$(\mathcal{H}, \mathcal{F}, \rho)$, $M := \operatorname{supp} \rho$

$x \in M$: $S_x := x(\mathcal{H}) \subset \mathcal{H}$ spin space

$\prec \psi | \phi \succ_x := -\langle \psi | x \phi \rangle_{\mathcal{H}}$ spin inner product

let $u \in \mathcal{H}$, $\psi^u(x) := \pi_x u \in S_x$ physical wave function

$\pi_x : \mathcal{H} \to S_x \subset \mathcal{H}$

orthogonal projection

<u>Def</u>: wave function

$\psi : M \to \mathcal{H}$ with $\psi(x) \in S_x \quad \forall x \in M$

continuity of wave function

<u>working definition</u>: ψ is continuous at $x \in M$ if

$\forall \varepsilon > 0 \;\; \exists$ open neighborhood $U \subset M$ of x s.t.

$$\| \psi(x) - \psi(y) \|_{\mathcal{H}} < \varepsilon \quad \forall y \in U.$$

better definition: $|x| = \sqrt{\underbrace{x^2}_{\geq 0}}$

$$\langle\!\langle \psi | \phi \rangle\!\rangle_x := \langle \psi | \, |x| \, \phi \rangle_{\mathcal{H}}$$

$$\| \psi \|_x := \sqrt{\langle \psi | \underbrace{|x|}_{= \sqrt{|x|}\sqrt{|x|}} \psi \rangle_{\mathcal{H}}} = \| \sqrt{|x|} \, \psi \|_{\mathcal{H}}$$

<u>Def</u>: wave function ψ is <u>continuous</u> at $x \in M$ if

$\forall \varepsilon > 0 \;\; \exists$ open neighborhood $U \subset M$ of x s.t.

$$\| \sqrt{|x|} \, \psi(x) - \sqrt{|y|} \, \psi(y) \|_{\mathcal{H}} < \varepsilon \quad \forall y \in U$$

The resulting vector space of all continuous wave functions is denoted by $C^0(M, SM)$.

Lemma: Any physical wave function ψ^u is continuous.

Proof:

$$\| \sqrt{|x|}\, \psi^u(x) - \sqrt{|y|}\, \psi^u(y) \|_{\mathcal{H}}$$

$$= \| \sqrt{|x|}\, \cancel{\Pi_x} u - \sqrt{|y|}\, \cancel{\Pi_y} u \|_{\mathcal{H}}$$

$$\leq \| \sqrt{|x|} - \sqrt{|y|} \| \quad \| u \|_{\mathcal{H}}$$

$$\overset{(*)}{\leq} \| y - x \|^{\frac{1}{4}} \; \| y + x \|^{\frac{1}{4}} \quad \| u \|_{\mathcal{H}} . \qquad \square$$

The inequality $(*)$ is proven in the next lemma.

Lemma: $\| \sqrt{|x|} - \sqrt{|y|} \| \leq \| y - x \|^{\frac{1}{4}} \; \| y + x \|^{\frac{1}{4}}$.

Proof: $\sqrt{|y|} - \sqrt{|x|}$ is symmetric and has finite rank.

$\Rightarrow \exists\, u \in \mathcal{H}$ s.t.

$$\left(\sqrt{|y|} - \sqrt{|x|} \right) u = \pm \| \sqrt{|y|} - \sqrt{|x|} \| \, u .$$

By exchanging x and y we can arrange the plus sign.

$$\Rightarrow \| \sqrt{|y|} - \sqrt{|x|} \| = \langle u | \left(\sqrt{|y|} - \sqrt{|x|} \right) u \rangle_{\mathcal{H}}$$

$$\leq \langle u | \left(\sqrt{|y|} + \sqrt{|x|} \right) u \rangle_{\mathcal{H}}$$

Multiply by $\| \sqrt{|y|} - \sqrt{|x|} \|$

$$\| \sqrt{|y|} - \sqrt{|x|} \|^2 \leq \frac{1}{2} \Big(\langle \left(\sqrt{|y|} - \sqrt{x} \right) u \,|\, \left(\sqrt{|y|} + \sqrt{|x|} \right) u \rangle_{\mathcal{H}}$$

$$+ \langle u \,|\, \left(\sqrt{|y|} + \sqrt{|x|} \right) \left(\sqrt{|y|} - \sqrt{|x|} \right) u \rangle_{\mathcal{H}} \Big)$$

$$= \frac{1}{2} \langle u, \{ \sqrt{|y|} - \sqrt{|x|} , \sqrt{|y|} + \sqrt{|x|} \} u \rangle$$

$$= \langle u, (|y| - |x|)\, u \rangle$$

We thus obtain

$$\| \sqrt{|y|} - \sqrt{|x|} \|^2 \leq \| |y| - |x| \|$$

Apply this inequality to $\begin{array}{l} x \to x^2 \\ y \to y^2 \end{array}$

$$\Rightarrow \| |y| - |x| \|^2 \leq \| y^2 - x^2 \| \leq \|y - x\| \, \|y + x\|$$

$$\| \sqrt{|y|} - \sqrt{|x|} \| \leq \| |y| - |x| \|^{\frac{1}{2}}$$

$$\leq \|y - x\|^{\frac{1}{4}} \, \|y + x\|^{\frac{1}{4}}. \qquad \square$$

This makes it possible to introduce the <u>wave evaluation operator</u>

$$\Psi : \mathcal{H} \longrightarrow C^0(M, SM)$$
$$u \longmapsto \psi^u$$

$x|_{S_x}$, $\quad |x||_{S_x} : S_x \circlearrowleft$

$$s_x := x|_{S_x}^{-1} \, |x| : S_x \circlearrowleft \; ; \quad s_x^2 = \mathbb{1}_{S_x}$$

Euclidean sign operator

$$|x| = s_x \, x = x \, s_x$$

$$\langle . | . \rangle_x \longrightarrow \langle\!\langle . | . \rangle\!\rangle_x = \langle \, . \, | \, |x| \, . \rangle_{\mathcal{H}}$$
$$= - \langle . | s_x \, . \rangle_x$$

$$\langle\!\langle \psi | \phi \rangle\!\rangle = \int_M \langle\!\langle \psi(x) | \phi(x) \rangle\!\rangle_x \, d\rho(x)$$

$$: C^0_0(M, SM) \times C^0(M, SM) \longrightarrow \mathbb{C}$$

$$\langle \psi \mid \phi \rangle = \int_M \langle \psi(x) \mid \phi(x) \rangle_x \, d\rho(x)$$

Krein inner product