The Dirac equation and its separation in the Reissner-Nordstroem geometry

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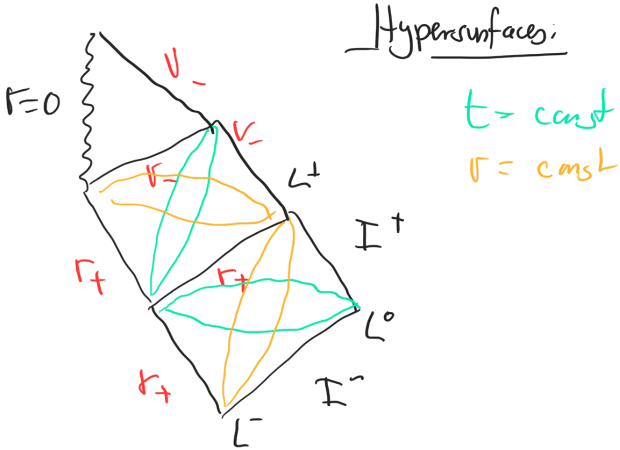
Overview

- Reissner-Nordstroem Metric
- Tetrad-Formalism
- Newman-Penrose-Formalism
- The Spin Frame
- Spin Coefficients and Dirac Equation
- Separation of the Dirac Equation

Reissner-Nordstroem Metric Basics

- Asymptotically flat, Lorentzian 4-Mfld. (M, g) w/ topology $S^2 \times \mathbb{R}^2$
- g stationary, radial sym. w/ signature $(+, -, -, -)$
- $g = \frac{\Delta}{r^2} \mathrm{d}t \otimes \mathrm{d}t - \frac{r^2}{\Delta} \mathrm{d}r \otimes \mathrm{d}r - r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta - r^2 \sin\theta \mathrm{d}\phi \otimes \mathrm{d}\phi$
- $\Delta(r) = r^2 - 2Mr + Q^2$, two roots at $r_\pm = M \pm \sqrt{M^2 - Q^2}$ when $Q < M$

Penrose Diagram



Eddington-Finkelstein-Coord

- Get rid of coord. singularities at $r_\pm$ by using tortoise coord. r_*
- $\frac{dr_*}{dr} = \frac{r^2}{\Delta} \Rightarrow r + A\ln(r - r_-) - B\ln(r - r_+) = r_*$
- Compute E-F-Coord. from tangent vectors associated to principal null geodesics - (png)
- $t = \pm r_* + C$, with " $-$ " ingoing and " $+$ " outgoing rays
- Define new time coord. $\tau := t + r_* - r$. Then τ is a Cauchy time function

Tetrad-Formalism Basics

- pointwise orthonormal vector fields - (VF)
- One VF is time-like, the other three are space-like
- At each point of the space-time (ST) set a basis of four contra-variant vectors $e^{\mu}_{(a)}$ with $a, \mu \in \{0, 1, 2, 3\}$
- $\eta_{(a)(b)} := e^{\mu}_{(a)} e_{(b)_\mu}$ constant, sym. matrix for lowering tetrad indices
- Project arbitrary tensor, i.e. $T^{\mu}_{\nu\alpha}$, onto the tetrad frame: $T^{(a)}_{(b)(c)} = e^{(a)}_{\mu} e^{\nu}_{(b)} e^{\alpha}_{(c)} T^{\mu}_{\nu\alpha}$

Ricci-Rotation Coefficients

$$\gamma_{(a)(b)(c)} := \left(\nabla_\nu e_{(b)\beta}\right) e^{\nu}_{(c)} e^{\beta}_{(a)}$$

- Intrinsic derivative is defined:
  $T_{(a)|(b)} := T_{(a),(b)} - \eta^{(m)(n)}\gamma_{(n)(a)(b)} T_{(m)}$
- $\gamma_{(a)(b)(c)}$ describes the Ricci rotation coefficients, also often called scalar fields
- When implying orthogonal tetrad in local Minkowski space $\gamma_{(a)(b)(c)} = -\gamma_{(b)(a)(c)}$. Therefore, 24 independent, real scalar fields
- Evaluation of Ricci rotation coeff. does not involve the evaluation of covariant derivatives.

Newman-Penrose-Formalism Basics

- Four null vectors $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ with normalization conditions $l^\mu n_\mu = 1$ and $m^\mu \bar{m}_\mu = -1$
- $e^\mu_{(0)} = l^\mu$, $e^\mu_{(1)} = n^\mu$, $e^\mu_{(2)} = m^\mu$ and $e^\mu_{(3)} = \bar{m}^\mu$
- Define 12 complex spin coefficients in the NP-Formalism due to the Ricci rotation coefficients.
  $\kappa = \gamma_{(2)(0)(0)} = \frac{1}{2}(\lambda_{(2)(0)(0)} + \lambda_{(0)(2)(0)} - \lambda_{(0)(0)(2)})$
- $\lambda_{(a)(b)(c)} := e_{(b)\mu,\nu}(e^\mu_{(a)} e^\nu_{(c)} - e^\nu_{(a)} e^\mu_{(c)})$

or by def: $\kappa = \ell^\nu (\nabla_\nu \ell_\alpha) m^\alpha$

Lorentz Transformations

- Most time ST has not enough local structure to define four vectors for a complete tetrad.
- In NP-Formalism: l^μ and n^μ are determined by pngs. m^μ and $\bar{m}^\mu$ are unit space-like VF, orthogonal to itself, l^μ and n^μ.
- Therefore, it exists a two dim. gauge freedom which is described by the two param. subgroup of the Lorentz Group - sometimes denoted by **rotations of class III**, leaving the direction of l^μ and n^μ unchanged.
- Generated by boosts $l^\mu \longrightarrow r l^\mu$, $n^\mu \longrightarrow r^{-1} n^\mu$ and rotations $m^\mu \longrightarrow e^{i\alpha} m^\mu$

The Spin Frame Basics

- ST of GR is locally Minkowskian. Therefore, define locally tetrad basis for spinors $\zeta^{A}_{(a)}$ and $\zeta^{A'}_{(a')}$
- A and A' are spinor indices where A corresponds to the fundamental and A' to the anti-fundamental representation
- Sometimes convenient use special symbols $\zeta^{A}_{(0)} = \sigma^{A}$ and $\zeta^{A}_{(1)} = \iota^{A}$
- ϵ_{AB} skew-symmetric metric: $\epsilon_{AB}\zeta^{A}_{(a)}\zeta^{B}_{(b)} = \zeta_{(a)B}\zeta^{B}_{(b)} = \epsilon_{(a)(b)}$

Generalized Pauli Spin-Matrices

- Spinors and their complex conjugate determine $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ by the correspondence
- $l^\mu \leftrightarrow \sigma^A \bar{\sigma}^{B'}$, $n^\mu \leftrightarrow \iota^A \bar{\iota}^{B'}$ $m^\mu \leftrightarrow \sigma^A \bar{\iota}^{B'}$ and $\bar{m}^\mu \leftrightarrow \iota^A \bar{\sigma}^{B'}$
- Due to this representation one can define the hermitian matrices **(generalized Pauli spin-matrices)** $\sigma^\mu_{AB'}$:
- $\sigma^\mu_{AB'} = \frac{1}{\sqrt{2}} \begin{bmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{bmatrix}, \qquad \sigma_{AB'\mu} = \frac{1}{\sqrt{2}} \begin{bmatrix} l_\mu & -m_\mu \\ -\bar{m}_\mu & n_\mu \end{bmatrix}$

Connection to NP-Formalism

- This null tetrad fulfils the normalization conditions:

$$l^\mu n_\mu = \underbrace{\sigma^\mu_{AB'}\sigma^{AB'\mu}}_{\delta^\mu_\mu = 1} \sigma^A \bar{\sigma}^{B'} \iota_A \bar{\iota}_{B'} = 1$$

$$m^\mu \bar{m}_\mu = \underbrace{\sigma^\mu_{AB'}\sigma^{AB'\mu}}_{\delta^\mu_\mu = 1} \sigma^A \bar{\iota}^{B'} \iota_A \bar{\sigma}_{B'} = -1$$

- Dyad basis determine four null vectors which can be used as a basis for the NP-Formalism

Dyad Spin Coefficients

- Covariant derivative of a spinor field satisfies the Leibnitz rule, is a real operator and based on correspondences:

$$\nabla_\mu \leftrightarrow \nabla_{AB'}, \qquad \nabla_\mu X_\nu \leftrightarrow \nabla_{AB'} X_{CD'}$$

- Define analogous an intrinsic derivative for the dyad components $\xi_{(a)}$ of a spinor along $(a)(b')$:

$$\begin{aligned} \xi_{(c)|(a)(b')} &= (\nabla_{AB'}\xi_C \zeta^C_{(c)})\zeta^A_{(a)}\zeta^{B'}_{(b)} \\ \Leftrightarrow \xi_{(c)|AB'} &= (\nabla_{AB'}\xi_C \zeta^C_{(c)}) \\ \Rightarrow \xi_{(a)|BC'} &= \xi_{(a),BC'} + \Gamma_{(d)(a)BC'}\xi^{(d)} \end{aligned}$$

Dyad Spin Coefficients II

- 12 independent complex coefficients
- One can show with one Lemma from Friedman (states an alternative rep. of $\Gamma_{(a)(b)CD'}$) following theorem:
  Theorem:
  It is possible to express the dyad spin coefficients $\Gamma_{(a)(b)CD'}$ *in terms of the covariant derivatives of the basis null vectors* $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ *and therefore show that they are in agreement with the coefficients* $\gamma_{(a)(b)(c)}$.

$$\Gamma_{000'0'} = \kappa = \gamma_{(2)(0)(0)} = \frac{1}{2}\left(\lambda_{(2)(0)(0)} + \lambda_{(0)(2)(0)} - \lambda_{(0)(0)(2)}\right)$$

Spin Coefficients in Reissner-Nordstroem

- Use tangent vectors associated to the png to define null tetrad
- Let (ψ, U) be a local param. of M with $p \in U \subset M$, $\psi(p) = (t, r, \theta, \phi)$. $\{\partial_t, \partial_r, \partial_\theta, \partial_\phi\}$ is the induced canonical basis of T_pM and $\{\mathrm{d}t, \mathrm{d}r, \mathrm{d}\theta, \mathrm{d}\phi\}$ of T_pM^*.
- $$l^\mu = \frac{1}{|\Delta|}(r^2\partial_t + \Delta\partial_r), \quad n^\mu = \frac{\mathrm{sign}(\Delta)}{2r^2}(r^2\partial_t - \Delta\partial_r)$$
$$m^\mu = \frac{1}{\sqrt{2}r}(\partial_\theta + i\csc(\theta)\partial_\phi), \quad \bar{m}^\mu = \frac{1}{\sqrt{2}r}(\partial_\theta - i\csc(\theta)\partial_\phi)$$

Spin Coefficients in Reissner-Nordstroem II

- Class III Lorentz transformation: $r = \sqrt{\frac{|\Delta|}{2r^2}}$ and $\alpha = 0$
- Re-write in Eddington-Finkelstein-Coordinates $\tau = t + r_* - r$
- Additionally Class III Lorentz transformation: $r^{'} = \frac{\sqrt{|\Delta|}}{r_+}$ and $\alpha^{'} = 0$

$$l^{''} = \frac{1}{\sqrt{2}rr_+}\left[\left(2r^2 - \Delta\right)\partial_\tau + \Delta\partial_r\right]$$

$$l_D^{''} = \frac{1}{\sqrt{2}rr_+}\left[\Delta d\tau + \left(\Delta - 2r^2\right)dr\right]$$

Spin Coefficients in Reissner-Nordstroem III

- Compute Spin coefficients in NP-Formalism for Reissner-Nordstroem ST. Six coefficients are distinct from zero.
- $$\pi'' = \tau'' = \kappa'' = \sigma'' = \nu'' = \lambda'' = 0$$
  $$\alpha'' = -\beta'' = \frac{1}{2^{3/2}} \frac{\cot(\theta)}{r}, \qquad \gamma'' = -\frac{r_+}{2^{3/2}} \frac{1}{r^2}$$
  $$\epsilon'' = \frac{1}{2^{3/2} r_+} \left(1 - \frac{Q^2}{r^2}\right), \qquad \rho'' = -\frac{1}{\sqrt{2} r_+} \frac{\Delta}{r^2}$$
  $$\mu'' = -\frac{r_+}{\sqrt{2}} \frac{1}{r^2}$$
- Consistent with results from C. Röken for a Kerr ST in the limit $a \longrightarrow 0$.

General Dirac Equation

- Got spin coefficients in the NP-Formalism. But where is the connection with the Dirac-Eq.?

- (M, g) is an arbitrary 4-dim. curved ST, $\mathcal{S} = P \times_\tau \Delta_4$ the associated spinor bundle, $\tau : \mathrm{Spin}(4) \longrightarrow \mathrm{GL}(\Delta_4)$ and (P, F) the spin structure. $\Psi \in \Gamma(\mathcal{S}) \simeq \mathbb{C}^4$ is a Dirac four spinor and m its invariant fermion rest mass.

- $\left[\gamma^\mu \nabla^S_\mu + im\right] \Psi(x^\mu) = 0$ with $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{1}^{\mathbb{C}}_{4\times 4}$

- γ^μ general relativist Dirac matrices and ∇^S being the metric connection in the spinor bundle

General Dirac Equation in Spinor Rep.

- Two spinor rep. of $\Psi = \begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix}$
- $\gamma^\mu = \gamma^\mu = \sqrt{2} \begin{bmatrix} 0^{\mathbb{C}}_{2\times 2} & \sigma^{\mu AB'} \\ \sigma^\mu_{AB'} & 0^{\mathbb{C}}_{2\times 2} \end{bmatrix}$ with $\sigma^\mu_{AB'}$ the generalized Pauli matrices
- $$\nabla_{AB'} P^A + \frac{im}{\sqrt{2}} \bar{Q}^{C'} \epsilon_{C'B'} = 0$$ $$\nabla_{AB'} Q^A + \frac{im}{\sqrt{2}} \bar{P}^{C'} \epsilon_{C'B'} = 0$$

General Dirac Equation in Spinor Rep. II

- Example calculation for $B' = 0$: $\nabla_{A0'}P^A + \frac{im}{\sqrt{2}}\bar{Q}^{1'}\underbrace{\epsilon_{1'0'}}_{=-1} = 0$

$$\nabla_{00'}P^0 + \nabla_{10'}P^1 = \frac{im}{\sqrt{2}}\bar{Q}^{1'}$$

$$\underbrace{\partial_{00'}}_{D}P^0 + \underbrace{\Gamma^0{}_{b00'}}_{\Gamma_{1b00'}}P^b + \underbrace{\partial_{10'}}_{\bar{\delta}}P^1 + \underbrace{\Gamma^1{}_{b10'}}_{-\Gamma_{0b10'}}P^b = \frac{im}{\sqrt{2}}\bar{Q}^{1'}$$

$$\boxed{D = \ell^\nu\partial_\nu = \sigma^\nu_{00'}\partial_\nu = \partial_{00'}}$$

$$(D - \Gamma_{1001'} - \Gamma_{0011'})P^0 + (\bar{\delta} + \Gamma_{1100'} - \Gamma_{0110})P^1 = \frac{im}{\sqrt{2}}\bar{Q}^{1'}$$

General Dirac Equation in Spinor Rep. II

$$(D+\epsilon-\rho)P^0+(\bar{\delta}+\pi-\alpha)P^1=\frac{im}{\sqrt{2}}\bar{Q}^{1'}$$

General Dirac Equation in Spinor Rep. III

- Substituting: $\mathcal{F}_0 = P^0$, $\mathcal{F}_1 = P^1$, $\mathcal{G}_0 = \bar{Q}^{1'}$ and $\mathcal{G}_1 = -\bar{Q}^{0'}$
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$$\left(D + \epsilon - \rho\right)\mathcal{F}_0 + \left(\bar{\delta} + \pi - \alpha\right)\mathcal{F}_1 = \frac{im}{\sqrt{2}}\mathcal{G}_0$$

$$\left(\delta + \beta - \tau\right)\mathcal{F}_0 + \left(\Delta + \mu - \gamma\right)\mathcal{F}_1 = \frac{im}{\sqrt{2}}\mathcal{G}_1$$

$$\left(D + \bar{\epsilon} - \bar{\rho}\right)\mathcal{G}_1 - \left(\delta + \bar{\pi} - \bar{\alpha}\right)\mathcal{G}_0 = \frac{im}{\sqrt{2}}\mathcal{F}_1$$

$$\left(\Delta + \bar{\mu} - \bar{\gamma}\right)\mathcal{G}_0 - \left(\bar{\delta} + \bar{\beta} - \bar{\tau}\right)\mathcal{G}_1 = \frac{im}{\sqrt{2}}\mathcal{F}_0$$

Separation of the Dirac Equation

- Ansatz: $\mathcal{F}_i = e^{i(\omega\tau + k\phi)}\mathcal{H}_i(r,\theta)$ and $\mathcal{G}_i = e^{i(\omega\tau + k\phi)}\mathcal{J}_i(r,\theta)$
- Define:

$$\chi(r,Q) := \frac{1}{2r^2}(Q^2 + r(3r - 4M))$$
$$\mathcal{L}_n := \partial_\theta + n\cot(\theta) + k\csc(\theta)$$

- Looking at the first ODE:
$\frac{1}{r_+}\left[i\omega(2r^2 - \Delta) + \Delta\partial_r + \chi\right]\mathcal{H}_0(r,\theta) + \mathcal{L}_{1/2}\mathcal{H}_1 = imr\mathcal{J}_0(r,\theta)$

Separation of the Dirac Equation II

- Separation Ansatz:

$$\mathcal{H}_0 = R_+(r)S_+(\theta), \qquad \mathcal{H}_1 = R_-(r)S_-(\theta)$$
$$\mathcal{J}_0 = R_-(r)S_+(\theta), \qquad \mathcal{J}_1 = R_+(r)S_-(\theta)$$

- These ansatz results in following equation:
$\left(\frac{1}{r_+}\left[i\omega(2r^2 - \Delta) + \Delta\partial_r + \chi(r,\theta)\right]R_+(r) - imrR_-(r)\right)S_+(\theta) + \mathcal{L}_{\frac{1}{2}}S_-(\theta)R_-(r) = 0$

- Therefore, following must be satisfied:

$$-\lambda S_+(\theta) = \mathcal{L}_{\frac{1}{2}}S_-(\theta)$$
$$\lambda R_-(r) = \frac{1}{r_+}\left[i\omega(2r^2 - \Delta) + \Delta\partial_r + \chi(r,\theta)\right]R_+(r) - imrR_-(r)$$

with λ being the constant of separation.

Separation of the Dirac Equation III

- Analogue for the other three equations. Resulting in four ODEs:

- Two radial equations:

$$\left[\Delta\partial_r + \chi(r,Q) + i\omega(2r^2 - \Delta)\right] R_+(r) = r_+(\lambda + imr)R_-(r)$$

$$r_+\left(\partial_r + \frac{1}{r} - i\omega\right) R_-(r) = (\lambda - imr)R_+(r)$$

- Two angular equations:

$$\mathcal{L}_{\frac{1}{2}} S_-(\theta) = -\lambda S_+(\theta)$$

$$\mathcal{L}^\dagger_{\frac{1}{2}} S_+(\theta) = \lambda S_-(\theta)$$

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