(x)$

$$\begin{aligned}
\langle u | F(x) \, v \rangle_{\mathcal{H}} &= -\prec \psi^u(x) \,|\, \psi^v(x) \succ_x \\
&= -\prec \pi_x u \,|\, \pi_x v \succ_x \\
&= \langle \pi_x u \,|\, x \, \pi_x v \rangle_{\mathcal{H}} \\
&= \langle u | x \, v \rangle_{\mathcal{H}} \qquad \forall u, v \in \mathcal{H}
\end{aligned}$$

$\pi_x = \pi_x^*$ and

$x = \pi_x x = x \pi_x$

$\Rightarrow \quad F(x) = x$

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wave evaluation operator $\Psi : \mathcal{H} \to C^0(M, SM)$

$\Psi u = \psi^u$

$$\Rightarrow \quad \langle u | F(x) \, v \rangle_{\mathcal{H}} = -\prec \Psi(x) \, u \,|\, \Psi(x) \, v \succ_x$$

$$= -\langle u \,|\, \Psi(x)^* \Psi(x) \, v \rangle_{\mathcal{H}} \quad \forall u, v \in \mathcal{H}$$

$$\begin{cases} \Psi(x) : \mathcal{H} \to S_x \\ \Psi(x)^* : S_x \to \mathcal{H} \end{cases}$$

$$\Rightarrow x = F(x) = -\Psi(x)^* \, \Psi(x)$$

compute $\Psi(x)^*$: For $u \in \mathcal{H}$ and $\psi \in S_x$,

$$\prec \psi \,|\, \Psi(x) \, u \succ_x = \langle \Psi(x)^* \psi \,|\, u \rangle_{\mathcal{H}}$$

$$\overset{\shortparallel}{-\langle \underset{\underset{\mathcal{H}}{\cap}}{\psi} | x \, \pi_x u \rangle_{\mathcal{H}}} = -\langle \psi | x \, u \rangle_{\mathcal{H}} = \langle (-x)|_{S_x} \psi \,|\, u \rangle_{\mathcal{H}}$$

$$\Rightarrow \quad \Psi(x)^* = (-x)|_{S_x} : S_x \to \mathcal{H}$$

$$\Psi(x)\,\Psi(y)^* : \quad S_y \to S_x$$

$$= \pi_x \; (-y)|_{S_y} = -\pi_x y|_{S_y} = -P(x,y)$$

$$\Rightarrow \quad P(x,y) = -\Psi(x)\,\Psi(y)^*$$

Finally, choosing an orthonormal basis (e_i)

$$\Rightarrow \quad P(x,y) = -\sum_i \Psi(x)\, e_i \, \langle e_i | \Psi(y)$$

$$= -\sum_i |\psi^{e_i}(x)\rangle \langle \psi^{e_i}(y)|$$

$(\mathcal{M}, \langle .,. \rangle)$ Minkowski space

$\mathcal{H} \subset \mathcal{H}_m$; $\quad \mathfrak{R}^\varepsilon : \mathcal{H} \to C^0(\mathcal{M}, S\mathcal{M})$

$$e_x^\varepsilon : \mathcal{H} \to S_x\mathcal{M}$$

$$u \mapsto (\mathfrak{R}^\varepsilon u)(x)$$

regularized wave evaluation operator

$$(e_x^\varepsilon)^* : S_x\mathcal{M} \to \mathcal{H}$$

$$\langle u | F^\varepsilon(x)\, v \rangle = -\overline{(\mathfrak{R}^\varepsilon u)(x)}\,(\mathfrak{R}^\varepsilon v)(x)$$

$$= -\overline{(e_x^\varepsilon u)}\,(e_x^\varepsilon v)$$

$$= -\langle u | (e_x^\varepsilon)^* e_x^\varepsilon \, v \rangle \quad \forall u, v \in \mathcal{H}$$

$$\Rightarrow \quad F^\varepsilon(x) = -(e_x^\varepsilon)^* e_x^\varepsilon$$

Lemma: $e^\varepsilon_x|_{S_{F^\varepsilon(x)}} : S_{F^\varepsilon(x)} \to S_x\mathcal{M}$

is an isometric embedding.

Proof: Let $\psi, \phi \in S_{F^\varepsilon(x)}$

$\overline{e^\varepsilon_x \psi}\, e^\varepsilon_x \phi = \langle \psi | (e^\varepsilon_x)^* e^\varepsilon_x \phi \rangle_{\mathcal{H}}$

$\in S_{F^\varepsilon(x)} \subset \mathcal{H}$

$= - \langle \psi | F^\varepsilon(x) \phi \rangle_{\mathcal{H}} = \langle \psi | \phi \rangle_{F^\varepsilon(x)}$.

Therefore, the operator $e^\varepsilon_x|_{S_{F^\varepsilon(x)}}$ is an isometric embedding. $\square$

$$e^\varepsilon_x|_{S_{F^\varepsilon(x)}} : S_{F^\varepsilon(x)} \hookrightarrow S_x\mathcal{M}$$

dim $S_{F^\varepsilon(x)}$ = rank $F^\varepsilon(x) \leq 4$

dimension 4

Def: $x \in M$ is regular if x has rank $2n$

If $F^\varepsilon(x)$ is regular, then the above embedding is bijective (from a dimensional argument), giving an isomorphism

$$S_{F^\varepsilon(x)} \simeq S_x\mathcal{M}$$

identify: $x \longleftrightarrow F^\varepsilon(x)$

$S_x\mathcal{M} \longleftrightarrow S_{F^\varepsilon(x)}$

Lemma Assume that $\mathcal{F}^\varepsilon(x)$ is regulr. Then

$$e^\varepsilon_x|_{\mathcal{F}^\varepsilon(x)} \, \psi^u(\mathcal{F}^\varepsilon(x)) = e^\varepsilon_x u$$

$$\left(e^\varepsilon_x|_{\mathcal{F}^\varepsilon(x)}\right)^{-1} : S_x\mathcal{M} \to S_{\mathcal{F}^\varepsilon(x)}$$

$$= \left(-\mathcal{F}^\varepsilon(x)|_{S_{\mathcal{F}^\varepsilon(x)}}\right)^{-1} \left(e^\varepsilon_x|_{S_{\mathcal{F}^\varepsilon(x)}}\right)^*$$

Proof:

$$e^\varepsilon_x|_{\mathcal{F}^\varepsilon(x)} \, \psi^u(\mathcal{F}^\varepsilon(x))$$

$$= e^\varepsilon_x|_{\cancel{\mathcal{F}^\varepsilon(x)}} \, \Pi_{\mathcal{F}^\varepsilon(x)} u = e^\varepsilon_x u$$

$$\left(-\mathcal{F}^\varepsilon(x)|_{S_{\mathcal{F}^\varepsilon(x)}}\right)^{-1} \left(e^\varepsilon_x|_{S_{\mathcal{F}^\varepsilon(x)}}\right)^* e^\varepsilon_x|_{S_{\mathcal{F}^\varepsilon(x)}}$$

$$\left(-\mathcal{F}^\varepsilon(x)|_{S_{\mathcal{F}^\varepsilon(x)}}\right)^{-1} \left(-\mathcal{F}^\varepsilon(x)\right)|_{S_{\mathcal{F}^\varepsilon(x)}} = \mathbb{1}_{S_{\mathcal{F}^\varepsilon(x)}}. \quad \square$$

$$e^{\varepsilon}_{x}\big|_{S_{F^{\varepsilon}(x)}} P(F^{\varepsilon}(x), F^{\varepsilon}(y)) \left(e^{\varepsilon}_{y}\big|_{S_{F^{\varepsilon}(y)}} \right)^{-1}$$

$$= e^{\varepsilon}_{x} \; \Pi_{F^{\varepsilon}(x)} \; F^{\varepsilon}(y) \left(- F^{\varepsilon}(y)\big|_{S_{F^{\varepsilon}(y)}} \right)^{-1} \left(e^{\varepsilon}_{y}\big|_{S_{F^{\varepsilon}(y)}} \right)^{*}$$

$$= - e^{\varepsilon}_{x} \left(e^{\varepsilon}_{y} \right)^{*} : S_y \mathcal{M} \longrightarrow S_x \mathcal{M},$$

$$= - \sum_{i} (R^{\varepsilon} e_i)(x) \, \overline{(R^{\varepsilon} e_i)(y)}$$