On the general-relativistic quantum-mechanical spectrum of hydrogenic ions

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Some Minor blunders spotted in the slides during and after the talk have been corrected (Nov. 7, 2020)

Outline

‘General-relativistic hydrogenic spectra’

is a research project jointly with:

Shadi Tahvildar-Zadeh   Ebru Toprak

“On general-relativistic hydrogen and hydrogenic ions,”
J. Math. Phys., **61** art.092303 22pp. (2020)
“On the Dirac operator for a test electron in a Reissner-Weyl-Nordström black hole spacetime,”
submitted to *Gen. Rel. Grav.*, Sept. 2020

A zero-Gravity refresher. Part I

Test Electron $(-e, m)$ in Galileian Spacetime; Nucleus (Ze) at origin

Electron rest energy as Remnant of SR (c)

- Schrödinger Hamiltonian $H = mc^2 - \frac{\hbar^2}{2m}\Delta - \frac{Ze^2}{r}$
- H is essentially self-adjoint on $C_c^\infty(\mathbb{R}^3\backslash\{0\})$ (Kato, 1944/1952)
- $\sigma_{ess}(H) = [mc^2, \infty) = \sigma_{ac}(H); \qquad \Longrightarrow \sigma_{sc}(H) = \emptyset$
- $\sigma_{pt}(H) = \sigma_{disc}(H)$
- $\sigma_{disc}(H) = \left\{ mc^2 \left(1 - \frac{1}{2n^2} \left(\frac{Ze^2}{\hbar c} \right)^2 \right) \right\}_{n=1}^{\infty}$ (Bohr)
- electron bound states $\sigma_{disc}(H)$ for all $Z > 0$.
- N.B.: $\frac{e^2}{\hbar c} = \alpha_S \approx \frac{1}{137.036} \Longrightarrow$ Bohr spectrum accurate for small Z
- But, $Z\alpha_S = 1 \Longrightarrow E_{n=1} = \frac{1}{2}mc^2$: SR is needed if $Z \approx 137$.

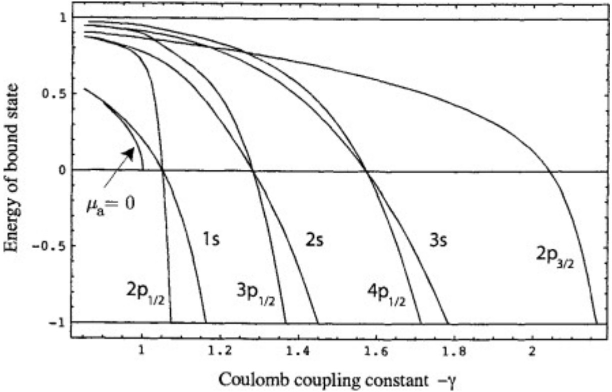
A zero-Gravity refresher. Part II

Test Electron ($-e$, m) in Minkowski Spacetime; Nucleus (Ze) at origin

- *Dirac Hamiltonian*: $H = -i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \frac{Ze^2}{r}\mathbf{1}_{4\times 4} + mc^2\beta$
- $\beta = \gamma^0$ and $\beta\alpha^k = \gamma^k$, with $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}_{(0)}\mathbf{1}_{4\times 4}$
- *Metric*: $g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$
- H essentially self-adjoint on $C_c^\infty(\mathbb{R}^3\backslash\{0\})^4$ iff $\frac{Ze^2}{\hbar c} \leq \frac{\sqrt{3}}{2}$ (Weidmann)
- H has analytical self-adjoint extension iff $\frac{Ze^2}{\hbar c} \leq 1$ (Narnhofer)
- $Z\alpha_S > 1 \implies$ electric attraction beats angular momentum barrier
- $\sigma_{ess}(H) = (-\infty, -mc^2] \cup [mc^2, \infty) = \sigma_{ac}(H)$; $\sigma_{sc}(H) = \emptyset$
- $\sigma_{disc}(H) = \left\{ mc^2 \Big/ \sqrt{1 + \frac{Z^2e^4/\hbar^2c^2}{\left(k+\sqrt{\kappa^2 - Z^2e^4/\hbar^2c^2}\right)^2}} \right\}_{|\kappa|\in\mathbb{N}\ \&\ k\in\{0,1,2,...\}}$ (Sommerfeld)
- Electron's anomalous magnetic moment μ_a restores esa $\forall\, Z > 0$
  Behncke (1980); Gesztezy-Simon-Thaller(1985); Thaller(1992)

A zero-*G*ravity refresher. Part II cont.'d

Test Electron ($-e$, m, μ_a) in Minkowski Spacetime; Nucleus (Ze) at origin



Low-lying energy eigenvalues of $H + H_{\rm anom}$ vs. $Z\alpha_{\rm S}$
From B. Thaller (2002)

How does intrinsic *G*ravity influence the spectra?

How does intrinsic *G*ravity influence the spectra?

An often heard, and **not unreasonable** opinion:

"Intrinsic gravity is too weak to be of significance in atoms!"

Here is a physicist's proof for why this should be true:

Intrinsic Newtonian gravity is weak inside atoms and ions.

Einstein gravity becomes Newtonian gravity for weak static fields.

Therefore it's true for Einstein gravity. QED

If you find this compelling, prepare for some surprises!

Since gravity enhances the attraction between electron and nucleus, at least we should expect a worsening of the self-adjointness criteria!

Schrödinger op. w. Coulomb & Newton interactions

Test Electron ($\mp e$, m) in Galileian Spacetime; Nucleus (Ze, M) at origin

Schrödinger op. w. Coulomb & Newton interactions

Test Electron ($\mp e$, m) in Galileian Spacetime; Nucleus (Ze, M) at origin

Electron rest energy as Remnant of SR (c)

Schrödinger op. w. Coulomb & Newton interactions

Test Electron ($\mp e$, m) in Galileian Spacetime; Nucleus (Ze, M) at origin

Electron rest energy as Remnant of SR (c)

Newtonian gravity as Remnant of GR (G)

Schrödinger op. w. Coulomb & Newton interactions

Test Electron ($\mp e$, m) in Galileian Spacetime; Nucleus (Ze, M) at origin

Newtonian gravity as Remnant of GR (G)

Electron rest energy as Remnant of SR (c)

- Schrödinger Hamiltonian $H^{\mp} = mc^2 - \frac{\hbar^2}{2m}\Delta + \frac{\pm Ze^2 - GmM}{r}$
- $H^{\mp}$ is essentially self-adjoint on $C_c^{\infty}(\mathbb{R}^3\backslash\{0\})$ (Kato, 1944/1952)
- $\sigma_{ess}(H^{\mp}) = [mc^2, \infty) = \sigma_{ac}(H)$; $\quad\sigma_{sc}(H^{\mp}) = \emptyset$
- $\sigma_{pt}(H^{\mp}) = \left\{ mc^2 \left(1 - \frac{1}{2n^2} \left(\pm\frac{Ze^2}{\hbar c} + \frac{GmM}{\hbar c}\right)_+^2 \right) \right\}_{n=1}^{\infty} = \sigma_{disc}(H^{\mp})$
- electron bound states $\sigma_{disc}(H^-)$ for all $M > 0$ and $Z > 0$;
- positron bound states $\sigma_{disc}(H^+)$ iff $GmM > Ze^2$ (hyper-heavy); INTERPRETATION: Hyper-heavy nuclei can bind a positron by gravitational attraction, overwhelming their electrical repulsion.
- N.B.: $\frac{GmM}{Ze^2} \approx 10^{-39}$ in the lab.

Dirac operator w. Coulomb & Newton interactions

Test Electron ($-e$, m) in Minkowski Spacetime; Nucleus (Ze, M) at origin

Newtonian gravity as Remnant of GR (G)

Dirac operator w. Coulomb & Newton interactions

Test Electron ($-e, m$) in Minkowski Spacetime; Nucleus (Ze, M) at origin
Newtonian gravity as Remnant of GR (G)

- *Dirac Hamiltonian*: $H = -i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \frac{Ze^2}{r}\mathbf{1}_{4\times4} + \left(mc^2 - \frac{GmM}{r}\right)\beta$
- $\beta = \gamma^0$ and $\beta\alpha^k = \gamma^k$, with $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}_{(0)}\mathbf{1}_{4\times4}$
- *Metric*: $g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$
- H essentially self-adjoint on $C^\infty_c(\mathbb{R}^3\backslash\{0\})^4$ iff $\frac{Z^2e^4}{\hbar^2c^2} \leq \frac{3}{4} + \frac{G^2m^2M^2}{\hbar^2c^2}$
- H has analytical self-adjoint extension iff $\frac{Z^2e^4}{\hbar^2c^2} \leq 1 + \frac{G^2m^2M^2}{\hbar^2c^2}$
- *Newtonian gravity regularizes hydrogenic Dirac operator*! (a tiny bit)
- $\sigma_{ess}(H) = (-\infty, -mc^2] \cup [mc^2, \infty)$; $\quad \sigma_{sc}(H) = \emptyset$
- $\sigma^{\mp}_{disc}(H) = \left\{ \pm mc^2 \frac{1 - \frac{1}{(k+g(\kappa))^2}\left(\frac{GmM}{\hbar c}\right)^2}{\sqrt{1 + \frac{1}{(k+g(\kappa))^2}\left(\frac{Z^2e^4}{\hbar^2c^2} - \frac{G^2m^2M^2}{\hbar^2c^2}\right) \pm \frac{Z\alpha_S}{(k+g(\kappa))^2}\frac{GmM}{\hbar c}}} \right\}$ with

$$g(\kappa) = \sqrt{\kappa^2 - \left(\frac{Z^2e^4}{\hbar^2c^2} - \frac{G^2m^2M^2}{\hbar^2c^2}\right)},\ |\kappa| \in \mathbb{N}\ \&\ k \in \{0, 1, 2, ...\}.$$

Dirac operator w. Coulomb & Newton interactions

Test Electron $(-e, m)$ in Minkowski Spacetime; Nucleus (Ze, M) at origin
Newtonian gravity as Remnant of GR (G)

- electron bound states $\sigma^-_{disc}(H)$ whenever $\frac{Z^2e^4}{\hbar^2c^2} \leq 1 + \frac{G^2m^2M^2}{\hbar^2c^2}$
- positron bound states $\sigma^+_{disc}(H)$ iff $Ze^2 < GmM$.
- N.B.: $\sigma^+_{disc}(H) < 0$ & $\sigma^-_{disc}(H) > \sigma^+_{disc}(H)$, but $\sigma^-_{disc}(H) \not> 0$ in general.
- For lab. ions TINY $O(10^{-39})$ G corr.^{s} to $G = 0$ Sommerfeld spectrum!
- Newtonian gravity is insignificant for Dirac hydrogen!
- Newtonian gravity seems significant for hyper-heavy nuclei, but ...
- $\{GmM > Ze^2 \text{ \& } M > Zm\} \implies GM^2 > Z^2e^2$ (BLACK HOLE !)

  N.B.: BLACK HOLE regime not reachable with $M \leq 3Zm_{pr}$,
  but neutron star parameters $(M \gg Zm_{pr})$ will do.
- To get a definitive answer, we cannot ignore GR! But first, ...

Dirac operator w. Coulomb & Newton interactions

Test Electron ($-e, m$) in Minkowski Spacetime; Nucleus (Ze, M) at origin

Newtonian gravity as Remnant of GR (G)

REMARKS and COMMENTS: (NB: Bullet pts. 2 & 3 corrected!)

- $\sigma^{\mp}_{disc}(H)$: adapted from Greiner, Müller, Rafelski (1985).
- $\sigma^{\mp}_{disc}(H)$ versus $\sigma^{\mp}_{BS}$ from Bohr-Sommerfeld quantization: Ignoring fine structure, the main lines of $\sigma^{\mp}_{BS} = \{\min_r V^{\mp}_n(r)\}_{n\in\mathbb{N}}$, where: $V^{\mp}_n(r) := mc^2\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{n^2}{r^2}} + \frac{\mp Ze^2 - GMm}{r}$
- $\inf_r V^-_n(r) > -\infty$ iff $\boxed{\frac{Ze^2}{\hbar c} + \frac{GmM}{\hbar c} \leq 1}$; [N.B.: $V^+_n(r) > V^-_n(r)$]
- analytic self-adjointness: adapted from Narnhofer(1974) ($G = 0$)
- essential self-adjointness: adapted from Weidmann(1971) ($G = 0$)
- Electron's anomalous magnetic moment restores e.s.a. for $G > 0$? TBA

General-Relativistic Hydrogenic Ions. I

Test Electron $(-e, m)$ in $\underbrace{\text{Reissner-Nordström}}_{\text{Weyl}}$ Spacetime of Nucleus (Ze, M)

$M < 10^{18} Z m_{\text{pr}}$: Naked Singularity Sector of RWN

- *Metric*: $g^{(G)}_{\mu\nu} dx^\mu dx^\nu = -f(r)^2 c^2 dt^2 + f(r)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$
  $$f(r)^2 \equiv 1 - \frac{2GM}{c^2 r} + \frac{GZ^2 e^2}{c^4 r^2}$$
  (N.B.: $e \to 0$: **Schwarzschild** $G \to 0$: **Minkowski**)
- Electromagnetic four-*Potential:* $A_\mu dx^\mu = -\frac{Ze}{r} c dt$
- 1924: Vallarta applies *Bohr–Sommerfeld* quantization to test electron in RWN spacetime of nucleus (Naked Singularity Sector)
- His goal: To confirm what everyone 'knew' already: the influence of gravity on the spectra of the known hydrogenic ions is immeasurably tiny and therefore negligible.
- Mission accomplished, he doesn't bother to compute any effects.
- However, had he not merely tried to confirm what everyone 'knew' already he could have made a startling discovery!

General-Relativistic Hydrogenic Ions. I

Test Electron $(-e, m)$ in $\overbrace{\text{Reissner-Nordström}}^{\text{Weyl}}$ Spacetime of Nucleus (Ze, M)

$M < 10^{18} Zm_{\text{pr}}$: Naked Singularity Sector of RWN

RWN spacetime regularizes hydrogenic Bohr-Sommerfeld theory!

- Ignoring fine structure, $\sigma_{pt}(U^{\mp}) = \left\{\min_r U_n^{\mp}(r)\right\}_{n\in\mathbb{N}}$, where:

$$U_n^{\mp}(r) := mc^2\sqrt{1+\frac{\hbar^2}{m^2c^2}\frac{n^2}{r^2}}\sqrt{1-\frac{2G}{c^4 r}\left[Mc^2-\frac{Z^2e^2}{2r}\right]} \mp \frac{Ze^2}{r}$$

- $\forall\, Z > 0$ & $Zm_{\text{pr}} \leq M < 10^{18} Zm_{\text{pr}}$: $\exists \min_r U_n^-(r)$ (NB: $U_n^+(r) > U_n^-(r)$)
- $\exists \min_r U_n^+(r)$ iff $GMm > Ze^2$ & $GM^2 < Z^2e^2$ (NOT nuclei)
- While Newton's G regularizes $\boxed{Z\alpha_S \leq 1} \mapsto \boxed{Z\alpha_S \leq 1 + 10^{-39}}$,
  Einstein's GR G regularizes $\boxed{Z\alpha_S \leq 1} \mapsto \boxed{Z\alpha_S < \infty}$.
- This suggests that RWN regularizes hydrogenic Dirac problem!

General-Relativistic Hydrogenic Ions. II

Test Electron $(-e, m)$ in $\overbrace{\text{Reissner-Nordström}}^{\text{Weyl}}$ Spacetime of Nucleus (Ze, M)

$M < 10^{18} Z m_{\text{pr}}$: Naked Singularity Sector of RWN

- *Dirac Equation:* $\gamma^\mu(-i\hbar c\nabla_\mu + eA_\mu)\Psi + mc^2\mathbf{1}_{4\times 4}\Psi = 0$;
- *Clifford algebra:* $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}_{(G)}\mathbf{1}_{4\times 4}$
- *Metric*: $g^{(G)}_{\mu\nu}dx^\mu dx^\nu = -f(r)^2c^2dt^2 + f(r)^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

$$f(r)^2 \equiv 1 - \frac{2GM}{c^2r} + \frac{Ge^2}{c^4r^2}$$

- Electromagnetic four-*Potential:* $A_\mu dx^\mu = -\frac{e}{r}cdt$
- Rewrite Dirac Eq. in Schrödinger form:

$$i\hbar\partial_t\Psi = H\Psi$$

and study *Dirac Hamiltonian* H on $C^\infty_c(\mathbb{R}^3\backslash\{0\})^4$.

General-Relativistic Hydrogen. II

Test Electron $(-e, m)$ in $\overbrace{\text{Reissner-Nordström}}^{\text{Weyl}}$ Spacetime of Nucleus (Ze, M)

Properties of Dirac Hamiltonian for the Naked Singularity Sector of RWN
Cohen and Powers (1982) Belgiorno-Martellini-Baldicchi (2000)

- H is NOT essentially self-adjoint.
- H has ∞ many self-adjoint extensions H_θ, $\theta \in [0, \pi]$. (Pick one?)
- $\sigma_{ess}(H_\theta) = (-\infty, -mc^2] \cup [mc^2, \infty)$; $\sigma_{sc}(H_\theta) = \emptyset$
- $\sigma_{pt}(H_\theta) \subset [-mc^2, mc^2]$ ∞ many Bound States

However, is $\sigma_{disc}(H_\theta) = \sigma_{\text{Sommerfeld}} + O(10^{-39})$?

EVEN IF TRUE, which one of the ∞ many extensions
of the Sommerfeld fine structure formula is correct, if any?

RWN spacetime has devastating effect on Dirac hydrogen!

General-Relativistic Hydrogen. II

Test Electron $(-e, m, \mu_a)$ in RWN Spacetime of Nucleus (Ze, M)

Properties of Dirac Hamiltonian for the Naked Singularity Sector of RWN cont.^{d}:

- Adding a sufficiently large anomalous magnetic moment operator H_{anom} to the formal Dirac Hamiltonian H that we just discussed does yield an essentially self-adjoint operator $H + H_{\mathrm{anom}}$ (Belgiorno-Martellini-Baldicchi, 2000).

  N.B.: The empirical anomalous magnetic moment is roughly 10^{18} times larger than the required minimal value!
- $\sigma_{ess}(H + H_{\mathrm{anom}}) = (-\infty, -mc^2] \cup [mc^2, \infty)$; $\sigma_{sc}(H + H_{\mathrm{anom}}) = \emptyset$
- $\sigma_{pt}(H + H_{\mathrm{anom}}) \subset [-mc^2, mc^2]$ ∞ many Bound States
- Detailed investigation of $\sigma_{pt}(H + H_{\mathrm{anom}})$ is underway

General-Relativistic Hydrogen. II

Test Electron $(-e, m)$ in RWN Spacetime of Nucleus (Ze, M)

Properties of Dirac H for the sub-extremal Black Hole Sector of RWN
Cohen and Powers (1982): study electron outside event horizon

- H essentially self-adjoint on $C_c^\infty([R_+, \infty) \times \mathbb{S}^2)^4$
- $\sigma_{ess}(H) = \mathbb{R}$; $\sigma_{sc}(H) = \emptyset$
- $\sigma_{pt}(H) = \emptyset$ NO Bound States (Not really surprising!)
- However, what about the static core region of RWN Black Hole?

General-Relativistic Hydrogen. II

Test Electron $(-e, m, \mu_a)$ in RWN Spacetime of Nucleus (Ze, M)

Properties of Dirac Hamiltonian for the Black Hole Sector of RWN cont.^{d}:

- K. & T.-Z. & T. (2020): study H on $C_c^\infty((0, R_-] \times \mathbb{S}^2)^4$.
- H is NOT essentially self-adjoint.
- H has ∞ many self-adjoint extensions H_θ, $\theta \in [0, \pi]$.
- Adding a sufficiently large anomalous magnetic moment operator H_{anom} to the formal Dirac Hamiltonian H that we just discussed does yield an essentially self-adjoint operator $H + H_{\text{anom}}$

  N.B.: The empirical anomalous magnetic moment $\approx \frac{1}{4\pi}\frac{e^3}{mc^2}$ is roughly 10^{18} times larger than the required minimal value!
- $\sigma_{ess}(H_{s.a.}) = \mathbb{R}$; $\sigma_{sc}(H_{s.a.}) = \emptyset$
- $\sigma_{pt}(H_{s.a.}) = \emptyset$: NO Bound States (SURPRISING!?)

Pathologies of Reissner-Weyl-Nordström spacetime

NOT ALL IS WELL, THOUGH!

Infinite Electrostatic Self-Energy of Nucleus!

Pathologies of Reissner-Weyl-Nordström spacetime

NOT ALL IS WELL, THOUGH!

Infinite Electrostatic Self-Energy of Nucleus!

- Gravitationally coupled to Spacetime Structure by Einstein's Eqs.

$$R_{\mu\nu}[\mathbf{g}] - \tfrac{1}{2}R[\mathbf{g}]g_{\mu\nu} = \tfrac{8\pi G}{c^4}T_{\mu\nu}[\mathbf{F},\mathbf{g}],$$

the non-integrable energy-momentum-stress tensor

$$T_{\mu\nu}[\mathbf{F},\mathbf{g}] = \tfrac{1}{4\pi}\left(F_\mu^\lambda F_{\nu\lambda} - g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

of Maxwell-Lorentz $\mathbf{F} = d\mathbf{A}$ causes strongly singular curvatures!

- $g_{\mu\nu}^{(G)}dx^\mu dx^\nu = -f(r)^2c^2dt^2 + f(r)^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

$$f(r)^2 \equiv 1 - \tfrac{2GM}{c^2 r} + \tfrac{GZ^2e^2}{c^4r^2} \equiv 1 - \tfrac{2G}{c^4 r}\underbrace{\left(Mc^2 - \frac{Z^2e^2}{2r}\right)}$$

NEGATIVE ($\downarrow -\infty$) quasi-local MASS: < 0 if $r < R_0 = \frac{Z^2e^2}{2Mc^2}$

Pathologies of Reissner-Weyl-Nordström spacetime

COMMENTS:

- For $Z = 1$: "Critical" R_0 deep inside the physical proton region:

$$R_0 = \frac{1}{2}\frac{m}{M}{\alpha_S}^2 R_{\mathrm{Bohr}} = 7.7 * 10^{-19}\mathrm{m} \ll 10^{-15}\mathrm{m} \approx R_{\mathrm{proton}}$$

(Compare: Sun's $R_{\mathrm{Schwarzschild}} \approx 3\mathrm{km} \ll 7 * 10^5\mathrm{km} \approx R_{\mathrm{sun}}$)

- This suggests that the problems arise because of the oversimplifying point proton model.
- BUT: Finite Size Proton is a "nuclear analog of an atom" made of charged point quarks! $\Longrightarrow$ ∞ EM Self-Energy!

- WAY OUT: More Regular Electromagnetic Field Equations!

Mie (1912): Nonlinear Electromagnetic Vacuum Laws

- General Maxwell's equations involve four fields: **E**, **B**, **D**, **H**.
- Need electromagnetic vacuum laws for closure:

$$\mathbf{E} = \mathbf{E}(\mathbf{D}, \mathbf{B}), \qquad \mathbf{H} = \mathbf{H}(\mathbf{D}, \mathbf{B})$$

- Maxwell's vacuum law:

$$\mathbf{E} = \mathbf{D}, \qquad \mathbf{H} = \mathbf{B}$$

(Yields infinite electromagnetic self-energy of a point charge.)

- Mie (1912): nonlinear vacuum laws [Lagrangian framework]
- Born–Infeld's (BI) vacuum law (1934) [b is Born's field constant]:

$$\mathbf{E} = \frac{\mathbf{D} - \frac{1}{b^2}\mathbf{B}\times(\mathbf{B}\times\mathbf{D})}{\sqrt{1 + \frac{|\mathbf{D}|^2+|\mathbf{B}|^2}{b^2} + \frac{|\mathbf{B}\times\mathbf{D}|^2}{b^4}}}, \quad \mathbf{H} = \frac{\mathbf{B} - \frac{1}{b^2}\mathbf{D}\times(\mathbf{D}\times\mathbf{B})}{\sqrt{1 + \frac{|\mathbf{D}|^2+|\mathbf{B}|^2}{b^2} + \frac{|\mathbf{B}\times\mathbf{D}|^2}{b^4}}}$$

(Yields finite Electrostatic Self-Energy of point charges!)

Hoffmann's Electrostatic Spacetime (1935)

Banesh Hoffmann found an $SO(3)$-symmetric solution family to the Einstein-Maxwell-Born-Infeld field equations with single point charge.

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For nucleus parameters (Ze, M) we can be either in the **black hole** or in the naked singularity sector, depending on Born's parameter b.

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For nucleus parameters (Ze, M) we can be either in the **black hole** or in the naked singularity sector, depending on Born's parameter b.

- The borderline case occurs for $b = \frac{M^2 c^4}{Z^3 e^3 (\frac{1}{6} B(\frac{1}{4}, \frac{1}{4}))^2} \equiv b_H$.

Hoffmann's Electrostatic Spacetime (1935)

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- $g^{(G)}_{\mu\nu}dx^\mu dx^\nu = -f(r)^2c^2dt^2 + f(r)^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

  $$f(r)^2 \equiv 1 - \frac{2G}{c^4 r}\underbrace{\left(Mc^2 - b^2\int_r^\infty\left(\sqrt{1+\frac{Z^2e^2}{b^2s^4}}-1\right)s^2ds\right)}$$

  POSITIVE quasi-local MASS: > 0 for all $r > 0$ if $b \leq b_H$.
  (N.B.: $b\uparrow\infty$: **RWN**; $e\downarrow 0$: **Schwarzschild**; $G\downarrow 0$: **Minkowski**)
- Mc^2 =Electrostatic energy of MBI field of point nucleus if $b = b_H$.
- Mild Conical Curvature Singularity at $r = 0$ if $b = b_H$ (T.-Z. 2011).

General-Relativistic Hydrogenic Ions. III

Test Electron $(-e, m)$ in Hoffmann Spacetime of Nucleus $(Ze, M, b = b_H)$
Moulik Balasubramanian (Ph.D. thesis, 2015; arXiv:2009.11986v1):

- Dirac Hamiltonian H on $C_c^\infty(\mathbb{R}^3\backslash\{0\})^4$ is essentially self-adjoint
- $\sigma_{ess}(H) = (-\infty, -mc^2] \cup [mc^2, \infty)$; $\sigma_{sc}(H) = \emptyset$
- $\sigma_{pt}(H) \subset [-mc^2, mc^2]$ ∞ many Bound states
- Conjecture: For normal ions
  $\sigma_{disc}(H) = \sigma_{\text{Sommerfeld}} + O(10^{-9})_b + O(10^{-39})_G$

  Sommerfeld fine-structure spectrum with Small $O(10^{-9})$ Born-Infeld $b = b_H$ corrections plus Tiny $O(10^{-39})$ Einsteinian G corrections
- Generalizes to a class of electrostatic S.T. with zero bare mass:
  Dirac H for test electron is well-defined already w/o μ_a.

General-Relativistic Hydrogenic Ions. III

NOT ALL IS WELL, THOUGH!

Test Electron $(-e, m)$ in Hoffmann Spacetime of Nucleus $(Ze, M, b = b_H)$

- b_H depends on Z and M, but b is 'constant of nature.'
- Thus, $m_{bare} = 0$-Hoffmann spacetime for only one kind of nuclei!
- Need to investigate: $b \neq b_H$
  N.B.: $b > b_H$ (naked singularity sector; negative bare mass)
  N.B.: $b < b_H$ (**black hole** sector) (Leave aside for now!)
- In naked singularity sector we can have any finite negative

$$m_{bare}c^2 = M_{\text{ADM}}c^2 - E^{\text{field}}$$

- Curvature singularity milder than for **RWN**, but worse than for $m_{bare} = 0$-Hoffmann. This raises the questions:

  Is Dirac H of 'RWN type' or of '$m_{bare} = 0$-Hoffmann type'?

  Or is there a critical $m_{bare}^{crit} < 0$ where the type changes?

General-Relativistic Hydrogenic Ions. III

Test Electron ($-e, m$) in Hoffmann Spacetime of Nucleus ($Ze, M, b > b_H$)
(strictly negative bare mass)

M.K., S.T.-Z., Ebru Toprak (2020):

- Dirac Hamiltonian H on $C_c^\infty(\mathbb{R}^3\backslash\{0\})^4$ NOT essentially self-adjoint
- H has ∞ many self-adjoint extensions H_θ, $\theta \in [0, \pi]$
- $\sigma_{ess}(H_\theta) = (-\infty, -mc^2] \cup [mc^2, \infty)$; $\sigma_{sc}(H_\theta) = \emptyset$
- $\sigma_{pt}(H_\theta) \subset [-mc^2, mc^2]$ ∞ many Bound states
- $+mc^2$ is accumulation point of $\sigma_{disc}(H_\theta)$.
- $-mc^2$ is accumulation point of $\sigma_{disc}(H_\theta)$ iff $GMm > Ze^2$
  & naked singularity (presumably NOT nuclei)

- Not investigated: $b < b_H$ (**black hole** sector)
  N.B.: Presumably no bound states.

General-Relativistic Hydrogenic Ions. III

Test Electron $(-e, m, \mu_a)$ in Hoffmann S.T. of Nucleus $(Ze, M, b > b_H)$
(strictly negative bare mass)

M.K., S.T.-Z., Ebru Toprak (2020):

- Dirac Hamiltonian $H + H_{\mathrm{anom}}$ on $C_c^\infty(\mathbb{R}^3\backslash\{0\})^4$ still NOT essentially self-adjoint despite anomalous magnetic moment of electron.
- $H + H_{\mathrm{anom}}$ has ∞ many self-adjoint extensions H_θ
- $\sigma_{ess}(H_\theta) = (-\infty, -mc^2] \cup [mc^2, \infty)$; $\sigma_{sc}(H_\theta) = \emptyset$
- $\sigma_{pt}(H_\theta) \subset [-mc^2, mc^2]$ ∞ many Bound states
- $+mc^2$ is accumulation point of $\sigma_{disc}(H_\theta)$.
- $-mc^2$ is accumulation point of $\sigma_{disc}(H_\theta)$ iff $GMm > Ze^2$
  & naked singularity (presumably NOT nuclei)

- Not investigated: $b < b_H$ (**black hole** sector)
  N.B.: Presumably no bound states.

General-Relativistic Hydrogenic Ions. III

Test Electron $(-e, m)$ or $(-e, m, \mu_a)$ in much more general electrostatic Spacetime of Naked Nucleus (Ze, M) with strictly negative bare mass

M.K., S.T.-Z., Ebru Toprak (2020):

- Dirac H on $C_c^\infty(\mathbb{R}^3\backslash\{0\})^4$ NEVER essentially self-adjoint,
  but H always has ∞ many self-adjoint extensions H_θ.
- Dirac Hamiltonian $H + H_{\mathrm{anom}}$ on $C_c^\infty(\mathbb{R}^3\backslash\{0\})^4$ MAY or MAY NOT be essentially self-adjoint. Electric field must blow up strongly.
  If not e.s.a., $H + H_{\mathrm{anom}}$ has ∞ many self-adjoint extensions, H_θ.

Let $\widetilde{H}$ be any self-adjoint realization of the above.

- $\sigma_{ess}(\widetilde{H}) = (-\infty, -mc^2] \cup [mc^2, \infty)$; $\sigma_{sc}(\widetilde{H}) = \emptyset$
- $\sigma_{pt}(\widetilde{H}) \subset [-mc^2, mc^2]$ ∞ many Bound states
- $+mc^2$ is accumulation point of $\sigma_{disc}(\widetilde{H})$.
- $-mc^2$ is accumulation point of $\sigma_{disc}(\widetilde{H})$ iff $GMm > Ze^2$
  & naked singularity (presumably NOT nuclei)

General-Relativistic Hydrogenic Ions. III

Test Electron $(-e, m)$ or $(-e, m, \mu_a)$ in much more general electrostatic Naked Spacetime of Nucleus (Ze, M) with strictly negative bare mass

COMMENTS:

- Spacetimes cover 'everything' from RWN to $m_{bare} = 0$-Hoffmann, generalizing those studied by S.T.-Z. (2011)
- Proofs use separation of variables (static, spherically symmetric)
- $H = \bigoplus H_k^{\mathrm{rad}}$, with H_k^{rad} acting on 2-D subspaces (Partial Wave Decomposition)
- Spectral problem for H_k^{rad}: system of two linear ODEs.
- Apply techniques of Weidmann & of Hinton-Mingarelli-Read-Shaw

Concluding remarks

- Far from being merely a quantitatively tiny deformation of the special relativistic Dirac equation, the general-relativistic Dirac equation (Schrödinger 1932; Brill & Cohen 1966) is a completely different beast.
- Trying to develop intuition is difficult: whenever you think you understand something well enough to dare make a prediction, the next theorem proves your intuition wrong.
- There are many tough conceptual questions to be sorted out before one arrives at a well-defined general-relativistic QM. Given the expected tininess of effects at laboratory scales, empirical input is not going to be of any help to achieve this feat.

This is a job for mathematical physicists!

THANK YOU FOR YOUR ATTENTION!