<u>Def</u>: Let V be a complex vector space.

A <u>scalar product</u> $\langle .|. \rangle$ is a mapping

$$\langle .|. \rangle : \quad V \times V \to \mathbb{C}$$

with the following properties: $\qquad u, v, w \in V$, $\alpha, \beta \in \mathbb{C}$

(i) linear in the second argument, i.e.

$$\langle u | \alpha v + \beta w \rangle = \alpha \langle u|v \rangle + \beta \langle u|w \rangle$$

(ii) $\overline{\langle u|v \rangle} = \langle v|u \rangle$.

(iii) positive definite:

$$\langle u|u \rangle \geq 0 \quad \text{and} \quad \langle u|u \rangle = 0 \Leftrightarrow u = 0$$

$(V, \langle .|. \rangle)$ is a <u>scalar product space.</u>

norm $\|.\|$, $\qquad \|u\| := \sqrt{\langle u|u \rangle}$

metric $d(u,v) := \|u - v\|$

completeness:

<u>Def</u>: $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence if $\forall \varepsilon > 0 \; \exists N$ s.t.

$$d(u_n, u_m) < \varepsilon \quad \forall n, m > N$$

$u_n \to u$ if $\forall \varepsilon > 0 \; \exists N$ s.t.

$$d(u_n, u) < \varepsilon \quad \forall n > N$$

V is called <u>complete</u> if every Cauchy sequence has a limit in V.

A complete scalar product space is called <u>Hilbert space.</u>

Def: Let $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ be a Hilbert space.
A mapping $A: \mathcal{H} \to \mathcal{H}$ is a
<u>bounded linear operator</u> if
(i) linear: $A(\alpha u + \beta v) = \alpha A(u) + \beta A(v)$
(ii) bounded: $\exists\, c > 0$ s.t.
$$\|Au\| \leq c\,\|u\| \qquad \forall u \in \mathcal{H}$$
The bounded linear operators form a complex vector space
$$(\alpha A + \beta B)(u) := \alpha A(u) + \beta B(u),$$
denoted by $L(\mathcal{H})$.

Remark On $L(\mathcal{H})$ one has a norm
$$\|A\| := \sup_{\substack{u \in \mathcal{H},\\ \|u\|=1}} \|Au\|$$
operator norm
sup norm

and $(L(\mathcal{H}), \|.\|)$ is a complete normed space
(also called Banach space).

Let $A \in L(\mathcal{H})$.

Def: A is symmetric if
$$\langle Au|v\rangle = \langle u|Av\rangle \qquad \forall u, v \in \mathcal{H}$$

Remark: $A \in L(\mathcal{H}) \Rightarrow A^* \in L(\mathcal{H})$ the adjoint is defined by
$$\langle A^* u|v\rangle = \langle u|Av\rangle \qquad \forall u, v \in \mathcal{H}$$

Thus A symmetric $\Longleftrightarrow$ $A = A^*$,
A is selfadjoint

Def: $A \in L(\mathcal{H})$ has <u>finite rank</u> if
$A(\mathcal{H})$ is finite-dimensional.

Let $A \in L(\mathcal{H})$ be symmetric and of finite rank.

$J = A(\mathcal{H}) \subset \mathcal{H}$ finite-dimensional

$J^{\perp} := \{ u \in \mathcal{H} \mid \langle u|v\rangle = 0 \ \ \forall v \in J \}$ orthogonal complement

Then: - $(J^{\perp}, \langle \cdot|\cdot \rangle|_{J^{\perp}\times J^{\perp}})$ is again a Hilbert space

- Any $u \in \mathcal{H}$ can be decomposed uniquely as

$$u \in \underset{\in J}{u^{\parallel}} + \underset{\in J^{\perp}}{u^{\perp}}$$

proof: $u^{\parallel} = \sum_{i=1}^{N} \langle e_i | u \rangle e_i$

where $(e_i)_{i=1,\dots,N}$ is an orthonormal basis of J.

<u>Lemma</u> $A|_{J^{\perp}} = 0$

<u>Proof</u>: Let $u \in J^{\perp}$, $v \in \mathcal{H}$. Then

$$\langle v | A u \rangle \underset{A \text{ symmetric}}{=} \langle \underbrace{A v}_{\in J} | \underbrace{u}_{\in J^{\perp}} \rangle = 0$$

Choose $v = Au$

$\Rightarrow 0 = \langle Au | Au \rangle \Longrightarrow Au = 0$. $\square$

$A|_J : J \longrightarrow J$ symmetric operator on J

can be diagonalized (linear algebra)

$$A = \begin{array}{cc} \begin{array}{cc} J & J^{\perp} \end{array} & \\ \left(\begin{array}{c|c} A|_J & 0 \\ \hline 0 & 0 \end{array}\right) & \begin{array}{c} J \\ J^{\perp} \end{array} \end{array}$$