Causal fermion systems and the causal action principle

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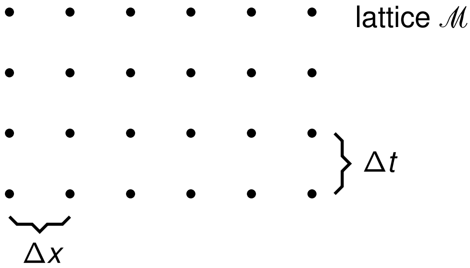
What is a causal fermion system?

- approach to fundamental physics
- novel mathematical model of space-time
- physical equations are formulated in generalized space-times

Motivation

- Planck scale gives natural length scale for "new physics"
- ultraviolet divergences of QFT
- Consider lattice system, for simplicity 2d



Motivation

Usual way to set up equations:

- Replace derivatives by difference quotients

$$0 = \Box\phi(t,x) := \frac{1}{(\Delta t)^2}\Big(\phi(t+\Delta t,x) - 2\phi(t,x) + \phi(t-\Delta t,x)\Big) - \frac{1}{(\Delta x)^2}\Big(\phi(t,x+\Delta x) - 2\phi(t,x) + \phi(t,x-\Delta x)\Big)$$

- Gives evolution equation, proceed time step by time step

Motivation

Drawback of this approach:

- Ad hoc: Why square lattice, why difference quotients?
- Is not background-free: What is lattice spacing?
- Not invariant under general coordinate transformations, not compatible with the equivalence principle

Basic question: Can one formulate equations without referring to the nearest neighbor relation and lattice spacing?

Motivation

- Consider wave functions $\psi_1, \ldots, \psi_f$ on lattice ($f < \infty$)
- Introduce scalar product; orthonormalize,

$$\langle \psi_k | \psi_l \rangle = \delta_{kl} \,,$$

gives f-dim Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.

important object: for any lattice point (t, x) introduce

local correlation operator $F(t,x) : \mathcal{H} \to \mathcal{H}$

- define matrix elements by

$$(F(t,x))^j_k = \overline{\psi_j(t,x)} \psi_k(t,x)$$

basis invariant:

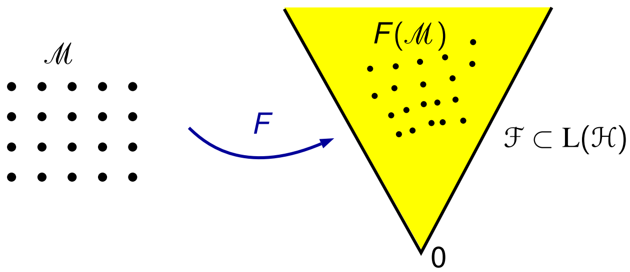
$$\langle \psi, F(t,x)\, \phi \rangle_{\mathcal{H}} = \overline{\psi(t,x)} \phi(t,x) \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

- Hermitian matrix
- Has rank at most one, is positive semi-definite

$$F(t,x) = e^* e \qquad \text{with} \qquad e : \mathcal{H} \to \mathbb{C} \,, \quad \psi \mapsto \psi(x)$$

Motivation

$$\mathcal{F} := \{F \text{ Hermitian, rank one, positive semi-definite}\}$$



general idea:

- disregard objects on the left
  (nearest neighbors, lattice spacing)
- work instead with the objects on the right
  (only local correlation operators)

Motivation

How to set up equations in this setting?
Explain idea in simple example:

- local correlation operators $F_1, \ldots, F_N \in \mathcal{F}$
- product $F_i \, F_j$ tells about correlation of wave functions at different space-time points
- $\mathrm{Tr}(F_i F_j)$ is real number
- minimize

$$\mathcal{S} = \sum_{i,j=1}^{N} \mathrm{Tr}(F_i F_j)^2$$

under suitable constraints.

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space

Given parameter $n \in \mathbb{N}$ ("spin dimension")

$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H})$ with the properties:

- x is self-adjoint and has finite rank
- x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$ ("universal measure")

$$\sum_{i=1}^{N} \cdots \rightsquigarrow \int_{\mathcal{F}} \cdots \, d\rho$$

Definition (Causal fermion system)

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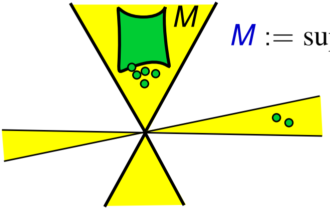


$M := \operatorname{supp} \rho$ space-time

Causal action principle

Let $x, y \in \mathcal{F}$. Then x and y are linear operators.

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not self-adjoint: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Causal action principle

Nontrivial eigenvalues of xy: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

Lagrangian $\quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \;\geq\; 0$

action $\quad \mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y) \;\in\; [0,\infty]$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $\quad \rho(\mathcal{F}) = \text{const}$

trace constraint: $\quad \int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$

boundedness constraint: $\quad \iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$

- F.F., “Causal variational principles on measure spaces,” *J. Reine Angew. Math.* **646** (2010) 141–194

Example: Dirac spinors in Minkowski space

Space-time is Minkowski space, signature $(+ - - -)$

- free Dirac equation $\quad (i\gamma^k \partial_k - m)\, \psi = 0$
- probability density $\psi^\dagger \psi = \overline{\psi} \gamma^0 \psi$,

  gives rise to a scalar product:

$$\langle \psi | \phi \rangle = \int_{t=\text{const}} (\overline{\psi} \gamma^0 \phi)(t, \vec{x})\, d\vec{x}$$

  time independent due to current conservation

Example: Dirac spinors in Minkowski space

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\text{span}(\psi_1, \ldots, \psi_f)}$$

For simplicity in presentation assume: ψ_i continuous.

- To $x \in \mathbb{R}^4$ associate a local correlation operator

$$\langle \psi | F(x) \phi \rangle = -\overline{\psi(x)} \phi(x) \qquad \forall \psi, \phi \in \mathcal{H}$$

Is self-adjoint, rank ≤ 4,
at most two positive and at most two negative eigenvalues

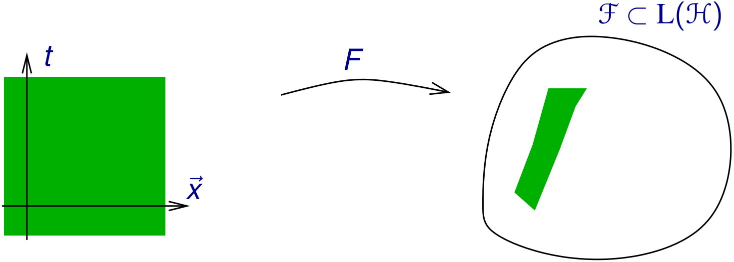
- Thus $F(x) \in \mathcal{F}$ where

$$\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

  - F is self-adjoint and has rank ≤ 4
  - F has at most 2 positive and at most 2 negative eigenvalues $\Big\}$

Example: Dirac spinors in Minkowski space

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$

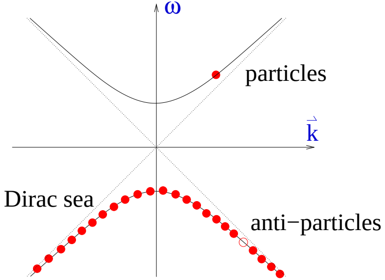


- push-forward measure $d\rho := F_*(d^4x)$, is measure on $\mathcal{F}$.

Example: the Minkowski vacuum

Specify vacuum:

- Choose $\mathcal{H}$ as the space of all negative-energy solutions, hence “Dirac sea”



Fixes length scale (“Compton length”)

- Introduce ultraviolet regularization

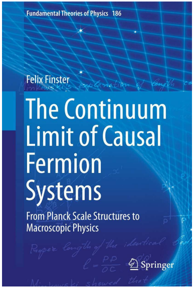
Fixes length scale ε (“Planck length”)

This is a minimizer of the causal action (in a well-defined sense).

Analysis in the Continuum Limit

This is the starting point for continuum limit analysis:

- Consider Dirac systems in a classical bosonic field,. Are measures critical points in the limit $\varepsilon \searrow 0$?



Fundamental Theories of Physics **186**
Springer, 2016
548+xi pages

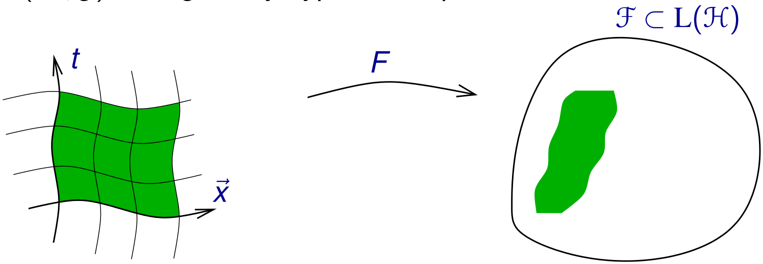
arXiv:1605.04742 [math-ph]

classical fields coupled to second-quantized Dirac field:

- interactions of the standard model (electroweak + strong)
- general relativity

Example: Dirac spinors in space-time

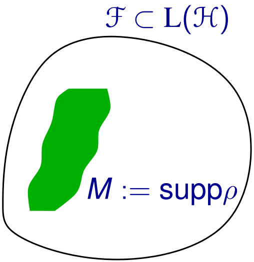
Let $(\mathscr{M}, g)$ be a globally hyperbolic space-time.



Take push-forward measure

$$\rho := F_*(\mu_{\mathscr{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathscr{M}}\big(F^{-1}(\Omega)\big)\text{)}$$

Example: Dirac spinors in space-time



In general, this is *not* a minimizer

general concept: “matter encodes geometry”

Underlying physical principles

- Pauli exclusion principle:
  Choose orthonormal basis $\psi_1, \ldots, \psi_f$ of $\mathcal{H}$. Set

$$\Psi = \psi_1 \wedge \cdots \wedge \psi_f \ ,$$

  gives equivalent description by Hartree-Fock state.
- local gauge principle:
  freedom to perform local unitary transformations.
- the "equivalence principle":
  symmetry under "diffeomorphisms" of M
  (note: M merely is a topological measure space)

locality, causality and time direction are emergent

Interpretation in terms of space-time events

- operators in $\mathcal{F}$ can be interpreted as "possible local correlation operators"
  or simply as possible events
- operators in M are the events realized in space-time
- space-time is made up of all the events
- the physical equations relate the events to each other

For details on this connection:

- F.F, J. Fröhlich, C. Paganini, C. and M. Oppio, "Causal fermion systems and the ETH approach to quantum theory," arXiv:2004.11785 [math-ph] (2020)

Inherent structures of a causal fermion system

Let $(\rho, \mathcal{F}, \mathcal{H})$ be a causal fermion of spin dimension n, space-time $M := \text{supp}\,\rho$.

space-time points are linear operators on $\mathcal{H}$

- For $x \in M$, consider eigenspaces of x.
- For $x, y \in M$,
  - consider operator products xy
  - project eigenspaces of x to eigenspaces of y

Gives rise to:

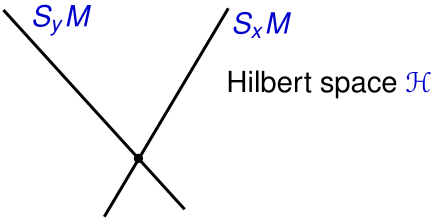
- quantum objects (spinors, wave functions)
- geometric structures (connection, curvature)
- causal structure, analytic structures

- Spinors

$$S_x M := x(\mathcal{H}) \subset \mathcal{H} \quad \text{"spin space"}, \; \dim S_x M \leq 2n$$

$$\prec u | v \succ_x := \langle u \,|\, x\, v \rangle_{\mathcal{H}} \quad \text{"spin scalar product"},$$

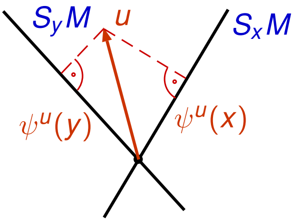
inner product of signature $(\leq n, \leq n)$



- Physical wave functions

$$\psi^u(x) = \pi_x u \text{ with } u \in \mathcal{H} \qquad \text{physical wave function}$$

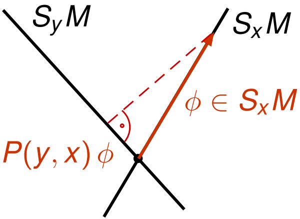
$$\pi_x : \mathcal{H} \to \mathcal{H} \qquad \text{orthogonal projection on } x(\mathcal{H})$$



Inherent structures in space-time

- The kernel of the fermionic projector:

$$P(y,x) = \pi_y \, x|_{S_xM} \;:\; S_xM \to S_yM$$



$$P(y,x) = -\sum_{i=1}^{f} |\psi^{e_i}(y) \succ \prec \psi^{e_i}(x)| \qquad \text{where } (e_i) \text{ ONB of } \mathcal{H}$$

Inherent structures in space-time

- Geometric structures
  - $P(x, y) \, : \, S_y M \to S_x M$ yields relations between spin spaces.
    Using a polar decomposition $(\ldots, \ldots)$ one gets:

    $$D_{x,y} \, : \, S_y M \to S_x M \text{ unitary} \qquad \text{“spin connection”}$$

  - tangent space T_x, carries Lorentzian metric,

    $$\nabla_{x,y} \, : \, T_y \to T_x \qquad \text{corresponding “metric connection”}$$

  - holonomy of connection gives curvature

    $$R(x, y, z) = \nabla_{x,y} \nabla_{y,z} \nabla_{z,x} \, : \, T_x \to T_x$$

Causal structure

Let $x, y \in M$. Then
$x \cdot y \in \mathrm{L}(H)$ has non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Definition (causal structure)

The points $x, y \in \mathcal{F}$ are called

$$\begin{cases} \text{spacelike separated} & \text{if } |\lambda_j^{xy}| = |\lambda_k^{xy}| \text{ for all } j, k = 1, \ldots, 2n \\ \text{timelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all real} \\ & \text{and } |\lambda_j^{xy}| \neq |\lambda_k^{xy}| \text{ for some } j, k \\ \text{lightlike separated} & \text{otherwise} \end{cases}$$

- Lagrangian is compatible with causal structure:

$$\text{Lagrangian} \quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \big(|\lambda_i^{xy}| - |\lambda_j^{xy}| \big)^2 \;\geq\; 0$$

thus x, y spacelike separated $\quad \Rightarrow \quad \mathcal{L}(x,y) = 0$

"points with spacelike separation do not interact"

A distinguished time direction

$$x(\mathcal{H}) \subset \mathcal{H} \qquad \text{subspace of dimension} \leq 2n$$

Introduce the functional

$$\mathcal{C} \;:\; M \times M \to \mathbb{R}\,, \qquad \mathcal{C}(x,y) := i\,\mathrm{tr}\big(y\,x\,\pi_y\,\pi_x - x\,y\,\pi_x\,\pi_y\big)$$

For timelike separated points $x, y \in M$,

$$\begin{cases} y \text{ likes in the future of } x & \text{if } \mathcal{C}(x,y) > 0 \\ y \text{ likes in the past of } x & \text{if } \mathcal{C}(x,y) < 0 \end{cases}$$

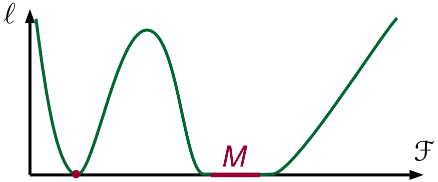
Analysis of the causal action principle

$$\ell(x) := \int_{\mathcal{F}} \Big(\mathcal{L}(x,y) + \kappa \sum_{i=1}^{2n} |\lambda_i^{xy}|^2 \Big)\, d\rho(y)$$

Lemma (Euler-Lagrange equations)

Let ρ be a minimizer of the causal action. Then

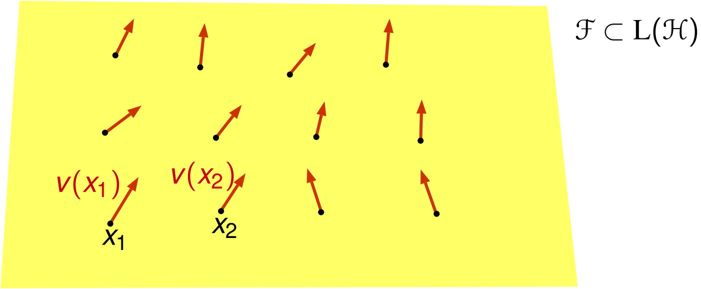
$$\ell|_M \equiv \inf_{\mathcal{F}} \ell$$



(Proof in tomorrow's talk)

Assume that ρ is a discrete minimizing measure, interpreted as describing the vacuum.

- What are linear perturbations of the measure?



Also a scalar weight function $b(x)$ comes into play

- jet $\mathfrak{v} := (b, v)$

Jet dynamics

The jet $\mathfrak{v} = (b, v)$ satisfies the linearized field equations

$$0 = \langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x)$$
$$:= \nabla_{\mathfrak{u}} \bigg(\int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \big) \mathcal{L}(x,y)\, d\rho(y) - \nabla_{\mathfrak{v}}\, \mathfrak{s} \bigg)$$

for all test jets $\mathfrak{u}$, where $\mathfrak{s} > 0$ is a Lagrange multiplier and

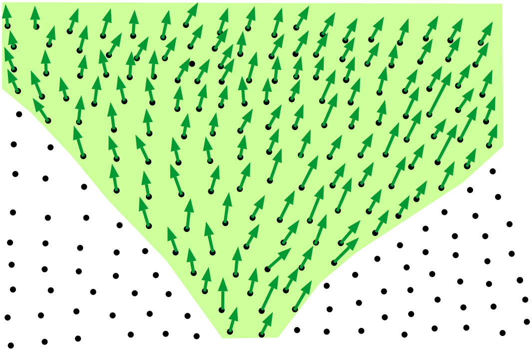
$$\nabla_{\mathfrak{v}} g(x) := a(x)\, g(x) + \big(D_v g \big)(x)$$

There are also corresponding nonlinear field equations.

- F.F., J. Kleiner, "A Hamiltonian formulation of causal variational principles," arXiv:1612.07192 [math-ph], *Calc. Var. Partial Differential Equations* **56:73** (2017)
- F.F., "Perturbation theory for critical points of causal variational principles," arXiv:1703.05059 [math-ph], *to appear in Adv. Theor. Math. Phys.* (2020)

Existence, Uniqueness, Finite Propagation Speed

for linearized fields



This holds "on the macroscopic scale"

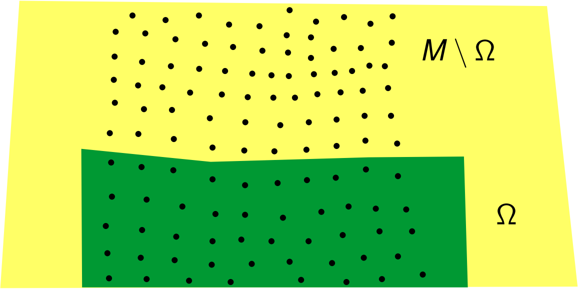
- C. Dappiaggi, F.F., "The Cauchy problem and the causal structure of linearized fields for causal variational principles," arXiv:1811.10587 [math-ph], *to appear in Methods and Applications of Analysis* (2020)

based on energy estimates

Surface Layer Integrals

For doing quantum theory, there is still missing

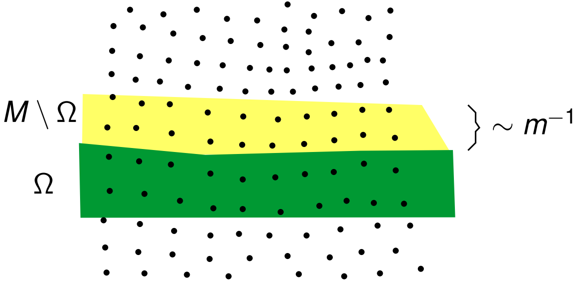
- probability density, scalar product is fixed time
- unitary time evolution



$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, (\cdots) \mathcal{L}(x,y)$$

Surface Layer Integrals

Typically: $\mathcal{L}(x,y)$ very small if x and y far apart
(decay on the Compton scale)



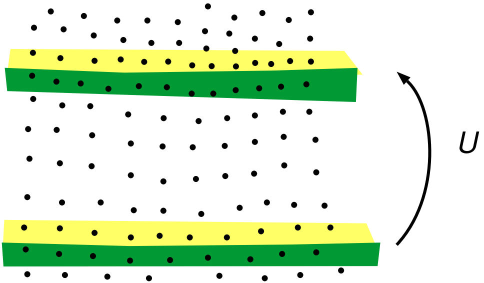
- F.F., J. Kleiner, "Noether-like theorems for causal variational principles," arXiv:1506.09076 [math-ph], *Calc. Var. Partial Differential Equations* **55:35** (2016)
- F.F., J. Kleiner, "A class of conserved surface layer integrals for causal variational principles," arXiv:1801.08715 [math-ph], *Calc. Var. Partial Differential Equations* **58:38** (2016)

Conservation laws for linearized fields

$$I_\rho^\Omega(\mathfrak{v}) := \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big) \mathcal{L}(x,y)$$

$$\sigma_\rho^\Omega(\mathfrak{u}, \mathfrak{v}) := \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{1,\mathfrak{v}} \nabla_{2,\mathfrak{u}}\big) \mathcal{L}(x,y)$$

Gives rise to: Complex structure, unitary time evolution



- F.F., N. Kamran, “Complex structures on jet spaces and bosonic Fock space dynamics for causal variational principles,” arXiv:1808.03177 [math-ph], *to appear in Pure Appl. Math. Q.* (2020)

Nonlinear conservation laws

Consider two measures ρ and $\tilde{\rho}$ and subsets $\Omega \subset M$, $\tilde{\Omega} \subset \tilde{M}$.

$$\begin{aligned} \gamma^\Omega(\tilde{\rho}, \rho) := &\int_{\tilde{\Omega}} d\tilde{\rho}(\tilde{x}) \int_{M \setminus \Omega} d\rho(y)\, \mathcal{L}(\tilde{x}, y) \\ &- \int_{\Omega} d\rho(x) \int_{\tilde{M} \setminus \tilde{\Omega}} d\tilde{\rho}(\tilde{y})\, \mathcal{L}(x, \tilde{y}) \end{aligned}$$

- Gives conservation laws for nonlinear fields

Used for definition of total mass (see talk tomorrow).

Fock space dynamics

- perform perturbation expansion
- rewrite p-multilinear mappings as linear mappings on the tensor product
- combine with conservation laws

Gives unitary time evolution on Fock spaces

Worked out for bosonic interactions in

- F.F., N. Kamran, “Complex structures on jet spaces and bosonic Fock space dynamics for causal variational principles,” arXiv:1808.03177 [math-ph], *to appear in Pure Appl. Math. Q.* (2020)

Outlook

- Connection to classical field theory and PDEs:
  - nonlinear fields, prove existence and uniqueness, ...
- Connection to quantum field theory:
  - interacting QFT with fermions:
    current project with Niky Kamran
  - linear field theories
    (with Claudio Dappiaggi and Marco Oppio)
- work out quantitative differences to standard QFT

www.causal-fermion-system.com

Thank you for your attention!

Gauß-like theorem

For simplicity leave out scalar components of jets.

$$(\Delta u)(x) = \int_M \big(D_{1,u} + D_{2,u}\big)\mathcal{L}(x,y)\, d\rho(y)$$

$$0 = D_u \ell = \int_M D_{1,u}\mathcal{L}(x,y)\, d\rho(y) \qquad \text{(EL eqns)}$$

Hence

$$(\Delta u)(x) = -\int_M \big(D_{1,u} - D_{2,u}\big)\mathcal{L}(x,y)\, d\rho(y)$$

$$\int_\Omega (\Delta u)(x)\, d\rho(x) = -\int_\Omega d\rho(x) \int_M d\rho(y)\, \big(D_{1,u} - D_{2,u}\big)\mathcal{L}(x,y)$$

$$= -\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(D_{1,u} - D_{2,u}\big)\mathcal{L}(x,y)$$

(volume integral) = (surface layer integral)

Energy estimates

- consider the energy as "smoothened" surface layer integral:

$$E(t) := (\mathfrak{v}, \mathfrak{v})_t = \int_U d\rho(x)\, \eta_t(x) \int_U d\rho(y)\, \big(1 - \eta_t(y)\big) \times \Big(\nabla_{1,\mathfrak{v}} \nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}} \nabla_{2,\mathfrak{v}} \Big) \mathcal{L}(x, y)$$

- energy identity

$$\frac{d}{dt} (\mathfrak{v}, \mathfrak{v})_t = 2 \int_U \langle \mathfrak{v}, \Delta \mathfrak{v} \rangle(x)\, d\rho_t(x) - 2 \int_U \Delta_2[\mathfrak{v}, \mathfrak{v}]\, d\rho_t(x) + \mathfrak{s} \int_U b(x)^2\, d\rho_t(x)$$

$$d\rho_t(x) := \theta_t(x)\, d\rho(x)\,, \qquad \theta_t(x) := \partial_t \eta_t(x)$$

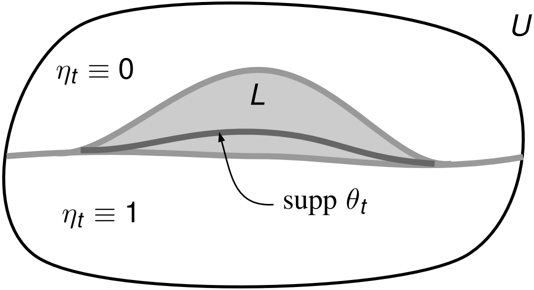
Energy estimates

- hyperbolicity conditions:

$$(\mathfrak{v}, \mathfrak{v})_t \geq \frac{1}{C^2} \int_U \Big(\|\mathfrak{v}(x)\|_x^2 + \big|\Delta_2[\mathfrak{v}, \mathfrak{v}]\big| \Big)\, d\rho_t(x) \qquad \text{for all } \mathfrak{v}$$

- Gives rise to weak solutions in lens-shaped regions:

$$\langle \Delta \mathfrak{u}, \mathfrak{v} \rangle_{L^2(L)} = \langle \mathfrak{u}, \mathfrak{w} \rangle_{L^2(L)} \qquad \text{for all test jets } \mathfrak{u}$$



$U \subset M := \text{supp}\ \rho$

Energy estimates

- strong solutions
- global solutions: exhaust space-time by lens-shaped regions
- gives existence of advanced and retarded Green's operators

What is a quantum field?

- What is a classical bosonic field?
  Recall concept of local correlation operator at the beginning:
  $$(F(t,x))^j_k = \overline{\psi_j(t,x)}\psi_k(t,x)$$
  This holds more generally: $x \in M := \text{supp}\,\rho \subset \mathcal{F}$,
  $$x = F(x)\,, \qquad F(x)^j_k = \prec\!\psi_j(x)|\psi_k(x)\!\succ_x$$
  The $\psi_k(x)$ are called physical wave functions. A vector field is the first variation of $F(x)$:
  $$v(x) = \delta F(x) = \prec\!\delta\psi_j(x)|\psi_k(x)\!\succ_x + \prec\!\psi_j(x)|\delta\psi_k(x)\!\succ_x$$

  - bosonic jet: vary physical wave functions collectively
    $$\delta\psi_j(x) = -(s_m \not{A}\,\psi_j)(x)$$

What is a quantum field?

Using bosonic classical jets alone, one cannot satisfy the EL equations.

- Additional transformations:

$$x \to U(x)\, x\, U(x)^{-1} \quad \text{with} \quad U(x) \in \mathrm{U}(\mathcal{H})$$

  where $U(x)$ "fluctuates" on Compton scale.
- Leaves local densities unchanged
- Changes many-particle correlations because

$$P(x, y) \to \pi_x U(x)^{-1}\, U(y) y|_{S_y M}$$

  In particular, oscillatory functions come into play.
- As a consequence, the system can no longer be described by Hartree-Fock states and by classical bosonic fields.