$$(\mathcal{D} - m)\psi = 0 \quad , \qquad \psi \in C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4)$$

$$\mathcal{D} = i\gamma^j \partial_j + \mathcal{B}$$

where $\quad \mathcal{B} = \gamma^j A_j$ in the presence of an electromagnetic potential

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define <u>spinor space</u> $(S, \langle.|.\rangle)$ by

- S is a 4-dim. complex vector space
- $\langle.|.\rangle$ is an inner product of signature $(2,2)$, the <u>spin inner product</u>

Choosing a pseudo-orthonormal basis,

$$S \simeq \mathbb{C}^4 \quad , \qquad \langle\psi|\phi\rangle = \sum_{\alpha=1}^{4} s_\alpha \overline{\psi^\alpha} \phi^\alpha$$

$s_1 = s_2 = 1$

$s_3 = s_4 = -1$

$$\psi \in C^\infty_{sc}(\mathcal{M}, S)$$

spinor bundle $\quad S\mathcal{M} := \mathcal{M} \times S$

$\qquad S_x\mathcal{M} = \{x\} \times S \qquad$ spinor space at x

$$S\mathcal{M} = \bigcup_{x\in\mathcal{M}} S_x\mathcal{M}$$

wave function

$$\psi : \mathcal{M} \longrightarrow S\mathcal{M}$$
$$x \longmapsto \psi(x) \in S_x\mathcal{M}$$

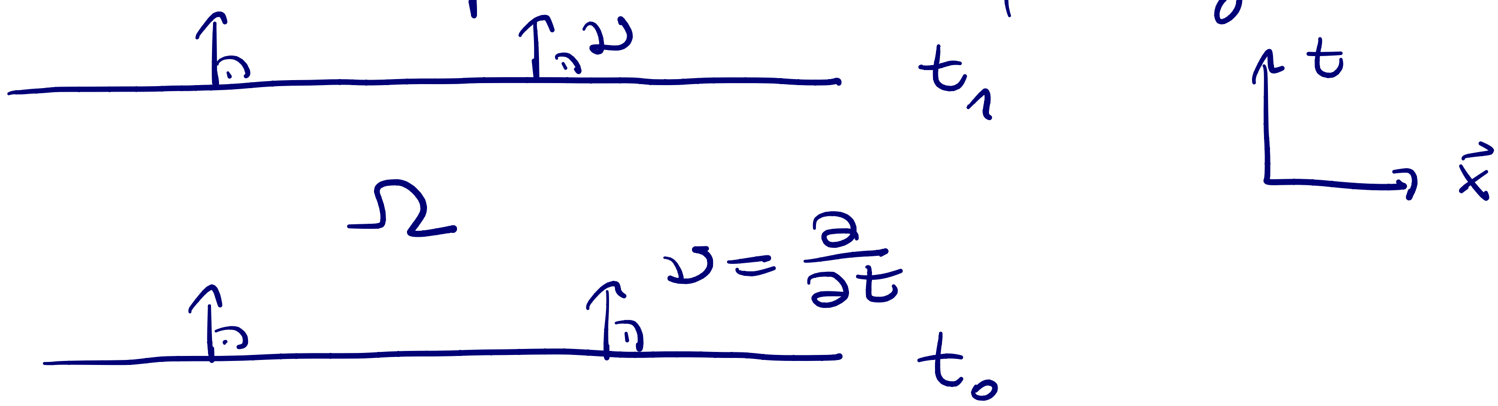
$\psi \in C^\infty_{sc}(\mathcal{M}, S\mathcal{M}) \qquad$ section in the spinor bundle

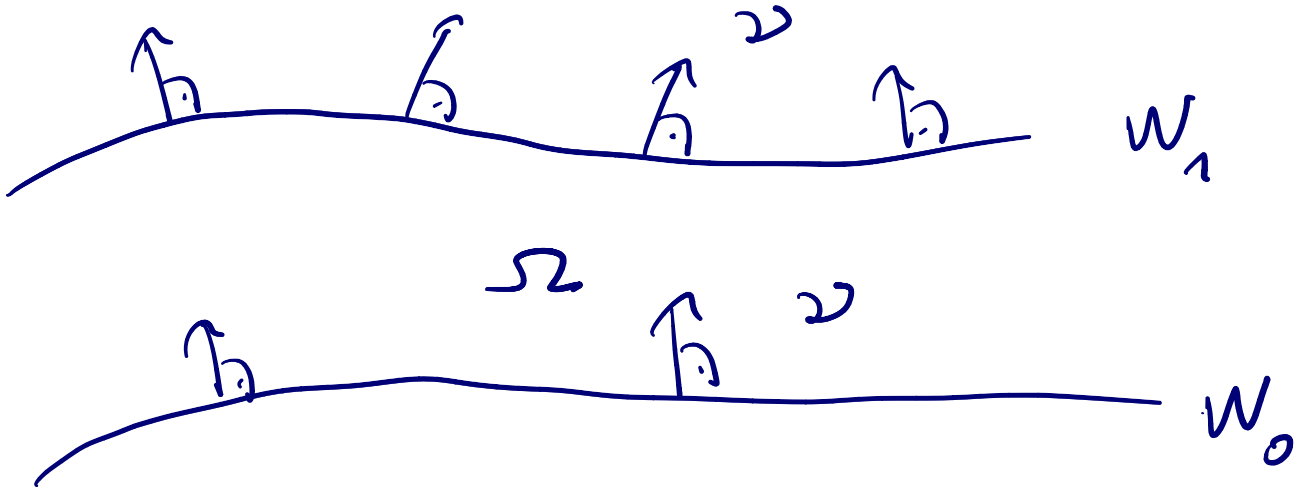
$j^k(x) := \langle \Psi(x) | \gamma^k \Psi(x) \rangle_x$ Dirac current

$\partial_k j^k(x) = 0$, provided that $\mathcal{D}(x)$ is symmetric w.r.t. $\langle . | . \rangle_x$, i.e.

$$\langle \mathcal{D}(x) \psi | \phi \rangle_x = \langle \psi | \mathcal{D}(x) \phi \rangle_x \quad \forall \psi, \phi \in S_x M$$

Current conservation follows from Gauß divergence theorem





$$0 = \int_\Omega \underbrace{\partial_k j^k}_{=0} \; \underset{d^4x}{\overset{}{d\mu}}$$

$$= \int_{\mathbb{R}^3} j^0(t_1, \vec{x}) \, d^3x - \int_{\mathbb{R}^3} j^0(t_0, \vec{x}) \, d^3x \leftarrow$$

$$\left(= \int_{W_1} j^k \nu_k \, d\mu_W - \int_{W_0} j^k \nu_k \, d\mu_W \right)$$

$$j^0(x) = \langle \psi(x) | \gamma^0 \psi(x) \rangle_x$$

$\langle \cdot | \gamma^0 \cdot \rangle_x$ is positive, is a scalar product

$\left(\langle \cdot | \not{v} \cdot \rangle_x \quad \text{—} \, '' \, \text{—} \right)$

This gives rise to a scalar product on solutions

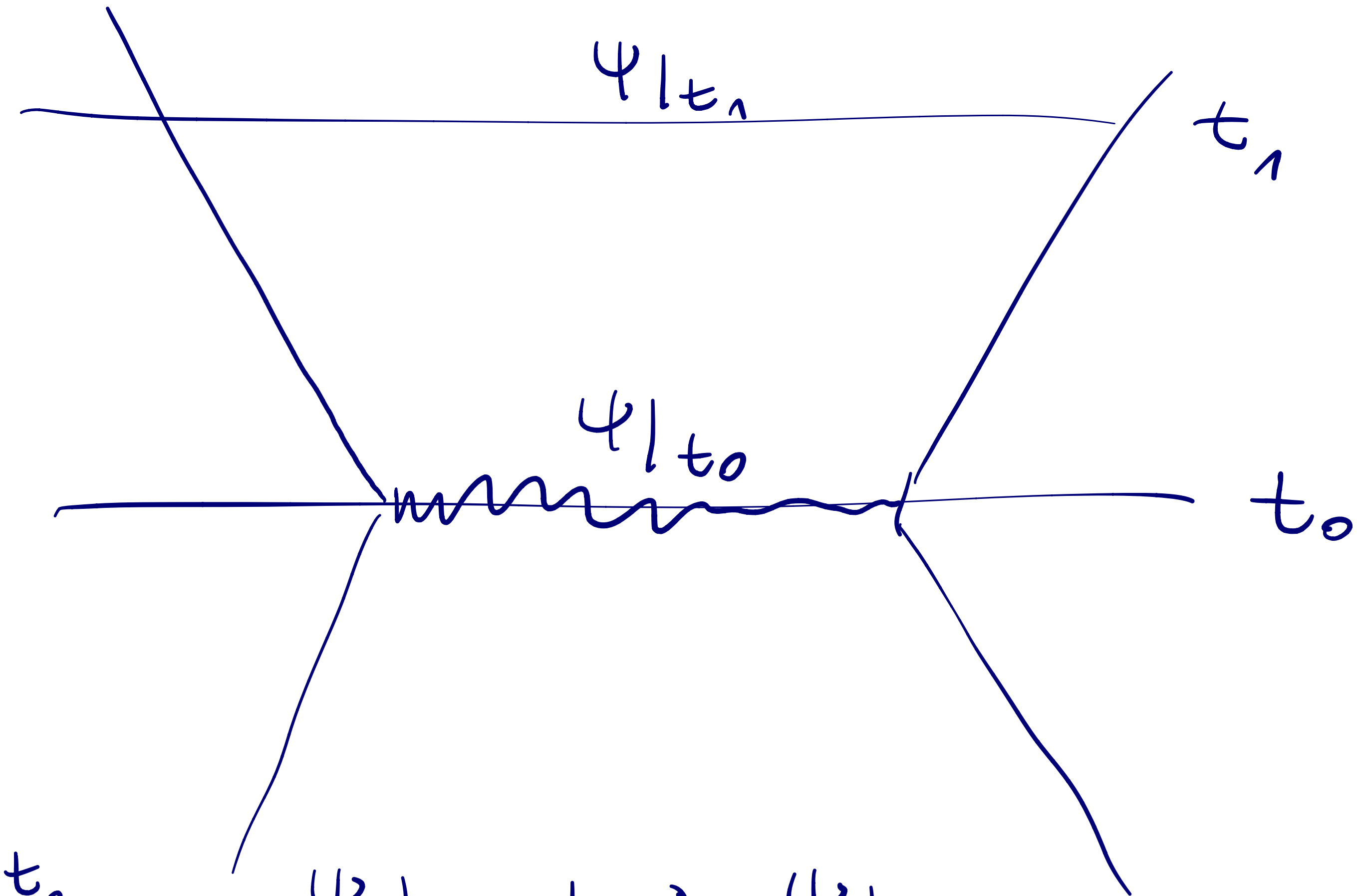
$$(\psi | \phi)_t := \int_{\mathbb{R}^3} \underbrace{\langle \psi | \gamma^0 \phi \rangle}_{\psi^\dagger \phi \,=\, \langle \psi , \phi \rangle_{\mathbb{C}^4}} (t, \vec{x}) \; d^3x$$

Taking the completion gives a Hilbert space

$$(\mathcal{H}_t, (\cdot|\cdot)_t) \simeq L^2(\mathbb{R}^3, \mathbb{C}^4)$$

The fact that $(\cdot|\cdot)_t$ is time independent gives rise to a unitary time evolution

$$U_{t_0}^{t_1} : \mathcal{H}_{t_0} \longrightarrow \mathcal{H}_{t_1}$$ unitary time evolution operator



$$U_{t_0}^{t_1} : \psi|_{t_0} \longmapsto \psi|_{t_1}$$

more global point of view

$(\psi|\phi)_m = (\psi|\phi)_t$ for any time $t \in \mathbb{R}$

$(\mathcal{H}_m, (.|.)_m)$ Hilbert space of all Dirac solutions

$$\psi \in H^{1,2}_{loc}(\mathcal{M}) \xrightarrow{\text{trace thm}} L^2_{loc}(\mathcal{N})$$

Krein structure

Let $\psi, \phi \in C^\infty_0(\mathcal{M}, S\mathcal{M})$ (not solution of the Dirac eqn)

$$<\psi|\phi>_{\mathcal{M}} := \int_{\mathcal{M}} \prec \psi(x)|\phi(x)\succ_x \, d\mu(x)$$

indefinite inner product

Gives rise to a Krein space

The Dirac operator is symmetric w.r.t. this inner product, i.e.

$$<\mathcal{D}\psi|\phi>_{\mathcal{M}} = <\psi|\mathcal{D}\phi>_{\mathcal{M}} \quad \forall \psi, \phi \in C^\infty_0(\mathcal{M}, S\mathcal{M})$$