Graded Geometry and Gravity

interaction via deformation

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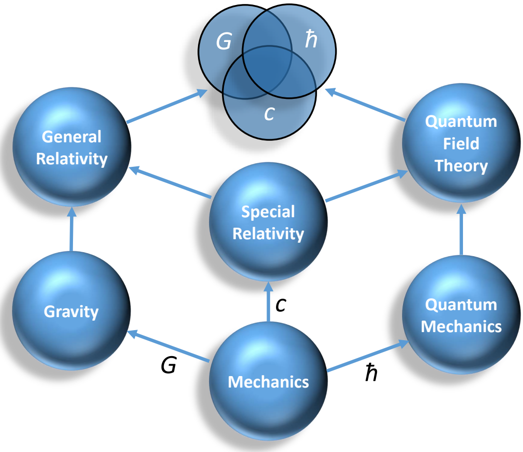
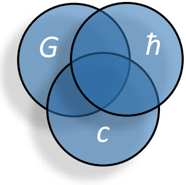
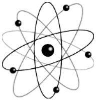
Motivation: quantum gravity, quantum geometry





Quantum spacetime



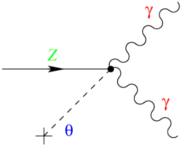
quantum + gravity $\Rightarrow$

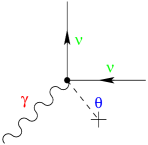
Quantized geometry: *apply the principles of QM to spacetime itself*

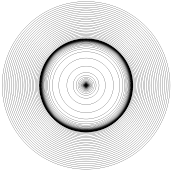
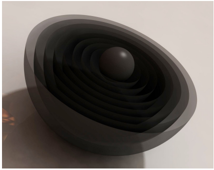
- intrinsic length scale, spacetime coarse-graining
- microscopic non-commutative/non-associative spacetime structures
- natural regularization, non-locality

Quantum fields on noncommutative spaces

- novel interactions, controlled Lorentz violation, UV/IR mixing
- NC Standard Model, NC GUTs, etc.
- Gravity on noncommutative spaces [,] $\neq 0$
  twisted tensor calculus, deformed Einstein equations





- space-time coarse graining $\Rightarrow$ higher structures
- beyond non-commutative geometry $\Rightarrow$ generalized geometry

Outline

- interaction via deformation ($\rightsquigarrow$ gauge theory, monopoles)
- aspects of quantization
- graded spacetime mechanics ($\rightsquigarrow$ general relativity)
- generalized geometry and gravity
- summary and outlook

Interaction via deformation

"Beyond gauge theory"

- gravity = free fall in curved spacetime 
  → extend this idea to all forces!
- free Hamiltonian, interaction via deformation:
  deformed symplectic structure (or operator algebra) [, ] ≠ 0
- gauge theory recovered via Moser's lemma:
  deformation maps are not unique ⇒ gauge symmetry
- more general than gauge theory, but just as simple to use as the good old gauge principle

Interaction via deformation

Analytical mechanics "warm-up" ...

Hamiltonian (first order) action "$S_H = \int \sum p dq - H(p,q) dt$" :

$$S_H = \int \alpha - H(X) d\tau + d\lambda \qquad \text{vary with } \delta X = 0 \text{ at boundary}$$

$$\mathcal{L}_{\delta X}(\alpha - H d\tau) = i_{\delta X} d\alpha + d(i_{\delta X}\alpha) - (i_{\delta X} dH) d\tau$$

$$\to \quad \boxed{\omega(-, \dot{X}) = dH} \quad \text{where} \quad \omega = d\alpha$$

$$\leftrightarrow \quad \boxed{\dot{X} = \theta(-, dH) \ \to \ \dot{f} = \{f, H\}} \quad \text{where} \quad \theta = \omega^{-1}$$

interaction, coupling to gauge field:

- *either* deform H ("minimal substitution"): $H' = H(p - A, q)$
- *or* deform ω and hence $\{\,,\,\}$: $\alpha' = \sum p dq + A \to \omega' = \omega + dA$

Interaction via deformation

example: relativistic particle in einbein formalism

$$S = \int d\tau \left(\frac{1}{2e} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - \frac{1}{2} e m^2 + A_\mu(x)\dot{x}^\mu \right) \rightsquigarrow p_\mu = \frac{1}{e} g_{\mu\nu}\dot{x}^\nu + A_\mu$$

$$S_H = \int p_\mu dx^\mu - \frac{1}{2} e \left((p_\mu - A_\mu)^2 + m^2 \right) d\tau \qquad \leftarrow p_\mu\text{: canonical momentum}$$

$$\boxed{S_H = \int \left(p_\mu + A_\mu \right) dx^\mu - \frac{1}{2} e \left(p_\mu^2 + m^2 \right) d\tau} \qquad \leftarrow p_\mu\text{: physical momentum}$$

$$\omega' = d\big(p_\mu + A_\mu\big) \wedge dx^\mu \ \rightsquigarrow$$

$$\boxed{\{p_\mu, p_\nu\}' = F_{\mu\nu},\ \{x^\mu, p_\nu\}' = \delta^\mu_\nu,\ \{x^\mu, x^\nu\}' = 0}$$

$$\boxed{\{p_\lambda, \{p_\mu, p_\nu\}'\}' + \text{cycl.} = (dF)_{\lambda\mu\nu} = (*j_m)_{\lambda\mu\nu}} \qquad \leftarrow \text{magnetic 4-current}$$

magnetic sources $\Leftrightarrow$ non-associativity

Interaction via deformation

Quantization

- path integral ✓
- deformation quantization ✓(→ details later)
- canonical? depends... (✓) :

Deformed CCR:

$$[p_\mu, p_\nu] = i\hbar F_{\mu\nu}, \quad [x^\mu, p_\nu] = i\hbar \delta^\mu_\nu, \quad [x^\mu, x^\nu] = 0, \quad [\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$$

Let $\mathrm{p} = \gamma^\mu p_\mu$ and $H = \frac{1}{2}\mathrm{p}^2 \rightsquigarrow$ correct coupling of fields to spin

$$H = \tfrac{1}{8}\big([\gamma^\mu, \gamma^\nu]_+[p_\mu, p_\nu]_+ + [\gamma^\mu, \gamma^\nu][p_\mu, p_\nu]\big) = \tfrac{1}{2}p^2 - \tfrac{i\hbar}{2}S^{\mu\nu}F_{\mu\nu}$$

Lorentz-Heisenberg equations of motion (ignoring spin)

$$\dot{p}_\mu = \tfrac{i}{\hbar}[H, p_\mu] = \tfrac{1}{2}(F_{\mu\nu}\dot{x}^\nu + \dot{x}^\nu F_{\mu\nu}) \quad \text{with} \quad \dot{x}^\nu = \tfrac{i}{\hbar}[H, x^\nu] = p^\nu$$

this formalism allows $dF \neq 0$: magnetic sources, non-associativity

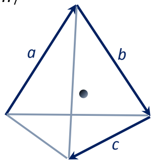
Interaction via deformation: monopoles

local non-associativity: $\frac{1}{3}[p_\lambda, [p_\mu, p_\nu]]\, dx^\lambda dx^\mu dx^\nu = \hbar^2 dF = \hbar^2 * j_m$

$j_m \neq 0 \Leftrightarrow$ no operator representation of the p_μ!

spacetime translations are still generated by p_μ, but magnetic flux Φ_m leads to path-dependence with phase $e^{i\phi}$; where $\phi = iq_e \Phi_m / \hbar$

globally:



$$\Phi_m = \int_S F = \int_{\partial S} A \qquad \leftrightarrow \text{ non-commutativity}$$

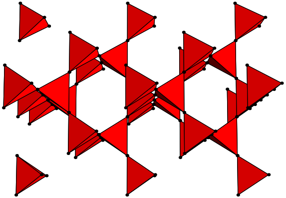
$$\Phi_m = \int_{\partial V} F = \int_V dF = \int_V * j_m = q_m \qquad \leftrightarrow \text{ non-associativity}$$

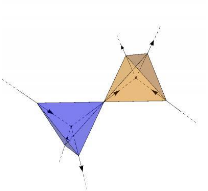
global associativity requires $\phi \in 2\pi\mathbb{Z} \;\Rightarrow\; \boxed{\frac{q_e q_m}{2\pi\hbar} \in \mathbb{Z}}$ Dirac quantization

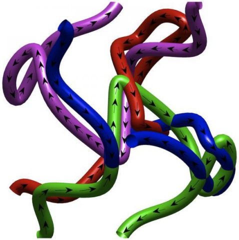
non-relativistic version of this: Jackiw 1985, 2002

Magnetic monopoles in the lab

spin ice pyrochlore and Dirac monopoles







Castelnovo, Moessner, Sondhi (2008)
Fennell; Morris; Hall, . . . (2009)

frustrated spin system $\leftrightarrow$ huge degeneracy of classical ground state

Lieb, PS (1999), (2000); PS (2001)

Aspects of quantization $\rightsquigarrow \star$

The operator-state formulation of QM cannot handle non-associative structures...

Phase-space formulation of QM

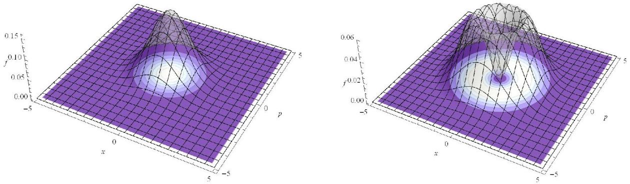
- Observables and states are (real) functions on phase space.
- Algebraic structure introduced by a star product,
  traces by integration.
- State function (think: "density matrix"): $S_\rho \geq 0$, $\int S_\rho = 1$.[1]
- Expectation values $\langle \mathcal{O} \rangle = \int \mathcal{O} \star S_\rho$.
- Schrödinger equation $H \star S_\rho - S_\rho \star H = i\hbar \dfrac{\partial S_\rho}{\partial t}$
- "Stargenvalue" equation: $H \star S_\rho = S_\rho \star H = E\, S_\rho$.

[1] Wick-Voros formulation yields non-negative state function; Moyal-Weyl leads instead to Wigner quasi-probability function that can be negative in small regions.

Aspects of quantization $\rightsquigarrow \star$

Popular choices of star products

- *Moyal-Weyl* (symmetric ordering, Wigner quasi-probability function)
  Weyl quantization associates operators to polynomial functions via symmetric ordering: $x^\mu \rightsquigarrow \hat{x}^\mu$, $x^\mu x^\nu \rightsquigarrow \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + x^\nu \hat{x}^\mu)$, etc.
  extend to functions, define star product $\widehat{f_1 \star f_2} := \widehat{f_1}\,\widehat{f_2}$.
- *Wick-Voros* (normal ordering, coherent state quantization)
  QHO states in Wick-Voros formulation:



- *xp-ordered star product:* $\star$-exponential $\equiv$ ordinary path integral

Aspects of quantization $\rightsquigarrow \star$

Deformation quantization of the point-wise product in the direction of a Poisson bracket $\{f, g\} = \theta^{ij}\partial_i f \cdot \partial_j g$:

$$f \star g = fg + \frac{i\hbar}{2}\{f, g\} + \hbar^2 B_2(f, g) + \hbar^3 B_3(f, g) + \dots \ ,$$

with suitable bi-differential operators B_n.

There is a natural (local) gauge symmetry: "equivalent star products"

$$\star \mapsto \star' \ , \quad Df \star Dg = D(f \star' g) \ ,$$

with $Df = f + \hbar D_1 f + \hbar^2 D_2 f + \dots$

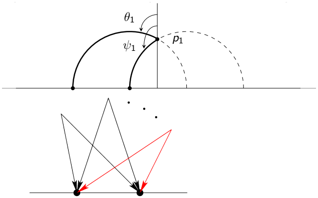
Dynamical non-associative star product:

$$f \star_p g = \cdot\Big[e^{\frac{\mathrm{i}\,\hbar}{2}\, R^{ijk}\, p_k\, \partial_i \otimes \partial_j}\, e^{\frac{\mathrm{i}\,\hbar}{2}\,\big(\partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\big)}(f \otimes g)\Big]$$

Kontsevich formality and star product

U_n maps n k_i-multivector fields to a $(2-2n+\sum k_i)$-differential operator

$$U_n(\mathcal{X}_1,\ldots,\mathcal{X}_n) = \sum_{\Gamma\in G_n} w_\Gamma\, D_\Gamma(\mathcal{X}_1,\ldots,\mathcal{X}_n)\ .$$



The star product for a given bivector θ is:

$$f\star g = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!}\, U_n(\Theta,\ldots,\Theta)(f,g)$$

$$= f\cdot g + \frac{i}{2}\sum \theta^{ij}\,\partial_i f\cdot\partial_j g - \frac{\hbar^2}{4}\sum \theta^{ij}\theta^{kl}\,\partial_i\partial_k f\cdot\partial_j\partial_l g$$

$$-\frac{\hbar^2}{6}\left(\sum \theta^{ij}\partial_j\theta^{kl}\,\left(\partial_i\partial_k f\cdot\partial_l g - \partial_k f\cdot\partial_i\partial_l g\right)\right) + \ldots$$

Kontsevich (1997)

Formality condition

The U_n define a quasi-isomorphisms of L_∞-DGL algebras and satisfy

$$\mathrm{d}.U_n(\mathcal{X}_1,\ldots,\mathcal{X}_n) + \frac{1}{2} \sum_{\substack{\mathcal{I}\sqcup\mathcal{J}=(1,\ldots,n)\\ \mathcal{I},\mathcal{J}\neq\emptyset}} \varepsilon_{\mathcal{X}}(\mathcal{I},\mathcal{J}) \left[U_{|\mathcal{I}|}(\mathcal{X}_{\mathcal{I}}),\, U_{|\mathcal{J}|}(\mathcal{X}_{\mathcal{J}})\right]_{\mathrm{G}}$$
$$= \sum_{i<j} (-1)^{\alpha_{ij}}\, U_{n-1}\big([\mathcal{X}_i,\mathcal{X}_j]_{\mathrm{S}}, \mathcal{X}_1,\ldots,\widehat{\mathcal{X}}_i,\ldots,\widehat{\mathcal{X}}_j,\ldots,\mathcal{X}_n\big)\ ,$$

relating Schouten brackets to Gerstenhaber brackets.

This implies in particular $\Phi(\mathrm{d}_\Theta\Theta) = \frac{1}{\mathrm{i}\,\hbar}\mathrm{d}_\star\Phi(\Theta)$, i.e.

θ (non-)Poisson $\quad\Leftrightarrow\quad$ $\star$ (non-)associative

Poisson sigma model

2-dimensional topological field theory, $E = T^*M$

$$S^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, \Theta^{ij}(X)\, \xi_i \wedge \xi_j \Big) \ ,$$

with $\Theta = \frac{1}{2}\, \Theta^{ij}(x)\, \partial_i \wedge \partial_j$, $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^* T^* M)$

perturbative expansion $\Rightarrow$ Kontsevich formality maps

valid on-shell ($[\Theta, \Theta]_S = 0$) as well as off-shell, e.g. twisted Poisson

Kontsevich (1997)
Cattaneo, Felder (2000)

Graded spacetime mechanics

Now try to do the same for gravity! Deformation maybe fine for curvature $R_{\mu\nu}$, however, the metric $g_{\mu\nu}$ is symmetric but $\{,\}$ is not.

- use graded geometry, i.e. odd variables and/or odd brackets
- or consider derived brackets

$$g^{\mu\nu} \sim \{\{x^\mu, H\}, x^\nu\}\ , \qquad \{H, H\} = 0$$

- $\rightsquigarrow$ algebraic approach to the geodesic equation, connections, curvature, etc. Properties like metricity follow from associativity. Local inertial coordinates are reinterpreted as Darboux charts
- the classical formulation requires graded variables ($\sim$ differentials), quantization leads to γ-matrices and Clifford algebras

$$\begin{array}{rcl}
\text{classical} & \leftrightarrow & \text{quantum} \\
\theta^\mu & \leftrightarrow & \gamma^\mu \\
\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu & \leftrightarrow & \frac{1}{2}[\gamma^\mu, \gamma^\nu]_- \\
\frac{1}{2}\{\theta^\mu, \theta^\nu\} = g^{\mu\nu} & \leftrightarrow & \frac{1}{2}[\gamma^\mu, \gamma^\nu]_+ = g^{\mu\nu}
\end{array}$$

Graded spacetime mechanics

Graded Poisson algebra

$$\{\underset{a}{\theta^\mu}, \underset{a}{\theta^\nu}\} = 2\underset{0}{g^{\mu\nu}}(x) \qquad \{\underset{c}{p_\mu}, \underset{0}{x^\nu}\} = \underset{0}{\delta^\nu_\mu} \qquad \{p_\mu, f(x)\} = \partial_\mu f(x)$$

Since $g^{\mu\nu}(x)$ has degree 0, the Poisson bracket must have degree $b = -2a$ for θ^μ of degree a, i.e. it is an even bracket.

Since $g^{\mu\nu}(x)$ is symmetric, we must have $-(-1)^{b+a^2} \overset{!}{=} +1$, i.e. a is odd.

wlog: $\{,\}$ is of degree $b = -2$, θ^μ are Grassmann variables of degree 1, $\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu$, and the momenta p_μ have degree $c = -b = 2$

$\Leftrightarrow$ a metric structur on TM and natural symplectic structure on T^*M, shifted in degree and combined into a graded Poisson structure on

$$\underset{p_\mu}{T^*[2]} \oplus \underset{\theta^\mu}{T[1]} \underset{x^\mu}{M}$$

Graded spacetime mechanics

Graded Poisson algebra

$$\{\underset{1}{\theta^\mu}, \underset{1}{\theta^\nu}\} = 2\underset{0}{g^{\mu\nu}(x)} \qquad \{\underset{2}{p_\mu}, \underset{0}{x^\nu}\} = \underset{0}{\delta^\nu_\mu} \qquad \{p_\mu, f(x)\} = \partial_\mu f(x)$$

Jacobi identity (i.e. associativity) $\Leftrightarrow$ metric connection

$$\{\underset{2}{p_\mu}, \underset{1}{\theta^\alpha}\} = \Gamma^\alpha_{\mu\beta}\underset{1}{\theta^\beta} =: \nabla_\mu \theta^\alpha$$

$$\{p_\mu, \{\theta^\alpha, \theta^\beta\}\} = 2\partial_\mu g^{\alpha\beta} = \{\{p_\mu, \theta^\alpha\}, \theta^\beta\} + \{\theta^\alpha, \{p_\mu, \theta^\beta\}\}$$

and curvature

$$\{\{p_\mu, p_\nu\}, \theta^\alpha\} = [\nabla_\mu, \nabla_\nu]\theta^\alpha = \theta^\beta R_\beta{}^\alpha{}_{\mu\nu}$$

$$\Rightarrow \quad \{\underset{2}{p_\mu}, \underset{2}{p_\nu}\} = \tfrac{1}{4}\underset{1}{\theta^\beta}\underset{1}{\theta^\alpha} R_{\beta\alpha\mu\nu}$$

Graded spacetime mechanics

symmetries = canonical transformations

- generator of degree 2 (degree-preserving):

$$v^\alpha(x)p_\alpha + \tfrac{1}{2}\Omega_{\alpha\beta}(x)\theta^\alpha\theta^\beta \quad \rightsquigarrow \quad \text{local Poincare algebra}$$

- generators of degree 1:

$$V = V_\alpha(x)\theta^\alpha \quad \rightsquigarrow \quad \{V, W\} = 2g(V, W) \quad \text{Clifford algebra}$$

- generators of degree 3:

$$\Theta = \theta^\alpha p_\alpha \quad (+\tfrac{1}{6}C_{\alpha\beta\gamma}\theta^\alpha\theta^\beta\theta^\gamma)$$

- generators of degree 4:

$$H = \tfrac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu + \tfrac{1}{2}\Gamma^{\beta}_{\mu\nu}(x)\theta^\mu\theta^\nu p_\beta + \tfrac{1}{16}R_{\alpha\beta\mu\nu}(x)\theta^\alpha\theta^\beta\theta^\mu\theta^\nu$$

$$\rightsquigarrow \quad \text{SUSY algebra} \quad \tfrac{1}{4}\{\Theta, \Theta\} = H$$

Graded spacetime mechanics

Graded Poisson algebra on $T^*[2]M \oplus T[1]M$: $\{p_\mu, x^\nu\} = \delta_\mu^\nu$

$$\{\theta^\mu, \theta^\nu\} = 2g^{\mu\nu}(x) \qquad \{p_\mu, \theta^\alpha\} = \Gamma^\alpha_{\mu\beta}\theta^\beta \qquad \{p_\mu, p_\nu\} = \tfrac{1}{2}\theta^\beta\theta^\alpha R_{\beta\alpha\mu\nu}$$

Equations of motion with Hamiltonian (Dirac op.) $\Theta = \theta^\mu p_\mu$

$$\frac{dA}{d\tau} = \tfrac{1}{2}\{\Theta, \{\Theta, A\}\} = \tfrac{1}{2}\{\{\Theta, \Theta\}, A\} - \tfrac{1}{2}\{\Theta, \{\Theta, A\}\} =: \{H, A\}$$

and derived Hamiltonian

$$H = \tfrac{1}{4}\{\Theta, \Theta\} = \tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu + \tfrac{1}{2}\theta^\mu\theta^\nu\Gamma^\beta_{\mu\nu}p_\beta + \tfrac{1}{16}\theta^\alpha\theta^\beta\theta^\mu\theta^\nu R_{\alpha\beta\mu\nu}$$

For a torsion-less connection, only the first term is non-zero.

Derived anchor map applied to $V = V_\alpha(x)\theta^\alpha$:

$$h(V)f = \{\{V, \Theta\}, f\} = V_\alpha(x)g^{\alpha\beta}\partial_\beta f$$

Graded spacetime mechanics

Equations of motion (cont'd)

$$\frac{dx^\mu}{d\tau} = \tfrac{1}{2}\{\Theta, \{\Theta, x^\mu\}\} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, x^\mu\} = g^{\mu\nu} p_\nu$$

$$\frac{dp_\nu}{d\tau} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, p_\nu\} = \tfrac{1}{2}(\partial_\mu g^{\alpha\beta}) p_\alpha p_\beta = {}^g\Gamma_\mu{}^{\alpha\beta} p_\alpha p_\beta$$

with *any* metric-compatible connection ${}^g\Gamma$; pick a WB connection...

Geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, g^{\mu\nu} p_\nu\} = -\frac{dx^\alpha}{d\tau} {}^{LC}\Gamma_{\alpha\beta}{}^\mu \frac{dx^\beta}{d\tau}$$

nice...supergravity and string effective actions? ... $\to$ double up ...

Geometric ladder to generalized geometry

hierarchie of actions, brackets, extended objects and algebras

AKSZ-model:	Poisson-sigma (open string) $T^*[1]M$	Courant-sigma (open membrane) $T^*[2]T[1]M$	...
derived bracket:	Poisson T^*M	Dorfman $TM \oplus T^*M$	...
	•		
object:	point particle	closed string	...
algebraic structure:	non-commutative	non-associative	...

AKSZ construction: action functionals in BV formalism of sigma model QFT's in $n+1$ dimensions for symplectic Lie n-algebroids E

Alexandrov, Kontsevich, Schwarz, Zaboronsky (1995/97)

Graded geometry

Graded Poisson manifold $T^*[2]T[1]M$

- degree 0: x^i "coordinates"
- degree 1: $\xi^\alpha = (\theta^i, \chi_i)$
- degree 2: p_i "momenta"

symplectic 2-form

$$\omega = dp_i \wedge dx^i + \frac{1}{2} G_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta = dp_i \wedge dx^i + d\chi_i \wedge d\theta^i$$

even (degree -2) Poisson bracket on functions $f(x, \xi, p)$

$$\{x^i, x^j\} = 0, \quad \{p_i, x^j\} = \delta_i^j, \quad \{\xi^\alpha, \xi^\beta\} = G^{\alpha\beta}$$

metric $G^{\alpha\beta}$: natural pairing of TM, T^*M:

$$\{\chi_i, \theta^j\} = \delta_i^j \ , \quad \{\chi_i, \chi_j\} = 0 \ , \quad \{\theta^i, \theta^j\} = 0$$

Graded geometry

degree-preserving canonical transformations

- infinitesimal, generators of degree 2:

$$v^\alpha(x)p_\alpha + \tfrac{1}{2}M^{\alpha\beta}(x)\xi_\alpha\xi_\beta \quad \rightsquigarrow \quad \text{diffeos and } o(d,d)$$

- finite, idempotent ("coordinate flip"): $(\tilde{\chi},\tilde{\theta}) = \tau(\chi,\theta)$ with $\tau^2 = \mathrm{id}$

  $\rightsquigarrow$ generating function F of type 1 with $F(\theta,\tilde{\theta}) = -F(\tilde{\theta},\theta)$:

$$F = \theta\cdot g\cdot\tilde{\theta} - \tfrac{1}{2}\,\theta\cdot B\cdot\theta + \tfrac{1}{2}\,\tilde{\theta}\cdot B\cdot\tilde{\theta}$$

$$\chi = \frac{\partial F}{\partial\theta} = \tilde{\theta}\cdot g + \theta\cdot B\ , \qquad \tilde{\chi} = -\frac{\partial F}{\partial\tilde{\theta}} = \theta\cdot g + \tilde{\theta}\cdot B$$

$$\Rightarrow \quad \tau(\chi,\theta) = (\chi,\theta)\cdot\begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1}\end{pmatrix}$$

  $\rightsquigarrow$ generalized metric

Generalized Geometry

Generalized tangent bundle E: $0 \to T^*M \to E \to TM \to 0$
e.g. $E = TM \oplus T^*M$, i.e. "vector fields plus differential forms"

Courant algebroid: vector bundle $E \xrightarrow{\pi} M$, anchor $h : E \to TM$, bracket $[-,-]$, pairing $\langle -,- \rangle$, s.t. for $e, e', e'' \in \Gamma E$:

$$2\langle [e,e'], e' \rangle \overset{(1)}{=} h(e)\langle e', e' \rangle \overset{(2)}{=} 2\langle [e',e'], e \rangle$$

$$[e,[e',e'']] = [[e,e'],e''] + [e',[e,e'']] \qquad (3)$$

Consequences:

$$[e, fe'] = h(e).f\, e' + f[e,e'] \qquad (L)$$

$$h([e,e']) = [h(e), h(e')]_{\mathrm{Lie}}$$

axioms 1, 2 can be polarized, axiom 3 and (L) define a Leibniz algebroid

Generalized geometry as a derived structure

Cartan's magic identy:

$$\mathcal{L}_X = [i_X, d] \equiv i_X d + d\, i_X$$

Lie bracket $[X, Y]_{\text{Lie}}$ of vector fields as a derived bracket:

$$[[i_X, d], i_Y] = [\mathcal{L}_X, i_Y] = i_{[X,Y]_{\text{Lie}}} \quad \text{with } [d, d] = d^2 = 0$$

Generalized geometry as a derived structure

degree 3 “Hamiltonian”: Dirac operator

$$\Theta = \xi^\alpha h^i_\alpha(x) p_i \; + \; \underbrace{\tfrac{1}{6} C_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma}_{\text{twisting/flux terms}}$$

For $e = e_\alpha(x)\xi^\alpha \in \Gamma(TM \oplus T^*M)$ (degree 1, odd):

- pairing: $\langle e, e' \rangle = \{e, e'\}$
- anchor: $h(e)f = \{\{e, \Theta\}, f\}$
- bracket: $[e, e']_D = \{\{e, \Theta\}, e'\}$

Generalized geometry as a derived structure

Courant algebroid axioms from associativity and $\{\Theta, \Theta\} = 0$:

$$
\begin{aligned}
h(\xi_1)\,\langle \xi_2, \xi_2 \rangle &= \{\{\Theta, \xi_1\}, \{\xi_2, \xi_2\}\} \\
&= 2\{\{\{\Theta, \xi_1\}, \xi_2\}, \xi_2\} = 2\,\langle [\xi_1, \xi_2]\,, \xi_2 \rangle \qquad \text{(axiom 1)} \\
&= 2\{\xi_1, \{\{\Theta, \xi_2\}, \xi_2\}\} = 2\,\langle \xi_1, [\xi_2, \xi_2] \rangle \qquad \text{(axiom 2)}
\end{aligned}
$$

$$
\begin{aligned}
[\xi_1, [\xi_2, \xi_3]] &= \{\{\Theta, \xi_1\}, \{\{\Theta, \xi_2\}, \xi_3\}\} \\
&= [[\xi_1, \xi_2], \xi_3] + [\xi_2, [\xi_1, \xi_3]] + \frac{1}{2}\{\{\{\{\Theta, \Theta\}, \xi_1\}, \xi_2\}, \xi_3\}.
\end{aligned}
$$

$\{\Theta, \Theta\} = 0 \quad \Leftrightarrow \quad [\,,\,]$-Jacobi identity (in 1st slot) (axiom 3)

Deformation

general (deformed) Poisson structure

$$\begin{aligned} \{v, f\} &= v.f \\ \{V, W\} &= G(V, W) \equiv \langle V, W \rangle \\ \{v, V\} &= \nabla_v V \qquad \leftarrow \text{connection metric wrt. } G \\ \{v, w\} &= [v.w]_{\text{Lie}} + R(v, w) \qquad \leftarrow \text{curvature of } \nabla \end{aligned}$$

with

- degree 0: $f(x)$
- degree 1: $V = V^\alpha(x)\xi_\alpha$ "generalized vectors"
- degree 2: $v = v^i(x)p_i$ "vector fields"

general Hamiltonian

$$\Theta = \tilde{\xi}^\alpha h(\xi_\alpha) + \frac{1}{6} C_{\alpha\beta\gamma}\tilde{\xi}^\alpha\tilde{\xi}^\beta\tilde{\xi}^\gamma \qquad \leftarrow \text{general flux (H,f,Q,R)}$$

Deformation

derived bracket

$$\{\{\{V,\Theta\},W\},X\} = \langle\nabla_V W, X\rangle - \langle\nabla_W V, X\rangle + \langle\nabla_X V, W\rangle + C(V,W,X)$$

$$\{\{\{\xi_\alpha,\Theta\},\xi_\beta\},\xi_\gamma\} = \underbrace{\Gamma_{\alpha\beta\gamma} - \Gamma_{\beta\alpha\gamma}}_{\text{torsion}} + \Gamma_{\gamma\alpha\beta} + C_{\alpha\beta\gamma} =: \Gamma^{\text{new}}_{\gamma\alpha\beta}$$

"mother of all brackets"

$$\begin{aligned}[V,W] &= \nabla_V W - \nabla_W V + \langle\nabla V, W\rangle + C(V,W,-)\\ &= [[V,W]] + T(V,W) + \langle\nabla V, W\rangle + C(V,W,-)\end{aligned}$$

In order to obtain a regular Courant algebroid, impose

$$\{\Theta,\Theta\} = 0 \quad \Leftrightarrow \quad \nabla C + \frac{1}{2}\{C,C\} = 0\,, \quad G^{-1}|_h = 0\,, \ldots$$

Generalized differential geometry

generalized Lie-bracket (involves anchor $h : E \to TM$)

$$[[V, W]] = -[[W, V]]\,, \quad [[V, fW]] = (h(V)f)W + f\,[[V, W]]$$

generalized connection "type I" and miraculous triple identity

$$\Gamma(V; fW, U) = (h(V)f)\langle W, U\rangle + f\,\Gamma(V; W, U)\,,$$

$$\boxed{\langle V, [W, Z]\rangle = \langle V, [[W, Z]]\rangle + \Gamma(V; W, Z)}$$

$$\langle \nabla_V W, U\rangle := \Gamma(V; W, U)$$

generalized curvature and torsion

$$R(V, W) = \nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[[V,W]]}$$

$$T(V, W) = \nabla_V W - \nabla_W V - [[V, W]]$$

Boffo, PS (2019/2020)

Graded/generalized geometry and gravity

cookbook recipe

- deform graded Poisson structure
- pick Hamiltonian Θ (e.g. canonical), compute derived brackets
- choose generalized Lie bracket $[[\, , \,]]$ (e.g. canonical)
- determine connection Γ from triple identity
- project (or rather embed) via non-isotropic splitting (e.g. canonical)

$$s : \Gamma(TM) \to \Gamma(E) \qquad \rho \circ s = \mathrm{id} \qquad \langle X, Y \rangle_{TM} := \langle s(X), s(Y) \rangle$$

$$\langle \nabla_Z X, Y \rangle_{TM} := \Gamma(s(Z); s(X), s(Y))$$

- compute Riemann and Ricci tensors, take trace with $g + B$, write action in terms of resulting Ricci scalar

Graded/generalized geometry and gravity

deformation by generalized vielbein E

$$\Omega = dx^i \wedge dp_i + d\theta^i \wedge d\chi_i$$

deformation by change of coordinates in the odd (degree 1) sector
two choices:

$$\begin{pmatrix} \theta \\ \chi \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ g+B & 1 \end{pmatrix} \cdot \begin{pmatrix} \theta \\ \chi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \Pi + G \\ -g+B & 1 \end{pmatrix} \cdot \begin{pmatrix} \theta \\ \chi \end{pmatrix}$$

Boffo, PS 1903.09112 and in preparation

now crank the "machine" (deformed derived bracket, connection, project, Riemann, Ricci) $\rightsquigarrow$ (effective) gravity actions ...

Graded/generalized geometry and gravity

generalized Koszul formula for nonsymmetric $\mathcal{G} = g + B$

$$
\begin{aligned}
2g(\nabla_Z X, Y) &= \langle Z, [X, Y]' \rangle' \\
&= X\mathcal{G}(Y, Z) - Y\mathcal{G}(X, Z) + Z\mathcal{G}(X, Y) \\
&\quad -\mathcal{G}(Y, [X, Z]_{\mathrm{Lie}}) - \mathcal{G}([X, Y]_{\mathrm{Lie}}, Z) + \mathcal{G}(X, [Y, Z]_{\mathrm{Lie}}) \\
&= 2g(\nabla_X^{LC} Y, Z) + H(X, Y, Z)
\end{aligned}
$$

$\Rightarrow$ non-symmetric Ricci tensor

$$R_{jl} = R_{jl}^{LC} - \frac{1}{2}\nabla_i^{LC} H_{jl}{}^i - \frac{1}{4} H_{lm}{}^i H_{ij}{}^m \qquad R = \mathcal{G}_{ij} g^{ik} g^{jl} R_{kl}$$

$\Rightarrow$ gravity action (closed string effective action) after partial integration:

$$S_{\mathcal{G}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left(R^{LC} - \frac{1}{12} H_{ijk} H^{ijk} \right)$$

Khoo, Vysoky, Jurco, Boffo, PS

Graded/generalized geometry and gravity

This formulation consistently combines all approaches of Einstein: Non-symmetric metric, Weitzenböck and Levi-Civita connections, without any of the usual drawbacks.

The dilaton $\phi(x)$ rescales the generalized tangent bundle. The deformation can be formulated in terms of vielbeins

$$E = e^{-\frac{\phi}{3}} \begin{pmatrix} 1 & 0 \\ g+B & 1 \end{pmatrix} \qquad E^{-1}\partial_i E = \begin{pmatrix} -\frac{1}{3}\partial_i\phi & 0 \\ \partial_i(g+B) & -\frac{1}{3}\partial_i\phi \end{pmatrix}$$

Going through the same steps as before we find in $d = 10$

$$S = \frac{1}{2\kappa}\int d^{10}x\, e^{-2\phi}\sqrt{-g}\Big(R^{\mathrm{LC}} - \tfrac{1}{12}H^2 + 4(\nabla\phi)^2\Big)$$

Boffo, PS

Quantization

$x^i, p_i, \theta^i, \chi_i \quad \rightsquigarrow$ differential ops on $\psi(x,\theta) \in \Lambda^\bullet T^*M$ (spinors):

$$p \rightsquigarrow \partial_x \qquad \chi \rightsquigarrow \partial_\theta = \mathrm{i}_\chi \qquad x \rightsquigarrow x\,\cdot \qquad \theta \rightsquigarrow \theta\wedge$$

θ, χ: finite dimensional representation by γ-matrices:

$$V \rightsquigarrow \gamma_V = V^\alpha(x)\gamma_\alpha\ , \qquad [\gamma_V, \gamma_W]_+ = G(V,W) \text{ etc.}$$

Symmetry Lie algebra generators: $M^\alpha{}_\beta \xi_\alpha \tilde{\xi}^\beta$
$M^i{}_j$ picks up $\mathrm{tr}M$ "anomaly" after quantization

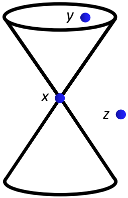
$$\Lambda^\bullet T^*M \rightsquigarrow \Lambda^\bullet T^*M \otimes \det^{\frac{1}{2}} TM$$

requiring the introduction of the dilaton field ϕ for covariance.

Interaction via deformation

Interaction via deformation as an alternative (slight generalization) of minimal coupling, covariant derivatives, gauge theory.

- classical: deformed Poisson structure
- quantum: deformed operator algebra (CCR)



$[\phi(x), \phi(z)] = 0, \quad [\phi(x), \phi(y)] \neq 0$

(non-)commutativity $\leftrightarrow$ causality

equal time CRs: $[\phi(x), \dot{\phi}(x')] \sim i\delta(x - x')$
single particle QM version: $[x_i, p_j] \sim i\delta_{ij}$

$\rightarrow$ deform these CCRs to introduce interactions

- gauge fields: recovered via Moser's lemma
- $U(1)$ case: closed expression for SW map, global NC line bundle
- here: adapt the approach to gravity

Link to gauge theory

deformation of symplectic form Ω' $\rightsquigarrow$ gauge field A:

Moser's lemma

Let $\Omega_t = \Omega + tF$, with Ω_t symplectic for $t \in [0,1]$.

$d\Omega_t = 0 \;\Rightarrow dF = 0 \;\Rightarrow$ locally $F = dA$

$\Omega' \equiv \Omega_1$ and Ω are related by a change of phase space coordinates generated by the flow of a vector field V_t defined up to gauge transformations by the gauge field $i_{V_t}\Omega_s = A$, i.e. $V_t = \theta_s(A, -)$.

Proof: $\mathcal{L}_{V_t}\Omega_t = i_{V_t} d\Omega_t + d\, i_{V_t}\Omega_t = 0 + dA = \frac{d}{dt}\Omega_t$. Moser 1965

Quantum and Poisson versions of the lemma exist based on equivalence of star products and formality maps: Jurco, PS, Wess 2000-2002

More deformation

our initial example:

deformation by a gauge field A

$$\Omega' = dx^i \wedge dp_i + \tfrac{1}{2} F_{ij}(x) dx^i \wedge dx^j,\ dF = 0, \text{ locally } F = dA$$

$$\boxed{\Omega_t = \Omega + t\, dA}, \quad A = A_i(x) dx^i$$

$$V_t = A_i(x) \frac{\partial}{\partial p_i}, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[A]}(p) = p + A$$

$$\{p_i, x^j\}_t = \delta_i^j$$

$$\{p_i, p_j\}_t = t\, F_{ij}(x)$$

gauge transformation $\delta A = d\lambda \leftrightarrow \delta\rho_{[A]}$: canonical transformation

non-abelian versions: $A_i^\alpha(x)\ell_\alpha dx^i$ and $A_{ia}^b(x)\theta^a \chi_b dx^i$

$\rightsquigarrow$ Abelian and non-abelian gauge theory

More deformation

deformation by a spin connection ω

$$\Omega = dx^i \wedge dp_i + \tfrac{1}{2}\eta_{ab} d\theta^a \wedge d\theta^b \qquad \theta^a = e^a_i \theta^i \,, \quad g_{ij} = e^a_i e^b_j \eta_{ab}$$

$$\boxed{\Omega_t = \Omega + t\, d\omega}\,, \quad \omega = \omega_i(x,\theta) dx^i = \tfrac{1}{2}\omega_{iab}(x)\theta^a\theta^b dx^i$$

$$V_t = \omega_i \partial_{p_i} \,, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[\omega]}(p) = p + \omega$$

$$\{p_i, x^j\}_t = \delta^j_i \qquad \{\theta^a, \theta^b\}_t = \eta^{ab}$$

$$\{p_i, \theta^a\} = t\, \eta^{ab} \omega_{ibc}(x)\, \theta^c \qquad \omega_{ibc} = -\omega_{icb}$$

$$\{p_i, p_j\}_t = t\, R_{ij} \qquad R = d\omega + t\omega \wedge \omega$$

gauge transformation $\delta\omega = d\lambda \leftrightarrow \delta\rho_{[\omega]}$: canonical transformation

$\rightsquigarrow$ Einstein-Cartan gravity

More deformation

deformation by a general connection Γ

$$\Omega = dx^i \wedge dp_i + d\theta^i \wedge d\chi_i$$

$$\boxed{\Omega_t = \Omega + t\, d\Gamma}\,, \quad \Gamma = \Gamma_i dx^i = \Gamma_{ij}{}^k(x)\theta^j \chi_k dx^i$$

$$V_t = \Gamma_i \partial_{p_i}\,, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[\Gamma]}(p) = p + \Gamma$$

$$\{p_i, x^j\}_t = \delta_i^j \qquad \{\chi_i, \theta^j\}_t = \delta_i^j$$

$$\{p_i, \theta^j\} = t\, \Gamma^j_{ik}\theta^k \qquad \{p_i, \chi_j\} = -t\, \Gamma^k_{ij}\chi_k$$

$$\{p_i, p_j\}_t = t\, R_k{}^l{}_{ij}\theta^k \chi_l \qquad R_k{}^l{}_{ij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}$$

gauge transformation $\delta\Gamma = d\Lambda \leftrightarrow \delta\rho_{[\Gamma]}$: canonical transformation

$\rightsquigarrow$ General relativity and alternative gravity theories

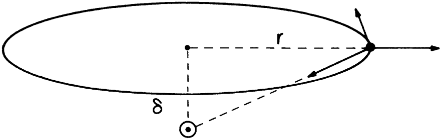
Non-associativity, non-metricity, gravitipoles ...

Non-associativity

The Jacobi identity playes a pivotal role; its violation has drastic effects:

- $\{p_\mu, \theta^\alpha, \theta^\beta\} \neq 0 \;\Rightarrow$ non-metricity of connection ∇
- $\{p_\alpha, p_\beta, p_\gamma\} \neq 0 \;\Rightarrow$ gravito-magnetic sources, mass quantization

Shifted orbit in the presence of a gravitipol:



mixed symmetry tensors, higher spin actions

Graded geometry is also a useful tool for mixed symmetry tensor theories: Consider *e.g.* a bi-partite tensor

$$\omega_{p,q} = \frac{1}{p!q!}\,\omega_{i_1\dots i_p}^{j_1\dots j_q}(x)\,\theta^{i_1}\dots\theta^{i_p}\chi_{j_1}\dots\chi_{j_q}$$

and the natural θ-χ duality transformation

$$\omega_{p,q} \mapsto \widetilde{\omega}_{q,p} \quad \text{via} \quad \theta^i \leftrightarrow \chi^i \equiv \eta^{ij}\chi_j\,.$$

Introduce two differentials

$$\mathrm{d} = \theta^i\partial_i \quad \text{and} \quad \widetilde{\mathrm{d}} = \chi^i\partial_i$$

and a generalized Hodge dual

$$(\star\,\omega)_{D-p,D-q} = \frac{1}{(D-p-q)!}\,\eta^{D-p-q}\,\widetilde{\omega}_{q,p} \quad \text{where} \quad \eta = \theta^i\chi_i\,.$$

Chatzistavrakidis, Khoo, Roest, PS (JHEP 2017)

spin ≤ 2 kinetic terms

$\rightsquigarrow$ natural and concise formalism for mixed symmetry tensor actions:

general kinetic term $\boxed{\mathcal{L}_{\text{kin}}(\omega_{p,q}) = \int_{\theta,\chi} \mathrm{d}\omega \star \mathrm{d}\omega}$ $\Rightarrow$

$$\mathcal{L}_{\text{scalar}}(\phi_{0,0}) = -\frac{1}{2(D-1)!}\int_{\theta,\chi} \eta^{D-1}\,\phi\,\mathrm{d}\widetilde{\mathrm{d}}\,\phi = \frac{1}{2}\phi\,\Box\,\phi$$

$$\mathcal{L}_{\text{Maxwell}}(A_{1,0}) = \frac{1}{2(D-2)!}\int_{\theta,\chi} \eta^{D-2}\,A\,\mathrm{d}\widetilde{\mathrm{d}}\,\widetilde{A} = -\frac{1}{4}F_{ij}F^{ij}$$

$$\mathcal{L}_{\text{LEH}}(h_{[1,1]}) = -\frac{1}{4}\left(h^i{}_i\,\Box\,h^j{}_j - 2h^k{}_k\partial_i\partial_j h^{ij} + 2h_{ij}\partial^j\partial_k h^{ik} - h_{ij}\,\Box\,h^{ij}\right)$$

$$\mathcal{L}_{\text{Curtright}}(\omega_{[2,1]}) = \frac{1}{2}\Big(\partial_i\omega_{jk|l}\partial^i\omega^{jk|l} - 2\partial_i\omega^{ij|k}\partial^l\omega_{lj|k} - \partial_i\omega^{jk|i}\partial^l\omega_{jk|l} - \\ - 4\omega_i{}^{j|i}\partial^k\partial^l\omega_{kj|l} - 2\partial_i\omega_j{}^{k|j}\partial^i\omega^l{}_{k|l} + 2\partial_i\omega_j{}^{i|j}\partial^k\omega^l{}_{k|l}\Big)$$

Chatzistavrakidis, Karagiannis, PS (CMP 2020)

spin ≤ 2 interaction and mass terms, higher spin

general interaction term with up to second order field equations:

$$\boxed{\mathcal{L}_{\mathsf{Gal}}(\omega_{p,q}) = \sum_{n=1}^{n_{\max}} \frac{1}{(D-k_n)!} \int_{\theta,\chi} \eta^{D-k_n}\, \omega\, (\mathrm{d}\widetilde{\mathrm{d}}\,\omega)^{n-1} (\mathrm{d}\widetilde{\mathrm{d}}\,\widetilde{\omega})^n}$$

higher gauge symmetry via higher Poincaré lemma: $\mathrm{d}\widetilde{\mathrm{d}}(\delta\omega) = 0$ implies

$$\delta\omega_{p,q} = \mathrm{d}\kappa_{p-1,q} + \widetilde{\mathrm{d}}\kappa_{p,q-1} + c_{i_1 \dots i_p k_0 k_1 \dots k_q} \theta^{i_1} \dots \theta^{i_p} x^{k_0} \chi^{k_1} \dots \chi^{k_q}$$

(locally, i.e. on a contractible patch)

mass term $\boxed{\mathcal{L}_{\mathsf{mass}}(\omega_{p,q}) = m^2 \int_{\theta,\chi} \omega \star \omega}$ $\rightsquigarrow$ Proca, Fierz-Pauli, etc.

---

application: standard and exotic dualizations "$[p, q] \leftrightarrow [D-p-2, q]$"

for higher spin ≥ 2: simply add further copies of $\theta\chi$-pairs...

Chatzistavrakidis, Karagiannis, PS (CMP 2020)

Conclusion

- deformation: combines best aspects of Lagrange and Hamilton
- graded/generalized geometry provides a perfect setting for the formulation of theories of gravity
- approach is based on deformed graded geometry is algebraic in nature: almost everything follows from associativity as unifying principle (which can be generalized)
- more traditional approaches are based on the generalized metric (with occasional covariance and uniqueness issues)
- non-associativity $\Rightarrow$ non-metricity, gravitipoles, mass quantization
- graded geometry provides a powerful formalism for higher spins

Thanks for listening!