Quantum states of causal fermion systems and collapse

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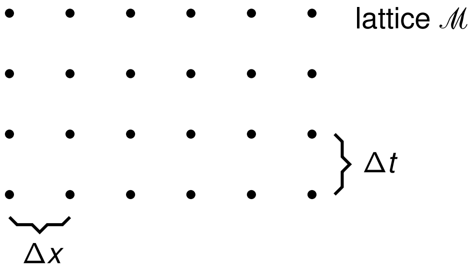
Motivation

- Planck scale gives natural length scale for “new physics”
- Physical equations become inconsistent
  - ultraviolet divergences of QFT
  - quantum fluctuations give rise to microscopic black holes, …, …
- Consider lattice system, for simplicity 2d



Motivation

Usual way to set up equations:

- Replace derivatives by difference quotients

$$0 = \Box\phi(t,x) := \frac{1}{(\Delta t)^2}\Big(\phi(t+\Delta t,x) - 2\phi(t,x) + \phi(t-\Delta t,x)\Big) - \frac{1}{(\Delta x)^2}\Big(\phi(t,x+\Delta x) - 2\phi(t,x) + \phi(t,x-\Delta x)\Big)$$

- Gives evolution equation, proceed time step by time step

Motivation

Drawback of this approach:

- Ad hoc: Why square lattice, why difference quotients?
- Is not background-free: What is lattice spacing?
- Not invariant under general coordinate transformations, not compatible with the equivalence principle

Basic question: Can one formulate equations without referring to the nearest neighbor relation and lattice spacing?

Motivation

- Consider wave functions $\psi_1, \ldots, \psi_f$ on lattice ($f < \infty$)
- Introduce scalar product; orthonormalize,

$$\langle \psi_k | \psi_l \rangle = \delta_{kl} \,,$$

gives f-dim Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.

important object: for any lattice point (t, x) introduce

local correlation operator $F(t,x) : \mathcal{H} \to \mathcal{H}$

- define matrix elements by

$$(F(t,x))^j_k = \overline{\psi_j(t,x)} \psi_k(t,x)$$

basis invariant:

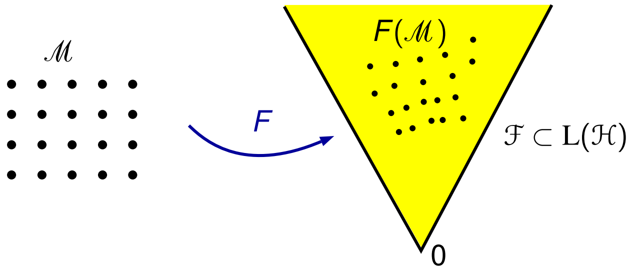
$$\langle \psi, F(t,x)\, \phi \rangle_{\mathcal{H}} = \overline{\psi(t,x)} \phi(t,x) \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

- Hermitian matrix
- Has rank at most one, is positive semi-definite

$$F(t,x) = e^* e \qquad \text{with} \qquad e : \mathcal{H} \to \mathbb{C} \,, \quad \psi \mapsto \psi(x)$$

Motivation

$$\mathcal{F} := \{F \text{ Hermitian, rank one, positive semi-definite}\}$$



general idea:

- disregard objects on the left
  (nearest neighbors, lattice spacing)
- work instead with the objects on the right
  (only local correlation operators)

Motivation

How to set up equations in this setting?
Explain idea in simple example:

- local correlation operators $F_1, \ldots, F_N \in \mathcal{F}$
- product $F_i \, F_j$ tells about correlation of wave functions at different spacetime points
- $\text{Tr}(F_i F_j)$ is real number
- minimize

$$\mathcal{S} = \sum_{i,j=1}^{N} \text{Tr}(F_i F_j)^2$$

under suitable constraints.

Causal fermion systems

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space
Given parameter $n \in \mathbb{N}$ ("spin dimension")
$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H})$ with the properties:

- x is self-adjoint and has finite rank
- x has at most n positive
  and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$ ("universal measure")

$$\sum_{i=1}^{N} \cdots \;\rightsquigarrow\; \int_{\mathcal{F}} \cdots \; d\rho$$

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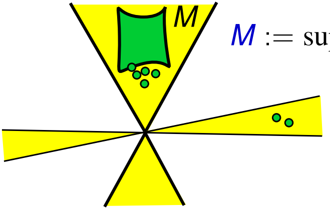


$M := \mathrm{supp}\, \rho$ spacetime

Causal action principle

Let $x, y \in \mathcal{F}$. Then x and y are linear operators.

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not self-adjoint: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Causal action principle

Nontrivial eigenvalues of xy: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

Lagrangian $\quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \;\geq\; 0$

action $\quad \mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y) \;\in\; [0,\infty]$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $\quad \rho(\mathcal{F}) = \text{const}$

trace constraint: $\quad \int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$

boundedness constraint: $\quad \iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$

- F.F., “Causal variational principles on measure spaces,” *J. Reine Angew. Math.* **646** (2010) 141–194

Example: Dirac spinors in Minkowski space

spacetime is Minkowski space, signature $(+ - - -)$

- free Dirac equation $\qquad (i\gamma^k \partial_k - m)\, \psi = 0$
- probability density $\psi^\dagger \psi = \overline{\psi} \gamma^0 \psi$,

  gives rise to a scalar product:

$$\langle \psi | \phi \rangle = \int_{t=\text{const}} (\overline{\psi} \gamma^0 \phi)(t, \vec{x})\, d\vec{x}$$

  time independent due to current conservation

Example: Dirac spinors in Minkowski space

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\text{span}(\psi_1, \ldots, \psi_f)}$$

For simplicity in presentation assume: ψ_i continuous.

- To $x \in \mathbb{R}^4$ associate a local correlation operator

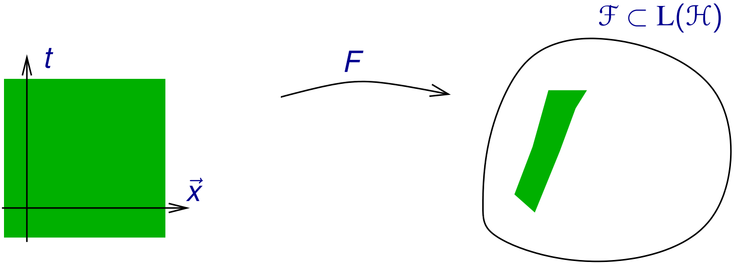
$$\langle \psi | F(x) \phi \rangle = -\overline{\psi(x)} \phi(x) \qquad \forall \psi, \phi \in \mathcal{H}$$

Is self-adjoint, rank ≤ 4,
at most two positive and at most two negative eigenvalues

- Thus $F(x) \in \mathcal{F}$ where
$\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H})$ with the properties:
  - F is self-adjoint and has rank ≤ 4
  - F has at most 2 positive and at most 2 negative eigenvalues $\Big\}$

Example: Dirac spinors in Minkowski space

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$

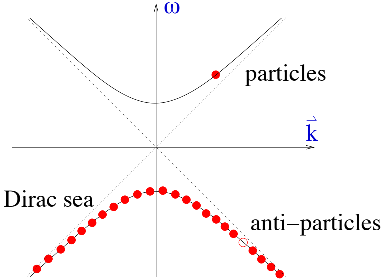


- push-forward measure $d\rho := F_*(d^4x)$, is measure on $\mathcal{F}$.

Example: the Minkowski vacuum

Specify vacuum:

- Choose $\mathcal{H}$ as the space of all negative-energy solutions, hence "Dirac sea"



Fixes length scale ("Compton length")

- Introduce ultraviolet regularization

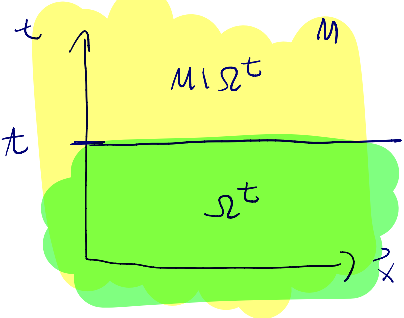
Fixes length scale ε ("Planck length")

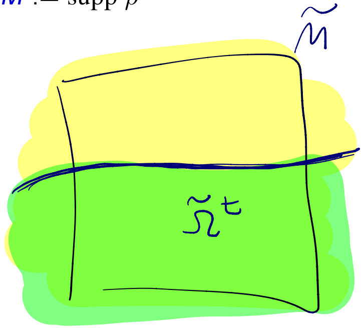
This is a minimizer of the causal action (in a well-defined sense).

- Two minimizing causal fermion systems
  - $(\mathcal{H}, \mathcal{F}, \rho)$ describing vacuum
  - $(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\rho})$ describing the interacting spacetime
  - corresponding spacetimes:

$$M := \text{supp}\, \rho\,, \quad \tilde{M} := \text{supp}\, \tilde{\rho}$$

- Goal: Compare $\tilde{\rho}$ and ρ at time t.

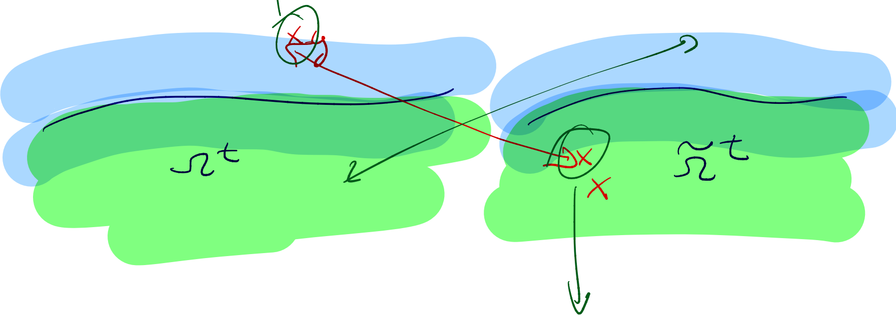




Nonlinear surface layer integrals

- Basic object: Nonlinear surface layer integral
  - identify Hilbert spaces by choosing $V : \mathcal{H} \to \tilde{\mathcal{H}}$ unitary

$$\gamma^t(\tilde{\rho}, \rho) := \int_{\tilde{\Omega}^t} d\tilde{\rho}(x) \int_{M \setminus \Omega^t} d\rho(y)\, \mathcal{L}(x,y) - \int_{\tilde{M} \setminus \tilde{\Omega}^t} d\tilde{\rho}(x) \int_{\Omega^t} d\rho(y)\, \mathcal{L}(x,y)$$



Freedom in identifying the Hilbert spaces

$V : \mathcal{H} \to \tilde{\mathcal{H}}$

$V \to \mathcal{U} V$

where $\mathcal{U} : \mathcal{H} \circlearrowleft$ unitary

- identification of Hilbert spaces:
  - Choose $V : \mathcal{H} \to \tilde{\mathcal{H}}$ unitary
  - Work exclusively in $\mathcal{H}$
  - But: identification is not canonical, gives freedom

$$\rho \to \mathcal{U}\rho\,, \quad (\mathcal{U}\rho)(\Omega) := \rho(\mathcal{U}^{-1}\Omega\,\mathcal{U})$$

- This freedom is treated by integrating over $\mathcal{U}$
  - Let $\mathcal{G} \subset \mathrm{U}(\mathcal{H})$ be compact subgroup
  - $\mu_{\mathcal{G}}$ normalized Haar measure on $\mathcal{G}$

The partition function

- symmetrized nonlinear surface layer integral

$$\gamma^t(\tilde{\rho}, \mathcal{U}\rho) = \int_{\tilde{\Omega}^t} d\tilde{\rho}(x) \int_{M \setminus \Omega^t} d\rho(y)\, \mathcal{L}(x, \mathcal{U}y\mathcal{U}^{-1}) - \int_{\Omega^t} d\rho(x) \int_{\tilde{M} \setminus \tilde{\Omega}^t} d\tilde{\rho}(y)\, \mathcal{L}(\mathcal{U}x\mathcal{U}^{-1}, y)$$

$$\gamma^t(\tilde{\rho}, \rho) = \fint_{\mathcal{G}} \gamma^t(\tilde{\rho}, \mathcal{U}\rho)\, d\mu_{\mathcal{G}}(\mathcal{U})$$

can be arranged to vanish for all t (Greene-Shiohama)
- *partition function*

$$Z^t(\beta, \tilde{\rho}) = \fint_{\mathcal{G}} \exp\Big(\beta\, \gamma^t(\tilde{\rho}, \mathcal{U}\rho)\Big)\, d\mu_{\mathcal{G}}(\mathcal{U})$$

where β free parameter (maybe discuss at the end)

How to “test” the interacting spacetime?

- Interacting spacetime can be arbitrarily complicated (interacting quantum fields, entanglement, collapse)
- describe by objects in the vacuum spacetime: free fields, wave functions, ...
- use insertions:

$$\frac{1}{Z^t} \int_{\mathcal{G}} (\cdots) \exp\Big(\beta\, \gamma^t(\tilde{\rho}, \mathcal{U}\rho)\Big)\, d\mu_{\mathcal{G}}(\mathcal{U})$$

  - formal analogy to path integral formalism

Bosonic Fields in the Vacuum

- linearized field equations: For all $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$,

$$\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}^{\text{test}}$$

$$\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) := \nabla_{\mathfrak{u}} \bigg(\int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \big) \mathcal{L}_\kappa(x,y)\, d\rho(y) - \nabla_{\mathfrak{v}}\, \mathfrak{s} \bigg)$$

- surface layer integrals:

$$\sigma^t_\rho \;:\; \mathfrak{J}_{\rho,} \times \mathfrak{J}_{\rho,} \to \mathbb{R} \qquad \text{(symplectic form)}$$

$$\sigma^t_\rho(\mathfrak{u}, \mathfrak{v}) = \int_{\Omega^t} d\rho(x) \int_{M \setminus \Omega^t} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{2,\mathfrak{u}} \nabla_{1,\mathfrak{v}} \big) \mathcal{L}(x,y)$$

$$(.,.)^t_\rho \;:\; \mathfrak{J}_{\rho,} \times \mathfrak{J}_{\rho,} \to \mathbb{R} \qquad \text{(surface layer inner product)}$$

$$(\mathfrak{u}, \mathfrak{v})^t_\rho = \int_{\Omega^t} d\rho(x) \int_{M \setminus \Omega^t} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}} \nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{u}} \nabla_{2,\mathfrak{v}} \big) \mathcal{L}(x,y)$$

- assume non-interacting

Bosonic Fields in the Vacuum

- give rise to complex structure:

$$\sigma(u,v) = (u, \mathcal{T}\, v)$$

$$J := -(-\mathcal{T}^2)^{-\frac{1}{2}}\, \mathcal{T}\,, \qquad J^* = -J,\; J^2 = -\mathbb{1}$$

  Complexify and decompose:

$$\mathbf{v} = \mathbf{v}^{\mathrm{hol}} + \mathbf{v}^{\mathrm{ah}}$$

  On holomorphic jets introduce scalar product

$$(.|.)^t_\rho := \sigma^t_\rho(\,.\,, J\,.\,) \;:\; \Gamma^{\mathrm{hol}}_\rho \times \Gamma^{\mathrm{hol}}_\rho \to \mathbb{C}$$

  Completion gives complex Hilbert space $(\mathfrak{h}, (.|.)^t_\rho)$.
- Cauchy problem: Existence and uniqueness proven.

  F.F. and N. Kamran, *"Complex Structures on Jet Spaces and Bosonic Fock Space Dynamics for Causal Variational Principles,"* arXiv:1808.03177 [math-ph], to appear in Pure Appl. Math. Q. (2021)

  C. Dappiaggi and F.F., *"Linearized Fields for Causal Variational Principles: Existence Theory and Causal Structure,"* arXiv:1811.10587 [math-ph], Methods Appl. Anal. **27** 1–56 (2020)

Fermionic Fields in the Vacuum

- dynamical wave equation:

$$\int_M Q^{\mathrm{dyn}}(x,y)\,\psi(y) = 0$$

- scalar product defined as surface layer integral:

$$\langle\psi|\phi\rangle_\rho^t = -2i \bigg(\int_{\Omega^t} d\rho(x) \int_{M\setminus\Omega^t} d\rho(y) - \int_{M\setminus\Omega^t} d\rho(x) \int_{\Omega^t} d\rho(y) \bigg) \times \;\prec\!\psi(x)\,|\,Q^{\mathrm{dyn}}(x,y)\,\phi(y)\!\succ_x$$

  is conserved in time,
  gives *extended Hilbert space* $\mathcal{H}_\rho \supset \mathcal{H}$.

- Cauchy problem: Existence and uniqueness proven.

  F.F., N. Kamran and M. Oppio, *"The Linear Dynamics of Wave Functions in Causal Fermion Systems,"* arXiv:2101.08673 [math-ph]

Field Operators in the Vacuum

- Canonical commutation/anti-commutation relations for $z, z' \in \mathfrak{h}$ and $\psi, \psi' \in \mathcal{H}^{\mathrm{f}}_\rho \subset \mathcal{H}_\rho$

$$\begin{aligned} \big[a(\overline{z}), a^\dagger(z')\big] &= (z|z')^t_\rho \\ \big[a(\overline{z}), a(\overline{z'})\big] &= 0 = \big[a^\dagger(z), a^\dagger(z')\big] \\ \big\{\Psi(\overline{\phi}), \Psi^\dagger(\phi')\big\} &= \langle \phi|\phi'\rangle^t_\rho \\ \big\{\Psi(\overline{\phi}), \Psi(\overline{\phi'})\big\} &= 0 = \big\{\Psi^\dagger(\phi), \Psi^\dagger(\phi')\big\} \end{aligned}$$

  - independent of time
  - generate unital $*$-algebra $\mathscr{A}$

Construction of the Quantum State

- Quantum state ω^t at time t:

$$\omega^t : \mathscr{A} \to \mathbb{C} \qquad \text{linear and positive, i.e.}$$
$$\omega^t(A^*A) \geq 0 \quad \text{for all } A \in \mathscr{A}$$

- More concretely, represented on Fock space:
  - With a density operator:

$$\omega^t(A) = \text{tr}_{\mathcal{F}}\big(\sigma^t\, A\big)$$

  - As an expectation value (pure state):

$$\omega^t(A) = \langle \Psi | A | \Psi \rangle_{\mathcal{F}}$$

- General structure:

$$\omega^t(\cdots) := \frac{1}{Z^t} \int_{\mathfrak{G}} (\cdots)\, e^{\beta\, \gamma^t(\tilde{\rho}, \mathfrak{U}\rho)}\, d\mu_{\mathfrak{G}}(\mathfrak{U})$$

How do the insertions look like?

Bosonic insertions

- physical picture:

  "Measurement" in $\tilde{M}$ with objects in M, using the identification given by $\mathcal{U}$

- associate z to a linearized field $\tilde{z}$ in $\tilde{M}$:

$$\mathcal{P}_\rho : U \subset \mathfrak{J}^{\text{lin}}_\rho \to \mathcal{B} \qquad \text{perturbation map}$$

$$D\mathcal{P}_\rho|_w : \mathfrak{J}^{\text{lin}}_{\rho,} \to \mathfrak{J}^{\text{lin}}_{\tilde{\rho},}$$

$$\tilde{z} := D\mathcal{P}_\rho|_w \, z \,, \qquad \overline{\tilde{z}} := D\mathcal{P}_\rho|_w \, \overline{z}$$

- perturb nonlinear surface layer integral:

$$D_{\tilde{z}} \gamma^t(\tilde{\rho}, \mathcal{U}\rho) \,, \qquad D_{\overline{\tilde{z}}} \gamma^t(\tilde{\rho}, \mathcal{U}\rho)$$

Fermionic insertions

- Work with scalar product $\langle . | . \rangle^t_\rho$ in vacuum.
- Map wave functions from $\tilde{M}$ to M:

$$\psi = \pi_{\rho,\tilde{\rho}}\, \tilde{\psi}\,, \quad \psi(x) := \frac{1}{\tilde{\mathfrak{t}}(x)} \int_{\tilde{M}} \pi_x\, \mathfrak{U}^{-1}\, \tilde{\psi}(y)\, |xy|^2\, d\tilde{\rho}(y)$$

$$\tilde{\mathfrak{t}}(x) := \int_{\tilde{M}} |xy|^2\, d\tilde{\rho}(y)$$

- Gives subspace $\pi^t_{\rho,\tilde{\rho}}\, \mathcal{H} \subset \mathcal{H}_\rho$,

$$\pi^t_{\mathfrak{U}} : \mathcal{H}^\rho \to \pi^t_{\rho,\tilde{\rho}}\, \mathcal{H} \quad \text{orthogonal projection}$$

- one-particle measurement: $\langle \psi \,|\, \pi^t_{\mathfrak{U}}\, \phi \rangle^t_\rho$
- multi-particle measurement:

$$\frac{1}{p!} \sum_{\sigma, \sigma' \in S_p} (-1)^{\text{sign}(\sigma) + \text{sign}(\sigma')}$$

$$\times\, \langle \tilde{f}_{\sigma(1)} \,|\, \pi^t_{\mathfrak{U}}\, \tilde{f}_{\sigma'(1)} \rangle^t_\rho \cdots \langle \tilde{f}_{\sigma(p)} \,|\, \pi^t_{\mathfrak{U}}\, \tilde{f}_{\sigma'(p)} \rangle^t_\rho$$

Pauli exclusion principle arises

Definition of the state

DEFINITION

The state ω^t at time t is defined by

$$\omega^t\Big(a^\dagger(z'_1) \cdots a^\dagger(z'_p)\, \Psi^\dagger(\phi'_1) \cdots \Psi^\dagger(\phi'_{r'})$$

$$\times\; a(\overline{z_1}) \cdots a(\overline{z_q})\, \Psi(\overline{\phi_1}) \cdots \Psi(\overline{\phi_r}) \Big)$$

$$:= \frac{1}{Z^t(\beta, \tilde{\rho})}\, \delta_{r'r}\, \frac{1}{p!} \sum_{\sigma, \sigma' \in S_r} (-1)^{\text{sign}(\sigma) + \text{sign}(\sigma')}$$

$$\times \int_{\mathcal{G}} \langle \tilde{\phi}_{\sigma(1)} \,|\, \pi^t_{\mathcal{U}}\, \tilde{\phi}'_{\sigma'(1)} \rangle^t_\rho \cdots \langle \tilde{\phi}_{\sigma(r)} \,|\, \pi^t_{\mathcal{U}}\, \tilde{\phi}'_{\sigma'(r)} \rangle^t_\rho$$

$$\times\; D_{\tilde{z}'_1} \gamma^t(\tilde{\rho}, \mathcal{U}\rho) \cdots D_{\tilde{z}'_p} \gamma^t(\tilde{\rho}, \mathcal{U}\rho)$$

$$\times\; D_{\overline{\tilde{z}}_1} \gamma^t(\tilde{\rho}, \mathcal{U}\rho) \cdots D_{\overline{\tilde{z}}_q} \gamma^t(\tilde{\rho}, \mathcal{U}\rho)\; e^{\beta\, \gamma^t(\tilde{\rho}, \mathcal{U}\rho)}\, d\mu_{\mathcal{G}}(\mathcal{U})$$

- fixed number of fermions.

Realization of insertions as functional derivatives

- Can the state be written as follows?

$$\omega^t(\cdots) = \frac{1}{\beta^k \, Z^t(\beta, \tilde{\rho})} \underbrace{D \cdots D}_{k \text{ derivatives}} \; Z^t(\beta, \tilde{\rho})$$

Short answer: Yes, up to rather subtle technical issues.

$$Z^t(\beta, \tilde{\rho}) = \fint_{\mathcal{G}} \exp\Big(\beta \, \gamma^t(\tilde{\rho}, \mathcal{U}\rho)\Big) \, d\mu_{\mathcal{G}}(\mathcal{U})$$

Positivity of the Quantum State

THEOREM

The state ω^t is positive, i.e.

$$\omega^t(A^* A) \geq 0 \qquad \textit{for all } t \in \mathbb{R} \textit{ and } A \in \mathscr{A}$$

The proof makes use of

- Canonical commutation/anti-commutation relations
- Positivity of $(.|.)^t_\rho$ and $\langle .|. \rangle^t_\rho$
- Positivity of insertions:

$$D_{\tilde{z}} \gamma^t(\tilde{\rho}, \mathfrak{U}\rho) \cdot D_{\overline{\tilde{z}}} \gamma^t(\tilde{\rho}, \mathfrak{U}\rho) = \big| D_{\tilde{z}} \gamma^t(\tilde{\rho}, \mathfrak{U}\rho) \big|^2 \geq 0$$

$$\langle \psi \,|\, \pi^t_{\mathfrak{U}} \, \psi \rangle^t_\rho \geq 0 \quad \text{and} \quad \langle \psi \,|\, (\mathbb{1} - \pi^t_{\mathfrak{U}}) \, \psi \rangle^t_\rho \geq 0$$

F.F., N. Kamran, *"Fermionic Fock spaces and quantum states for causal fermion systems,"* arXiv:2101.10793 [math-ph]

Representations of the Quantum State

- GNS representation.
  - Introduce scalar product on $\mathscr{A}$ by
    $$\langle A|A'\rangle := \omega^t(A^* A') \;:\; \mathscr{A} \times \mathscr{A} \to \mathbb{C}$$
    Forming the completion gives a Hilbert space.
  - $\mathscr{A}$ has a natural representation on this Hilbert space.
  - Setting $\Phi = \mathbb{1}$,
    $$\langle \Phi \,|\, A\, \Phi \rangle = \omega^t(\mathbb{1}^*\, A\, \mathbb{1}) = \omega^t(A)$$
  - always exists, but in general not a Fock representation
- Representation on the Fock space of vacuum
  - choose $\mathcal{F}$ as the Fock space generated by acting with $\mathscr{A}$ on vacuum state (Dirac sea vacuum)
  - construct density operator σ^t on $\mathcal{F}$ with
    $$\omega^t(A) = \text{tr}_{\mathcal{F}}\big(\sigma^t\, A\big)$$
  - inductive construction for states involving *finite number of particles and anti-particles*
  - in general diverges (inequivalent Fock vacua, ...)
  - makes connection to perturbative description

Outlook: Dynamics of the quantum state

- Construction so far gives ω^t for all t
- Next steps:
  - Construct time evolution for the density operator

$$\mathfrak{L}^t_{t_0} \;:\; \sigma^{t_0} \to \sigma^t$$

  - Is there a unitary time evolution on the Fock space?

$$\omega^t = U^t_{t_0}\, \omega^{t_0}\, \big(U^t_{t_0}\big)^{-1}$$

- Will be objective of follow-up paper

Outlook: Connection to collapse models

General structure:

- Nonlinear dynamics of $\tilde{\rho}$ (from causal action principle)
- Conservation laws hold
  (current conservation, conserved symplectic form)
- Causality holds in the sense
  "pairs of points with spacelike separation do not interact"
  in particular: no superluminal signalling
- In approximation ("approximation of inhomogeneous fluctuating fields") one gets
  linear and unitary time evolution

$$U_{t_0}^t \;:\; \mathcal{F} \to \mathcal{F}$$

Outlook: Connection to collapse models

As observed by Johannes Kleiner, this seems to indicate that causal fermion systems are an effective collapse theory.

A. Bassi, D. Dürr, G. Hinrichs, *"Uniqueness of the equation for quantum state vector collapse,"* Phys. Rev. Lett. **111**, 210401 (2013)

- No faster-than-light signalling
- Time evolution Markovian and homogeneous in time

$\Longrightarrow$ collapse theory

Can this be adapted to causal fermion systems?

www.causal-fermion-system.com

Thank you for your attention!

Outlook: Holographic Mixing

- $\Psi : \mathcal{H} \to C^0(M, SM)$ wave evaluation operator describing Minkowski vacuum,

$$(i\partial\!\!\!/ - m)\,\Psi = 0$$

- Decompose into holographic components:

$$\Psi_\alpha(x) := \Psi(x)\, B_\alpha \qquad \text{with} \qquad B_\alpha \in \mathrm{L}(\mathcal{H})$$

- Perturb each holographic component by electromagnetic potential A_α,

$$\Delta\Psi_\alpha = s_m\, A\!\!\!/_\alpha\, \Psi\, B_\alpha$$

- Gives rise to *microscopic fluctuations*
  - scaling behavior can be computed explicitly
- *Approximation of inhomogeneous fluctuating fields* gives bosonic loop diagrams

Quantum Entanglement

- Holographic components can be decoherent
- Choosing different $\mathcal{U}$ makes different holographic components "visible"

$$\omega^t(\cdots) := \frac{1}{Z^t} \int_{\mathcal{G}} (\cdots)\, e^{\beta\, \gamma^t\left(\tilde{\rho}, \mathcal{U}\rho\right)}\, d\mu_{\mathcal{G}}(\mathcal{U})$$

- $\mathcal{U}$-dependence gives correlations between insertions
- This gives rise to entangled state.