A positive mass theorem for static causal fermion systems

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The mass of a static, asymptotically flat spacetime

Joint work with A. Platzer:

- F.F, A. Platzer,
  "A positive mass theorem for static causal fermion systems,"
  arXiv:1912.12995 [math-ph] (2019)

The mass of a static, asymptotically flat spacetime

- Let $\mathscr{M}$ be a globally hyperbolic Lorentzian manifold, always four-dimensional
- static: $\mathscr{M} = \mathbb{R} \times \mathscr{N} \ni (t, x)$
  ∂_t is Killing, orthogonal to $\mathscr{N}_t := \{(t, x) \,|\, x \in \mathscr{N}\}$
- g induced Riemannian metric on $\mathscr{N}$
- asymptotically flat:
  - $\exists$ diffeomorphism $\phi : \mathscr{N} \setminus K \to \mathbb{R}^3 \setminus \overline{B_R(0)}$
  - in corresponding chart,

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} + a_{\alpha\beta}(x) \,, \qquad x \in \mathbb{R}^3 \setminus \overline{B_R(0)}$$

$$a_{\alpha\beta} = \mathcal{O}(1/|x|), \quad \partial_\gamma a_{\alpha\beta} = \mathcal{O}(1/|x|^2) \quad \text{and} \quad \partial_{\gamma\delta} a_{\alpha\beta} = \mathcal{O}(1/|x|^3)$$

Then the *total mass* or *ADM mass* is defined by

$$\mathfrak{M}_{\text{ADM}} = \frac{1}{16\pi} \lim_{R \to \infty} \sum_{\alpha,\beta=1}^{3} \int_{S_R} \left(\partial_\beta g_{\alpha\beta} - \partial_\alpha g_{\beta\beta} \right) \nu^\alpha \, d\Omega$$

- S_R coordinate sphere with normal ν and area measure $d\Omega$

The total mass abstractly

- Let $\mathcal{G}$ be a locally compact and σ-compact Hausdorff space
- Let $\mathcal{L} : \mathcal{G} \times \mathcal{G} \to \mathbb{R}_0^+$ be
  - symmetric: $\mathcal{L}(x, y) = \mathcal{L}(y, x)$
  - continuous and bounded (for simplicity of presentation)
  - of compact range (for simplicity of presentation), i.e. $\mathcal{L}(x, .)$ has compact support $\forall x \in \mathcal{G}$.
- Let μ and $\tilde{\mu}$ be Radon measures on $\mathcal{G}$ (i.e. positive regular Borel measure, $\mu(K) < \infty$ for compact $K \subset \mathcal{G}$)
- Denote the supports of the measures by

$$N := \text{supp}\, \mu \,, \qquad \tilde{N} := \text{supp}\, \tilde{\mu}$$

$$\text{supp}\, \mu := \big\{ x \in \mathcal{G} \mid \rho(U) \neq 0 \text{ for every open neighborhood } U \subset N \text{ of } x \big\}$$

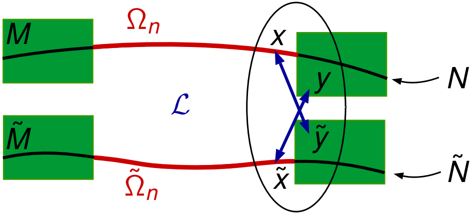
The total mass abstractly

Idea: "Compare μ and $\tilde{\mu}$ asymptotically near infinity"

- Let $(\Omega_n)_{n\in\mathbb{N}}$ be exhaustion of N by compact sets, $(\tilde{\Omega}_n)_{n\in\mathbb{N}}$ exhaustion of $\tilde{N}$ with

$$\mu(\Omega_n) = \tilde{\mu}(\tilde{\Omega}_n) \quad \forall n$$

$$\mathfrak{M} := \lim_{n\to\infty} \bigg(\int_{\tilde{\Omega}_n} d\tilde{\mu}(\tilde{x}) \int_{N\setminus\Omega_n} d\mu(y)\, \mathcal{L}(\tilde{x},y) - \int_{\Omega_n} d\mu(x) \int_{\tilde{N}\setminus\tilde{\Omega}_n} d\tilde{\mu}(\tilde{y})\, \mathcal{L}(x,\tilde{y}) \bigg)$$



- Has structure of a surface layer integral

Rewriting the surface layer integral as a volume term

$$A := \int_{\tilde{\Omega}_n} d\tilde{\mu}(\tilde{x}) \int_{N \setminus \Omega_n} d\mu(y) \mathcal{L}(\tilde{x}, y) - \int_{\Omega_n} d\mu(x) \int_{\tilde{N} \setminus \tilde{\Omega}_n} d\tilde{\mu}(\tilde{y}) \mathcal{L}(x, \tilde{y})$$

Moreover, using the symmetry of $\mathcal{L}$,

$$\int_{\tilde{\Omega}_n} d\tilde{\mu}(\tilde{x}) \int_{\Omega_n} d\mu(y)\, \mathcal{L}(\tilde{x}, y) - \int_{\Omega_n} d\mu(x) \int_{\tilde{\Omega}_n} d\tilde{\mu}(\tilde{y})\, \mathcal{L}(x, \tilde{y}) = 0$$

Add to obtain

$$\begin{aligned} A &= \int_{\tilde{\Omega}_n} d\tilde{\mu}(\tilde{x}) \int_{N} d\mu(y)\, \mathcal{L}(\tilde{x}, y) - \int_{\Omega_n} d\mu(x) \int_{\tilde{N}} d\tilde{\mu}(\tilde{y})\, \mathcal{L}(x, \tilde{y}) \\ &= \int_{\tilde{\Omega}_n} \tilde{n}(\tilde{x})\, d\tilde{\mu}(\tilde{x}) - \int_{\Omega_n} n(x)\, d\mu(x) \end{aligned}$$

with

$$n(x) := \int_N \mathcal{L}(x, \tilde{y})\, d\tilde{\mu}(\tilde{y})\,, \qquad \tilde{n}(\tilde{x}) := \int_N \mathcal{L}(\tilde{x}, y)\, d\mu(y)\,.$$

- analog of Gauß divergence theorem, but nonlinear

The total mass abstractly

Thus the total mass becomes

$$\mathfrak{M} = \lim_{n\to\infty} \bigg(\int_{\tilde{\Omega}_n} \tilde{n}(\tilde{x})\, d\tilde{\mu}(\tilde{x}) - \int_{\Omega_n} n(x)\, d\mu(x) \bigg)$$

Use that the volumes of Ω_n and $\tilde{\Omega}_n$ are equal,

$$\begin{aligned} &= \lim_{n\to\infty} \bigg(\int_{\tilde{\Omega}_n} \big(\tilde{n}(\tilde{x}) - \mathfrak{s}\big)\, d\tilde{\mu}(\tilde{x}) - \int_{\Omega_n} \big(n(x) - \mathfrak{s}\big)\, d\mu(x) \bigg) \\ &= \int_{\tilde{N}} \big(\tilde{n}(\tilde{x}) - \mathfrak{s}\big)\, d\tilde{\mu}(\tilde{x}) - \int_{N} \big(n(x) - \mathfrak{s}\big)\, d\mu(x)\,, \end{aligned}$$

provided that the integrals exist, i.e.

$$n(x) - \mathfrak{s} \in L^1(N, d\mu)\,, \qquad \tilde{n}(\tilde{x}) - \mathfrak{s} \in L^1(\tilde{N}, d\tilde{\mu})\,.$$

The total mass abstractly

Definition

The measures $\tilde{\mu}$ and μ are asymptotically close if they are both σ-finite with infinite total volume (i.e. $\tilde{\mu}(\tilde{N}) = \mu(N) = \infty$), and for a suitable constant $\mathfrak{s} \geq 0$,

$$\int_N \big| n(x) - \mathfrak{s} \big| \, d\mu(x) < \infty \qquad \text{and} \qquad \int_{\tilde{N}} \big| \tilde{n}(\tilde{x}) - \mathfrak{s} \big| \, d\tilde{\mu}(\tilde{x}) < \infty$$

$$\text{where} \quad n(x) = \int_{\tilde{N}} \mathcal{L}(x, \tilde{y}) \, d\tilde{\mu}(\tilde{y}) \,, \quad \tilde{n}(\tilde{x}) = \int_N \mathcal{L}(\tilde{x}, y) \, d\mu(y) \,.$$

Lemma

Under these assumptions, the total mass is well-defined and finite and

$$\mathfrak{M} = \int_{\tilde{N}} \big(\tilde{n}(\tilde{x}) - \mathfrak{s} \big) \, d\tilde{\mu}(\tilde{x}) - \int_N \big(n(x) - \mathfrak{s} \big) \, d\mu(x) \,.$$

The causal variational principle

The *causal variational principle* is to minimize the action

$$\mathcal{S}(\mu) = \int_{\mathcal{G}} d\mu(x) \int_{\mathcal{G}} d\mu(y)\, \mathcal{L}(x,y)$$

- Vary μ in the class of all regular Borel measures,
- keeping the total volume $\mu(\mathcal{G})$ fixed. (*volume constraint*).

Existence of minimizers is proven in this generality in

- F.F., C. Langer, "Causal variational principles in the σ-locally compact setting: Existence of minimizers," arXiv:2002.04412 [math-ph] (2020)

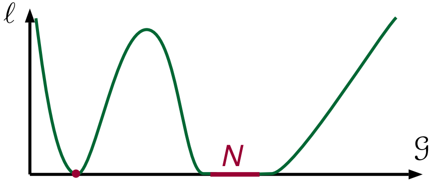
The Euler-Lagrange equations

$$\ell : \mathcal{G} \to \mathbb{R} \,, \qquad \ell(x) := \int_N \mathcal{L}(x,y)\, d\mu(y) - \mathfrak{s}$$

Lemma

Let ρ be a minimizer of the causal action. Then, for a suitable value $\mathfrak{s} \geq 0$,

$$\ell|_N \equiv \inf_{\mathcal{G}} \ell = 0 \,.$$



Proof.

Given $x_0 \in \text{supp}\, \mu$, choose open neighborhood $U \subset N$ of x_0 with $0 < \mu(U) < \infty$. Consider variation

$$\tilde{\mu}_\tau = \chi_{N \setminus U}\, \mu + (1-\tau)\, \chi_U\, \mu + \tau\, \mu(U)\, \delta_y$$

with $\tau \in [0,1)$ and $y \in \mathcal{G}$ (where δ_y is the Dirac measure). Then $S(\tilde{\mu}_\tau) - S(\mu)$ is well-defined and finite. Moreover,

$$0 \leq \frac{d}{d\tau} \mathcal{S}(\tilde{\mu}_\tau)\big|_{\tau=0} = 2 \int_{\mathcal{G}} d\dot{\tilde{\mu}}_\tau\big|_{\tau=0} \int_{\mathcal{G}} d\mu\, \mathcal{L}(x,y)$$
$$= 2 \left(\mu(U)\, \ell(y) - \int_U \ell(x)\, d\mu(x) \right)$$

$$\Longrightarrow \quad \ell(y) \geq \frac{1}{\mu(U)} \int_U \ell(x)\, d\mu(x)$$

□

"Asymptotically close" revisited

Rewrite the above definition using ℓ:

Definition

The measures $\tilde{\mu}$ and μ are asymptotically close if they are both σ-finite with infinite total volume (i.e. $\tilde{\mu}(\tilde{N}) = \mu(N) = \infty$), but for a suitable constant $\mathfrak{s} \geq 0$,

$$\int_N \big|\tilde{\ell}(x)\big|\, d\mu(x) < \infty \qquad \text{and} \qquad \int_{\tilde{N}} \big|\ell(\tilde{x})\big|\, d\tilde{\mu}(\tilde{x}) < \infty$$

If μ and $\tilde{\mu}$ are minimizing measures, then

$$\tilde{\ell}(x) \geq \tilde{\ell}|_{\tilde{N}} \equiv 0\,, \qquad \ell(\tilde{x}) \geq \ell|_N \equiv 0$$

- Now measures are asymptotically close if N and $\tilde{N}$ "approach each other near infinity"

Example: Dirac spinors in space-time

Let $\mathscr{M}$ be a Lorentzian space-time,
for simplicity 4-dimensional, globally hyperbolic,
then automatically spin,

$$(S\mathscr{M}, \prec.|.\succ) \qquad \text{spinor bundle}$$

- $S_p\mathscr{M} \simeq \mathbb{C}^4$
- spin inner product

$$\prec.|.\succ_p \;:\; S_p\mathscr{M} \times S_p\mathscr{M} \to \mathbb{C}$$

is indefinite of signature (2,2)

$$(\mathcal{D} - m)\,\psi_m = 0 \qquad \text{Dirac equation}$$

Example: Dirac spinors in space-time

- Cauchy problem well-posed, global smooth solutions (for example symmetric hyperbolic systems)
- finite propagation speed

$C^\infty_{\text{sc}}(\mathcal{M}, S\mathcal{M})$ spatially compact solutions

$$(\psi_m|\phi_m)_m := 2\pi \int_{\mathcal{N}} \prec \psi_m | \not{\nu} \phi_m \succ_x \, d\mu_{\mathcal{N}}(x) \quad \text{scalar product}$$

completion gives Hilbert space $(\mathcal{H}_m, (.|.)_m)$

Example: Dirac spinors in space-time

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\text{span}(\psi_1, \ldots, \psi_f)}$$

- To $x \in \mathbb{R}^4$ associate a local correlation operator

$$\langle \psi | F(x) \phi \rangle = - \prec \psi(x) | \phi(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

  Is self-adjoint, rank ≤ 4
  at most two positive and at most two negative eigenvalues
- Here ultraviolet regularization may be necessary:

$$\langle \psi | F(x) \phi \rangle = - \prec (\mathfrak{R}_\varepsilon \psi)(x) | (\mathfrak{R}_\varepsilon \phi)(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

$$\mathfrak{R}_\varepsilon \,:\, \mathcal{H} \to C^0(\mathscr{M}, S\mathscr{M}) \qquad \text{regularization operators}$$

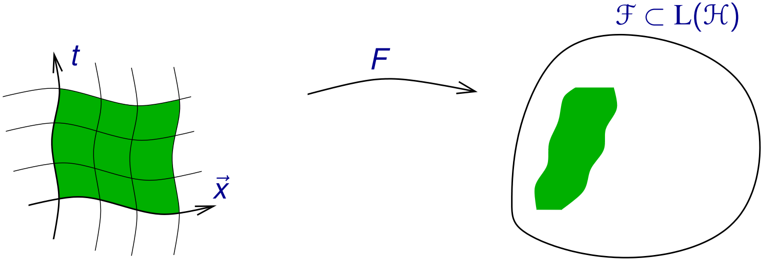
$$\varepsilon > 0 : \text{regularization scale (Planck length)}$$

Example: Dirac spinors in space-time

- Thus $F(x) \in \mathcal{F}$ where
  $\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H})$ with the properties:
  - ▷ F is self-adjoint and has rank ≤ 4
  - ▷ F has at most 2 positive
    and at most 2 negative eigenvalues $\Big\}$

Example: Dirac spinors in space-time

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$



- push-forward measure $\rho := F_*(\mu_{\mathcal{M}})$, is measure on $\mathcal{F}$,

$$\rho(U) := \mu_{\mathcal{M}}\big(F^{-1}(U)\big)$$

- support of the measure is closure of image of F.

Causal fermion systems

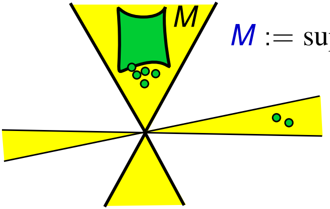
Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space
Given parameter $n \in \mathbb{N}$ ("spin dimension")
$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H})$ with the properties:

- x is self-adjoint and has finite rank
- x has at most n positive

  and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$ ("universal measure")



$M := \text{supp}\, \rho$ space-time

Static Causal Fermion Systems

- Assume that $\mathscr{M}$ is a static globally hyperbolic spacetime. Then

$$M := \text{supp}\,\rho = \mathbb{R} \times N$$

$$d\rho = dt\, d\mu\,, \qquad N = \text{supp}\,\mu\,.$$

- On the level of causal fermion systems,
  - one-parameter unitary group $(\mathcal{U}_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$
  - is a symmetry of ρ,

$$\rho\big(\mathcal{U}_t\,\Omega\,\mathcal{U}_t^{-1}\big) = \rho(\Omega)\,.$$

  - $\mathcal{G} := \mathcal{F}/\mathbb{R}$

There is an explicitly given static Lagrangian

$$\mathcal{L} : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^+_0 \quad \text{(more details later)}$$

Correspondence to the ADM mass

- $(\mathcal{H}, \mathcal{F}, \rho)$ CFS describing Minkowski vacuum
  ($\mathcal{H}$ all negative energy solutions, regularization on scale ε)
- $(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\rho})$ CFS describing a static, asymptotically Schwarzschild spacetime
- identify $\mathcal{H}$ and $\tilde{\mathcal{H}}$ unitarily.
  - Arrange that measures are asymptotically close
  - Apart from this, identification is irrelevant

Theorem

$$\mathfrak{M} = c\, \mathfrak{M}_{\mathrm{ADM}}$$

with the constant c given by

$$c = \frac{1}{4\pi} \int_{\mathbb{R}^3} |y|^2 \, \mathcal{L}(0, y) \, d^3(y) \; > \; 0\,.$$

Correspondence to the ADM mass

Remarks on the proof:

- Volume constraint $\mu(\Omega_n) = \tilde{\mu}(\tilde{\Omega}_n)$ implies that the leading contribution $\sim \mathfrak{s}$ drops out.
- It remains to compute the next-to-leading order $\sim c$
  - independent of the volumes of the inner regions
  - only involves the metric near infinity
  - described by a linear surface layer integral, $w \in \Gamma(N, T\mathcal{G})$

$$\mathfrak{M} = \lim_{\Omega \nearrow N} \int_\Omega d\mu(x) \int_{N \setminus \Omega} d\mu(y) \, \big(D_{1,w} - D_{2,w}\big) \mathcal{L}(x,y)$$

- use perturbative methods (linearized gravity)
  - Compute the perturbation of all Dirac wave functions
  - Compute first variations of $\mathcal{L}$
  - Compute the surface layer integral asymptotically on large spheres

The causal action principle

Let $x, y \in \mathcal{F}$. Then x and y are linear operators.

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not self-adjoint: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

The causal action principle

Nontrivial eigenvalues of xy: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

Lagrangian $\quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$

action $\quad \mathcal{S} = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y) \in [0, \infty]$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $\quad \rho(\mathcal{F}) = \text{const}$

trace constraint: $\quad \int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$

boundedness constraint: $\quad \iint_{\mathcal{F} \times \mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$

The static causal action principle

- Choose unitary group $(\mathcal{U}_t)_{t \in \mathbb{R}}$ acting on $\mathcal{H}$.
- Vary in the class of static measures.
- Treat the boundedness constraint with a Lagrange multiplier $\kappa > 0$,

$$\mathcal{L}_\kappa(x,y) := \mathcal{L}(x,y) + \kappa \sum_{i=1}^{2n} |\lambda_i^{xy}|^2 \,.$$

- Then the static causal action principle is to minimize

$$\mathcal{S}(\mu) = \int_{\mathcal{G}} d\mu(x) \int_{\mathcal{G}} d\mu(y)\, \mathcal{L}(x,y)$$

  where

$$\mathcal{L}(x,y) := \int_{-\infty}^{\infty} \mathcal{L}_\kappa\big((t_0,x),\,(t,y)\big)\, dt$$

  (independent of t_0 due to time symmetry)
- $\kappa = \kappa(C)$ dimensionless parameter

The positive mass theorem

Definition

The measure μ is κ-extendable if the following conditions hold:

(i) There is a family of measures $(\mu_\tau)_{\tau \in (-1,1)}$ of the form

$$\mu_\tau = (F_\tau)_* \mu \,,$$

each of which satisfies the EL equations with a parameter $\kappa(\tau)$ and

$$F_0 = \mathrm{id}_N \qquad \text{and} \qquad \kappa'(0) = -1 \,.$$

(ii) For every $x \in N$, the curve $F_\tau(x)$ is differentiable at $\tau = 0$, giving rise to a vector field

$$v := \frac{d}{d\tau} F_\tau \Big|_{\tau=0} \;\in\; \Gamma(N, T\mathcal{G}) \,.$$

The positive mass theorem

- Assume that μ and $\tilde{\mu}$ are κ-scalable. Then

$$\mathfrak{M} = \lim_{\Omega \nearrow N} \int_\Omega d\mu(x) \int_{N \setminus \Omega} d\mu(y) \; \big(D_{1,w} - D_{2,w} \big) \mathcal{L}(x,y)$$

with

$$w = g \, \big(\tilde{v} - v \big)$$

and $g \in \mathbb{R}$, called the gravitational coupling constant.

The positive mass theorem

Theorem (Positive mass theorem)

The total mass can be written as

$$\mathfrak{M} = g \int_{\tilde{N}} \big(\tilde{\ell} - \tilde{\ell}_\infty \big)\, d\tilde{\mu}$$

If $\tilde{\mu}$ satisfies the local energy condition

$$\tilde{\ell}(x) \geq \tilde{\ell}_\infty \qquad \text{for all } x \in \tilde{N}$$

and the gravitational coupling constant g is positive, then the total mass is non-negative,

$$\mathfrak{M} \geq 0 \,.$$

- No smoothness assumptions. Definition of total mass works similarly for discrete spacetimes or generalized "quantum spacetimes."

Outlook

- Rigidity statement: $\mathfrak{M} = 0 \quad \Longrightarrow \quad \tilde{\ell} \equiv \tilde{\ell}_\infty$
  Which measures have this property? Are they unique? In which sense?
- Time-dependent setting: Ongoing work with J. Wurm
- Penrose inequality: Seems difficult; spinor methods do not seem to apply
- Connection between area and black hole entropy: Ongoing work with E. Curiel, J. Isidro, M. Lottner
- ……

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Thank you for your attention!

Inherent structures of a causal fermion system

Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion of spin dimension n, space-time $M := \text{supp}\rho$.

space-time points are linear operators on $\mathcal{H}$

- For $x \in M$, consider eigenspaces of x.
- For $x, y \in M$,
  - consider operator products xy
  - project eigenspaces of x to eigenspaces of y

Gives rise to:

- quantum objects (spinors, wave functions)
- geometric structures (connection, curvature)
- causal structure, analytic structures

Causal structure

Let $x, y \in M$. Then
$x \cdot y \in \mathrm{L}(H)$ has non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Definition (causal structure)

The points $x, y \in \mathcal{F}$ are called

$$\begin{cases} \text{spacelike separated} & \text{if } |\lambda_j^{xy}| = |\lambda_k^{xy}| \text{ for all } j, k = 1, \ldots, 2n \\ \text{timelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all real} \\ & \text{and } |\lambda_j^{xy}| \neq |\lambda_k^{xy}| \text{ for some } j, k \\ \text{lightlike separated} & \text{otherwise} \end{cases}$$

- Lagrangian is compatible with causal structure:

$$\text{Lagrangian} \quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$$

thus x, y spacelike separated $\Rightarrow \mathcal{L}(x,y) = 0$

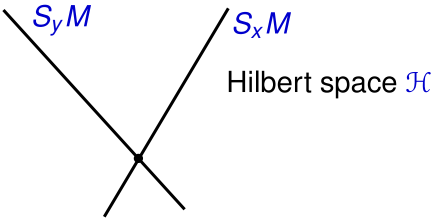
"points with spacelike separation do not interact"

- Spinors

$$S_xM := x(\mathcal{H}) \subset \mathcal{H} \quad \text{“spin space”}, \; \dim S_xM \leq 2n$$

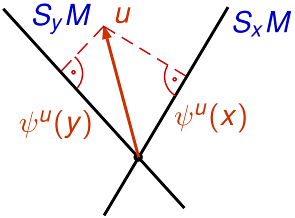
$$\prec u | v \succ_x := \langle u \,|\, x\, v \rangle_{\mathcal{H}} \quad \text{“spin scalar product”},$$

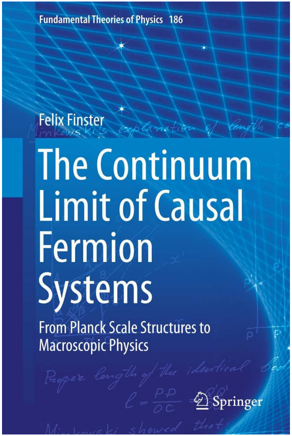
inner product of signature $(\leq n, \leq n)$



- Physical wave functions

$$\psi^u(x) = \pi_x\, u \;\text{ with } u \in \mathcal{H} \qquad \text{physical wave function}$$
$$\pi_x : \mathcal{H} \to \mathcal{H} \qquad \text{orthogonal projection on } x(\mathcal{H})$$





Fundamental Theories
of Physics **186**
Springer, 2016
548+xi pages

arXiv:1605.04742 [math-ph]

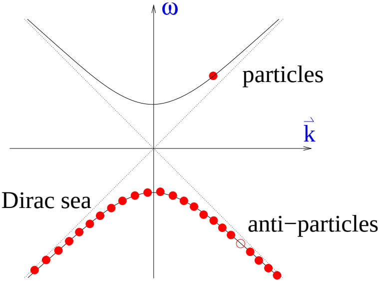
In this limiting case one gets:

- interactions of the standard model
- classical gravity: Einstein equations modulo higher order curvature corrections

The continuum limit in Minkowski space

Specify vacuum:

- Choose $\mathcal{H}$ as the space of all negative-energy solutions, hence "Dirac sea"



Fixes length scale ("Compton length")

- Introduce ultraviolet regularization

Fixes length scale δ ("Planck length")

Fixes length scale ε ("regularization length")

This is a minimizer of the causal action (in a well-defined sense).

The continuum limit in Minkowski space

- Construct causal fermion system in gravitational field (as outlined above)
- Consider the Euler-Lagrange equations of causal action principle
- Analyze the asymptotics as $\varepsilon \searrow 0$
- One gets a statement of the form

$$\begin{aligned} &\text{EL equations are satisfied as } \varepsilon \searrow 0 \\ \Longleftrightarrow \quad &\text{linearized Einstein equations hold} \end{aligned}$$

Total mass, area and matter flux

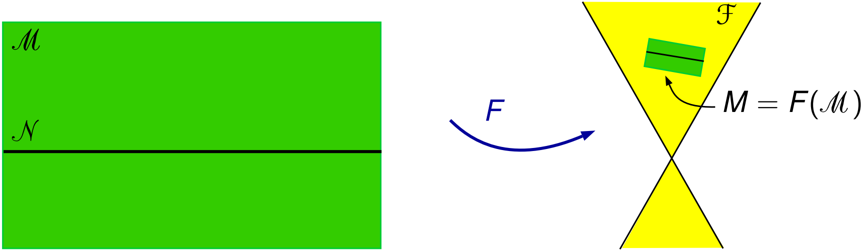
General question:

How does the causal action principle relate matter to the geometry of space-time?

based on two papers:

- Erik Curiel, F.F., José M. Isidro, “Two-dimensional area and matter flux in the theory of causal fermion systems,” arXiv:1910.06161 [math-ph] (2019)
- F.F., Andreas Platzer, “A positive mass theorem for static causal fermion systems,” PhD thesis defended in July 2019, paper in preparation

Static causal fermion systems



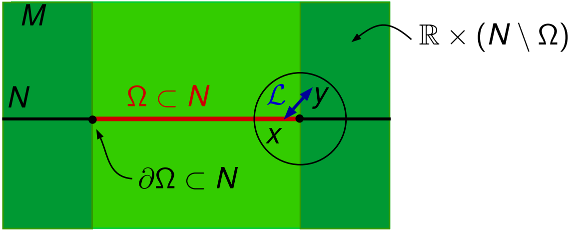
Time translations realized by unitary group on $\mathcal{H}$,

$$F(t + \Delta t, \mathbf{x}) = U(\Delta t)\, F(t, \mathbf{x})\, U(\Delta t)^{-1}$$

again work on the right side

decompose the space-time measure: $d\rho = dt\, d\mu$

Two-dimensional area in the static case



$$A(\partial\Omega) := \int_\Omega d\mu(x) \int_{\mathbb{R}\times(N\setminus\Omega)} d\rho(y)\, \mathcal{L}(x,y)$$

- Make use of the fact that $\mathcal{L}(x,y)$ is of short range (Compton scale)
- Is example of surface layer integral (as developed with Johannes Kleiner 2014-17)

The ADM mass

- static Lorentzian space-time, induced Riemannian metric g on hypersurface $g = \text{const.}$
- $g_{\alpha\beta} = \mathcal{O}_2\left(\frac{1}{r}\right)$

$$m_{\text{ADM}} = \frac{1}{16\pi} \lim_{R\to\infty} \sum_{\alpha,\beta=1}^{3} \int_{S_R} \left(\partial_\beta g_{\alpha\beta} - \partial_\alpha g_{\beta\beta}\right) \nu^\alpha \, d\Omega$$

$$m_{\text{iso}} = \limsup_{r\to\infty} \frac{2}{A(r)} \left(V(r) - \frac{1}{6\sqrt{\pi}} \, A(r)^{\frac{3}{2}} \right)$$

The total mass in the static case

Consider two jointly static measures

- $d\rho = dt\, d\mu$: vacuum
- $d\tilde{\rho} = dt\, d\tilde{\mu}$: asymptotically flat, static space-time

The total mass in the static case

- Definition very general, no smoothness assumptions!

THEOREM

For causal fermion systems constructed from Dirac solutions in a static, asymptoticaly flat space-time,

$$\mathfrak{M} = C\, M_{ADM}$$

THEOREM

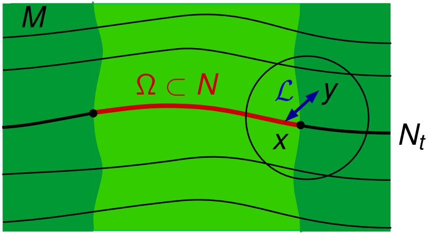
Under suitable assumptions (asymptotic flat and κ-scalable),

$$\mathfrak{M} = g \int_{\tilde{N}} \big(\tilde{\ell} - \tilde{\ell}_\infty \big)\, d\tilde{\mu}$$

- uses EL equations of causal action
- gives rise to a positive mass theorem

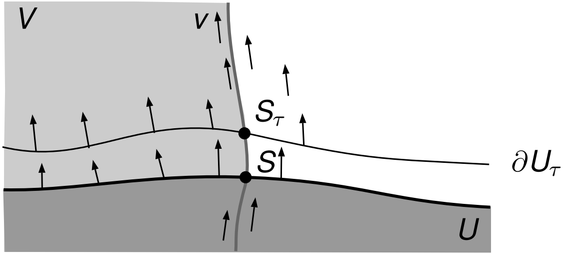
Two-dimensional area in the dynamical case

- Choose local time function $T : U \subset M \to \mathbb{R}$, gives foliation
$$N_t := T^{-1}(t) .$$
- Decompose the measure as
$$d\rho = dt \, d\mu_t$$
- For $\Omega \subset N_t$ define area of its boundary by
$$A(\partial\Omega) := \int_\Omega d\mu_t(x) \int_{\mathbb{R}\times(N_t\setminus\Omega)} d\rho(y) \, \mathcal{L}(x, y)$$



Area and area change

More convenient: consider flow by vector field v:



$$A = \int_{U \cap V} d\rho(x) \, \nabla_{\mathfrak{v}} \int_{M \setminus V} d\rho(y) \, \mathcal{L}_\kappa(x,y)$$
$$= \int_{U \cap V} d\rho(x) \int_{M \setminus V} d\rho(y) \, \big(\nabla_{1,\mathfrak{v}} \pm \nabla_{2,\mathfrak{v}}\big) \mathcal{L}_\kappa(x,y)$$

Area and area change

- jet $\mathfrak{v} := (b, v)$
- jet derivative $\nabla_{\mathfrak{v}} g(x) := a(x)\, g(x) + \big(D_v g\big)(x)$
- choose b as the divergence of the vector field,

$$b = \mathrm{div}\, v := \frac{1}{h}\, \partial_j \big(h\, v^j\big) \quad \text{where} \quad d\rho = h(x)\, d^4x\,.$$

$$\begin{aligned} A &= \int_{U \cap V} d\rho(x)\, \nabla_{\mathfrak{v}} \int_{M \setminus V} d\rho(y)\, \mathcal{L}_\kappa(x,y) \\ &= \int_{U \cap V} d\rho(x) \int_{M \setminus V} d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} \pm \nabla_{2,\mathfrak{v}}\big) \mathcal{L}_\kappa(x,y) \end{aligned}$$

Area and area change

Now one can compute the time derivative:

$$
\begin{aligned}
&\frac{d}{d\tau} A(S_\tau)\Big|_{\tau=0} \\
&= \int_{U\cap V} d\rho(x) \int_{M\setminus V} d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}_\kappa(x,y) \\
&\quad + \int_{U\cap V} d\rho(x) \int_{M\setminus V} d\rho(y)\, \mathcal{L}_\kappa(x,y)\, \Big(D_v \mathrm{div}\, v(x) - D_v \mathrm{div}\, v(y)\Big) \\
&\quad + \int_{U\cap V} d\rho(x) \int_{M\setminus V} d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}_\kappa(x,y)\, \big(\mathrm{div}\, v(x) + \mathrm{div}\, v(y)\big)
\end{aligned}
$$

Symmetries and Killing fields

DEFINITION

A vector field v on M is called **Killing field** *of the causal fermion system if the following conditions hold:*

(i) *The divergence of v vanishes,*

$$\mathrm{div}\, v = 0$$

(ii) *The directional derivative of the Lagrangian is small in the sense that*

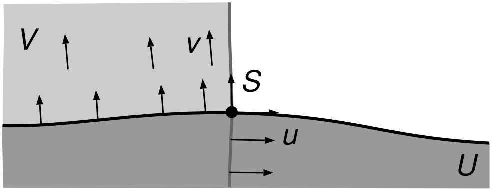
$$\big(D_{1,v} + D_{2,v}\big)\mathcal{L}_\kappa(x,y) \lesssim \frac{m^4}{\varepsilon^4\, \delta^4}$$

- in the last inequality the EL equations of the causal action principle are used!

Matter flux

Now consider

- v Killing field
- $\mathfrak{u} = (\operatorname{div} u, u)$ with u tangential to ∂U



Then the matter flux can be introduced by

$$F(S_\tau) := \int_{\Omega \cap V} d\rho(x) \int_{M \setminus V} d\rho(y) \big(\nabla_{1,\mathfrak{u}} - \nabla_{2,\mathfrak{u}}\big) \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathcal{L}_\kappa(x,y).$$

Limiting case of lightlike propagation

If v is a Killing field, then

$$\frac{d}{d\tau} A(S_\tau)\Big|_{\tau=0}$$
$$= \int_{U\cap V} d\rho(x) \int_{M\setminus V} d\rho(y) \; (\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}})(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}) \mathcal{L}_\kappa(x,y)$$
$$F(S_\tau)$$
$$= \int_{\Omega\cap V} d\rho(x) \int_{M\setminus V} d\rho(y) \; (\nabla_{1,\mathfrak{u}} - \nabla_{2,\mathfrak{u}})(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}) \mathcal{L}_\kappa(x,y)$$

In the limiting case when v becomes timelike, u and v coincide. Thus

$$\frac{d}{d\tau} A(S_\tau)\Big|_{\tau=0} = F(S_\tau)$$

This generalizes a formula by Ted Jacobson (1995) to the setting of causal fermion systems.

Summary

- Area, area change, matter flux and total mass can be defined intrinsically for a causal fermion system
- agreement with classical notions (ADM mass, Jacobson's area law)
- conclusion: causal action principle describes gravitational effects in a sensible way
- gives an intuitive and direct understanding of the causal action principle