$(M, \langle \cdot, \cdot \rangle)$ $\qquad i, j \in \{0, \dots, 3\}$

$$g_{ij} = s_i \delta_{ij} \qquad s_0 = 1, \quad s_1 = s_2 = s_3 = -1$$

$$\langle x, y \rangle = g_{ij} x^i y^j$$

g_{ij} is the Minkowski metric

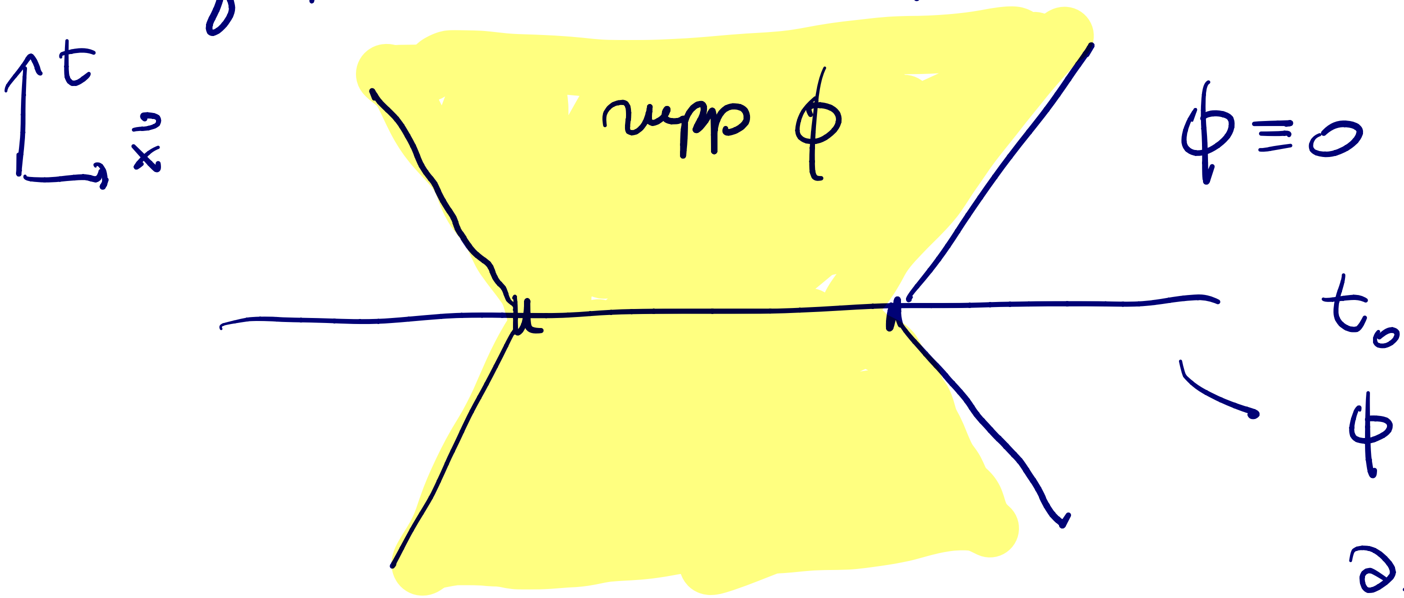
$$g^{ij} = (g_{ij})^{-1} \quad \text{i.e.} \quad g^{ij} g_{jk} = \delta^i_k$$

$$= \operatorname{diag}(1, -1, -1, -1)$$

$$\Box = g^{ij} \partial_i \partial_j = \partial_t^2 - \Delta_{\mathbb{R}^3} \quad \text{wave operator}$$

$$\Box \phi = 0 \quad \text{scalar wave equation}$$

Cauchy problem is well-posed



$$\phi|_{t_0} = \phi_0 \in C_0^\infty(\mathbb{R}^3)$$

$$\partial_t \phi|_{t_0} = \phi_1 \in C_0^\infty(\mathbb{R}^3)$$

ϕ smooth solution with spatially compact support (i.e. $\phi|_{t = \text{const}}$ has compact support $\forall t$)

$$C^\infty_{sc}(M, \mathbb{R})$$

symplectic form:

Let $\phi, \tilde{\phi} \in C^\infty_{sc}(M, \mathbb{R})$ be two solution

$$\sigma_t(\phi, \tilde{\phi}) := \int_{\mathbb{R}^3} \Big((\partial_t \phi)\tilde{\phi} - \phi\,(\partial_t \tilde{\phi}) \Big)(t, \vec{x})\, d^3x$$

is conserved in time:

$$\frac{d}{dt}\sigma_t(\phi, \tilde{\phi})$$

$$= \int_{\mathbb{R}^3} \Big((\partial_t^2 \phi)\tilde{\phi} + \cancel{(\partial_t \phi)(\partial_t \tilde{\phi})} - \cancel{(\partial_t \phi)(\partial_t \tilde{\phi})} - \phi\, \partial_t^2 \tilde{\phi} \Big)\, d^3x$$

$$= \int_{\mathbb{R}^3} \big(\Delta_{\mathbb{R}^3} \phi\; \tilde{\phi} - \phi\; \Delta_{\mathbb{R}^3} \tilde{\phi} \big)\, d^3x$$

$= 0$ due to Green's formula.

Klein-Gordon equation

$$(-\Box - m^2)\psi = 0, \qquad \psi \in C^\infty_{sc}(M, \mathbb{C})$$

Conserved sesquilinear form

$$\langle \psi | \tilde{\psi} \rangle = i \int_{\mathbb{R}^3} \Big(\big(\overline{\partial_t \psi}\big)\, \tilde{\psi} - \overline{\psi}\,(\partial_t \tilde{\psi}) \Big)\, d^3x$$

$$\langle \psi | \psi \rangle = i \int_{\mathbb{R}^3} \big(\overline{\partial_t \psi}\; \psi - \overline{\psi}\; \partial_t \psi \big)\, d^3x$$

$$= -2\,\mathrm{Im} \int_{\mathbb{R}^3} \overline{\partial_t \psi}\; \psi$$

Dirac equation

$$\mathcal{D}^2 = -\Box \, \mathbb{1}$$

ansatz: $\mathcal{D} = i \gamma^j \partial_j$

$$\mathcal{D}^2 = -(\gamma^j \partial_j)(\gamma^k \partial_k)$$
$$= -\gamma^j \gamma^k \partial_{jk}$$
$$= -\frac{1}{2}\{\gamma^j, \gamma^k\} \partial_{jk}$$

$\{\gamma^j, \gamma^k\} := \gamma^j \gamma^k + \gamma^k \gamma^j$

$$\Rightarrow \quad \{\gamma^j, \gamma^k\} = 2 g^{jk}$$ anti-commutation relations

Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}$$ Dirac matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Pauli matrices

$$(i\gamma^j \partial_j - m\mathbb{1})\psi = 0 \quad ; \quad \psi \in C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4)$$

Dirac equation in the vacuum

multiply the equation by $(i\gamma^j \partial_j + m)$

$$\Rightarrow ((i\gamma^j \partial_j)^2 - m^2)\psi = 0$$

$$\Leftrightarrow \quad (-\Box - m^2)\psi = 0.$$

In the presence of an electromagnetic potential A_j

$$(i\gamma^j \partial_j + A_j \gamma^j - m)\psi = 0$$

short notation $\qquad \not{\partial} = \gamma^j \partial_j$

$$\not{A} = \gamma^j A_j$$

$$(i\not{\partial} + \not{A} - m)\psi = 0$$

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spin inner product $\quad \overline{\psi(x)}\,\phi(x), \qquad \overline{\psi(x)} = \psi(x)^\dagger \gamma^0$

$$\langle \psi(x) | \phi(x) \rangle_x = \overline{\psi(x)}\,\phi(x)$$

$$= \sum_{\alpha=1}^{4} s_\alpha \, \overline{\psi^\alpha(x)}\, \phi^\alpha(x)$$

$$s_1 = s_2 = 1$$

$$s_3 = s_4 = -1$$

The Dirac matrices are symmetric w.r.t. the spin inner product,

$$\langle \gamma^j \psi | \phi \rangle_x = \langle \psi | \gamma^j \phi \rangle_x$$

Let $\psi, \phi \in C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4)$ be two solutions of the Dirac equation

$$J^k(x) := \langle \psi(x) | \gamma^k \phi(x) \rangle$$

In the case $\psi = \phi$, this is the Dirac current

$$\partial_k J^k = \langle \partial_k \psi | \gamma^k \phi \rangle + \langle \psi | \gamma^k \partial_k \phi \rangle$$

$$= \langle \not{\partial}\psi | \phi \rangle + \langle \psi | \not{\partial} \phi \rangle$$

$$= \langle (\cancel{-im} + i\not{A})\psi \mid \phi\rangle + \langle \psi \mid (\cancel{-im} + i\not{A})\phi\rangle = 0$$

$$(i\not{\partial} + \not{A} - m)\psi = 0 \quad \Rightarrow \quad \not{\partial}\psi = (-im + i\not{A})\psi$$

<u>Lemma</u> $\displaystyle\int_{\mathbb{R}^3} \langle \psi(x) \mid \gamma^0 \phi(x)\rangle(t, \vec{x})\, d^3x$ is conserved in time

<u>Proof</u>: $$\frac{d}{dt}\int_{\mathbb{R}} j^0\, d^3x = \int_{\mathbb{R}} \partial_0 j^0\, d^3x$$

Gauß divergence theorem

$$\overset{\downarrow}{=} \int_{\mathbb{R}} (\partial_0 j^0 + \partial_\alpha j^\alpha)\, d^3x$$

$$= \int_{\mathbb{R}} \partial_k j^k\, d^3x = 0. \qquad \Box$$

Note: $$\langle \psi \mid \gamma^0 \phi\rangle = \overline{\psi}\gamma^0\phi$$
$$= (\psi^\dagger\gamma^0)\gamma^0\phi = \psi^\dagger\phi$$
is positive definite

We thus obtain a conserved scalar product

$$(\psi \mid \phi)_t := \int_{\mathbb{R}^3} \langle \psi \mid \gamma^0 \phi\rangle(t, \vec{x})\, d^3x$$

$$(\psi \mid \psi)_t = \int_{\mathbb{R}^3} \underbrace{\langle \psi \mid \gamma^0 \psi\rangle(t, \vec{x})}_{\text{probability density}}\, d^3x$$

$$(i\gamma^j \partial_j + \cancel{A} - m)\psi = 0$$

$$\Leftrightarrow \quad i\partial_t \psi = H\psi$$

$$H = -i\gamma^0 (\gamma^\alpha \partial_\alpha) - \gamma^0 \cancel{A} + \gamma^0 m$$

Dirac Hamiltonian

symmetric on $L^2(\mathbb{R}^3, \mathbb{C}^4)$

Lorentz transformations

$$x^i \longrightarrow \tilde{x}^j = \Lambda^j{}_k \, x^k$$

$$\psi \longrightarrow \tilde{\psi} = U\psi$$

U is unitary with respect to $\langle . | . \rangle_x$

$$\langle U\psi | U\phi \rangle_x = \langle \psi | \phi \rangle_x \qquad \forall \psi, \phi \in \mathbb{C}^4$$