Asymptotic equivalence of two strict deformation quantizations and applications to the classical limit.

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Research interests

Transition between quantum and classical theories

- Quantum mechanics $\to$ classical mechanics.
- Statistical mechanics of a quantum spin system $\to$ classical thermomechanics of a spin system.
- Statistical mechanics of a quantum spin system on a finite lattice $\to$ statistical mechanics of an infinite quantum spin system.

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Main focus

- **Classical limit** of quantum (spin) systems & Schrödinger operators.
  ▷ Prof. Valter Moretti (University of Trento)
  ▷ Dr. Simone Murro (University of Paris-Saclay)
- **Emergence**, e.g. Spontaneous Symmetry Breaking (SSB).
  ▷ Prof. Klaas Landsman (Radboud University Nijmegen)
  ▷ Dr. Robin Reuvers (University of Rome 3)

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Approach

Strict deformation quantization

• Ingredients: sequence of C^*-algebras $(A_\hbar)_{\hbar\in I}$ over locally compact Hausdorff space I, $A_0 = C_0(X)$ where X a smooth Poisson manifold (possibly with boundary).

• Consider class of elements $a := \{a_0, a_\hbar\}_\hbar \in \Pi_{\hbar\in I} A_\hbar$ that is closed w.r.t. pointwise sums, products, the adjoint, and such that

$$||a|| := \sup_{\hbar\in I}\{||a_\hbar||_\hbar\} < \infty, \quad (1)$$
$$||aa^*|| = ||a||^2. \quad (2)$$

• By construction the set

$$A = \left\{ a = \{a_0, a_\hbar\}_\hbar \;\middle|\; \text{all conditions above are satisfied} \right\}, \quad (3)$$

is a C^*- algebra with norm (1).

• A continuous bundle of C^*-algebras over I consists of a C^*- algebra A (constructed by (3)), a collection of C^*-algebras $(A_\hbar)_{\hbar\in I}$ and surjective homomorphisms $\phi_\hbar : A \to A_\hbar$, such that $A \ni a := \{a_0, a_\hbar\}_\hbar$ satisfies

$$\phi_\hbar(a) = a_\hbar. \quad (4)$$

• Moreover, we require that for any $f \in C_0(I)$ one has $\{f(\hbar)a_\hbar\}_\hbar \in A$.

• We furthermore demand the continuity property for the norm, in that for each $a \in A$ one has

$$I \ni \hbar \mapsto ||a_\hbar||_\hbar \in C_0(I), \quad (5)$$

• If all these conditions are satisfied, the continuous cross-sections are then maps $I \ni \hbar \mapsto a_\hbar \in A_\hbar$, i.e., elements of A.

Definition (Strict deformation quantization of a Poisson manifold X)

- Continuous bundle of C^*-algebras $(A_\hbar)_{\hbar \in I}$ over I with $A_0 = C_0(X)$;

- A dense Poisson subalgebra $\tilde{A}_0 \subset C^\infty(X) \subset A_0$ (on which $\{\cdot, \cdot\}$ is defined);

- Quantization maps $Q_\hbar : \tilde{A}_0 \to A_\hbar$ such that Q_0 is the inclusion map $\tilde{A}_0 \to A_0$, each $Q_\hbar$ is linear, and the next conditions $(1) - (4)$ hold:

1. $Q_\hbar(1_X) = 1_{A_\hbar}$ (if unital) .
2. $Q_\hbar(f^*) = Q_\hbar(f)^*$.
3. For each $f \in \tilde{A}_0$ the following map

$$0 \mapsto f;$$
$$\hbar \mapsto Q_\hbar(f), \quad (\hbar > 0)$$

is a continuous section of the bundle.

4. For all $f, g \in \tilde{A}_0$ one has the Dirac-Groenewold-Rieffel condition:

$$\lim_{\hbar \to 0} || \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\})||_\hbar = 0.$$

- Consider

$$A_0 = C_0(\mathbb{R}^{2n}) \ (\hbar = 0);$$
$$A_\hbar = B_\infty(L^2(\mathbb{R}^n)) \ (\hbar > 0),$$

where $\mathbb{R}^{2n}$ is equipped with thet standard symplectic Poisson structure.

$\Rightarrow$ A_0 and $A_\hbar$ form fibers of a continuous bundle of C^*- algebras over $I = [0,1]$.

- Quantization maps: for any $\hbar \in (0,1]$ define

$$Q_\hbar : C_c^\infty(\mathbb{R}^{2n}) \to B_\infty(L^2(\mathbb{R}^n));$$
$$Q_\hbar(f) = \int_{\mathbb{R}^{2n}} \frac{d^n p d^n q}{(2\pi\hbar)^n} f(p,q) |\phi_\hbar^{(p,q)}\rangle\langle\phi_\hbar^{(p,q)}|,$$

where for each $\hbar \in I$ the operator $|\phi_\hbar^{(p,q)}\rangle\langle\phi_\hbar^{(p,q)}|$ is the projection onto the subspace spanned by the unit vector $\phi_\hbar^{(p,q)} \in L^2(\mathbb{R}^n)$, also called a Schrödinger coherent state.

- Consider

$$A'_0 = C(S^2),\ (1/N = 0);$$
$$A'_{1/N} = M_{N+1}(\mathbb{C}),\ (1/N > 0).$$

$\Rightarrow A'_0$ and $A'_{1/N}$ form fibers of a continuous bundle of C^*- algebras over $I = 1/\mathbb{N} \cup \{0\}$.

- Poisson structure: $\{f, g\}(x) = \sum_{a,b,c=1}^{3} \epsilon_{abc} x_c \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}$ $(x \in S^2))$, with f, g restrictions of smooth functions to $S^2 \rightarrow$ dense subspace $\tilde{A}'_0 \subset A'_0$ made of polynomials in three real variables restricted to S^2.

- Quantizations maps: for any $1/N \in 1/\mathbb{N}$:

$$Q'_{1/N} : \tilde{A}'_0 \rightarrow M_{N+1}(\mathbb{C});$$
$$Q'_{1/N}(p) = \frac{N+1}{4\pi} \int_{S^2} d\mu(\Omega) p(\Omega) |\Omega_N\rangle\langle\Omega_N|.$$

$|\Omega_N\rangle\langle\Omega_N|$ is the projection onto the linear span of the vector Ω_N, called a spin coherent state.

• Consider

$$A_0 = C(S(M_2(\mathbb{C}))) \simeq C(B^3),\ (1/N = 0);$$
$$A_{1/N} = \bigotimes_{n=1}^{N} M_2(\mathbb{C}),\ (1/N > 0).$$

$\Rightarrow$ A_0 and $A_{1/N}$ are the fibers of a continuous bundle of C^*- algebras over $I = 1/\mathbb{N} \cup \{0\}$.

• Poisson structure on $S(M_2(\mathbb{C})) \simeq B^3$: $\{f, g\}(x) = \sum_{a,b,c=1}^{3} \epsilon_{abc} x_c \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}$ $(x \in B^3)$, with f, g restrictions of smooth functions to B^3.

• It can be shown that the continuous cross-sections of the bundle with fibers $(A_0, A_{1/N})$ are precisely given by the quasi-symmetric sequences which uniquely identify this bundle (Landsman, 2017).

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Quantizations maps must be defined by (quasi)-symmetric sequences.

• Quasi-symmetric seqeunces $\leftrightarrow$ macroscopic observables. These can start in any finite way, but their infinite tails consist of averaged observables, and therefore they asymptotically commute.

• Symmetrization operator $S_N : A_{1/N} \to A_{1/N}$, defined as the unique linear continuous extension of the following map on elementary tensors:

$$S_N(a_1 \otimes \cdots \otimes a_N) = \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(N)}. \tag{6}$$

• For $N \geq M$ define a bounded operator $S_{M,N} : A_{1/M} \to A_{1/N}$, by linear and continuous extension of

$$S_{M,N}(b) = S_N(b \otimes \underbrace{I \otimes \cdots \otimes I}_{N-M \text{times}}), \quad b \in A_{1/M}. \tag{7}$$

• Sequences $A \ni a = (a_0, a_{1/N})_{N \in \mathbb{N}}$ are called symmetric if there exist $M \in \mathbb{N}$ and $a_{1/M} \in A_{1/M}$ such that

$$a_{1/N} = S_{M,N}(a_{1/M}) \text{ for all } N \geq M, \tag{8}$$

• They are called quasi-symmetric if $a_{1/N} = S_N(a_{1/N})$ if $N \in \mathbb{N}$, and for every $\epsilon > 0$, there is a symmetric sequence $(b_{1/N})_{N \in \mathbb{N}}$ as well as $M \in \mathbb{N}$ such that

$$\|a_{1/N} - b_{1/N}\| < \epsilon \text{ for all } N > M. \tag{9}$$

• Subspace $Z \subset \bigoplus_{M=0}^{\infty} M_2(\mathbb{C})^{\otimes M}$ made of symmetric tensor products $\rightarrow$ map $\chi : Z \rightarrow C(S(M_2(\mathbb{C})))$ defined by linear extension of the map

$$\chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L})(\omega) = \omega^N(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L}) = \omega(b_{j_1}) \cdots \omega(b_{j_L}),$$

where ib_1, ib_2, ib_3 form a basis of the Lie algebra of $SU(2)$, where $\omega \in S(M_2(\mathbb{C}))$ and $\omega(b_{j_i}) = x_{j_i}$ $(j_1, \ldots, j_L \in \{1, 2, 3\})$.

• χ is a well-defined linear injective map $\rightarrow$ $\chi(Z) \subset C(S(M_2(\mathbb{C})))$ is dense, and elements of $\chi(Z)$ are polynomials.

• Hence, each polynomial p of degree L uniquely corresponds to a polynomial of symmetric elementary tensors of the form $b_{j_1} \otimes_s \cdots \otimes_s b_{j_L}$.

• Define $\tilde{A}_0 := \chi(Z)$, and for $p_L = \chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L})$ the quantization maps $Q_{1/N} : \tilde{A}_0 \subset C(B^3) \rightarrow M_2(\mathbb{C})^{\otimes N}$ are defined as the unique continuous and linear extensions of the maps

$$Q_{1/N}(p_L) = \begin{cases} S_{L,N}(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L}), & \text{if } N \geq L, \\ 0, & \text{if } N < L, \end{cases}$$
$$Q_{1/N}(1) = \underbrace{I_2 \otimes \cdots \otimes I_2}_{N\ times}. \tag{10}$$

• Note that the quantization maps indeed define symmetric (hence macroscopic) observables. No coherent states involved!

• Existence of invariant $(N+1)$-dimensional symmetric subspace $\text{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2$ for operators $Q_{1/N}(p) \in \bigotimes_{n=1}^N M_2(\mathbb{C})$.

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$$Q_{1/N}(p)|_{\text{Sym}^N(\mathbb{C}^2)} \in B(\text{Sym}^N(\mathbb{C}^2)) \simeq M_{N+1}(\mathbb{C}).$$

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Theorem (M, v.d. V, 2020)

For any polynomial $p \in \tilde{A}_0$ (the complex vector space of polynomials in three real variables on the closed unit ball $S(M_2(\mathbb{C})) \cong B^3$), one has

$$||Q'_{1/N}(p|_{S^2}) - Q_{1/N}(p)|_{Sym^N(\mathbb{C}^2)}||_N \to 0, \text{ as } N \to \infty, \tag{11}$$

the (operator) norm being the one on $B(Sym^N(\mathbb{C}^2))$.

• Consider collection of N two-level atoms corresponding to a spin chain of N sites described by a mean-field Hamiltonian H_N.

• Example: quantum Curie-Weiss spin Hamiltonian defined on $\mathcal{H}_N = \bigotimes_{n=1}^N \mathbb{C}^2$:

$$H_N \equiv H_N^{CW} = -\frac{J}{2N} \sum_{i,j=1}^N \sigma_3(i)\sigma_3(j) - B \sum_{i=1}^N \sigma_1(i), \tag{12}$$

with B magnetic field and J a coupling constant .

• H_N typically leaves the subspace $\mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2$ invariant.

• $(H_N)_N$ defines a quasi-symmetric sequence $\Rightarrow$ relation with SDQ of $S(M_2(\mathbb{C})) \simeq B^3$:

$$\lim_{N\to\infty} ||H_N - Q_{1/N}(h)||_N = 0, \tag{13}$$

for some polynomial $h \in C(B^3)$ (called the classical CW model).

• By the theorem $\lim_{N\to\infty} ||H_N|_{\mathrm{Sym}^N(\mathbb{C}^2)} - Q'_{1/N}(h|_{S^2})||_N = 0$, $\rightarrow$ the restricted mean-field spin system is represented by quantization of the Bloch sphere in the semiclassical limit $1/\hbar := N \to \infty$.

Study: classical limit of quantum theories and SSB

• Study: asymptotic properties of vectors in Hilbert space $\mathcal{H}_\hbar$; e.g. think of eigenvectors $(\psi_\hbar)_\hbar$ of quantum operators $(H_\hbar)_\hbar$, as $\hbar \to 0$.

Difficulty: behaviour of $(\psi_\hbar)_\hbar$ in $\mathcal{H}_\hbar$ is hard to capture

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Algebraic approach helpful

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Algebraic vector states $\omega_\hbar(\cdot) := \langle \psi_\hbar, (\cdot)\psi_\hbar \rangle$.

• Question: *Which set of physical observables makes the sequence* $(\omega_\hbar)$ *'converge' as* $\hbar \to 0$*?*

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Strict deformation quantization

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Observables defined by quantization maps $Q_\hbar(f), (f \in C_0(X))$.

• Existence of **classical limit**, does

$$\omega_0(f) := \lim_{\hbar \to 0} \omega_\hbar(Q_\hbar(f)), \quad (f \in C_0(X)); \tag{14}$$

exists as a **state** ω_0 on $A_0 = C_0(X)$?

• SSB: natural **emergent** phenomenon typically occuring in thermodynamic/classical limit.

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Difficulty: proving existence of SSB in such limits

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Strict deformation quantization

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Existence of classical limit

Rigorous notion of SSB in the classical limit: pure ground states are not invariant, whilst invariant ground states are not pure.

- 1-dimensional Schrodinger operator $h_\hbar = -\hbar^2 \frac{d^2}{dx^2} + V(x)$, with V a symmetric double well potential, $h_\hbar \psi_\hbar^{(0)} = \lambda_\hbar^{(0)} \psi_\hbar^{(0)}$ where $\lambda_\hbar^{(0)}$ minimal.
- One can show that the Berezin quantization on $\mathbb{R}^2$ induces the existence of the classical limit on $C_0(\mathbb{R}^2)$:

$$\lim_{\hbar \to 0} \langle \psi_\hbar^{(0)}, Q_\hbar(f)\psi_\hbar^{(0)} \rangle = \frac{1}{2}(\omega_+^{(0)}(f) + \omega_-^{(0)}(f)). \tag{15}$$

where $\omega_\pm^{(0)}$ are Dirac measures localized in the minima of both wells (Lansdman 2017).

Theorem (L, M, v.d.V)

Let H_N^{CW} be the Curie-Weiss quantum spin model defined on a chain of N sites. Then the sequence of unique (up to phase) ground state eigenvectors $(\psi_N^{(0)})_N$ admits a classical limit on $X_2 := S(M_2(\mathbb{C})) \cong B^3$, in the sense that

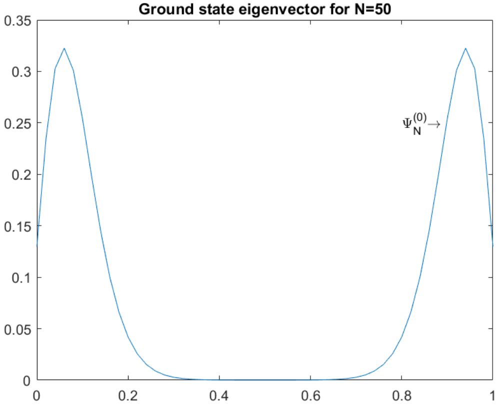
$$\lim_{N \to \infty} \langle \psi_N^{(0)}, Q_{1/N}(f)\psi_N^{(0)} \rangle = \frac{1}{2}(f(\Omega_-) + f(\Omega_+)), \quad (f \in C_0(X_2)); \tag{16}$$

where $\Omega_\pm$ denote the minima of the classical CW Hamiltonian h^{CW} on B^3:

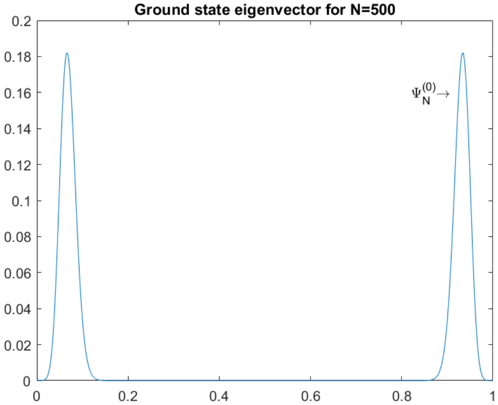
$$h^{CW}(x, y, z) = -(\frac{J}{2}z^2 + Bx), \quad ((x, y, z) \in B^3). \tag{17}$$

- Proof Idea: **Localization** of eigenvectors $\psi_N^{(0)}$ of H_N^{CW} ($N \to \infty$) only depends on properties of h^{CW}.

• Existence of spontaneous symmetry breaking (SSB) in the classcial limit: pure ground states are not invariant, whilst invariant ground states are not pure.



• The (pure) ground state eigenvector $\Psi_N^{(0)}$ of the quantum Curie-Weiss model is invariant under $\mathbb{Z}_2$- reflexion symmetry for any N.



- In the limit $N \to \infty$ the ground state eigenvector $\Psi_N^{(0)}$ 'decomposes' into two parts corresponding to the invariant (but not pure) state $\frac{1}{2}(\omega_+^{(0)}(f) + \omega_-^{(0)}(f))$.

Some results: classical limit and SSB

Theorem (v.d.V)

If $(\lambda_N^{(i)})_N$ *is a sequence of eigenvalues corresponding to a mean-field quantum spin Hamiltonian* H_N *such that* $\lambda_N^{(i)}$ *converges to some energy* E*, as* $N \to \infty$*. Then the corresponding sequence of eigenvectors* $\psi_N^{(i)}$ *of* H_N *admits a classical limit, in that*

$$\lim_{N\to\infty} \langle \psi_N^{(i)}, Q'_{1/N}(f)\psi_N^{(i)} \rangle = \frac{1}{n}\sum_{i=1}^{n} f(\Omega_i), \quad (f \in C(S^2)); \tag{18}$$

where Ω_i *are distinct points in* $h_0^{-1}(E) \subset S^2$ *and* h_0 *is the 'classical' analog of the operator* H_N*, i.e. a polynomial on* S^2*.*

- Joint work with Murro: **'Injective tensor products in strict deformation quantization'.**

  $\to$ Natural frame work for many-body quantum systems.

  $\to$ Application to quantum spin systems with nearest neighbor interactions.

  $\to$ Application to Schrödinger operators affiliated with the resolvent algebra.

- Joint work with Landsman, Groenenboom, Reuvers: **'Quantum spin systems versus Schroedinger operators: A case study in spontaneous symmetry breaking (Scipost, 2019)'.**

  $\to$ Spontaneous symmetry breaking: small perturbations of quantum system should yield a pure state for finite but large N $\to$ explanation for symmetry breaking in real materials: only pure states (i.e. physical states) are found. (Generalization of work by Barry Simon, Jona-Lasinio, Martinelli and Scoppola)

Research in progress

◇ Which states admit a classical limit? (Think e.g. of pure (vector) states, local Gibbs states, β-KMS states).

◇ Generalize methods to more complicated many-body quantum systems and prove existence of SSB in the classical/thermodynamic limit:

- Spin systems with nearest neighbor interactions, e.g. Heisenberg model.
- Schrödinger operators and potentials with continuous symmetry, e.g. $SO(2) \to$ Publication in preparation [Moretti, v.d. Ven]

◇ Small perturbations in many body quantum systems $\to$ model symmetry breaking in real materials.

◇ Quantization of 'commutative' resolvent algebra $\to$ model unbounded operators (Buchholz & Grundling, 2008).

◇ SDQ $\to$ quantum versus classical dynamics.

◇ Not every classical theory is related to a underlying quantum theory. Only few pairs of a classical and a quantum C^*-algebra are known to connect in this way $\to$ open topic.

Thank you for your attention!

- Consider

$$A_0 = C_0(\mathbb{R}^{2n}) \ (\hbar = 0);$$
$$A_\hbar = B_\infty(L^2(\mathbb{R}^n)) \ (\hbar > 0),$$

$\Rightarrow$ fibers of a continuous bundle of C^*- algebras over $I = [0,1]$. Quantization maps $Q_\hbar$: for any $\hbar \in (0,1]$

$$Q_\hbar : C_c^\infty(\mathbb{R}^{2n}) \to B_\infty(L^2(\mathbb{R}^n));$$
$$Q_\hbar(f) = \int_{\mathbb{R}^{2n}} \frac{d^n p d^n q}{(2\pi\hbar)^n} f(p,q) |\phi_\hbar^{(p,q)}\rangle\langle\phi_\hbar^{(p,q)}|.$$

- Consider

$$A_0' = C(S^2), \ (1/N = 0);$$
$$A_{1/N}' = M_{N+1}(\mathbb{C}), \ (1/N > 0),$$

$\Rightarrow$ fibers of a continuous bundle of C^*- algebras over $I = 1/\mathbb{N} \cup \{0\}$. Quantizations maps $Q_{1/N}$: for any $1/N \in 1/\mathbb{N}$

$$Q_{1/N}' : \tilde{A}_0' \to M_{N+1}(\mathbb{C});$$
$$Q_{1/N}'(p) = \frac{N+1}{4\pi} \int_{S^2} d\mu(\Omega) p(\Omega) |\Omega_N\rangle\langle\Omega_N|.$$

• Ingredients: sequence of C^*-algebras $(A_\hbar)_{\hbar\in I}$ over locally compact Hausdorff space I, $A_0 = C_0(X)$ where X a smooth Poisson manifold (possibly with boundary).

• Consider class of elements $a := \{a_0, a_\hbar\}_\hbar$ that is closed w.r.t. pointwise sums, products, the adjoint, and such that

$$||a|| := \sup_{\hbar\in I}\{||a_\hbar||_\hbar\} < \infty, \tag{19}$$

$$||aa^*|| = ||a||^2. \tag{20}$$

• By construction the set

$$A = \left\{a = \{a_0, a_\hbar\}_\hbar \;\middle|\; \text{all conditions above are satisfied}\right\}, \tag{21}$$

is a C^*- algebra with norm (19).

• A continuous bundle of C^*-algebras over I consists of a C^*- algebra A (constructed by (21)), a collection of C^*-algebras $(A_\hbar)_{\hbar\in I}$ and surjective homomorphisms $\phi_\hbar : A \to A_\hbar$, such that $A \ni a := \{a_0, a_\hbar\}_\hbar$ satisfies

$$\phi_\hbar(a) = a_\hbar. \tag{22}$$

• Moreover, we require that for any $f \in C_0(I)$ one has $\{f(\hbar)a_\hbar\}_\hbar \in A$.

• We furthermore demand the continuity property for the norm, in that for each $a \in A$ one has

$$I \ni \hbar \mapsto ||a_\hbar||_\hbar \in C_0(I), \tag{23}$$

• If all these conditions are satisfied, the continuous cross-sections are then maps $I \ni \hbar \mapsto a_\hbar \in A_\hbar$, i.e., elements of A.

Poisson bracket

$$\{f, g\}(x) = \sum_{a,b,c=1}^{n} C_{a,b}^{c} x_c \frac{\partial f(x)}{\partial x_a} \frac{\partial g(x)}{\partial x_b},$$

with structure constants coming from the Lie- algebra of $SU(k)$.

• The non-degenerate states $(\psi_N^{(0)}, \psi_N^{(1)})$ converge (in algebraic sense) to mixed classical states, i.e.,

$$\lim_{N\to\infty} \psi_N^{(0)} = \lim_{N\to\infty} \psi_N^{(1)} = \omega_0^{(0)},$$

where $\omega_0^{(0)} = \frac{1}{2}(\omega_0^+ + \omega_0^-)$.

• In contrast, the localized pure ground states

$$\psi_N^{\pm} = \frac{1}{\sqrt{2}}(\psi_N^{(0)} + \psi_N^{(1)}),$$

converge (in algebraic sense) to pure classical states, i.e.,

$$\lim_{N\to\infty} \psi_N^{\pm} = \omega_0^{\pm}.$$

Continuous bundle of C^*-algebras

Definition

Let I be a locally compact Hausdorff space. A continuous bundle of C^*-algebras over I consists of a C^*-algebra A, a collection of C^*-algebras $(A_\hbar)_{\hbar \in I}$ with norms $||\cdot||_\hbar$, and surjective homomorphisms $\varphi_\hbar : A \to A_\hbar$ for each $\hbar \in I$, such that

1. The function $\hbar \mapsto ||\varphi_\hbar(a)||_\hbar$ is in $C_0(I)$ for all $a \in A$.

2.The norm for any $a \in A$ is given by

$$||a|| = \sup_{\hbar \in I} ||\varphi_\hbar(a)||_\hbar. \tag{24}$$

3. For any $f \in C_0(I)$ and $a \in A$, there is an element $fa \in A$ such that for each $\hbar \in I$,

$$\varphi_\hbar(fa) = f(\hbar)\varphi_\hbar(a). \tag{25}$$

• A continuous (cross-) section of the bundle in question is a map $\hbar \mapsto a(\hbar) \in A_\hbar$, ($\hbar \in I$), for which there exists an $a \in A$ such that $a(\hbar) = \varphi_\hbar(a)$ for each $\hbar \in I$.

Algebraic ground states and SSB

Definition

Let A be a C^*-algebra with time evolution, i.e., a continuous homomorphism $\alpha : \mathbb{R} \to \text{Aut}(A)$. A *ground state* of (A, α) is a state ω on A such that:

1. ω is time independent, i.e., $\omega(\alpha_t(a)) = \omega(a)\ \forall a \in A\ \forall t \in \mathbb{R}$.
2. The generator h_ω of the ensuing continuous unitary representation

$$t \mapsto u_t = e^{ith_\omega} \tag{26}$$

of $\mathbb{R}$ on $\mathcal{H}_\omega$ has positive spectrum, i.e., $\sigma(h_\omega) \subset \mathbb{R}_+$, or equivalently $\langle \psi, h_\omega \psi \rangle \geq 0$ $(\psi \in D(h_\omega))$.

• The set of ground states forms a compact convex subset of $S(A)$, and we denote this set by $S_0(A)$. We moreover assume that pure ground states are pure states as well as ground states.

Definition

Suppose we have a C^*-algebra A, a time evolution α, a group G, and a homomorphism $\gamma : G \to \text{Aut}(A)$, which is a symmetry of the dynamics α in that

$$\alpha_t \circ \gamma_g = \gamma_g \circ \alpha_t \quad (g \in G, t \in \mathbb{R}). \tag{27}$$

The G-symmetry is said to be spontaneously broken (at temperature $T = 0$) if

$$(\partial_e S_0(A))^G = \emptyset, \tag{28}$$

• Here $\mathscr{S}^G = \{\omega \in \mathscr{S} | \ \omega \circ \gamma_g = \omega \ \forall g \in G\}$, defined for any subset $\mathscr{S} \in S(A)$, is the set of G-invariant states in $\mathscr{S}$. (28) means that there are no G-invariant pure ground states. This means also that if spontaneous symmetry breaking occurs, then invariant ground states are not pure.