$\mathcal{F}$ smooth manifold, $\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0)$

$\nabla_{\underline{u}}\, \ell|_M = 0$ weak EL-eqns

$$\ell(x) := \int_M \mathcal{L}(x,y)\, d\rho(y) - s$$

ingredient: families of solutions of the weak EL eqns.

two parameters $s, t \in (-\delta, \delta)$

$$\tilde{\rho}_{s,t} = (F_{s,t})_* (f_{s,t}\, \rho)$$

where $f_{s,t} \in C^\infty(M, \mathbb{R}^+)$, $F_{s,t} \in C^\infty(M, \mathcal{F})$

$f_{0,0} = 1_M$, $F_{0,0}(x) = x \quad \forall x \in M$.

(r1) For all $x \in M$, $p, q \geq 0$ and $r \in \{0,1\}$,

$$\int_M \partial_{s'}^r\, \partial_s^p\, \partial_t^q\, \mathcal{L}\big(F_{s+s',t}(x), F_{s,t}(y)\big)\, d\rho(y)\Big|_{s'=s=t=0}$$

$$= \partial_{s'}^r\, \partial_s^p\, \partial_t^q \int_M \mathcal{L}\big(F_{s+s',t}(x), F_{s,t}(y)\big)\, d\rho(y)\Big|_{s'=s=t=0}$$

Thm Let $f_{s,t}$ and $F_{s,t}$ as above, satisfy (r1).
Assume that $\tilde{\rho}_{s,t}$ all satisfy the weak EL eqn.
Then for every compact $\Omega \subset M$,

$$\gamma^\Omega_{k+1} := \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)$$

$$\times (\partial_{1,s} - \partial_{2,s})(\partial_{1,t} + \partial_{2,t})^k$$

$$\times \Big(f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y) \Big)\Big|_{\substack{s=\\t=0}}$$

$$= \int_\Omega \partial_s \partial_t^k f_{s,t}(x)\Big|_{s=t=0} d\rho(x) ,$$

<u>Proof</u> $L(x_{s,t}, y_{s,t}) := f_{s,t}(x) \, \mathcal{L}(F_{s,t}(x), F_{s,t}(y)) \, f_{s,t}(y)$

$$\int_{\tilde{M}_{s,t}} \mathcal{L}(\tilde{x}, y) \, d\tilde{\rho}_{s,t}(y)$$

$$= \int_M \mathcal{L}(\underset{\substack{\| \\ F_{s,t}(x)}}{\tilde{x}}, F_{s,t}(y)) \, f_{s,t}(y) \, d\rho(y)$$

The weak EL eqns state

$$\nabla_{\tilde{u}} \Big(\int_M \mathcal{L}(F_{s,t}(x), F_{s,t}(y)) \, f_{s,t}(y) \, d\rho(y) - s \Big) = 0$$

Multiply by $f_{s,t}(x)$ $\qquad \forall s,t$

$$\Rightarrow \nabla_{\underline{u}} \Big(\int_M L(x_{s,t}, y_{s,t}) \, d\rho(y) - f_{s,t}(x) \, s \Big) = 0$$

Choose $v = v^s \partial_s + v^t \partial_t$, $v^s, v^t \in \mathbb{R}$

Choose $\underline{u} = (0, \partial_s)$ and differentiate k times

$$\int_M \partial_{1,s} (\partial_{1,v} + \partial_{2,v})^k L(x_{s,t}, y_{s,t})\Big|_{s=t=0} d\rho(y)$$

$$= s \, \partial_s \partial_v^k f_{s,t}(x)\Big|_{s=t=0} \qquad \Big| \times 2$$

Choose $\underline{u} = (1, 0)$ and differentiate $k+1$ times

$$\int_M (\partial_{1,v} + \partial_{2,v})^{k+1} L(x_{s,t}, y_{s,t})\Big|_{s=t=0} d\rho(y)$$

$$= s \, \partial_v^{k+1} f_{s,t}(x)\Big|_{s=t=0} .$$

In the last term, differentiate w.r.t. v^s and divide by $k+1$.

$$\Rightarrow \int_M (\partial_{1,s} + \partial_{2,s})\,(\partial_{1,v} + \partial_{2,v})^k \; L(x_{s,t}, y_{s,t})\Big|_{s=t=0} \, d\rho(y)$$

$$= s\, \partial_s \partial_v^k f_{s,t}(x) \qquad \Big| \times (-1)$$

Combining these equations gives

$$\int_M (\partial_{1,s} - \partial_{2,s})\,(\partial_{1,v} + \partial_{2,v})^k \; L(x_{s,t}, y_{s,t})\Big|_{s=t=0} \, d\rho(y)$$

$$= S\, \partial_s \partial_v^k f_{s,t}(x).$$

Integrate over Ω.

$$\int_\Omega d\rho(x) \int_M d\rho(y)\, (\partial_{1,s} - \partial_{2,s})\,(\partial_{1,v} + \partial_{2,v})^k\, L$$

$$= S \int_\Omega \partial_s \partial_v^k f_{s,t}(x)\, d\rho(x)$$

$$\int_\Omega d\rho(x) \int_\Omega d\rho(y)\, (\partial_{1,s} - \partial_{2,s})\,(\partial_{1,v} + \partial_{2,v})^k\, L = 0 \qquad \Big| (-1)$$

because the integrand is anti-symmetric.

Subtracting these equations gives

$$\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, (\partial_{1,s} - \partial_{2,s})\,(\partial_{1,v} + \partial_{2,v})^k\, L$$

$$= S \int_\Omega \partial_s \partial_v^k f_{s,t}(x)\, d\rho(x). \qquad \square$$