$$\int_{-\infty}^{\infty} f(x)\, dx$$

$U \subset \mathbb{R}$, $\quad \chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} \chi_U(x)\, dx = \mu(U) \qquad \text{measure of } U$$

μ can be defined only on certain subsets of $\mathbb{R}$, the so-called measurable sets

Work with at most countable set operations

<u>Def</u>: Let E be a set.

$\mathcal{M} \subset \mathcal{P}(E)$ is a <u>σ-algebra</u> if

(i) $\emptyset \in \mathcal{M}$

(ii) $A \in \mathcal{M} \Rightarrow CA \in \mathcal{M}$

(iii) $(A_n)_{n \in \mathbb{N}}$, $A_n \in \mathcal{M}$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}.$$

The elements of $\mathcal{M}$ are called <u>measurable sets</u>

<u>Def</u>: A <u>measure</u> μ is a mapping

$$\mu: \mathcal{M} \longrightarrow \mathbb{R}_0^+ \cup \{\infty\}$$

with the following properties:

(i) $\mu(\emptyset) = 0$

(ii) σ-additivity: $(A_n)_{n \in \mathbb{N}}$, $A_n \in \mathcal{M}$ and pairwise disjoint

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Def: A function $f: E \to \mathbb{R}$ is called <u>measurable</u> if $f^{-1}(U) \in \mathcal{M}$ $\forall U \subset \mathbb{R}$ open

For measurable function,

$$\int_E |f(x)|\, d\mu(x) \in \mathbb{R}_0^+ \cup \{\infty\} \quad \text{is well-defined.}$$

If this integral is finite, also

$$\int_E f(x)\, d\mu(x) \in \mathbb{R} \qquad \text{is well-defined}$$

Such functions are called <u>integrable</u>

$L^1(E, d\mu)$ vector space of all integrable functions

$$\|f\|_{L^1} := \int_E |f(x)|\, d\mu(x)$$

$$L^2(E, d\mu) := \left\{ f \text{ measurable},\ \int_E |f(x)|^2 d\mu(x) < \infty \right\}$$

$$\langle f, g \rangle_{L^2} := \int_E \overline{f(x)}\, g(x)\, d\mu(x)$$

Let $(E, \mathcal{O})$ be a topological space.

The <u>Borel algebra</u> $\mathcal{B}$ is the smallest σ-algebra with $\mathcal{O} \subset \mathcal{B}$.

$\mathcal{B} = \{$ sets which can be generated from $\mathcal{O}$ with at most countable set operations $\}$

The elements of $\mathcal{B}$ are the <u>Borel sets</u>

A measure $\mu: \mathcal{B} \to \mathbb{R}_0^+ \cup \{\infty\}$ is a <u>Borel measure</u>.