Causal Fermion Systems

An Introduction to Fundamental Structures, Methods and Applications

Felix Finster, Sebastian Kindermann and Jan-Hendrik Treude

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Felix Finster
Fakultät für Mathematik
Universität Regensburg
Regensburg, Germany
finster@ur.de

Sebastian Kindermann
Comenius Gymnasium Deggendorf
Deggendorf, Germany
sebastian.kindermann@schule.bayern.de

Jan-Hendrik Treude
Fachbereich Mathematik und Statistik
Universität Konstanz
Konstanz, Germany
jan-hendrik.treude@uni-konstanz.de

Preface

The theory of causal fermion systems is an approach to fundamental physics. In different limiting cases, causal fermion systems give rise to the standard model of particle physics and gravity on the level of classical field theory [**45**] as well as to quantum field theory [**62, 23**]. In view of these results, causal fermion systems are a promising candidate for a unified physical theory. The dynamics of a causal fermion system is described by a novel variational principle: the causal action principle. From the mathematical perspective, causal fermion systems provide a general framework for describing non-smooth geometries and for formulating and analyzing dynamical equations in this non-smooth setting.

This book is intended as an easily accessible introduction to the theory of causal fermion systems. After giving the physical and mathematical background (Part 1), the theory of causal fermion systems is introduced (Part 2). We proceed by providing mathematical methods which can be regarded as a toolbox for analyzing causal fermion systems (Part 3). We conclude with an outlook on the applications (Part 4).

In order to address as large an audience as possible, the book contains extensive preliminaries which cover both physical and mathematical aspects. We have two typical audiences in mind when writing these preliminaries: physicists with only basic knowledge of mathematics and mathematicians without physical background.

The book is based on three main resources: First, the lecture notes of the spring school "Relativistic Fermion Systems" held in Regensburg in April 2013, adapted for the spring school "Causal Fermion Systems" held in Regensburg in March 2016. Second, the lecture "Causal Variational Principles" given at the University of Regensburg in the summer semester 2017. Finally, the online course "An Introduction to Causal Fermion Systems" held in the summer semester 2021.

We would like to thank the participants of the spring schools and the students in the above lectures for valuable feedback. In particular, we are grateful to Jonas Bierler, David Cherney, Franz Gmeineder, Stefan Lippoldt, Marcin Napiórkowski, Simon Reinhardt, Julien Sabin and Andrea Schätzl for valuable feedback. Moreover, we are grateful to Sami Abdallah, Marvin Becker, Shane Farnsworth, Patrick Fischer, Christoph Krpoun, Magdalena Lottner, Valter Moretti, Heiko von der Mosel, Marco van den Beld Serrano and Johannes Wurm for helpful comments on the manuscript. A special thanks goes to Johannes Kleiner and Marco Oppio for helping with the lecture notes and providing many exercises. We are grateful to the Deutsche Forschungsgemeinschaft (DFG) for financial support. Finally, we would like to thank Nicholas Gibbons and the publishing team of Cambridge University Press for the excellent collaboration.

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How to use this book

This book is addressed to both mathematicians and physicists interested in the subject. We want to address young students and researchers on the master or graduate level. But the book should be helpful to the senior researcher as well.

The book is divided into four parts, each consisting of several chapters. Part 1 provides the necessary physical and mathematical preliminaries. Here the presentation is quite brief, and we refer to the standard textbooks for more details. We selected the material with the focus on what is most essential for causal fermion systems. We also introduce the conventions and notation which will be used later in the book. The content of Chapter 4 can be omitted by a reader who wants to concentrate on systems without gravity in Minkowski space.

Part 2 introduces the main concepts and structures. In Chapter 5 we motivate and define causal fermion systems and explain the fundamental structures. This chapter is essential for all the later parts of the book and should be read first. In the following Chapters of Part 2 the structures of a causal fermion system are explained in more detail, also setting the state for the later analysis.

In Part 3 we introduce the mathematical methods for the analysis of causal fermion systems. The different methods can be understood as a toolbox, from which the reader may choose depending on her interests and needs. The chapters in this part are self-contained, except for obvious dependencies (for example, the energy methods for the linearized field equations in Chapter 14 build on similar methods for symmetric hyperbolic systems in Chapter 13). We note that the methods presented in this book are by no means exhaustive; we concentrate on the main methods which have been fruitful so far.

Part 4 provides additional examples and gives an outlook on the physical applications. Here the presentation is a bit more sketchy than in Parts 2 and 3. The reason is that, after being familiar with Parts 2 and 3 of the present book, the reader should be well-prepared for delving into the research articles. Moreover, the content of Chapter 21 is covered in detail in the textbook [**45**]. Therefore, the purpose of this chapter merely is to give a non-technical overview. The content of Chapter 22, on the other hand, is a field of active research. Therefore, it seems preferable to present this material systematically and in more detail at a later stage in a separate textbook.

Every chapter is supplemented by a section with exercises. Studying these exercises is important for getting familiar and deepening the understanding of the material. Hints on how to solve the problems should simplify the self-study.

We finally note that part of the material of this book is complemented by videos of an online course, which are available on the website

www.causal-fermion-system.com/online_course

We hope that the reader will enjoy reading and learning from this book. Feedback is always welcome.

Contents

Part 1

Physical and Mathematical Background

CHAPTER 1

Physical Preliminaries

In this chapter we summarize some basics on quantum mechanics and relativity theory as needed in order to understand the physical content and context of the theory of causal fermion systems. We also fix our conventions and introduce the notation which will be used consistently throughout this book. Clearly, reading this summary cannot replace studying quantum mechanics and relativity theory in detail. To this end, we will cite various standard physics textbooks along the way.

1.1. The Schrödinger Equation

We begin by recalling a few basics of non-relativistic quantum mechanics. For more details, we refer to standard textbooks like [**139, 141, 111**].

The state of a quantum mechanical particle without spin is described by its *wave function* $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, $(t, \vec{x}) \mapsto \psi(t, \vec{x})$, where $t \in \mathbb{R}$ describes time and $\vec{x} \in \mathbb{R}^3$ position. Its absolute square $|\psi(t, \vec{x})|^2$ has the interpretation as the *probability density* of the particle to be located at the position $\vec{x}$ at time t. For this interpretation to be sensible, the integral over the probability density must be equal to one,

$$\int_{\mathbb{R}^3} |\psi(t, \vec{x})|^2 \, \mathrm{d}^3x = 1 \,. \tag{1.1.1}$$

This equation must hold at any time $t \in \mathbb{R}$. This entails that the dynamical equations must preserve the integral in (1.1.1), as will be discussed in more detail shortly.

A basic tenet of quantum mechanics is the *superposition principle*. It states that for any wave functions ψ and ϕ, also their complex linear combination

$$\tilde{\psi} = \alpha\psi + \beta\phi \qquad \text{with } \alpha, \beta \in \mathbb{C} \tag{1.1.2}$$

(defined pointwise by $\tilde{\psi}(t, \vec{x}) = \alpha\psi(t, \vec{x}) + \beta\phi(t, \vec{x})$) is a physically admissible wave function. Thus, in more mathematical terms, the physical wave functions form a complex vector space. Evaluating (1.1.1) for the wave function $\tilde{\psi}$ and using that the probability integral must be preserved in time for all α and β, one concludes that the integral

$$\int_{\mathbb{R}^3} \overline{\phi(t, \vec{x})} \, \psi(t, \vec{x}) \, \mathrm{d}^3x \tag{1.1.3}$$

must be time independent for any wave functions ψ and ϕ. The procedure to deduce (1.1.3) from (1.1.1) is sometimes referred to as *polarization*. The integral (1.1.3) defines a scalar product on the wave functions, which we denote by

$$\langle \phi | \psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^3} \overline{\phi(t, \vec{x})} \, \psi(t, \vec{x}) \, \mathrm{d}^3x \,. \tag{1.1.4}$$

From the mathematical point of view, the most natural complex vector space for this scalar product is the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C})$ of square-integrable functions, which we also denote by $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ (for basics on Hilbert spaces see Section 2.2 below). This Hilbert space contains for instance all smooth functions with compact support (i.e. which

vanish outside a compact set; here *smooth* means that the function is differentiable to every order).

The dynamics of the wave function is described by a linear evolution equation on $\mathcal{H}$, the *Schrödinger equation*, which is first-order in time and whose general form is

$$\mathrm{i}\,\partial_t \psi = H\psi\,. \tag{1.1.5}$$

Here H, the so-called *Hamiltonian*, is a linear operator acting on the Hilbert space $\mathcal{H}$. The linearity of the Schrödinger equation is essential in order to ensure that the time evolution is compatible with the superposition principle. The requirement that the scalar product (1.1.4) must be time independent implies that

$$0 = \partial_t \langle \phi | \psi \rangle_{\mathcal{H}} = -\mathrm{i}\left(\langle H\phi | \psi \rangle_{\mathcal{H}} - \langle \phi | H\psi \rangle_{\mathcal{H}} \right) \tag{1.1.6}$$

for all wave functions ψ, ϕ. In other words, the Hamiltonian must be a *symmetric* operator on the Hilbert space $\mathcal{H}$ (for mathematical basics see Definition 2.2.5 below; all mathematical issues like domains and the distinction between symmetric and selfadjoint operators are postponed to Section 3.2).

In the simplest setting (more precisely, for a particle without spin), the Hamiltonian has the form

$$H = -\frac{1}{2m}\,\Delta + V\,,$$

where we chose units where Planck's constant and the speed or light are equal to one,

$$\hbar = c = 1$$

(we will do so throughout this book). Here, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplacian on $\mathbb{R}^3$, and $V(t, \vec{x})$ is a real-valued potential which acts on wave functions by multiplication. The parameter $m > 0$ is the rest mass of the particle.

The Schrödinger equation can be analyzed using various methods. If the potential is time independent, it can be solved by exponentiating,

$$\psi(t) = \mathrm{e}^{-\mathrm{i}Ht}\,\psi(0)\,,$$

where the exponential may be defined using the spectral theorem (for details see Section 3.2 below). In this case the dynamics of ψ can be related to spectral properties of the Hamiltonian. Another method, which has the advantage that it also applies if the potential depends on time, is to make use of the fact that the time evolution forms a strongly continuous semigroup of operators (see for example [**116**, Section 34]). Alternatively, one can analyze the Schrödinger equation as a parabolic partial differential equation. Since our focus are the relativistic equations, these methods are not covered in this book. But we refer the interested reader to the textbooks [**143**, Chapter 6] or [**32**, Section II.7.1].

1.2. Special Relativity and Minkowski Space

We now give a brief introduction to special relativity. For more details, in particular on the physical background, we recommend the textbooks [**134, 121, 112**].

In special relativity, space and time are combined into a four-dimensional *spacetime*. Mathematically, this four-dimensional spacetime is described by Minkowski space $(\mathscr{M}, \langle .,. \rangle)$, a real four-dimensional vector space endowed with an inner product $\langle .,. \rangle$ of signature $(+---)$. For $\mathscr{M}$ one may always choose a basis $(e_i)_{i=0,\ldots,3}$ satisfying $\langle e_0, e_0 \rangle = 1$ and $\langle e_i, e_i \rangle = -1$ for $i = 1, 2, 3$. Such a basis is called pseudo-orthonormal basis or *reference frame*, since the corresponding coordinate system (x^i) describes time and space for

an observer in a system of inertia. We also refer to $t := x^0$ as time and denote spatial coordinates by $\vec{x} = (x^1, x^2, x^3)$. Representing two vectors $\xi, \eta \in \mathscr{M}$ in such a basis as $\xi = \sum_{i=0}^3 \xi^i e_i$ and $\eta = \sum_{i=0}^3 \eta^i e_i$, the inner product takes the form

$$\langle \xi, \eta \rangle = \sum_{j,k=0}^{3} g_{jk}\, \xi^j\, \eta^k \,, \tag{1.2.1}$$

where g_{jk}, the *Minkowski metric*, is the diagonal matrix $g = \text{diag}\,(1, -1, -1, -1)$. We note that the origin of Minkowski space is not distinguished (apart from the fact that it can be regarded as the origin of the observer). This could be formalized mathematically by regarding $\mathscr{M}$ as an *affine vector space*. Here we prefer to regard $\mathscr{M}$ simply as a vector space, noting that the translation $\mathscr{M} \to \mathscr{M} + u$ by a vector $u \in \mathscr{M}$ corresponds to a symmetry of spacetime.

In what follows we usually use *Einstein's summation convention*, according to which one omits the sign for sums and always sums over any pair of indices appearing twice, one being an upper and one a lower index. For instance, with this convention, the relation (1.2.1) is written simply as $\langle \xi, \eta \rangle = g_{jk} \xi^j \eta^k$. By g^{ij} we denote the inverse of the Minkowski metric, which in a pseudo-orthonormal basis is again the diagonal matrix $\text{diag}(1, -1, -1, -1)$. We *raise and lower indices* using the Minkowski metric and its inverse, meaning that for a vector $\xi = \xi^i e_i$ we set $\xi_i := g_{ij} \xi^j$ for any $i = 0, \ldots, 3$, and we also write $\partial^j = g^{jk} \partial_k$. Finally, we sometimes abbreviate the Minkowski inner product by writing $\xi\eta := \langle \xi, \eta \rangle$ and $\xi^2 := \langle \xi, \xi \rangle$.

The sign of the Minkowski metric encodes the causal structure of spacetime. Namely, a vector $\xi \in \mathscr{M}$ is said to be

$$\begin{cases} \textit{timelike} & \text{if } \langle \xi, \xi \rangle > 0 \\ \textit{spacelike} & \text{if } \langle \xi, \xi \rangle < 0 \\ \textit{lightlike} & \text{if } \langle \xi, \xi \rangle = 0 \,. \end{cases} \tag{1.2.2}$$

Lightlike vectors are also referred to as *null* vectors. Moreover, the term *non-spacelike* refers to timelike or lightlike vectors. The timelike and null vectors form a double cone. Its boundary

$$L := \{ \xi \in \mathscr{M} \,|\, \langle \xi, \xi \rangle = 0 \}$$

is referred to as the *light cone*. Physically speaking, the light cone is formed of all light rays through the origin of $\mathscr{M}$. Similarly, the timelike vectors correspond to velocities slower than the speed of light; they form the

$$\textit{interior light cone} \qquad I := \{ \xi \in \mathscr{M} \,|\, \langle \xi, \xi \rangle > 0 \} \,.$$

Likewise, the non-spacelike vectors form the

$$\textit{closed light cone} \qquad J := \{ \xi \in \mathscr{M} \,|\, \langle \xi, \xi \rangle \geq 0 \} = I \cup L \,.$$

We denote the future and past light cones by superscripts $\vee$ and $\wedge$, i.e.

$$J^\vee := \{ \xi \in \mathscr{M} \,|\, \langle \xi, \xi \rangle \geq 0,\ \xi^0 \geq 0 \}$$
$$J^\wedge := \{ \xi \in \mathscr{M} \,|\, \langle \xi, \xi \rangle \geq 0,\ \xi^0 \leq 0 \} \,,$$

and similarly for I. These notions are illustrated in Figure 1.1.

The spacetime trajectory of a moving object is described by a curve $q(\tau)$ in Minkowski space (with τ an arbitrary parameter). We always assume that the parametrization is *regular*, meaning that the tangent vector $\mathrm{d}q/\,\mathrm{d}\tau$ to the spacetime curve is non-zero for all τ. We say that the curve $q(\tau)$ is timelike if its tangent vector is everywhere timelike.

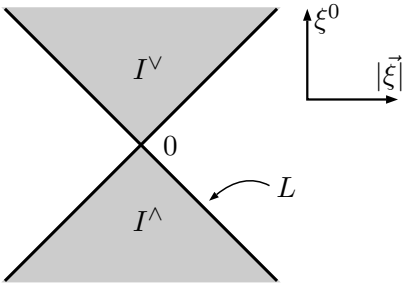


FIGURE 1.1. The causal structure of Minkowski space.

Spacelike, null, and non-spacelike curves are defined analogously. The usual statement of causality, which says that no information can travel faster than the speed of light, can then be expressed as follows:

> *Causality*: Information can be transmitted only along non-spacelike curves.

In view of this notion, a non-spacelike curve is also referred to as a *causal curve*. The set of all points which can be joined to a given spacetime point x by a non-spacelike curve is precisely the closed light cone centered at x, denoted by $J_x := J - x$. It is the union of the two single cones

$$J_x^\vee = \{y \in \mathscr{M} \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \geq 0\}$$
$$J_x^\wedge = \{y \in \mathscr{M} \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \leq 0\}\,,$$

interpreted as the points in the causal future and past of x, respectively. Therefore, we refer to $J_x^\vee$ and $J_x^\wedge$ as the closed *future* and *past light cones* centered at x, respectively. The sets $I_x^\vee$, $I_x^\wedge$ and $L_x^\vee$, $L_x^\wedge$ are introduced similarly. We remark that the resulting relations like "lies in the timelike future of" or "lies in the causal future of" are transitive (see Exercise 1.1).

Special relativity demands that physical equations be *Lorentz invariant*. Qualitatively speaking, this means that they must be formulated in a manner independent of the choice of reference frame. More concretely, this independence can be formulated in terms of transformation laws: Recall that a reference frame is an orthonormal basis of Minkowski space. Any two reference frames $(e_i)_{i=0,\ldots,3}$ and $(\widetilde{e}_i)_{i=0,\ldots,3}$ are related to each other by a linear transformation $\Lambda \in \mathrm{L}(\mathscr{M})$ (here $\mathrm{L}(\mathscr{M})$ are the linear transformations on $\mathscr{M}$; clearly, Λ can be written as a 4×4-matrix with real-valued entries) which preserves the Minkowski metric, i.e. one has (again using the Einstein summation convention)

$$\widetilde{e}_i = \Lambda_i^j\, e_j \qquad \text{and} \qquad \Lambda_j^\ell\, \Lambda_k^m\, g_{\ell m} = g_{jk}\,. \tag{1.2.3}$$

The coordinates in the old and new reference frames and the corresponding partial derivatives are related to each other by

$$\widetilde{x}^i = \Lambda_j^i\, x^j \qquad \text{and} \qquad \frac{\partial}{\partial \widetilde{x}^i} = \Lambda_i^j\, \frac{\partial}{\partial x^j}\,. \tag{1.2.4}$$

The Lorentz transformations form a group (with the group operation being the composition of the linear transformations or, equivalently, matrix multiplication of the corresponding matrices), the so-called *Lorentz group*. The Lorentz transformations which preserve both the time direction and the spatial orientation form a subgroup of the Lorentz group, the *orthochronous proper Lorentz group*.

If one wants to formulate a physical equation, the *principle of Lorentz invariance* demands that its explicit form must be invariant under the joint transformations (1.2.3) and (1.2.4). The simplest example is the *Klein-Gordon equation*

$$(-\Box - m^2)\,\psi = 0\,, \tag{1.2.5}$$

where $\Box := \partial_j \partial^j = g^{jk}\partial_k \partial_j$ is the scalar wave operator and $\psi : \mathscr{M} \to \mathbb{C}$.

Using (1.2.4), one verifies by a short computation that this equation takes the same form in any reference frame. The Klein-Gordon equation describes a scalar particle (i.e. a particle without spin) of mass m. If the particle has an electric charge e, one also has to take into account the interaction with the electromagnet field. One finds empirically that the correct equation to describe the influence of the field on the particle is

$$-(\partial_k - \mathrm{i}eA_k)(\partial^k - \mathrm{i}eA^k)\,\psi = m^2\psi\,, \tag{1.2.6}$$

where A is the electromagnetic potential.

1.3. The Dirac Equation

The Schrödinger equation 1.1.5 is not Lorentz invariant. Therefore, it is not suitable to describe a relativistic quantum particle. Although being Lorentz-invariant, the Klein-Gordon equation is also not suitable for this purpose because the interpretation of the absolute value of its solutions as probability density is not sensible (for example because for general solutions of (1.2.6), the spatial integral of $|\psi(t,\vec{x})|^2$ is not conserved in time). The correct relativistic generalization of the Schrödinger equation is the so-called Dirac equation, which will now be introduced. More on its physical background can be found for example in the classic textbooks [**14, 112, 128**].

In order to describe a relativistic particle with spin, Dirac had the idea to work with a first order differential operator $\gamma^j \partial_j$ whose square is the wave operator. The coefficients of this operator are the *Dirac matrices* γ^j, which are 4×4-matrices characterized by the *anti-commutation relations*

$$2\, g^{jk}\, \mathbb{1} \stackrel{!}{=} \{\gamma^j,\, \gamma^k\} := \gamma^j\gamma^k + \gamma^k\gamma^j\,. \tag{1.3.1}$$

Using these relations, one finds that the square of the operator $\gamma^j\partial_j$ indeed gives the wave operator,

$$(\gamma^j\partial_j)^2 = \gamma^j\gamma^k\,\partial_j\partial_k = \frac{1}{2}\,\{\gamma^j, \gamma^k\}\,\partial_{jk} = \Box \tag{1.3.2}$$

(of course, here the operator $\gamma^j\partial_j$ acts on wave functions with four components, also called *spinorial wave functions*, and the wave operator has to be understood as acting on each component separately). There are different possible choices for 4×4-matrices γ^j satisfying (1.3.1) which are all related by a change of spinor basis. For convenience, we shall always work in the *Dirac representation*, i.e. we choose (see Exercise 1.4)

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1.3.3}$$

where σ^i are the three Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3.4}$$

Including the mass $m \geq 0$, the Dirac equation in the vacuum (i.e. without any interaction) reads

$$\left(\mathrm{i} \gamma^k \frac{\partial}{\partial x^k} - m \right) \psi(x) = 0 \,, \tag{1.3.5}$$

where the *Dirac spinor* $\psi : \mathscr{M} \to \mathbb{C}^4$ has four complex components. If we multiply (1.3.5) by the operator $(\mathrm{i} \gamma^j \partial_j + m)$ and use (1.3.2), we find that each component of ψ satisfies the Klein-Gordon equation (1.2.5).

Following the standard conventions in physics, we denote contractions with Dirac matrices by a slash, i.e. $\u\!\!\!/ = \gamma^j u_j$ for u a vector of Minkowski space and $\partial\!\!\!/ = \gamma^j \partial_j$. The action of the matrix $\u\!\!\!/$ on a spinor ψ as described by the mapping $u \mapsto \u\!\!\!/ \psi$ is often referred to as *Clifford multiplication* by the vector u.

A quantum particle described by a solution of the Dirac equation is called Dirac particle. The leptons and quarks in the standard model are Dirac particles. Thus, on the fundamental level, all matter is described by the Dirac equation.

In the presence of an electromagnetic field with electromagnetic potential A, the Dirac equation is modified to

$$\mathrm{i} \gamma^k (\partial_k - \mathrm{i} A_k) \psi = m \psi \tag{1.3.6}$$

(here for convenience we absorbed the electromagnetic coupling constant into the potential). Similar as mentioned for the Klein-Gordon equation after (1.2.6), the coupling to the electromagnetic field can again be understood from the compatibility with local gauge transformations of electrodynamics (see Exercise 1.3). Multiplying by the operator $(\mathrm{i} \gamma^j (\partial_j - \mathrm{i} A_j) + m)$ and using again the anti-commutation relations, we obtain the equation

$$\Big(- (\partial_k - \mathrm{i} A_k)(\partial^k - \mathrm{i} A^k) + \frac{\mathrm{i}}{2} F_{jk} \gamma^j \gamma^k - m^2 \Big) \psi = 0 \,,$$

where $F_{jk} = \partial_j A_k - \partial_k A_k$ (see Exercise 1.6). This differs from the Klein-Gordon equation (1.2.6) by the extra term $\frac{\mathrm{i}}{2} F_{jk} \gamma^j \gamma^k$, which describes the coupling of the spin to the electromagnetic field.

A Dirac spinor takes value in $\mathbb{C}^4$. This four-dimensional complex vector space is also referred to as the *spinor space*, and its elements are referred to as *spinors*. An important structure on the spinor space is an indefinite inner product of signature $(2,2)$, which we call *spin inner product* and denote by

$$\prec \psi | \phi \succ := \sum_{\alpha=1}^{4} s_\alpha \, (\psi^\alpha)^\dagger \, \phi^\alpha \,, \qquad s_1 = s_2 = 1, \; s_3 = s_4 = -1 \,, \tag{1.3.7}$$

where for a spinor $\psi \in \mathbb{C}^4$ we denote by $\psi^\dagger$ the componentwise complex conjugate. In physics textbooks, the spin inner product is often written as $\overline{\psi} \phi$ with the so-called adjoint spinor $\overline{\psi} := \psi^\dagger \gamma^0$. By the *adjoint* A^* of an operator A acting on spinors we always mean the adjoint with respect to the spin inner product. Thus it is defined by the relation

$$\prec A^* \psi \,|\, \phi \succ = \prec \psi \,|\, A \phi \succ \qquad \text{for all } \psi, \phi \in \mathbb{C}^4 .$$

In an obvious way, this definition of the adjoint gives rise to the the notions of a *symmetric*, *anti-symmetric* and *unitary operator*. With these notions, the Dirac matrices are symmetric, meaning that

$$\prec \gamma^l \psi \,|\, \phi \succ = \prec \psi \,|\, \gamma^l \phi \succ \qquad \text{for all } \psi, \phi \in \mathbb{C}^4 . \tag{1.3.8}$$

From this it follows that also Clifford multiplication $\u\!\!\!/$ by a vector $u \in \mathscr{M}$ is symmetric. We note for clarity that, in the setting of finite-dimensional matrices considered here,

symmetric operators can be referred to equivalently as *selfadjoint* operators. We usually prefer the notion of a symmetric operator, leaving the notion of a selfadjoint operator to the setting of densely defined operators on infinite-dimensional Hilbert spaces.

To every solution ψ of the Dirac equation we can associate a vector field J^k by

$$J^k = \prec\psi \,|\, \gamma^k \,\psi\succ\,, \tag{1.3.9}$$

referred to as the *Dirac current*. It is either timelike or lightlike (see Exercise 1.7). Moreover, it is divergence-free, as the following computation shows,

$$\begin{aligned} \partial_k J^k &= \partial_k \prec\psi \,|\, \gamma^k \,\psi\succ = \prec\partial_k\psi \,|\, \gamma^k \,\psi\succ + \prec\psi \,|\, \gamma^k \partial_k \,\psi\succ \\ &= \mathrm{i}\,(\prec\mathrm{i}\partial\!\!\!/\psi \,|\, \psi\succ - \prec\psi \,|\, \mathrm{i}\partial\!\!\!/\psi\succ) \\ &= \mathrm{i}\,\big(\prec(\mathrm{i}\partial\!\!\!/ + A\!\!\!/ - m)\psi \,|\, \psi\succ - \prec\psi \,|\, (\mathrm{i}\partial\!\!\!/ + A\!\!\!/ - m)\psi\succ\big) = 0\,. \end{aligned} \tag{1.3.10}$$

This property is referred to as *current conservation*.

Current conservation is closely related to the probabilistic interpretation of the Dirac wave function, as we now explain. Suppose that ψ is a smooth solution of the Dirac equation with suitable decay at spatial infinity (for example of spatially compact support; see Section 1.4). Then current conservation allows us to apply the Gauß divergence theorem in a spacetime-region $[t_1, t_2] \times \mathbb{R}^3$ to obtain

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}^3x\; \partial_k \prec\psi \,|\, \gamma^k \psi\succ(t, \vec{x}) \\ &= \int_{\mathbb{R}^3} \prec\psi \,|\, \gamma^0 \psi\succ(t_2, \vec{x})\, \mathrm{d}^3x - \int_{\mathbb{R}^3} \prec\psi \,|\, \gamma^0 \psi\succ(t_1, \vec{x})\, \mathrm{d}^3x \end{aligned} \tag{1.3.11}$$

We remark that this argument works similarly on a region $\Omega \subset M$ whose boundary consists of two space-like hypersurfaces. Polarizing similar as explained after (1.1.2), we conclude that for any two solutions ϕ, ψ of the Dirac equation, the spatial integral

$$(\phi|\psi) := \int_{\mathbb{R}^3} \prec\phi \,|\, \gamma^0 \psi\succ(t, \vec{x})\, \mathrm{d}^3x \tag{1.3.12}$$

is time independent. Since the inner product $\prec.|\gamma^0.\succ$ is positive definite, the integral (1.3.12) defines a scalar product. We denote the Hilbert space corresponding to this scalar product by $\mathcal{H} = L^2(\mathbb{R}^3)^4$. In analogy to the integrand in (1.1.4) in non-relativistic quantum mechanics, the quantity $\prec\psi|\gamma^0\psi\succ(t, \vec{x})$ can be interpreted as the probability density of the particle being located at the spacetime point $(t, \vec{x})$. Current conservation (1.3.11) ensures that the probability integral is time independent.

The previous considerations generalize immediately to the situation in the presence of an *external potential*. To this end, we replace the operator $A\!\!\!/$ in the Dirac equation (1.3.6) by a multiplication operator $\mathcal{B} : \mathscr{M} \to \mathbb{C}^{4\times 4}$, which may even depend (smoothly) on the spacetime coordinates and which we assume to be symmetric with respect to the spin inner product, i.e.

$$\prec\mathcal{B}(x)\psi|\phi\succ = \prec\psi|\mathcal{B}(x)\phi\succ \qquad \text{for all } \psi, \phi \in \mathbb{C}^4, x \in \mathscr{M}\,. \tag{1.3.13}$$

We then write the Dirac equation with a Dirac operator $\mathcal{D}$ as

$$(\mathcal{D} - m)\,\psi = 0 \qquad \text{where} \qquad \mathcal{D} := \mathrm{i}\partial\!\!\!/ + \mathcal{B}\,. \tag{1.3.14}$$

The symmetry assumption (1.3.13) is needed for current conservation to hold (as one sees immediately if in (1.3.10) one replaces $A\!\!\!/$ by $\mathcal{B}$).

Similar as the Schrödinger equation (1.1.5), also the Dirac equation can be rewritten with a symmetric operator H acting on the Hilbert space $\mathcal{H}$. To this end, we multiply equation (1.3.14) by γ^0 and isolate the t-derivative on one side of the equation,

$$\mathrm{i}\partial_t \psi = H\psi \qquad \text{where} \qquad H := -\gamma^0(\mathrm{i}\vec{\gamma}\vec{\nabla} + \mathcal{B} - m) \tag{1.3.15}$$

Note here that $\gamma^j \partial_j = \gamma^0 \partial_0 + \vec{\gamma}\vec{\nabla}$. We refer to (1.3.15) as the Dirac equation in the *Hamiltonian form*. Now we can again apply (1.1.6) to conclude that the Hamiltonian is a symmetric operator on $\mathcal{H}$.

We remark that in the Hamiltonian formulation, one often absorbs the prefactor γ^0 in (1.3.15) into the other Dirac matrices and instead works with the new matrices

$$\beta := \gamma^0 \qquad \text{and} \qquad \vec{\alpha} := \gamma^0 \vec{\gamma}\,.$$

This is convenient because these new matrices are Hermitian with respect to the standard scalar product on $\mathbb{C}^4$. In this book however, we shall not work with α and $\vec{\beta}$. We prefer the notation (1.3.15), because it is more visible which parts of the operators are Lorentz invariant. For calculations using β and $\vec{\alpha}$ we refer for example to the monograph [**146**].

In addition to integrating over space (1.3.12), one can also introduce an inner product on spinorial wave functions by integrating the spin inner product over all of spacetime,

$$<\psi|\phi> = \int_{\mathcal{M}} \prec \psi|\phi \succ_x \mathrm{d}\mu_{\mathcal{M}}\,. \tag{1.3.16}$$

This inner product will in general not be well-defined on solutions of the Dirac equation, because (even for "normalized" solutions for which the spatial integrals are finite) the time integral may diverge. But the inner product can be considered for example on spinorial wave functions which are compactly supported in spacetime (but are no solutions of the Dirac equation). This *spacetime inner product* will be important for the constructions in Chapter 15. In this context, it is very useful that the Dirac operator is symmetric with respect to the spacetime inner product, meaning that

$$<\mathcal{D}\psi|\phi> = <\psi|\mathcal{D}\phi> \tag{1.3.17}$$

for all spinorial wave functions which decay sufficiently fast at spatial infinity and for large times. Indeed, the symmetry property (1.3.17) holds in curved spacetime as well (see the explanation after (4.2.33) below).

So far, Dirac spinors were introduced in a given reference frame. Let us verify that our definitions are in fact independent of the choice of reference frame. To this end we consider two reference frames (x^j) and $(\tilde{x}^l)$ with the same orientation of time and space. Then the reference frames are related to each other by an orthochronous proper Lorentz transformation Λ, i.e. in components

$$\tilde{x}^l = \Lambda^l_j\, x^j\,, \qquad \frac{\partial}{\partial x^j} = \frac{\partial \tilde{x}^l}{\partial x^j} \frac{\partial}{\partial \tilde{x}^l} = \Lambda^l_j\, \frac{\partial}{\partial \tilde{x}^l}\,,$$

and Λ leaves the Minkowski metric invariant,

$$\Lambda^l_j\, \Lambda^m_k\, g_{lm} = g_{jk}\,. \tag{1.3.18}$$

Under this change of spacetime coordinates, the Dirac operator $\mathrm{i}\gamma^j(\frac{\partial}{\partial x^j} - \mathrm{i}A_j)$ transforms to

$$\mathrm{i}\tilde{\gamma}^l \left(\frac{\partial}{\partial \tilde{x}^l} - \mathrm{i}\tilde{A}_l \right) \qquad \text{with} \qquad \tilde{\gamma}^l = \Lambda^l_j \gamma^j \quad \text{and} \quad \tilde{A}_l = \Lambda^k_l A_k\,. \tag{1.3.19}$$

This transformed Dirac operator does not coincide with the Dirac operator $\mathrm{i}\gamma^l(\frac{\partial}{\partial \tilde{x}^l} - \mathrm{i}\tilde{A}_l)$ as defined in the reference frame $(\tilde{x}^l)$ because the new Dirac matrices have a different form. However, the next lemma shows that the two Dirac operators do coincide after a suitable unitary transformation of the spinors.

LEMMA 1.3.1. *For any orthochronous proper Lorentz transformation* Λ *there is a unitary matrix* $U(\Lambda)$ *(unitary with respect to the spin inner product* (1.3.7)*) such that*

$$U(\Lambda)\, \Lambda^l_j \gamma^j\, U(\Lambda)^{-1} = \gamma^l\,.$$

PROOF. Since Λ is orthochronous and proper, we can write it in the form $\Lambda = \exp(\lambda)$, where λ is a suitable generator of a rotation and/or a Lorentz boost. Then $\Lambda(s) := \exp(s\lambda)$ with $s \in \mathbb{R}$, is a family of Lorentz transformations, and differentiating (1.3.18) with respect to s as $s = 0$, we find that

$$\lambda^l_j\, g_{lk} = -g_{jm}\, \lambda^m_k$$

(note that $\Lambda(s)^l_j = \delta^l_j + s\,\lambda^l_j + \cdots$). Using this identity together with the fact that the Dirac matrices are symmetric, it is straightforward to verify that the matrix

$$u := \frac{1}{4}\, \lambda^l_k\, \gamma_l\, \gamma^k$$

is anti-symmetric. As a consequence, the family of matrices

$$U(s) := \exp(su)$$

is unitary. We now consider for a fixed index l the family of matrices

$$A(s) := U(s)\, \Lambda(s)^l_j\, \gamma^j\, U(s)^{-1}\,.$$

Clearly, $A(0) = \gamma^l$. Furthermore, differentiating with respect to s gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\, A(s) = U\, \Lambda^l_j \left\{ u\, \gamma^j - \gamma^j\, u + \lambda^j_k \gamma^k \right\} U^{-1}\,,$$

and a short calculation using the commutation relations (see Exercise 1.8)

$$\left[\gamma_l\, \gamma^k, \gamma^j\right] = 2\left(\gamma_l\, g^{kj} - \delta^j_l\, \gamma^k\right) \tag{1.3.20}$$

shows that the curly brackets vanish. We conclude that $A(1) = A(0)$, proving the lemma. □

Applying this lemma to the Dirac operator in (1.3.19), one sees that the Dirac operator is invariant under the joint transformation of the spacetime coordinates and the spinors

$$x^j \;\longrightarrow\; \Lambda^j_k x^k\,, \qquad \psi \;\longrightarrow\; U(\Lambda)\,\psi\,.$$

Moreover, since the matrix $U(\Lambda)$ is unitary, the representation of the spin inner product (1.3.7) is valid in any reference frame. We conclude that our definition of spinors is indeed Lorentz invariant.

For what follows, it is important to keep in mind that, in contrast to the spin inner product, the combination $\psi^\dagger\phi = \prec\!\psi|\gamma^0\phi\!\succ$ is *not* Lorentz invariant. Instead, it is the zero component of a Minkowski vector. Consequently, the integrand in (1.3.12) is not a scalar. Its spatial integral, on the other hand, is again Lorentz invariant due to current conservation.

As a combination of all the Dirac matrices one can form the so-called *pseudo-scalar matrix* Γ by

$$\Gamma = \frac{\mathrm{i}}{4!}\, \epsilon_{jklm} \gamma^j \gamma^k \gamma^l \gamma^m = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{1.3.21}$$

(In the physics literature, this matrix is usually denoted by γ^5). Here ϵ_{jklm} is the totally anti-symmetric symbol (i.e. ϵ_{jklm} is equal to ± 1 if (j,k,l,m) is an even and odd permutation of $(0,1,2,3)$, respectively, and vanishes otherwise). A short calculation shows that the pseudo-scalar matrix is anti-symmetric and that $\Gamma^2 = \mathbb{1}$ (see Exercise 1.8). As a consequence, the matrices

$$\chi_L = \frac{1}{2}(\mathbb{1} - \Gamma) \;, \qquad \chi_R = \frac{1}{2}(\mathbb{1} + \Gamma) \tag{1.3.22}$$

satisfy the relations (see again Exercise 1.8)

$$\begin{gathered} \chi_{L/R}^2 = \chi_{L/R} \;, \qquad \Gamma\, \chi_L = -\chi_L \;, \qquad \Gamma\, \chi_R = \chi_R \;, \\ \chi_L^* = \chi_R \;, \qquad \chi_L + \chi_R = \mathbb{1} \;. \end{gathered} \tag{1.3.23}$$

They can be regarded as the spectral projectors of the matrix Γ and are called the *chiral projectors*. The projections $\chi_L \psi$ and $\chi_R \psi$ are referred to as the *left-* and *right-handed* components of the spinor, respectively. A matrix is said to be *even* and *odd* if it commutes or anti-commutes with Γ, respectively. It is straightforward to verify that the Dirac matrices are odd, and therefore

$$\gamma^j\, \chi_{L/R} = \chi_{R/L}\, \gamma^j \;. \tag{1.3.24}$$

By multiplying the Dirac equation (1.3.6) from the left by $\chi_{L/R}$, one can rewrite it as a system of equations for the left- and right-handed components of ψ,

$$\mathrm{i}\gamma^k (\partial_k - \mathrm{i}A_k)\, \chi_L \psi = m\, \chi_R \psi \;, \qquad \mathrm{i}\gamma^k (\partial_k - \mathrm{i}A_k)\, \chi_R \psi = m\, \chi_L \psi \;.$$

If $m = 0$, these two equations decouple, and we get separate equations for the left- and right-handed components of ψ. This observation is the starting point of the 2-component Weyl spinor formalism. Here we shall not use this formalism. Instead, we will describe chiral massless particles (like massless neutrinos) by the left- or right-handed components of a Dirac spinor.

1.4. The Hilbert Space of Dirac Solutions

We now express the structures of Dirac theory in a convenient notation, which harmonizes with the structures of causal fermion systems to be introduced later on (in Chapter 5) and also generalizes to curved spacetime (in Chapter 4). In Minkowski space, the Dirac wave functions are four-component complex wave functions. More generally, one can consider them as sections of a vector bundle. In view of these general concepts (which will be introduced in Section 2.5 below), we denote the Cartesian product

$$S\mathscr{M} := \mathscr{M} \times \mathbb{C}^4 \;,$$

as the *spinor bundle of* $\mathscr{M}$. For every $x \in \mathscr{M}$, the subset $S_x\mathscr{M} := \{x\} \times \mathbb{C}^4$ is referred to as the *spinor space at the spacetime point* x. Clearly, the spinor bundle is the disjoint union of all the spinor spaces,

$$S\mathscr{M} = \bigcup_{x \in \mathscr{M}} S_x\mathscr{M} \;,$$

and in the more general language of vector bundles the spinor spaces at the individual spacetime points are referred to as *fibers* of the spinor bundles. The spin inner product (1.3.7) can now be regarded as an inner product on each fiber $S_x\mathscr{M}$, as we often clarify by an additional subscript x (although here the inner product does not actually depend on x),

$$\prec .|. \succ_x \;:\; S_x\mathscr{M} \times S_x\mathscr{M} \to \mathbb{C} \;.$$

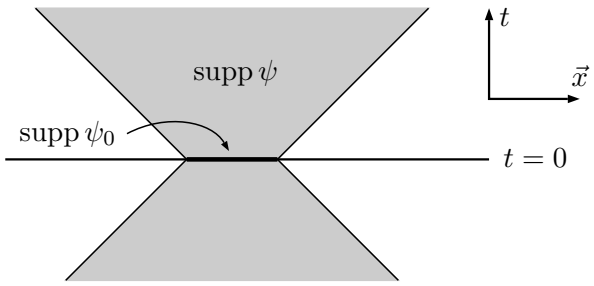


FIGURE 1.2. A spatially compact Dirac solution.

A Dirac wave function can be considered as a mapping $\psi : \mathcal{M} \to S\mathcal{M}$ with the property that for every $x \in \mathcal{M}$ one has

$$\psi(x) \in S_x\mathcal{M} \simeq \mathbb{C}^4 .$$

Such a mapping is referred to as a *section* of the spinor bundle.

We next highlight the obtained analytic structures, again anticipating concepts and results to be introduced later in this book. The scalar product (1.3.12) on the Dirac solutions gives rise to a Hilbert space structure on an appropriate class of solutions (for a mathematical introduction to Hilbert spaces see Section 2.2 below). In order to construct this appropriate class of solutions, one can begin by solving the Cauchy problem for smooth initial data ψ_0 of compact support given for example on the hypersurface $\{t = 0\}$. Rewriting the Dirac equation as a linear symmetric hyperbolic system (see Chapter 13 below), one sees that this Cauchy problem has a unique global solution in Minkowski space. Moreover, this solution is smooth and, due to finite propagation speed, has compact support on any other hypersurface $\{t = \text{const}\}$ (see Figure 1.2). One says that the solution has *spatially compact support*. More generally, the set of all smooth and spatially compact sections of the spinor bundle (not necessarily being solutions of the Dirac equation) is denoted by $C^\infty_{\text{sc}}(\mathcal{M}, S\mathcal{M})$. Clearly, for spatially compact solutions, the scalar product (1.3.12) is well-defined and finite. Taking the completion of the set of all spatially compact solutions with respect to the scalar product (1.3.12), one obtains a Hilbert space denoted by $(\mathcal{H}_m, (.|.))$, where m denotes the mass parameter of the Dirac equation (for details on the completion and Hilbert spaces see Section 2.2 and Exercise 2.6). By construction, we know that

$$C^\infty_{\text{sc}}(\mathcal{M}, S\mathcal{M}) \cap \mathcal{H}_m \qquad \text{is dense in } \mathcal{H}_m .$$

We note for clarity that the wave functions in the completion $\mathcal{H}_m$ are not necessarily differentiable. Therefore, they do not satisfy the Dirac equations. But they are *weak solutions* in the sense that the Dirac equation holds after taking the spacetime inner product (1.3.16) with a test wave function ϕ and formally integrating by parts, i.e.

$$\int_{\mathcal{M}} \prec (\mathcal{D} - m)\phi \,|\, \psi \succ \, \mathrm{d}\mu_{\mathcal{M}} = 0 \qquad \text{for all } \phi \in C^\infty_0(\mathcal{M}, S\mathcal{M}) ,$$

where $C^\infty_0(\mathcal{M}, S\mathcal{M})$ denotes the space of smooth wave functions with compact support. For the reader familiar with the theory of partial differential equations, we finally remark that the solutions in $\mathcal{H}_m$ can also be characterized in terms of Sobolev spaces (see for example [**32**, Section II.5]). More precisely, the vectors in $\mathcal{H}_m$ are weak solutions in $H^{1,2}_{\text{loc}}(\mathcal{M}, S\mathcal{M})$. By the trace theorem (see for example [**32**, Section II.5.2]), their restriction to a hypersurface $\{t = \text{const}\}$ is in $L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{C}^4)$. As a consequence, the integrand

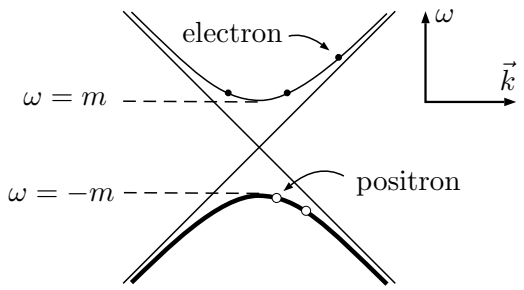


FIGURE 1.3. The Dirac sea with particles and anti-particles.

of the spatial integral in (1.3.12) is locally integrable. The solutions in $\mathcal{H}_m$ have the additional property that their restriction to the hypersurfaces are even in $L^2(\mathbb{R}^3, \mathbb{C}^4)$, so that the integral in (1.3.12) exists and is finite.

1.5. Dirac's Hole Theory and the Dirac Sea

The Dirac theory gives rise to anti-matter and pair creation, as we now briefly explain. For the present purposes, it suffices to consider the Dirac equation in the vacuum (1.3.5). It can be solved by the plane wave ansatz (for more details see for example [**14**, Section 3.1], [**128**, Section 3.3] or [**146**, Section 1.4.1])

$$\psi(x) = \chi(k)\, \mathrm{e}^{-\mathrm{i}kx}\,,$$

where $k \in \mathscr{M}$ is the four-momentum, and $kx = \langle k, x\rangle$ is the Minkowski inner product (for a mathematically precise treatment in terms of the Fourier transform see Section 2.4 below). Using this ansatz in (1.3.5) yields the (zeroth order) linear system

$$(\not{k} - m)\, \chi(k) = 0 \tag{1.5.1}$$

for the vector $\chi(k) \in \mathbb{C}^4$. Multiplying by the matrix $\not{k}+m$ and using the anti-commutation relations (1.3.1) gives the necessary condition

$$k^2 = m^2\,. \tag{1.5.2}$$

If this condition is satisfied, the matrix $\not{k}-m$ has a two-dimensional kernel, which coincides with the image of the matrix $\not{k} + m$. Thus the general solution of (1.5.1) can be written as

$$\chi(k) = (\not{k} + m)\, \phi \qquad \text{with} \qquad \phi \in \mathbb{C}^4\,.$$

The zero component $\omega := k^0$ of the four-momentum is physically interpreted as 2π times the frequency of the wave. Equation (1.5.2), also referred to as the *dispersion relation*, can then be written as

$$\omega^2 = \left|\vec{k}\right|^2 + m^2 \qquad \text{or} \qquad \omega = \pm\sqrt{\left|\vec{k}\right|^2 + m^2}\,.$$

Here the plus and the minus sign correspond to positive and negative frequency, respectively. The corresponding solutions are said to be *on the upper and lower mass shell* (see Figure 1.3). Using Planck's relation $E = \hbar\omega$, the frequency can be related to the *energy* of the solution. Thus the solutions on the upper and lower mass shell have positive respectively negative energy.

At first sight, the occurrence of solutions of negative energy seems problematic, because particles of negative energy have never been observed. Moreover, at least in a naive

consideration, solutions of arbitrarily large negative energy should make the physical system unstable, because by bringing a particle into a state of larger and larger negative energy, one could extract more and more positive energy from the system by the principle of conservation of energy. This problem was resolved by Dirac in 1930 [**28**] and led to the prediction of *particle creation* and *anti-matter*, as we now outline (an excellent and more detailed explanation can be found in [**14**, Section 5.1]). We work in the setting of non-interacting many-particle quantum mechanics, where the many-particle wave function is described by a product of one-particle wave functions. In other words, the quantum state is described by occupying many one-particle states. Dirac's concept is that in the vacuum, all the states of negative energy should be occupied, forming the so-called *Dirac sea*. According to the *Pauli exclusion principle*, each state may be occupied by at most one electron. Therefore, adding particles to the system, the additional particles must occupy states of positive energy, giving rise to *electrons*. By convention, the electrons have *negative* electric charge. Moreover, one can create "holes" in the Dirac sea. The resulting "hole in a sea of negative energy" appears as a particle of again positive energy, but with the opposite and thus *positive* electric charge. These "holes" can be observed as *positrons*. Furthermore, starting from the completely filled Dirac sea, one can "excite" a particle of the sea by a transition from a state of negative energy to a state of positive energy. As a result, one obtains a particle (=electron) plus a hole (=positron). This explains why matter and anti-matter can be created in pairs in a process called *pair creation*.

The above intuitive picture of the Dirac sea has important observable consequences, because it explains fundamental physical phenomena like anti-matter and pair creation. Nevertheless, the naive picture suffers from the problems that the Dirac sea has an *infinite negative charge density* and an *infinite energy density*. In modern quantum field theory, these problems are bypassed by introducing a suitable vacuum state and working "relative" to this vacuum state. Here we shall not enter these constructions. Instead, we shall take Dirac's concept of a "sea of interacting particles" seriously, as will be explained in more detail in Section 5.9.

1.6. Exercises

EXERCISE 1.1. Show that the relations "lies in the timelike future of" and "lies in the causal future of" are transitive in the following sense,

$$\begin{aligned} y \in I_x^\vee \text{ and } z \in I_y^\vee \quad &\Longrightarrow \quad z \in I_x^\vee \\ y \in J_x^\vee \text{ and } z \in J_y^\vee \quad &\Longrightarrow \quad z \in J_x^\vee \,. \end{aligned}$$

EXERCISE 1.2. *(Local gauge transformations I)* Show that the Klein-Gordon equation (1.2.6) is invariant under joint transformations of the electromagnetic potential and the wave function according to

$$A_j(x) \to A_j(x) + \partial_j \Lambda(x)\,, \qquad \psi(x) \to \mathrm{e}^{-\mathrm{i}\Lambda(x)}\, \psi(x)\,. \tag{1.6.1}$$

Moreover, show that the electromagnetic field tensor $F_{jk} := \partial_j A_k - \partial_k A_j$ remains unchanged under these transformations.

We remark that the transformations (1.6.1) are the classical gauge transformations of electrodynamics. They give rise to local phase transformations of the quantum mechanical wave functions, which have no physical significance because all measurable quantities involve the product of the wave function with its complex conjugate.

EXERCISE 1.3. *(Local gauge transformations II)* Show that the Dirac equation 1.3.6 is invariant under joint transformations (1.6.1) of the electromagnetic potential and the Dirac wave function.

EXERCISE 1.4. *(Anti-commutation relations)*

(a) Verify by direct computation that the Dirac matrices (1.3.3) satisfy the anti-commutation relations (1.3.1).
(b) Why is it not possible to satisfy these anti-commutation relations with 2×2- or 3×3-matrices? *Hints:* The case of odd-dimensional matrices can be ruled out by computing the square and the trace of the matrix $\gamma^0 \gamma^1$. For 2×2-matrices, a similar argument shows that the matrix $\gamma^0 \gamma^1$ is diagonalizable, making it possible to proceed in an eigenvector basis.

EXERCISE 1.5. Show that the Dirac matrices in the Dirac representation (1.3.3) are symmetric with respect to the spin inner product (1.3.7). Show that this symmetry property is equivalent to the statement that the matrices $\gamma^0 \gamma^j$ are Hermitian.

EXERCISE 1.6. Show that, multiplying the Dirac equation (1.3.6) by the operator $(\mathrm{i}\gamma^j(\partial_j - \mathrm{i}A_j) + m)$ and using the anti-commutation relations, we obtain the equation

$$\Big(-(\partial_k - \mathrm{i}A_k)(\partial^k - \mathrm{i}A^k) + \frac{\mathrm{i}}{2} F_{jk}\gamma^j\gamma^k - m^2 \Big)\psi = 0\,.$$

This differs from the Klein-Gordon equation (1.2.6) by the extra term $\frac{\mathrm{i}}{2}F_{jk}\gamma^j\gamma^k$, which describes the coupling of the spin to the electromagnetic field.

EXERCISE 1.7. In this exercise, we shall verify that for any non-zero spinor ψ, the corresponding Dirac current vector $J^k = \prec\psi|\gamma^k\psi\succ$ is non-spacelike.

(a) Show that the matrix $\gamma^0\gamma^1$ is Hermitian and has eigenvalues ± 1. Deduce that

$$\langle \psi, \gamma^0\gamma^1\psi\rangle_{\mathbb{C}^4} \leq \|\psi\|^2_{\mathbb{C}^4}\,.$$

(b) Show that the last inequality implies that $|J^1| \leq J^0$.
(c) Use the rotational symmetry of the Dirac equation to conclude that $J^0 \geq |\vec{J}|$ (where $\vec{J} = (J^1, J^2, J^3) \in \mathbb{R}^3$).

EXERCISE 1.8. This exercise has the purpose of getting more familiar with the computation rules for Dirac matrices.

(a) Derive (1.3.20) from the anti-commutation relations.
(b) Derive from the anti-commutation relations and the symmetry of the Dirac matrices that the pseudo-scalar matrix Γ in (1.3.21) is anti-symmetric and that $\Gamma^2 = \mathbb{1}$.
(c) Show that the chiral projectors χ_L and χ_R defined by (1.3.22) satisfy the relations (1.3.23) and (1.3.24). Show that the Dirac equation in the presence of an external field (1.3.6) can be rewritten as a system of equations for the left- and right-handed components of ψ,

$$\mathrm{i}\gamma^k(\partial_k - \mathrm{i}A_k)\,\chi_L\psi = m\,\chi_R\psi\,, \qquad \mathrm{i}\gamma^k(\partial_k - \mathrm{i}A_k)\,\chi_R\psi = m\,\chi_L\psi\,.$$

What happens in the limiting case $m = 0$?

EXERCISE 1.9. This exercise explains how the *causal structure* of Minkowski space is encoded in the Dirac matrices. This method generalizes to partial differential equations of so-called *symmetric hyperbolic type* as will be introduced later in this book (see Chapter 13). For this exercise no knowledge on symmetric hyperbolic systems is needed.

But we use the same notions to be introduced in Chapter 13. Show that the Dirac equation $(\mathrm{i}\partial\!\!\!/ - m)\psi = 0$ can be rewritten as a so-called *symmetric hyperbolic system*, i.e. in the form

$$\big(A^0(x)\,\partial_0 + A^\alpha(x)\,\partial_\alpha + B(x)\big)\psi = 0, \quad \text{with} \;\; (A^i)^\dagger = A^i \;\; \text{and} \;\; A^0(x) \geq c\,\mathbb{1}$$

with a constant $c > 0$ (where the dagger means transposition and complex conjugation). For such systems a notion of *causality* can be introduced as follows. A vector $\xi \in \mathbb{R}^4$ is said to be *time-like* or *light-like* at $x \in \mathbb{R}^4$, if the matrix $A(x,\xi) := A^i(x)\,\xi_i$ is definite (either positive or negative) or singular, respectively.

Find the matrices A^i and B for the Dirac equation and show that the above notions of time-like and light-like vectors coincide with the corresponding notions in Minkowski space. *Hint:* Do not be surprised if the naive choice $A^j = \gamma^j$ does not work.

EXERCISE 1.10. *(Decomposition of Dirac solutions into positive and negative energy)* For a momentum $\vec{k} \in \mathbb{R}^3$ we define the energy $\omega(\vec{k}) := \sqrt{\vec{k}^2 + m^2}$ and the matrices

$$p_\pm(\vec{k}) := \frac{k\!\!\!/ + m}{2\,k^0}\,\gamma^0\bigg|_{k^0 = \pm\omega(\vec{k})} \in \mathrm{Mat}(4,\mathbb{C}) \quad (\text{with } k\!\!\!/ := k^0\gamma^0 - \vec{k}\cdot\vec{\gamma}).$$

(i) Referring to the standard scalar product of $\mathbb{C}^4$, show that the matrices $p_\pm(\vec{k})$ are symmetric, idempotent, add up to the identity and have orthogonal images. Conclude that the spinor space $\mathbb{C}^4$ can be decomposed into the orthogonal direct sum

$$\mathbb{C}^4 = W^+_{\vec{k}} \oplus W^-_{\vec{k}}, \quad \text{with} \quad W^\pm_{\vec{k}} := \mathrm{Im}\, p_\pm(\vec{k}).$$

(ii) Let $\varphi \in C^\infty_{\mathrm{sc}}(\mathbb{R}^4, \mathbb{C}^4)$ be a smooth solution of the Dirac equation with spatially compact support, i.e.

$$(\mathrm{i}\gamma^0\partial_0 + \mathrm{i}\gamma^\alpha\partial_\alpha - m)\varphi = 0, \quad \text{with} \;\; \varphi(t,\,\cdot\,) \in C^\infty_0(\mathbb{R}^3, \mathbb{C}^4) \text{ for all } t \in \mathbb{R}.$$

Let $\hat{\varphi}$ be the smooth function on $\mathbb{R}^4$ defined by taking the Fourier transform of φ in the spatial variables only. Find $h \in C^\infty(\mathbb{R}^3, \mathrm{Mat}(4,\mathbb{C}))$ such that

$$\mathrm{i}\,\partial_t\,\hat{\varphi}(t,\vec{k}) = h(\vec{k})\cdot\hat{\varphi}(t,\vec{k}) \quad \text{for all } t \in \mathbb{R},\; \vec{k} \in \mathbb{R}^3.$$

Show that $h(\vec{k})$ is also symmetric with respect to the standard scalar product of $\mathbb{C}^4$ and satisfies

$$h(\vec{k})p_\pm(\vec{k}) = \pm\omega(\vec{k})p_\pm(\vec{k}).$$

In particular, $\pm\omega(\vec{k})$ form the spectrum of $h(\vec{k})$.

(iii) Referring to point (ii), conclude that

$$\varphi(t,\vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{3/2}} \left(p_-(\vec{k})\hat{\varphi}(0,\vec{k})\,\mathrm{e}^{\mathrm{i}\omega(\vec{k})t} + p_+(\vec{k})\hat{\varphi}(0,\vec{k})\,\mathrm{e}^{-\mathrm{i}\omega(\vec{k})t}\right)\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}$$

Hint: You can use the fact that the Cauchy problem admits unique smooth solutions.

From a mathematical point of view, the *Dirac sea* is described by the Hilbert space generated by all the smooth solutions with spatially compact support and the property that $p_-(\vec{k})\,\hat{\varphi}(0,\vec{k}) = \hat{\varphi}(0,\vec{k})$.

CHAPTER 2

Mathematical Preliminaries

In this chapter we summarize some mathematical preliminaries which we believe are important for studying causal fermion systems. In particular, the basic definitions of measure theory and functional analysis are needed already for the very definition of a causal fermion system. For mathematicians, the material covered in this chapter will probably be rather familiar. For physicists, depending on ones background, this might be different, and we hope that this chapter helps to make the core contents of this book accessible. The presentation is mostly a summary without detailed proofs, but we do provide explanations, examples and exercises to get acquainted with the material. For a more thorough coverage, we again provide references to various standard textbooks throughout the chapter.

2.1. Basics on Topology

We here recall a few basic concepts from topology. A more systematic treatment can be found in many good elementary textbooks like for example [**140**] or [**106**]. In a topological space the fundamental concept is that of an open set. Since topological spaces are a rather abstract concepts, we prefer to begin with metric spaces.

DEFINITION 2.1.1. *Let E be a set. A mapping*

$$d : E \times E \to \mathbb{R}^+_0$$

is called **metric on** E *if it has the following properties:*

(i) *Positivity: For all $x, y \in E$,*

$$d(x,y) \geq 0 \qquad \text{and} \qquad d(x,y) = 0 \quad \Longleftrightarrow \quad x = y\,.$$

(ii) *Symmetry: For all $x, y \in E$,*

$$d(x,y) = d(y,x)\,.$$

(iii) *Triangle inequality: For all $x, y, z \in E$,*

$$d(x,y) \leq d(x,z) + d(z,y)\,.$$

If d is a metric on E, then the pair (E,d) is called a **metric space**.

A simple example of a metric space is $E = \mathbb{R}^3$ with the Euclidean distance function

$$d(x,y) = \|x-y\| := \left(\sum_{\alpha=1}^{3} \left| x^\alpha - y^\alpha \right|^2 \right)^{\frac{1}{2}},$$

or also $\mathbb{R}^n$ with the analogously defined distance function. In view of this example, $d(x,y)$ is sometimes also referred to as the *distance* between x and y. More examples of metric spaces will be given in the following section.

A metric gives rise to a corresponding topology:

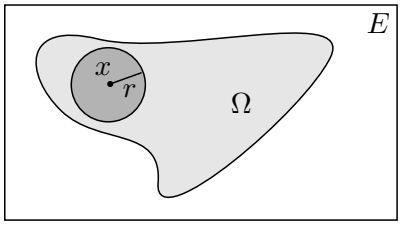


FIGURE 2.1. An open set $\Omega \subset E$.

DEFINITION 2.1.2. *Let (E, d) be a metric space. For any $x \in E$ and $r > 0$, the set*

$$B_r(x) := \big\{ y \in E \,\big|\, d(x, y) < r \big\}$$

is referred to as the **open ball of radius r centered at x**. *A subset $\Omega \subset E$ is* **open** *if for every $x \in \Omega$ there is some radius $r > 0$ such that $B_r(x) \subset \Omega$ (see Figure 2.1). The* **metric topology** $\mathcal{O}$ *of (E, d) is defined as the family of all open subsets,*

$$\mathcal{O} := \big\{ \Omega \subset E \mid \Omega \text{ is open} \big\} \subset \mathfrak{P}(E)$$

(here $\mathfrak{P}(E)$ denotes the power set of E, i.e. the set of all subsets of E).

The open sets in a metric space satisfy certain properties (we omit the proof, which is an easy exercise and can also be found in many textbooks):

LEMMA 2.1.3. *Given a metric space (E, d), the corresponding metric topology $\mathcal{O}$ has the following properties:*

(i) $\varnothing, E \in \mathcal{O}$

(ii) *Closedness under finite intersections: For any $n \in \mathbb{N}$ and $\Omega_1, \ldots, \Omega_n \subset E$,*

$$\Omega_1, \ldots, \Omega_n \in \mathcal{O} \quad \Longrightarrow \quad \Omega_1 \cap \cdots \cap \Omega_n \in \mathcal{O} \,.$$

(iii) *Closedness under arbitrary unions: For any (possibly infinite) family $(\Omega_\lambda)_{\lambda \in \Lambda}$ of subsets of E,*

$$\Omega_\lambda \in \mathcal{O} \text{ for all } \lambda \in \Lambda \quad \Longrightarrow \quad \bigcup_{\lambda \in \Lambda} \Omega_\lambda \in \mathcal{O} \,.$$

A topological space is now defined by turning exactly these properties into a definition.

DEFINITION 2.1.4. *A set E together with a distinguished family of subset $\mathcal{O} \subset \mathfrak{P}(E)$ satisfying the properties* (i)–(iii) *in Lemma 2.1.3 is referred to as a* **topological space**. *The family of subsets $\mathcal{O}$ is called the* **topology** *of E. The sets in $\mathcal{O}$ are called* **open subsets of** *E (with respect to $\mathcal{O}$). A topology $\mathcal{O}$ on E with the additional property that for any distinct point $x, y \in E$, there are disjoint open sets $U, V \in \mathcal{O}$ with $x \in U$ and $y \in V$ is called* **Hausdorff** *(see Figure 2.2).*

Clearly, topological spaces are a general and abstract concept. In particular, the topology of a topological space does not necessarily need to come from an underlying metric. Note that the topology coming from a metric is always Hausdorff, as one sees immediately by choosing $U = B_r(x)$ and $V = B_r(y)$ with $r = d(x, y)/3$.

The significance of the definition of a topological space lies in the fact that many notions from analysis can be formulated purely in topological terms and thus be generalized to arbitrary topological space. We conclude by recalling a few of such topological definitions:

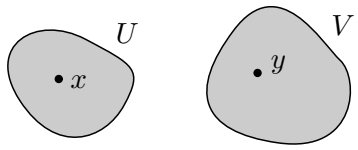


FIGURE 2.2. The Hausdorff property.

A set $A \subset E$ is called *closed* if its complement $E \setminus A$ is open. The properties in Lemma 2.1.3 can be restated for closed sets by saying that the empty set and E are closed, and that finite unions as well as arbitrary intersections of closed sets are again closed (see Exercise 2.1). The *closure* $\overline{A}$ of a subset $A \subset E$ is defined by

$$\overline{A} := \bigcap \{B \subset E \mid A \subset B \text{ and } B \text{ is closed}\} .$$

It is by definition the smallest closed set containing A. Similarly, the *interior* $\overset{\circ}{A}$ of a set $A \subset E$ is defined as the largest open set contained in A, i.e.

$$\overset{\circ}{A} := \bigcup \{B \subset E \mid B \subset A \text{ and } B \text{ is open}\} .$$

A subset $K \subset E$ is called *compact* if for every collection $\{U_i\}_{i\in I}$ of open sets of E with $K \subset \bigcup_{i\in I} U_i$ there exist finitely many $i_1, \dots, i_n \in I$ such that still $K \subset \bigcup_{k=1}^{n} U_{i_k}$ (every open cover of K has a finite subcover).

A sequence $(x_n)_{n\in\mathbb{N}}$ in E *converges to a point* $x \in E$ if for any open set $U \subset E$ with $x \in E$ there is some $N \in \mathbb{N}$ with $x_n \in U$ for all $n \geq N$. In this notion of convergence the Hausdorff property is important because it guarantees uniqueness of the limit point x (if it exists).

The *support of a function* $f : E \to \mathbb{R}$ (or, more generally, mapping to a vector space) is defined as the closure of the set where it is non-zero,

$$\operatorname{supp} f := \overline{\{x \in E \mid f(x) \neq 0\}} . \tag{2.1.1}$$

In the applications one often encounters functions with *compact support*. Finally, a mapping $f : E \to F$ between two topological spaces $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ is *continuous* if the pre-image of any open set is open, i.e. if for any $\Omega \subset F$

$$\Omega \in \mathcal{O}_F \quad \Longrightarrow \quad f^{-1}(\Omega) \in \mathcal{O}_E . \tag{2.1.2}$$

A continuous mapping which is invertible and whose inverse is also continuous is referred to as a *homeomorphism*.

In metric spaces, this definition of continuity is equivalent to the usual ε-δ-criterion (see Exercise 2.2). The topological definition has the advantage that in proofs it fits together nicely with other topological notions. Many important theorems from real analysis have topological generalizations. For example, every real-valued continuous function on a compact topological space attains its maximum (see Exercise 2.3).

2.2. Banach Spaces, Hilbert Spaces and Linear Operators

In this section, we consider complex vector spaces equipped with additional structures like a norm and a scalar product (most definitions can be adapted in a straightforward way to real vector spaces). We also recall the notion of completeness and introduce linear operators. For more details and further reading we recommend the textbooks [**136, 131, 116**].

DEFINITION 2.2.1. *Let V be a complex vector space. A* **norm on** V *is a mapping*

$$\|\cdot\| \,:\, V \to \mathbb{R}_0^+$$

with the following properties:

(i) *Homogeneity: For all $x \in V$ and $\lambda \in \mathbb{C}$,*

$$\big\|\lambda x\big\| = |\lambda|\, \|x\| \,.$$

(ii) *Definiteness: For all $x \in V$,*

$$\|x\| = 0 \iff x = 0 \,.$$

(iii) *Triangle inequality: For all $x, y \in V$,*

$$\|x + y\| \leq \|x\| + \|y\| \,.$$

If $\|\cdot\|$ is a norm on V, then the pair $(V, \|\cdot\|)$ is called a **normed space**.

Every normed space is naturally a metric space (see Definition 2.1.1) with the metric defined by $d(x, y) := \|x - y\|$ for all $x, y \in V$.

To give a concrete example, on $\mathbb{C}^n$ for any $p \in [1, \infty) \cup \{\infty\}$ one obtains a norm $\|\cdot\|_p$ as

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \quad \text{for } p < \infty \qquad \text{and} \qquad \|x\|_\infty := \max\{|x_1|, \ldots |x_p|\} \,. \tag{2.2.1}$$

The same norms can be considered on $\mathbb{R}^n$ instead of $\mathbb{C}^n$, and in the case $p = 2$, this gives the Euclidean length of a vector $x \in \mathbb{R}^n$. As a related, but infinite-dimensional example, on the vector space of compactly supported continuous functions $C_0^0(\mathbb{R}^n)$ one defines the integral norms

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} \quad \text{for } p < \infty \qquad \text{and} \qquad \|f\|_\infty := \sup_{x \in \mathbb{R}^n} |f(x)| \,. \tag{2.2.2}$$

For more details on these examples see Exercise 2.4. The norm $\|.\|\|_p$ will be introduced on an abstract measure space in Section 2.3.

We next recall the notion of completeness, which is a property about convergence of sequences. First recall that a sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space E *converges* to a point $x \in E$ if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. Similarly, a sequence $(x_n)_{n \in \mathbb{N}}$ in E is a *Cauchy sequence* if for any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$. Any sequence converging to a point of E is a Cauchy sequence. If conversely also any Cauchy sequence converges to a point of E, one calls E a *complete metric space*. A normed space which as metric space is complete is referred to as a *Banach space*. A few examples of Banach spaces are given in Exercise 2.5. Completeness is an important and very useful property. Therefore, we would like to restrict attention to complete metric spaces. This is no major restriction because any non-complete metric space can be regarded as subset of a corresponding complete metric space, its so-called *completion* (for details see Exercise 2.6).

We next specialize the setting by turning from a norm to a scalar product.

DEFINITION 2.2.2. *Let V be a complex vector space. A* **scalar product on** V *is a mapping*

$$\langle .|. \rangle \,:\, V \times V \to \mathbb{C}$$

with the following properties:

(i) *Linearity in the second argument: For all* $u, v, w \in V$ *and* $\alpha, \beta \in \mathbb{C}$,

$$\langle u \,|\, \alpha v + \beta w \rangle = \alpha \, \langle u|v\rangle + \beta \, \langle u|w\rangle \, .$$

(ii) *Hermitian symmetry: For all* $u, v \in V$,

$$\overline{\langle u|v\rangle} = \langle v|u\rangle \, .$$

(iii) *Positive definiteness: For all* $u \in V$,

$$\langle u|u\rangle \geq 0 \qquad \textit{and} \qquad \langle u|u\rangle = 0 \iff u = 0 \, .$$

If $\langle .|. \rangle$ *is a scalar product on* V, *the pair* $(V, \langle .|. \rangle)$ *is a* **scalar product space**.

Every scalar product space $(V, \langle .|. \rangle)$ is also a normed space with the norm being defined by $\|u\| := \sqrt{\langle u|u\rangle}$ for all $u \in V$ (see Exercise 2.7). The *Cauchy-Schwarz inequality*

$$\big|\langle u|u\rangle\big| \leq \|u\| \, \|v\|$$

bounds the scalar product in terms of the corresponding norms; it is a direct consequence of the above properties of a scalar product. A scalar product space which is a Banach space, i.e. which is complete, is called a *Hilbert space*. We usually denote a Hilbert space by $\mathcal{H}$ or $(\mathcal{H}, \langle .|. \rangle)$.

A simple example of a scalar product space is $\mathbb{C}^n$ with the scalar product defined by

$$\langle u|v\rangle := \sum_{i=1}^{n} \overline{u_i} \, v_i \, .$$

An infinite-dimensional example is the space $C_0(\mathbb{R}^n, \mathbb{C})$ of complex-valued test functions with the scalar product

$$\langle f|g\rangle_{L^2} := \int_{\mathbb{R}^n} \overline{f(x)} \, g(x) \mathrm{d}x \, .$$

The corresponding norms give us back (2.2.1) or (2.2.2), respectively, in the case $p = 2$.

Throughout this book, all Hilbert spaces will be *separable*, meaning that there is a countable subset $D \subset \mathcal{H}$ which is *dense* in the sense that its closure is the whole Hilbert space, $\overline{D} = \mathcal{H}$. In a separable Hilbert space, one can choose an *orthonormal Hilbert space basis* $(e_i)_{i \in I}$ characterized by the following properties:

(i) The index set I is at most countable.

(ii) The system $(e_i)_{i \in I}$ is *orthonormal*, i.e.

$$\langle e_i|e_j\rangle = \delta_{ij} \, ,$$

where δ_{ij} is the *Kronecker delta* defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \, . \end{cases}$$

(iii) The system $(e_i)_{i \in I}$ is *complete*, meaning that every vector $u \in \mathcal{H}$ has the representation as a (possibly infinite) linear combination

$$u = \sum_{i \in I} c_i \, e_i \qquad \text{with} \qquad c_i \in \mathbb{C} \, ,$$

where the series converges in $\mathcal{H}$.

Using property (ii), the coefficients c_i in the representation in (iii) can be computed by $c_i = \langle e_i|u\rangle$. Thus every vector $u \in \mathcal{H}$ can be written as

$$u = \sum_{i\in I} \langle e_i|u\rangle \, e_i \, .$$

The convergence of this series is guaranteed by *Bessel's inequality*

$$\sum_{i\in I} \big|\langle e_i|u\rangle\big|^2 \leq \|u\| \, ,$$

which holds for any orthonormal system $(e_i)_{i\in I}$ (even if not complete) as a direct consequence of the properties of the scalar product. Moreover, the cardinality of the index set I does not depend on the choice of the basis, making it possible to define the *dimension* of the Hilbert space

$$\dim \mathcal{H} := \#I \ \in \ \mathbb{N}_0 \cup \{\infty\} \, .$$

Now we turn our attention to linear operators.

DEFINITION 2.2.3. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. A mapping*

$$A : V \to W$$

is a **bounded linear operator** *from V to W if it has the following properties:*

(i) *Linearity: For all $u, v \in V$ and $\alpha, \beta \in \mathbb{C}$,*

$$A\big(\alpha u + \beta v\big) = \alpha \, A(u) + \beta \, A(v) \, .$$

(ii) *Boundedness: There is a constant $c > 0$ such that for all $u \in V$,*

$$\big\|A(u)\big\|_W \leq c \, \|u\|_V \, .$$

Usually, one also writes $A(u)$ simply as Au. We remark that for linear operators, boundedness is equivalent to continuity (see Exercise 2.8).

The set of all bounded linear operators between two complex vector spaces V and W forms again a complex vector space with the vector operations defined pointwise. Thus for any bounded linear operators $A, B : V \to W$ and $\alpha, \beta \in C$, the operator $\alpha A + \beta B : V \to W$ is defined by

$$\big(\alpha A + \beta B\big)(u) := \alpha \, Au + \beta \, Bu \qquad \text{for any } u \in V \, .$$

The resulting vector space of bounded linear operators from V to W is denoted by $\mathrm{L}(V, W)$. One obtains a norm on this vector space by setting

$$\|A\| := \sup_{u\in V, \|u\|_V = 1} \big\|Au\big\|_W \, ,$$

referred to as the sup-norm or the *operator norm*. With this norm, the space $\mathrm{L}(V, W)$ is complete if and only if W is. For details we refer to Exercise 2.9.

We will be concerned mainly with bounded linear operators acting on a Hilbert space $(\mathcal{H}, \langle .|. \rangle)$. Two cases are of specific interest: mappings from $\mathcal{H}$ to the complex numbers, and mappings from $\mathcal{H}$ back to itself. In the first case, the resulting operator is also referred to as a *bounded linear form*. These operators form the so-called *dual space* $\mathcal{H}^*$,

$$\mathcal{H}^* := \mathrm{L}(\mathcal{H}, \mathbb{C}) \, .$$

An example of a bounded linear form is the mapping $\mathcal{H} \ni u \mapsto \langle v|u\rangle \in \mathbb{C}$ obtained by taking the scalar product with a fixed vector $v \in \mathcal{H}$. In fact, every bounded linear form

can be written in this way, making it possible to canonically identify the dual space of a Hilbert space with the Hilbert space itself:

THEOREM 2.2.4. **(Fréchet-Riesz)** *Let $\mathcal{H}$ be a Hilbert space. Then for any bounded linear functional $\phi \in \mathcal{H}^*$ there is a unique vector $v \in \mathcal{H}$ such that*

$$\phi(u) = \langle v|u\rangle \qquad \textit{for all } u \in \mathcal{H}\,. \tag{2.2.3}$$

In the case of a separable Hilbert space of interest here, this theorem can be understood in simple terms by expanding vectors in an orthonormal Hilbert space basis $(e_i)_{i\in I}$. Writing a vector $u \in \mathcal{H}$ as $u = \sum_{i\in I}\langle e_i|u\rangle\, e_i$ we have

$$\phi(u) = \sum_{i\in I}\langle e_i|u\rangle\, \phi(e_i)\,,$$

where in the infinite-dimensional case we have to use continuity (boundedness) of ϕ to see that we can pull out the (infinite) sum. Now the idea is to rewrite the last term as

$$\sum_{i\in I}\langle e_i|u\rangle\, \phi(e_i) = \Big\langle \sum_{i\in I} \overline{\phi(e_i)}\, e_i \,\Big|\, u\Big\rangle$$

(where we used linearity and continuity of the scalar product), and then simply define the vector $v \in \mathcal{H}$ we are looking for by

$$v = \sum_{i\in I} \overline{\phi\big(e_i\big)}\, e_i\,. \tag{2.2.4}$$

In finite dimensions, this computation is fine. In infinite dimensions, however, one must show that the series defining v converges in $\mathcal{H}$ *before* one can compute as above. Fortunately, with a little trick this convergence is not hard to see: Let $I = \mathbb{N}$ and note first that for $v_n := \sum_{i=1}^n \phi(e_i)\, e_i$ we get

$$\sum_{i=1}^n \big|\phi(e_i)\big|^2 = |\phi(v_n)| \le \|\phi\|\cdot\|v_n\| = \|\phi\|\cdot\Big(\sum_{i=1}^n |\phi(e_i)|^2\Big)^{\frac12}\,,$$

and therefore $\sum_{i=1}^n |\phi(e_i)|^2 < \|\phi\|^2$. Using completeness of $\mathcal{H}$, one readily finds that the series in (2.2.4) also converges as desired. It is also straightforward to verify that $\|v\| = \|\phi\|$.

The Fréchet-Riesz theorem is the mathematical justification for the *bra/ket notation* commonly used in quantum mechanics. In this notation elements of $\mathcal{H}$ are denoted by $|v\rangle$ and referred to as *kets*, and elements of the dual space $\mathcal{H}^*$ are denoted by $\langle v|$ and referred to as *bra*s. As a consequence of the Fréchet-Riesz theorem, for any ket $|v\rangle$ one may form the corresponding bra $\langle v|$ and vice-versa. This notation resembles the role of the inner product $\langle\cdot|\cdot\rangle$, as for any bra $\langle v|$ and any ket $|w\rangle$ one may form the bra-(c)ket $\langle v|w\rangle$.

Another application of the Fréchet-Riesz theorem is to define the *adjoint* of a bounded linear operator: Given two Hilbert spaces $(\mathcal{H}_1, \langle.|.\rangle_{\mathcal{H}_1})$ and $(\mathcal{H}_2, \langle.|.\rangle_{\mathcal{H}_2})$ as well as an operator $A \in \mathrm{L}(\mathcal{H}_1, \mathcal{H}_2)$, for every $u \in \mathcal{H}_2$ we can the define the linear functional on $\mathcal{H}_1$

$$\phi \,:\, \mathcal{H}_1 \to \mathbb{C}\,, \qquad v \mapsto \langle u\,|\,Av\rangle_2\,.$$

The estimate $|\phi(v)| \le \|u\|_{\mathcal{H}_2}\,\|A\|\,\|v\|_{\mathcal{H}_1}$ shows that this functional is bounded, making it possible to represent it uniquely by a vector $w \in \mathcal{H}_1$, i.e.

$$\langle u|Av\rangle_{\mathcal{H}_2} = \langle w|v\rangle_{\mathcal{H}_1} \qquad \text{for all } v \in \mathcal{H}_1\,.$$

By direct computation one verifies that the resulting mapping $v \mapsto w$ is linear and bounded. The resulting operator $A^* \in \mathrm{L}(\mathcal{H}_2, \mathcal{H}_1)$, referred to as the **adjoint** of A, satisfies and is uniquely determined by the relation

$$\langle u | Av \rangle_{\mathcal{H}_2} = \langle A^* u | v \rangle_{\mathcal{H}_1} \qquad \text{for all } u \in \mathcal{H}_2 \text{ and } v \in \mathcal{H}_1 \,. \tag{2.2.5}$$

We finally consider the space $\mathrm{L}(\mathcal{H}, \mathcal{H})$ of bounded linear endomorphisms. For brevity, this space is also denoted by $\mathrm{L}(\mathcal{H})$. In the context of such linear endomorphisms, the following additional notions are important.

DEFINITION 2.2.5. *A bounded linear operator $A \in \mathrm{L}(\mathcal{H})$ is* **symmetric** *if*

$$\langle Au \,|\, v \rangle = \langle u \,|\, Av \rangle \qquad \textit{for all } u, v \in \mathcal{H} \,.$$

An operator $U \in \mathrm{L}(\mathcal{H})$ is **unitary** *if it has a bounded inverse and if*

$$\langle Uu \,|\, Uv \rangle = \langle u \,|\, v \rangle \qquad \textit{for all } u, v \in \mathcal{H} \,.$$

It has **finite rank** *if its image $A(\mathcal{H})$ is a finite-dimensional subspace of $\mathcal{H}$.*

For clarity, we mention that there is also the notion of an operator being **selfadjoint**. For bounded linear operators, this is equivalent to being symmetric. For unbounded operators, however, being selfadjoint is a stronger property than being symmetric. Self-adjointness is required for instance in the proof of the spectral theorem (see Section 3.2). With this in mind, when talking about bounded operators, in this book we usually prefer the notion of a *symmetric* operator.

If the Hilbert space $\mathcal{H}$ is finite-dimensional, symmetric and unitary operators can be diagonalized by choosing an orthonormal basis of eigenvectors (for details see standard textbooks on linear algebra like [**100, 142**]). The eigenvalues of a symmetric operator are all real, whereas the eigenvalues of a unitary operator have modulus one. The generalization of this result to the infinite-dimensional setting is provided by the *spectral theorem*. The general spectral theorem for bounded symmetric operators, which will be needed mainly in Section 15, will be treated in Section 3.2 below. In large parts of this book, however, we only deal with symmetric operators having *finite rank* (on possibly infinite-dimensional Hilbert spaces). For such operators the results of linear algebra carry over in a straight-forward manner, as we now explain.

Thus let $A \in \mathrm{L}(\mathcal{H})$ be a symmetric bounded operator of finite rank. Then by definition its image $I := A(\mathcal{H})$ is finite-dimensional. Its orthogonal complement

$$I^\perp := \big\{ v \in \mathcal{H} \mid \langle v, u \rangle = 0 \text{ for all } u \in I \big\} \tag{2.2.6}$$

is a closed subspace of $\mathcal{H}$, and every vector $u \in \mathcal{H}$ can be decomposed uniquely as

$$u = u^{||} + u^\perp \qquad \text{with} \qquad u^{||} \in I, u^\perp \in I^\perp \tag{2.2.7}$$

(see Exercise 2.12). Moreover, for any $u \in I^\perp$ the computation

$$0 = \langle A^2 u \,|\, u \rangle = \langle Au \,|\, Au \rangle$$

shows that $Au = 0$. Therefore, A vanishes identically on $I^\perp$ and so it suffices to consider the restriction $A|_I : I \to I$. Being an operator on a finite-dimensional vector space, it can be diagonalized as in linear algebra.

When taking products of symmetric operators $A_1, \ldots, A_n$ of finite rank, one chooses I as the finite-dimensional vector space spanned by the images of all the operators. Then the restrictions $A_k|_I$ with $k = 1, \ldots, n$ all map I to itself, making it possible to work again in a finite-dimensional subspace of $\mathcal{H}$.

2.3. Basics on Abstract Measure Theory

The basic object of a causal fermion system is a measure. For this reason, we now provide the necessary basics on abstract measure theory. For more details we refer to good standard textbooks like [**136, 101, 15, 87, 8**].

Let $\mathcal{F}$ be a set. A measure is a mapping which to a subset of $\mathcal{F}$ associates a non-negative number, which can be thought of as the "volume" of the set. In order to get into a mathematically sensible setting, one cannot define the measure on any subset, but only on a distinguished family of subsets of $\mathcal{F}$. This family must form a σ-algebra, which we now define.

DEFINITION 2.3.1. *A system $\mathfrak{M}$ of subsets of $\mathcal{F}$ is a* **σ-algebra** *if it has the following properties:*

(i) $\varnothing \in \mathfrak{M}$

(ii) $\mathfrak{M}$ *is closed under taking complements: For any subset $A \subset \mathcal{F}$,*

$$A \in \mathfrak{M} \quad \Longrightarrow \quad \mathcal{F} \setminus A \in \mathfrak{M}\,.$$

(iii) $\mathfrak{M}$ *is closed under at most countable unions: For any sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathcal{F}$,*

$$A_n \in \mathfrak{M} \text{ for all } n \in \mathbb{N} \quad \Longrightarrow \quad \bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{M}\,.$$

The sets in $\mathfrak{M}$ are also referred to as the **measurable** *sets.*

Using De Morgan's laws, it follows that a σ-algebra is closed even under at most countable intersections and, more generally, under at most countable set operations.

We next introduce a measure as a mapping which to every measurable set associates its "volume." This mapping is compatible with at most countable set operations, as is made precise by the notion of σ-additivity.

DEFINITION 2.3.2. *A* **measure** *ρ is a mapping from a σ-algebra to the non-negative numbers or infinity,*

$$\rho \,:\, \mathfrak{M} \to \mathbb{R}^+_0 \cup \{\infty\}\,,$$

which has the following properties:

(i) $\rho(\varnothing) = 0$

(ii) ρ *is* **σ-additive***: For any sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint measurable sets,*

$$\rho\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \rho(A_n)\,.$$

The structure $(\mathcal{F}, \mathfrak{M}, \rho)$ is a **measure space***.*

By choosing almost all of the sets A_n in (ii) as empty sets, one sees that σ-additivity implies *finite additivity*. In particular $\rho(A \cup B) = \rho(A) + \rho(B)$ for any disjoint measurable sets A and B. As a consequence, a measure is *monotone* in the sense that $\rho(A) \leq \rho(B)$ for any measurable sets $A \subset B$ (see Exercise 2.15).

A set of measure zero is also referred to as a *null set*. By monotonicity, every measurable subset of a null set is again a null set. If every subset of a null set is measurable, then the measure is called *complete*.

On a measure space $(\mathcal{F}, \mathfrak{M}, \rho)$ a notion of *integration* is introduced as follows. We begin with complex-valued functions which take only a finite number of values, also

referred to as *step functions*. A step function $f : \mathcal{F} \to \mathbb{C}$ can be written as

$$f = \sum_{n=1}^{N} c_n \chi_{A_n}$$

with $N \in \mathbb{N}$, coefficients $c_n \in \mathbb{C}$ and measurable sets $A_n \in \mathfrak{M}$ (here χ_A is the characteristic function defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise). Its integral is defined in the natural way by

$$\int_{\mathcal{F}} f \, \mathrm{d}\rho := \sum_{n=1}^{N} c_n \, \rho\big(A_n\big) \, .$$

This integral can be extended to more general functions as follows. A function $f : \mathcal{F} \to \mathbb{C}$ is *measurable* if the pre-image of any open set is measurable. For a measurable function $f : \mathcal{F} \to [0, \infty]$ taking real, nonnegative values or the value plus infinity, one defines

$$\int_{\mathcal{F}} f \, \mathrm{d}\rho := \sup \left\{ \int_{\mathcal{F}} s \, \mathrm{d}\rho \,\middle|\, s : \mathcal{F} \to [0, \infty) \text{ is a step function with } s \leq f \right\} \in \mathbb{R}_0^+ \cup \{\infty\} \, .$$

This expression is allowed to be infinite. If it is finite, one calls f *integrable*. From here on the further generalization is straightforward: For a measurable function $f : \mathcal{F} \to \mathbb{R}$ one defines its positive and negative $f^+, f^- : \mathcal{F} \to [0, \infty)$ by $f^+(x) := \max\{f(x), 0\}$ and $f^-(x) := \max\{-f(x), 0\}$. Then clearly $f = f^+ - f^-$ and one can show that f^+, f^- are again measurable. If at least one of them is integrable, one defines

$$\int_{\mathcal{F}} f \mathrm{d}\rho := \int_{\mathcal{F}} f^+ \mathrm{d}\rho - \int_{\mathcal{F}} f^- \mathrm{d}\rho \in \mathbb{R} \cup \{-\infty, \infty\} \, .$$

Again, f is defined to be integrable if this integral is finite (note that no cancellation between positive and negative terms can happen, as positive and negative part of f are considered separately). Finally, for a complex-valued measurable function $f : \mathcal{F} \to \mathbb{C}$ whose real and imaginary part are both integrable (they are always measurable), one defines

$$\int_{\mathcal{F}} f \mathrm{d}\rho := \int_{F} \mathrm{Re}(f) \mathrm{d}\rho + \mathrm{i} \int_{\mathcal{F}} \mathrm{Im}(f) \mathrm{d}\rho \in \mathbb{C} \, .$$

A complex-valued measurable function on $\mathcal{F}$ whose real and imaginary part are both integrable is again called integrable. One can combine all these notions of integrability by demanding that the absolute value is integrable,

$$\int_{\mathcal{F}} |f(x)| \, \mathrm{d}\rho(x) \;\in\; \mathbb{R}_0^+ \cup \{\infty\} \, . \tag{2.3.1}$$

This condition can be understood immediately from the requirement that in integrals one must always avoid expressions of the form "$\infty - \infty$."

The integrable functions form a vector space denoted by $L^1(\mathcal{F}, \mathrm{d}\rho)$. Similarly, the measurable functions f whose power $|f|^p$ with $p \in (1, \infty)$ is integrable, form a vector space $L^p(\mathcal{F}, \mathrm{d}\rho)$. Finally, the space $L^\infty(\mathcal{F}, \mathrm{d}\rho)$ is defined as the functions which are essentially bounded in the sense that there is a number $c > 0$ such that the pre-image $|f|^{-1}((c, \infty])$ has ρ-measure zero. The spaces $L^p(\mathcal{F}, \mathrm{d}\rho)$ with $p \in [1, \infty]$ are almost

normed spaces if endowed with the corresponding norms

$$\|f\|_p := \Big(\int_{\mathcal{F}} |f|^p \, \mathrm{d}\rho \Big)^{\frac{1}{p}} \qquad \text{if } p \in [0, \infty)$$
$$\|f\|_\infty := \inf \Big\{ c \geq 0 \;\Big|\; \rho\Big(|f|^{-1}\big((c,\infty]\big) \Big) = 0 \Big\} .$$

The only issue here is that functions which vanish almost everywhere (i.e. which are non-zero only on a set of ρ-measure zero) have norm zero. In order to resolve this issue, in the L^p-spaces one quotients out these functions. We thus obtains a normed space denoted by $L^p(\mathcal{F}, \mathrm{d}\rho)$, which even turn out to be Banach spaces. Although the vectors in these L^p-spaces are equivalence classes of functions which differ on sets of measure zero, for simplicity one usually refers to vectors in $L^p(\mathcal{F}, \mathrm{d}\rho)$ as *functions* and understands implicitly that they may be changed arbitrarily on sets of measure zero. The space $L^2(\mathcal{F}, \mathrm{d}\rho)$ is even a Hilbert space, endowed with the scalar product

$$\langle f | g \rangle_{L^2(\mathcal{F}, \mathrm{d}\rho)} := \int_{\mathcal{F}} \overline{f(x)}\, g(x) \, \mathrm{d}\rho(x) \, . \tag{2.3.2}$$

We remark that these constructions generalize immediately to functions taking values in a Banach or Hilbert spaces, if one simply replaces the absolute value in (2.3.1) by the norm on the Banach space and the inner product $\overline{f} g$ in (2.3.2) by the Hilbert space scalar product. Finally, we remark that integration over a subset $A \subset \mathcal{F}$ is defined as

$$\int_A f \, \mathrm{d}\rho := \int_{\mathcal{F}} f \chi_A \, \mathrm{d}\rho \, ,$$

where χ_A is the characteristic function of A (defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise).

We now specialize the setting by considering a class of measures which will be of major importance in this book, namely Borel measures. To this end, we assume that $(\mathcal{F}, \mathcal{O})$ is a topological space. Then the *Borel algebra* is defined as the smallest σ-algebra which contains all the open sets (see Exercise 2.16). An element of the Borel algebra is a *Borel set*. A measure on the Borel algebra is referred to as a *Borel measure*. The Borel measures of relevance to us will typically harmonize with the topology in the following sense.

Definition 2.3.3. *A Borel measure ρ on $\mathcal{F}$ is called* **regular** *if for any measurable set A,*

$$\rho(A) = \sup_{K \subset A \text{ compact}} \rho(K) = \inf_{\Omega \supset A \text{ open}} \rho(\Omega) \, .$$

It is **locally finite** *if every point of $\mathcal{F}$ has an open neighborhood Ω with $\rho(\Omega) < \infty$. Regular, locally finite Borel measures are also referred to as* **Radon measures**.

The Lebesgue measure on $\mathbb{R}^n$ is a Radon measure. We remark that a Borel measure in general is *not complete*, because a Borel null set may have subsets which are not Borel sets. One could improve the situation by forming the *completion* of the measure (see Exercise 2.17). However, completeness of the measure is not important for most applications, and is will often be more convenient to work with Borel measures.

One of the advantages of the notion of integration introduced above (compared to for example the Riemann integral) is that various (easy to use and prove) results regarding convergence of integrals hold. For instance, for a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable functions $f_n : \mathcal{F} \to \mathbb{C}$ which converge pointwise to a function $f : \mathcal{F} \to \mathbb{C}$, the function f is

again measurable and one has

$$\int_{\mathcal{F}} f_n \,\mathrm{d}\rho \to \int_{\mathcal{F}} f \,\mathrm{d}\rho$$

if all functions only take nonnegative values and $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ (*Lebesgue's monotone convergence theorem*), or if there exists an integrable function $g : \mathcal{F} \to [0, \infty)$ with $|f_n| \leq g$ for all $n \in \mathbb{N}$ (*Lebesgue's dominated convergence theorem*). Another important result in integration theory is *Fubini's theorem*, which is about iterated integrals. If $(\mathcal{F}, \mathfrak{M}, \rho)$ and $(\mathcal{G}, \mathfrak{N}, \nu)$ are two measure spaces, then on the product space $\mathcal{F} \times \mathcal{G}$ there is a natural σ-algebra containing all product sets $M \times N$ with $M \in \mathfrak{M}$, $N \in \mathfrak{N}$, and a measure $\rho \times \nu$ on this σ-algebra such that $(\rho \times \nu)(M \times N) = \rho(M)\nu(N)$ for all $M \in \mathfrak{M}$, $N \in \mathfrak{N}$. Now Fubini's theorem now states that if $f : \mathcal{F} \times \mathcal{G} \to \mathbb{C}$ is integrable with respect to this measure, then

$$\int_{\mathcal{F}\times\mathcal{G}} f \,\mathrm{d}(\rho \times \nu) = \int_{\mathcal{G}} \left(\int_{\mathcal{F}} f(x, y) \,\mathrm{d}\rho(x) \right) \mathrm{d}\nu(y) = \int_{\mathcal{F}} \left(\int_{\mathcal{G}} f(x, y) \,\mathrm{d}\nu(y) \right) \mathrm{d}\rho(x) \,. \quad (2.3.3)$$

A variant of Fubini's theorem, referred to as *Tonelli's theorem*, states that if f is measurable and non-negative (but not necessarily integrable), then again (2.3.3) holds, but now the integrals could take the value plus infinity.

We conclude this section by introducing a few other notions which will be needed later on. The first notion is the support of a measure.

Definition 2.3.4. *Let $(\rho, \mathfrak{M})$ be a measure on the topological space $(\mathcal{F}, \mathcal{O})$. The* **support** *of ρ is defined as the complement of the largest open set of measure zero, i.e.*

$$\operatorname{supp} \rho := \mathcal{F} \setminus \bigcup \big\{ \Omega \subset \mathcal{F} \;\big|\; \Omega \text{ is open and } \rho(\Omega) = 0 \big\} \,. \quad (2.3.4)$$

Note that the support is by definition a closed subset of $\mathcal{F}$. In integrals, one can always restrict to the support of a measure in the sense that the identity

$$\int_{\mathcal{F}} f \,\mathrm{d}\rho = \int_{\operatorname{supp}\rho} f \,\mathrm{d}\rho$$

holds for any integrable function f on $\mathcal{F}$.

Suppose we want to compare two Radon measures ρ and $\tilde{\rho}$ on $\mathcal{F}$. A natural idea is to consider the difference of the measures $\rho - \tilde{\rho}$. The difficulty is that for a measurable set $A \subset \mathcal{F}$, its measures $\rho(A)$ and $\tilde{\rho}(A)$ could both take the value $+\infty$, in which case their difference would be ill-defined. In order to avoid this problem, we use the regularity of a Radon measure and exhaust by compact sets. Assuming that $\mathcal{F}$ is locally compact, the fact that Radon measures are locally finite implies that Radon measures of compact sets are always finite. This leads us to the following definition:

Definition 2.3.5. *Given two Radon measures ρ and $\tilde{\rho}$ on a locally compact topological space $\mathcal{F}$, we define the Borel measures $\mu^\pm$ by*

$$\mu^+(A) = \sup_{K \subset A \text{ compact}} \big(\tilde{\rho}(K) - \rho(K) \big)$$
$$\mu^-(A) = \sup_{K \subset A \text{ compact}} \big(\rho(K) - \tilde{\rho}(K) \big)$$

for any Borel subset $A \subset \mathcal{F}$. The difference of measures $\rho - \tilde{\rho}$ is said to have **bounded total variation** *if the measures $\mu^\pm$ are finite, i.e. if*

$$\mu_+(\mathcal{F}), \mu_-(\mathcal{F}) < \infty \,.$$

If this is the case, the **total variation measure** $|\rho - \tilde{\rho}|$ *is defined by*

$$|\rho - \tilde{\rho}| = \mu^+ + \mu^- \,.$$

We remark that $\rho - \tilde{\rho}$ can also be defined in the context of signed measures; we refer the interested reader to [**101**, §28] or [**136**, Section 6.1].

Another notion of measure theory which we will use frequently is the push-forward measure, which we now define (for more details see for example [**15**, Section 3.6] or Exercise 2.18). To this end, let $(\mathcal{F}, \mathfrak{M}, \rho)$ be a measure space, and suppose we are given a mapping $f : \mathcal{F} \to X$, where X is any set. Then f induces a measure on X as follows: Let $\mathfrak{M}_X$ be the set of all subsets $\Omega \subset X$ whose pre-image $f^{-1}(\Omega)$ is ρ-measurable. Using the elementary identities for inverse images of unions and complements, one verifies that $\mathfrak{M}_X$ is indeed a σ-algebra on X. On this σ-algebra, the *push-forward measure* $f_*\rho$ is defined by

$$(f_*\rho)(\Omega) := \rho\big(f^{-1}(\Omega)\big) \,.$$

Using again the above-mentioned identities for inverse images, one verifies that $f_*\rho$ is indeed a measure.

2.4. Distributions and Fourier Transform

We now recall a few basics on distribution theory and the Fourier transform. For more details, we recommend the textbook [**89**], [**131**, Sections V.3 and IX] or [**130**, §2.1, §2.2 and Appendix A].

The theory of distributions describes a generalization of the concept of a function on $\mathbb{R}^n$ (or, similarly, on a bounded domain or smooth manifold). Moreover, the differential calculus for smooth functions is extended to objects which are more singular than functions. The desire for such objects can be motivated for instance by the classical problem in electrostatics to determine the electric field generated by a distribution of charges. In the continuum formulation, a distribution of charges is described by a charge density $\rho : \mathbb{R}^3 \to \mathbb{R}$ (typically compactly supported) having the interpretation that for any domain $\Omega \subset \mathbb{R}^3$ the integral $\int_\Omega \rho(x) \mathrm{d}^3x$ describes the total charge contained inside of Ω. The electromagnetic field $E : \mathbb{R}^3 \to \mathbb{R}^3$ generated by ρ can then be computed as $E = -\nabla\phi$, where ϕ is a (suitable) solution of Poisson's equation $\Delta\phi = -\rho$. Now suppose that in this formulation one wants to deal with a *point particle* whose complete charge Q is concentrated at a single point, say the origin. Then the corresponding density ρ would need to satisfy

$$\int_\Omega \rho(x)\, \mathrm{d}^3x = \begin{cases} Q & 0 \in \Omega \,, \\ 0 & 0 \notin \Omega \end{cases} \qquad \text{for any } \Omega \subset \mathbb{R}^3 \,. \tag{2.4.1}$$

It is not difficult to see that such a function ρ cannot exist (see Exercise 2.19). Intuitively speaking, this function would need to vanish outside of the origin. At the origin, however, its value would have to be "so (infinitely) large" that an integral over a region containing the origin still gives a nonzero contribution. The most common way to rigorously deal with such singular objects is to understand them as linear functionals on certain spaces of smooth functions on $\mathbb{R}^3$, referred to as *test functions*. In order to motivate this functional, we write (2.4.1) more generally as

$$\int_{\mathbb{R}^3} \rho(x)\, f(x)\, \mathrm{d}^3x = Q\, f(0) \,.$$

Indeed, choosing f as the characteristic function $f = \chi_\Omega$ (defined by $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ otherwise), we recover (2.4.1). But now f can be a more general function. Restricting to smooth function gives rise to Dirac's δ distribution (or simply δ distribution), as is explained in more detail in the next example (for simplicity in one dimension).

EXAMPLE 2.4.1. **(The δ distribution)** The prime example of a distribution is Dirac's δ *distribution* which in physics textbook is introduced as a "function" $\delta(x)$ which is zero everywhere except at the origin, where it takes the value ∞. The infinite contribution at the origin is "normalized" by demanding that its integral is equal to one. These properties can be summarized by saying that

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, \mathrm{d}x = f(0)\,. \tag{2.4.2}$$

There are various ways to make mathematical sense of this equation. One method is to regard the combination $\delta(x)\, \mathrm{d}x$ as a measure δ_0 supported at the origin of total volume one, i.e. $\delta_0(\mathbb{R}) = 1$. In this way, the equation (2.4.2) makes sense if f is any continuous function. An alternative method is to take (2.4.2) as the definition of a linear functional on a space of suitable test functions f. The latter method has the advantage that it makes it possible to even define the derivative of the δ distribution by

$$\int_{-\infty}^{\infty} f(x)\, \delta'(x)\, \mathrm{d}x = -f'(0)\,. \tag{2.4.3}$$

In order to allow for distributions to include an arbitrary number of derivatives, we choose a space of *smooth* test functions. Distributions will be defined as linear functionals on this space of test functions. Derivatives of distributions can be defined similar as in (2.4.3) by "formally integrating by parts." Since a function g defines a linear functional T_g by integrating,

$$T_g(f) := \int_{-\infty}^{\infty} g(x)\, f(x)\, \mathrm{d}x\,, \tag{2.4.4}$$

every function gives rise to a corresponding distribution. In this sense, distributions are generalized functions. In order to make sure that the integral in (2.4.4) exists and is finite, it is a good idea to assume that the test function f has suitable decay properties at infinity.

In order to make these ideas mathematically precise, we need to specify the space of test functions. Moreover, we need to endow this space of test function with a topology. Then we can introduce distributions as the space of linear functionals on the test functions. As we shall see, working with the right space of test functions, one can make mathematical sense of the Fourier transform for distributions. ◊

After this motivation, we now turn to the mathematical definition of distributions. In preparation of our constructions, we recall the *multi-index notation* in $\mathbb{R}^n$. A *multi-index* is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ of non-negative integers. For such a multi-index α we define the corresponding monomial x^α and combination of partial derivatives ∂^α by

$$x^\alpha := \big(x_1\big)^{\alpha_1} \cdots \big(x^n\big)^{\alpha_n} \qquad \text{and} \qquad \partial^\alpha := \partial_x^\alpha := \Big(\frac{\partial}{\partial x^1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x^n}\Big)^{\alpha_n}\,.$$

The *order* $|\alpha|$ of the multi-index α is defined by

$$|\alpha| := \alpha_1 + \cdots + \alpha_n\,.$$

We say that a function $f : \mathbb{R}^n \to \mathbb{C}$ is *smooth* if all its partial derivatives exist to every order. The space of smooth, complex-valued functions is denoted by $C^\infty(\mathbb{R}^n, \mathbb{C})$. For a smooth function $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$ we define its *Schwartz norms* $\|f\|_{p,q}$ with $p, q \in \mathbb{N}_0$ by

$$\|f\|_{p,q} := \max_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leq p}} \max_{\substack{\beta \in \mathbb{N}_0^n \\ |\beta| \leq q}} \sup_{x \in \mathbb{R}^n} \left| x^\alpha \, \partial^\beta f(x) \right| .$$

The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ is formed of all smooth functions for which all the Schwartz norms are finite, i.e.

$$\mathcal{S}(\mathbb{R}^n) := \Big\{ f \in C^\infty\big(\mathbb{R}^n, \mathbb{C}\big) \;\Big|\; \|f\|_{p,q} < \infty \text{ for all } p, q \in \mathbb{N}_0 \Big\} .$$

We always consider complex-valued functions, but the constructions work similarly for real-valued functions. Defining the vector operators pointwise, $\mathcal{S}(\mathbb{R}^n)$ is a complex vector space. The functions in $\mathcal{S}(\mathbb{R}^n)$ are referred to as *Schwartz functions*. These functions have the property that they as well as all their partial derivatives have *rapid decay* in the sense that multiplying them by a polynomial of arbitrary order still gives a bounded function. In particular, one has $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$, where both inclusions are strict. An example of a Schwartz function without compact support is a Gaussian $f : \mathbb{R}^n \to \mathbb{R}, x \mapsto \mathrm{e}^{-x^2}$.

The Schwartz norms induce a *topology* on $\mathcal{S}(\mathbb{R}^n)$ as follows. We say a set $\Omega \subset \mathcal{S}(\mathbb{R}^n)$ is *open* if for every $f \in \Omega$ there exists $p, q \in \mathbb{N}_0$ and $r > 0$ such that the open r-ball corresponding to the norm $\|.\|_{p,q}$ is contained in Ω, i.e.

$$\big\{ g \in \mathcal{S}(\mathbb{R}^n) \;\big|\; \|f - g\|_{p,q} < r \big\} \subset \Omega . \tag{2.4.5}$$

For many purposes, it is sufficient to have in mind how convergence is expressed concretely by the Schwartz norms. For a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$ one has (see Exercise 2.20)

$$f_n \to f \text{ in } \mathcal{S}(\mathbb{R}^n) \quad \Longleftrightarrow \quad \|f_n - f\|_{p,q} \to 0 \text{ for all } p, q \in \mathbb{N}_0 . \tag{2.4.6}$$

Here we point out that for a set $\Omega \subset \mathcal{S}(\mathbb{R}^n)$ to be an open neighborhood of $f \in \mathcal{S}(\mathbb{R}^n)$, it suffices that condition (2.4.5) is satisfied for *some* $p, q \in \mathbb{N}_0$. If instead one uses the stronger condition that (2.4.5) must hold for *all* $p, q \in \mathbb{N}_0$, one obtains a *coarser* topology on $\mathcal{S}(\mathbb{R}^n)$, meaning that there are fewer open sets. In contrast, demanding (2.4.5) merely for some $p, q \in \mathbb{N}_0$ gives a finer topology. Working with a *finer topology* has the following purpose: For a finer topology, fewer sequences converge (as is obvious in (2.4.6), where the sequence must converge for *all* Schwartz norms). As a consequence, there are more continuous linear functionals, simply because sequential continuity must be verified for fewer sequences. In other words, choosing a finer topology on a vector space has the effect that its dual space becomes larger (where the dual space is defined as the space of all continuous linear functionals). Since distributions will now be defined as such a dual space, this is desirable because it will ensure a sufficiently rich and general class of objects.

Definition 2.4.2. *The space of* **tempered distributions** *denoted by* $\mathcal{S}'(\mathbb{R}^n)$ *is defined as the dual space of the Schwartz space,*

$$\mathcal{S}'(\mathbb{R}^n) := \mathcal{S}^*(\mathbb{R}^n) = \mathrm{L}\big(\mathcal{S}(\mathbb{R}^n), \mathbb{C}\big) .$$

For a linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ continuity means that there are $p, q \in \mathbb{N}_0$ and a constant $c > 0$ such that (see again Exercise 2.20)

$$\big|T(f)\big| \leq c \, \|f\|_{p,q} \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n) . \tag{2.4.7}$$

As first example we return to the δ *distribution* from our motivating Example 2.4.1. It is the tempered distribution $\delta \in \mathcal{S}'(\mathbb{R}^n)$ given by $\delta(f) := f(0)$. Linearity of δ is obvious and the estimate

$$|\delta(f)| = |f(0)| \leq \|f\|_{0,0}$$

shows that δ is continuous.

Next we explain how any (bounded, measurable) function can be naturally viewed as a distribution, thus explaining why tempered distributions can be regarded as generalized functions. To this end, let $g \in L^\infty(\mathbb{R}^n)$ be a bounded, measurable function. We define a linear functional T_g on $\mathcal{S}(\mathbb{R}^n)$ by

$$T_g(f) := \int_{\mathbb{R}^n} g(x) f(x) \mathrm{d}^n x \,.$$

This functional is continuous because

$$\begin{aligned} \big|T_g(f)\big| &\leq \int_{\mathbb{R}^n} |g(x) f(x)| \mathrm{d}^n x = \int_{\mathbb{R}^n} \frac{|g(x)|}{\big(1+|x|^2\big)^{\frac{n+1}{2}}} \, |f(x)| \, \big(1+|x|^2\big)^{\frac{n+1}{2}} \, \mathrm{d}^n x \\ &\leq C(n) \, \|g\|_{L^\infty(\mathbb{R}^n)} \, \|f\|_{n+1,0} \int_{\mathbb{R}^n} \frac{\mathrm{d}^n x}{\big(1+|x|^2\big)^{\frac{n+1}{2}}} \\ &\leq C'(n) \, \|g\|_{L^\infty(\mathbb{R}^n)} \, \|f\|_{n+1,0} \,. \end{aligned} \tag{2.4.8}$$

Since the remaining integral is finite, it follows that the inequality (2.4.7) holds for a suitable constant $c > 0$, $p = n+1$ and $q = 0$. Therefore, $T_g \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution.

In this way, every function $g \in L^\infty(\mathbb{R}^n)$ gives rise to a corresponding tempered distribution T_g. Let us verify that the corresponding linear mapping

$$T \,:\, L^\infty(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \,, \quad g \mapsto T_g \qquad \text{is injective} \,. \tag{2.4.9}$$

(Note that strictly speaking we are dealing with equivalence classes, identifying two functions which differ only on a set of measure zero.) To this end, let $g \in L^\infty(\mathbb{R}^n)$ be non-zero. Then $A := \{x \in \mathbb{R}^n \mid |g(x)| \geq \|g\|_\infty/2\}$ has nonzero measure. By inner regularity of the Lebesgue measure, the same is true for $A \cap B_R(0)$ for $R > 0$ sufficiently large. Choose $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta|_{B_R(0)} \equiv 1$. Then ηf is bounded and compactly supported, hence square integrable, and one easily checks that $\|\eta g\|_{L^2} \neq 0$. Using that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we conclude that there is a function $f \in C_0^\infty(\mathbb{R}^n)$ with

$$0 \neq \langle f, \eta g \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x) \, \big(\eta g\big)(x) \mathrm{d}^n x = T_g(\eta f) \,.$$

Hence $T_g \neq 0$, and we conclude that the mapping T in (2.4.9) is indeed injective.

The fact that the mapping (2.4.9) is an embedding means that distributions can be regarded as "generalized functions." Distributions which can be represented in the form T_g with $g \in L^\infty(\mathbb{R}^n)$ are referred to as *regular distributions*. We finally remark that T_g can be defined more generally for functions g which increase at most polynomially at infinity. But we do not need this generalization here.

In order to speak about convergence of distributions and related things, one needs to endow the space $\mathcal{S}'(\mathbb{R}^n)$ with a topology. Being defined as a dual space of functionals on $\mathcal{S}(\mathbb{R}^n)$, the usual choice is the so-called *weak*-topology*, which is the coarsest topology such that for every $f \in \mathcal{S}(\mathbb{R}^n)$ the evaluation map $\mathcal{S}'(\mathbb{R}^n) \ni T \mapsto T(f) \in \mathbb{C}$ is continuous.

This means that for a sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$ one has

$$T_n \to T \text{ in } \mathcal{S}'(\mathbb{R}^n) \quad \Longleftrightarrow \quad T_n(f) \to T(f) \text{ for all } f \in \mathcal{S}(\mathbb{R}^n)\,.$$

With respect to this topology, the map (2.4.9) is continuous.

In the applications, it is important to differentiate distributions. For example, one wants to construct distributional solutions of partial differential equations (like the Poisson equation with a δ distribution or the Green's kernels of the Dirac equation to be considered in Chapters 14 and 16.1). It turns out that distributions can always be differentiated, as we now explain. The idea behind the definition of the derivative of a distribution is to generalize the integration-by-parts formula, which for two Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^n)$ states that

$$T_{\partial^\alpha f}(g) = \int_{\mathbb{R}^n} (\partial^\alpha f)(x)\, g(x)\, \mathrm{d}^n x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\, (\partial^\alpha g(x))\, \mathrm{d}^n x = (-1)^{|\alpha|}\, T_f(\partial^\alpha g)$$

(note that there are no boundary terms due to the rapid decay of f and g at infinity).

Definition 2.4.3. *For a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ and multi-index $\alpha \in \mathbb{N}_0^n$, we define the* **distributional derivative** *$\partial^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$ by*

$$\big(\partial^\alpha T\big)(f) := (-1)^{|\alpha|}\, T\big(\partial^\alpha f\big) \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n)\,.$$

Using the continuity estimate (2.4.7) for T we have

$$\big|\big(\partial^\alpha T\big)(f)\big| = \big|T\big(\partial^\alpha f\big)\big| \leq c\, \big\|\partial^\alpha f\big\|_{p,q} \leq c\, \big\|f\big\|_{p,q+|\alpha|} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n)\,,$$

which shows that $\partial^\alpha T$ is indeed a *continuous* linear functional again.

We now come to the *Fourier transformation*. We first introduce it for Schwartz functions.

Definition 2.4.4. *For $f \in \mathcal{S}(\mathbb{R}^n)$, we define the Fourier transform $(\mathcal{F}f) : \mathbb{R}^n \to \mathbb{C}$ and the adjoint Fourier transform $(\mathcal{F}^* f) : \mathbb{R}^n \to \mathbb{C}$ by*

$$(\mathcal{F}f)(p) = \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{\mathrm{i}px}\, \mathrm{d}^n x \tag{2.4.10}$$

$$(\mathcal{F}^* f)(x) = \int_{\mathbb{R}^n} f(p)\, \mathrm{e}^{-\mathrm{i}px}\, \frac{\mathrm{d}^n p}{(2\pi)^n}\,, \tag{2.4.11}$$

where $x, p \in \mathscr{M}$, and $px = \langle p, x\rangle$ denotes the Minkowski inner product.

Lemma 2.4.5. *The Fourier transform and its adjoint map Schwartz functions to Schwartz functions and yield continuous linear maps*

$$\mathcal{F}, \mathcal{F}^* \,:\, \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,. \tag{2.4.12}$$

Proof. In order to prove (2.4.12), we differentiate (2.4.10) to obtain

$$\begin{aligned} p^\alpha\, \partial_p^\beta (\mathcal{F}f)(p) &= p^\alpha \int_{\mathbb{R}^n} f(x)\, x^\beta\, \mathrm{e}^{\mathrm{i}px}\, \mathrm{d}^n x = (-\mathrm{i})^{|\alpha|} \int_{\mathbb{R}^n} f(x)\, x^\beta\, \big(\partial_x^\alpha \mathrm{e}^{\mathrm{i}px}\big)\, \mathrm{d}^n x \\ &= \mathrm{i}^{|\alpha|} \int_{\mathbb{R}^n} \big(\partial_x^\alpha f(x)\big)\, x^\beta\, \mathrm{e}^{\mathrm{i}px}\, \mathrm{d}^n x\,, \end{aligned}$$

where in the last step we integrated by parts (to justify differentiation under the integral one can use the dominated convergence theorem). Taking the absolute value, we obtain the estimate

$$\big|p^\alpha\, \partial_p^\beta (\mathcal{F}f)(p)\big| \leq \int_{\mathbb{R}^n} \big|\partial^\alpha f(x)\big|\, \big|x^\beta\big|\, \mathrm{d}^n x \overset{(\star)}{\leq} C(n) \cdot \|\partial^\alpha f\|_{|\beta|+n+1,0} \leq \|f\|_{|\beta|+n+1,|\alpha|}\,,$$

where in $(\star)$ the integral can be estimated similar as in (2.4.8). This estimate shows that the Fourier transform of a Schwartz function is again a Schwartz function. It also follows directly from this estimate that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous (as linearity clearly holds). The estimate for the inverse Fourier transform is similar. □

THEOREM 2.4.6. **(Fourier inversion formula)** *The Fourier transform and its adjoint on Schwartz functions are inverses of each other,*

$$\mathcal{F} \circ \mathcal{F}^* = \mathcal{F}^* \circ \mathcal{F} = \mathbb{1}_{\mathcal{S}(\mathbb{R}^n)} .$$

A detailed proof of this lemma can be found in [**130**, Theorem 2.2.4] or [**89**, Theorem 8.2.2]. With this in mind, we only give a sketch of the proof in one dimension. Writing everything out explicitly, for $f \in \mathcal{S}(\mathbb{R}^n)$ we get

$$\mathcal{F}^*\big(\mathcal{F}f\big)(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) \mathrm{e}^{\mathrm{i}py} \mathrm{d}y \right) \mathrm{e}^{-\mathrm{i}px} \, \frac{\mathrm{d}p}{2\pi} .$$

The basic idea is to exchange the order of integration. However, the problem is that the resulting function is not integrable in the p-variable. One way to make this mathematically sound is by inserting a convergence-generating factor $\mathrm{e}^{-\varepsilon p^2}$. More precisely, using dominated convergence and Fubini one computes

$$\begin{aligned} \mathcal{F}^*\big(\mathcal{F}f\big)(x) &= \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \big(\mathcal{F}f\big)(p) \, \mathrm{e}^{-\varepsilon p^2} \, \mathrm{e}^{-\mathrm{i}px} \, \frac{\mathrm{d}p}{2\pi} \\ &= \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) \, \mathrm{e}^{\mathrm{i}py} \, \mathrm{d}y \right) \mathrm{e}^{-\varepsilon p^2} \, \mathrm{e}^{-\mathrm{i}px} \, \frac{\mathrm{d}p}{2\pi} \\ &= \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(y) \left\{ \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}p(y-x) - \varepsilon p^2} \, \frac{\mathrm{d}p}{2\pi} \right\} \mathrm{d}y . \end{aligned} \tag{2.4.13}$$

The integral inside the curly brackets is Gaussian and can be computed explicitly. The resulting family of Gaussians tends to the δ distribution $\delta(x-y)$ (see Exercise 2.22, and thus altogether one obtains $f(x)$ in the limit $\varepsilon \searrow 0$ as desired.

Having given the proof of the Fourier inversion formula, we return once more to the formal computation from the beginning,

$$\mathcal{F}^*\big(\mathcal{F}f\big)(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) \mathrm{e}^{\mathrm{i}py} \mathrm{d}y \right) \mathrm{e}^{-\mathrm{i}px} \frac{\mathrm{d}p}{2\pi} = \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}p(y-x)} \frac{\mathrm{d}p}{2\pi} \right) \mathrm{d}x .$$

Although the right-hand side is ill-defined as an integral, knowing that $\mathcal{F}^*\big(\mathcal{F}f\big)(x) = f(x)$ holds, one may interpret it as the distributional identity

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}p(x-y)} \, \mathrm{d}p = 2\pi \, \delta(x-y) . \tag{2.4.14}$$

Having the Fourier transform for Schwartz functions at our disposal, we can now introduce the Fourier transform of tempered distributions. Similar as in the definition of the distributional derivative, the idea is to let the Fourier transform act on the test function. To see how to do this concretely, we again first consider a regular distribution $T_{\mathcal{F}g}$ corresponding to the Fourier transform of a Schwartz function g. Then for any $f \in \mathcal{S}(\mathbb{R}^n)$,

using Fubini we have

$$T_{\mathcal{F}g}(f) = \int_{\mathbb{R}^n} (\mathcal{F}g)(p)\, f(p)\, \mathrm{d}^n p = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} g(x)\, \mathrm{e}^{\mathrm{i}px}\, \mathrm{d}^n x \Big)\, f(p)\, \mathrm{d}^n p$$
$$= \int_{\mathbb{R}^n} g(x) \Big(\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}px}\, f(p)\, \mathrm{d}^n p \Big)\, \mathrm{d}^n x = T_g\big(\mathcal{F}f\big)\,.$$

The right side can now be used to *define* the Fourier transform of a tempered distribution.

Definition 2.4.7. *The Fourier transform and the adjoint Fourier transform of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ are defined by*

$$\mathcal{F}, \mathcal{F}^* \,:\, \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,, \qquad \big(\mathcal{F}T\big)(f) = T\big(\mathcal{F}f\big)\,, \quad \big(\mathcal{F}^*T\big)(f) = T\big(\mathcal{F}^*f\big)\,.$$

Note that for a tempered distribution T, the maps $\mathcal{F}T = T \circ \mathcal{F}$ and $\mathcal{F}^*T = T \circ \mathcal{F}^*$ are again linear and continuous as composition of two linear and continuous maps. Hence they are tempered distributions again. A direct computation shows that $\mathcal{F}^*$ is the inverse of $\mathcal{F}$ (see Exercise 2.21). Examples for how to compute Fourier transforms of distributions can be found in the exercises (see Exercise 2.25).

We now come to an operation on functions and distributions which we will use a few times in this book: the *convolution*. For two Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^n)$, the convolution $f * g \in \mathcal{S}(\mathbb{R}^n)$ is defined by (see Exercise 2.34)

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, \mathrm{d}y\,. \tag{2.4.15}$$

One immediately notes that convolution is commutative, i.e. $f * g = g * f$ (it is also associative). Interestingly, taking the Fourier transform, the convolution goes over to multiplication, i.e.

$$\mathcal{F}\big(f * g\big)(p) = (\mathcal{F}f)(p)\, (\mathcal{F}g)(p) \tag{2.4.16}$$

(for the derivation see Exercise 2.35). Conversely, the Fourier transform of a product is the convolution of the individual Fourier transforms. The convolution can be extended to an operation involving tempered distributions by interpreting the right hand side of (2.4.15) as the action of a regular distribution T_g. Namely, given $T \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, one defines

$$\big(f * T\big)(x) := T(f_x) \qquad \text{with} \qquad f_x(y) := f(y-x)\,. \tag{2.4.17}$$

This is even a smooth function (see Exercise 2.37). With this in mind, a convolution can be used to "smoothen" or "mollify" functions and distributions. More details and examples can be found for example in [**89**, Chapter 5].

We close with two remarks. First, it is often very useful to consider the Fourier transform on L^2-functions, where the Fourier transform is unitary:

Theorem 2.4.8. **(Plancherel)** *For any $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}}\, \|f\|_{L^2(\mathbb{R}^n)}\,.$$

Furthermore, the Fourier transform and the adjoint Fourier transform extend to isomorphisms $\mathcal{F}, \mathcal{F}^* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ which are inverse to each other.

We again omit the proof, which can be found in [**130**, Theorem 2.3.4] or [**89**, Theorem 9.2.2]. On a formal level, Plancherel's formula is obtained by a direct computation using again (2.4.14). Similar to (2.4.13), this computation can be made mathematically sound by introducing a convergence-generating factor. The extension to $L^2(\mathbb{R}^n)$ then simply follows using that $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is dense.

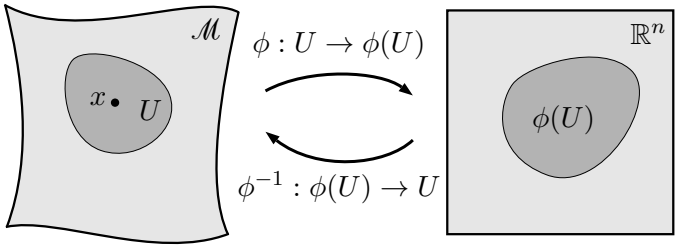


FIGURE 2.3. A chart (ϕ, U) around a point $x \in \mathscr{M}$.

Finally, we remark that in some applications (for example when working in local charts on a manifold or in a bounded domain), it is not feasible to work with functions defined in all of $\mathbb{R}^n$ having suitable decay properties. In this case, instead of the Schwartz functions, one considers the space of *test functions*, i.e. smooth functions with compact support, denoted by

$$\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)\,,$$

endowed with the topology induced by the family of norms $\|.\|_q$ with $q \in \mathbb{N}_0$ given by

$$\|f\|_q := \max_{\substack{\beta \in \mathbb{N}_0^n \\ |\beta| \leq q}} \sup_{x \in \mathbb{R}^n} \left|\partial^\beta f(x)\right|.$$

Its dual space $\mathcal{D}'(\mathbb{R}^n)$ is referred to as the space of *distributions*. Here the dual space is again defined as the space of all continuous linear functionals, where continuity of a linear functional $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ means in analogy to (2.4.7) that there is $q \in \mathbb{N}_0$ and a constant $c > 0$ such that

$$\left|T(f)\right| \leq c\,\|f\|_q \qquad \text{for all } f \in \mathcal{D}(\mathbb{R}^n)\,.$$

Differentiation of distributions can be defined similarly as for tempered distributions. However, the Fourier transform cannot be be defined on all of $\mathcal{D}'(\mathbb{R}^n)$ since the Fourier transform of a test function need not be compactly supported again. Test functions and distributions can also be defined on any open subset $U \subset \mathbb{R}^n$ instead of all of $\mathbb{R}^n$ (and with a little bit of extra work also on manifolds).

2.5. Manifolds and Vector Bundles

We now recall the basic definitions of a manifold and a vector bundle. Since the machinery of differential geometry is loaded with many definitions and quite subtle (notational) conventions, we must restrict to the very basics. For more details we refer to good textbooks like [**113, 118**] or to the basic definitions in [**120**, §1 and §2].

DEFINITION 2.5.1. *A* **topological manifold** *of dimension $n \in \mathbb{N}$ is a Hausdorff topological space $\mathscr{M}$ which is σ-compact (i.e. which can be written as an at most countable union of compact subsets) and has the property that every point in $\mathscr{M}$ has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.*

More specifically, for every $p \in \mathscr{M}$ there is an open neighborhood $U \subset \mathscr{M}$ of p and a map

$$\phi \;:\; U \to \mathbb{R}^n$$

such that the image $\phi(U)$ is an open subset of $\mathbb{R}^n$ and the mapping $\phi : U \to \phi(U)$ is a homeomorphism (i.e. it is continuous, invertible, and its inverse is also continuous). We refer to (ϕ, U) as a *chart* around p (see Figure 2.3). A collection of charts is an *atlas* $\mathcal{A}$.

We always assume that the atlas is *complete* in the sense that every point of $\mathscr{M}$ lies in the domain of a chart of the atlas.

A chart (ϕ, U) can be seen as an identification of the open subset $U \subset \mathscr{M}$ with the open subset $\phi(U) \subset \mathbb{R}^n$, any $p \in U$ being identified with the coordinates of its image

$$\phi(p) = (x^1, \dots, x^n) \in \mathbb{R}^n \,. \tag{2.5.1}$$

In this way a chart introduces *local coordinates* $x^1, \dots, x^n$ on $U \subset M$. Note that by (2.5.1) the coordinates can be understood as the component functions of the map ϕ and thus as functions on U. Also maps between manifolds can locally be identified with maps between open subset of Euclidean spaces. Concretely, if $F : \mathscr{M} \to \mathscr{N}$ is a map between two manifolds $\mathscr{M}$ and $\mathscr{N}$, and if (ϕ, U) and (ψ, V) are charts on $\mathscr{M}$ and $\mathscr{N}$, respectively, with $F(U) \subset V$, then we can consider the map

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V) \,, \tag{2.5.2}$$

called *coordinate representation* of F.

One would like to use these local identifications to carry over the machinery of differential calculus to maps between manifolds. To this end, the structure of a topological manifold alone is not sufficient. Rather, one needs additional compatibility conditions on the transition maps between charts: Given two charts (ϕ, U) and $(\tilde\phi, \tilde U)$ on a manifold $\mathscr{M}$ with $U \cap \tilde U \neq \varnothing$, the two mappings

$$\phi|_{U\cap\tilde U},\ \tilde\phi|_{U\cap U'} \ : \ U \cap \tilde U \to \mathbb{R}^n$$

are both homeomorphism onto open subsets of $\mathbb{R}^n$. We define the *transition map* by (see Figure 2.4)

$$\tilde\phi|_{U\cap\tilde U} \ \circ \big(\phi|_{U\cap\tilde U}\big)^{-1} \ : \ \phi\big(U \cap \tilde U\big) \to \tilde\phi\big(U \cap \tilde U\big) \,.$$

Being a mapping between two open subsets of $\mathbb{R}^n$, it is clear what "differentiability" of this mapping means. A *differentiable manifold* is a topological manifold together with a complete atlas with the property that all transition maps are differentiable. Likewise, a *smooth manifold* is defined by the requirement that all transition maps are smooth.

The differentiability of all transition maps now allows us to introduce a notion of differentiable maps: A map $F : \mathscr{M} \to \mathscr{N}$ between two differentiable manifolds $\mathscr{M}$ and $\mathscr{N}$ is called *differentiable map* if all of its coordinate representations (2.5.2) are differentiable maps between open subsets of Euclidean spaces. It turns out that this is actually a local condition, and in order to prove differentiability of F in a point $p \in \mathscr{M}$ it suffices to verify that *one* coordinate representation around p is differentiable.

Having introduced the notion of a differentiable map, the next question is how to compute its derivative. We begin with the case of a real-valued differentiable function $f : \mathscr{M} \to \mathbb{R}$. If (ϕ, U) is a chart of $\mathscr{M}$, then by definition the coordinate representation $f\circ\phi^{-1} : \phi(U) \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable. (Note that we take the identity as chart of $\mathbb{R}$.) We can therefore take partial derivatives $\frac{\partial(f\circ\phi^{-1})}{\partial x^i}(x)$ or also the total derivative $D_x(f \circ \phi^{-1})$ in a point $x \in \phi(U)$. These, of course, depend on the chart ϕ, and if $(\tilde\phi, U)$ is another chart with the same domain, then by the chain rule we have

$$D_x(f \circ \phi^{-1}) = D_x(f \circ \tilde\phi^{-1} \circ \tilde\phi \circ \phi^{-1}) = D_{\tilde\phi(\phi^{-1}(x))}(f \circ \tilde\phi^{-1}) \cdot D_x(\tilde\phi \circ \phi^{-1}) \,,$$

or, written in terms of partial derivatives (note that we are using the Einstein summation convention),

$$\frac{\partial(f \circ \phi^{-1})}{\partial x^i} = \frac{\partial(f \circ \tilde\phi^{-1} \circ \tilde\phi \circ \phi^{-1})}{\partial x^i} = \frac{\partial(f \circ \tilde\phi^{-1})}{\partial x^j} \cdot \frac{\partial(\tilde\phi \circ \phi^{-1})^j}{\partial x^i} \,.$$

This shows that the partial derivatives with respect to the two charts are linearly related via the Jacobian matrix of the transition map between the two charts.

It is desirable to introduce a notion of derivative which is independent of the choice of local charts. There are different approaches one can take, and here we only sketch the approach which geometrically seems the most intuitive. To this end, let again $f : \mathscr{M} \to \mathbb{R}$ be a differentiable function and let $p \in \mathscr{M}$. If $c : (-\varepsilon, \varepsilon) \to \mathscr{M}$ is a differentiable curve with $c(0) = p$, then it is easy to check that the composition $f \circ c : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is differentiable, so one can take the derivative $(f \circ c)'(0)$. For $\mathscr{M} = \mathbb{R}^n$ this would yield precisely the directional derivative of f in p in direction $c'(0)$, but in the general case such an interpretation is not possible yet (as we have not defined $c'(0)$ for a curve in a manifold). However, if (ϕ, U) is a local chart around p, then we can write

$$(f \circ c)'(0) = (f \circ \phi^{-1} \circ \phi \circ c)'(0) = D_{\phi(p)}(f \circ \phi^{-1})((\phi \circ c)'(0)) \,. \tag{2.5.3}$$

From this it follows that for any other differentiable curve $\tilde{c} : (-\tilde{\varepsilon}, \tilde{\varepsilon}) \to \mathscr{M}$ with $\tilde{c}(0) = p$ we will have $(f \circ c)'(0) = (f \circ \tilde{c})(0)$ if we have $(\phi \circ c)'(0) = (\phi \circ \tilde{c})'(0)$. It is easy to check that the latter condition on the two curves c and $\tilde{c}$ is actually independent of the chart in the sense that if it holds in one chart, then it also holds in any other chart. Moreover, it is clear that it defines an equivalence relation on the set of differentiable curves passing through p. The corresponding equivalence classes are called *tangent vectors* to $\mathscr{M}$ in p, and the set of all equivalence classes is called the *tangent space* to $\mathscr{M}$ in p and is denoted by $T_p\mathscr{M}$. Any chart (ϕ, U) around p defines a map

$$\phi_{*,p} : T_p\mathscr{M} \to \mathbb{R}^n \,, \quad \phi_{*,p}([c]) := (\phi \circ c)'(0) \,.$$

It is not difficult to verify that this map is a bijection, so it can be used to pull back the vector space structure of $\mathbb{R}^n$ to $T_p\mathscr{M}$ by

$$[c_1] + [c_2] := \phi_{*,p}^{-1}\big((\phi \circ c_1)'(0) + (\phi \circ c_2)'(0)\big) \,,$$

and similarly for scalar multiplication. The induced vector space structure on $T_p\mathscr{M}$ is actually independent of the chart ϕ, as follows from the observation that for any other chart $(\tilde{\phi}, U)$ around p we have

$$\tilde{\phi}_{*,p} \circ \phi_{*,p}^{-1} = D_{\phi(p)}(\tilde{\phi} \circ \phi^{-1}) \in \mathrm{GL}(\mathbb{R}^n) \,. \tag{2.5.4}$$

Having defined the tangent space, we can finally define the *derivative in a point* $p \in \mathscr{M}$ of a differentiable map $f : \mathscr{M} \to \mathbb{R}$ as the map

$$D_p f : T_p\mathscr{M} \to \mathbb{R} \,, \quad D_p f([c]) := (f \circ c)'(0) \,. \tag{2.5.5}$$

One easily verifies that this is indeed a linear map.

We note that, for any local chart (ϕ, U), by (2.5.3) the derivative can be expressed in terms of the Jacobian of the coordinate representation $f \circ \phi^{-1}$. This can be understood a bit more systematically by defining the particular tangent vectors

$$\frac{\partial}{\partial x^i}\Big|_p := \phi_{*,p}^{-1}(e_i) = [t \mapsto \phi^{-1}(\phi(p) + te_i)] \in T_p\mathscr{M} \,. \tag{2.5.6}$$

They form a basis of $T_p\mathscr{M}$. (As will be considered in more detail below, they do depend on ϕ, even though the notation does not seem to reflect this directly.) Now observe that

$$D_p f\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(f \circ \phi^{-1})(\phi(p) + te_i) = D_{\phi(p)}(f \circ \phi^{-1})(e_i) = \frac{\partial(f \circ \phi^{-1})}{\partial x^i}(\phi(p)) \,.$$

This shows that the Jacobian of $f \circ \phi^{-1}$ is precisely the matrix representation of $D_p f$ in the basis $\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$. We now come back to the point that the basis vectors (2.5.6)

depend on the chart ϕ, and to how one usually resembles this in the notation. To this end, recall that one says that the chart ϕ introduces local coordinates $x^1, \dots, x^n$ on its domain. A different chart $\tilde{\phi}$ on the same domain introduces different coordinates. One usually denotes these by $\tilde{x}^1, \dots, \tilde{x}^n$, and denotes the basis of $T_p\mathscr{M}$ induced by $\tilde{\phi}$ by

$$\frac{\partial}{\partial \tilde{x}^1}\Big|_p, \dots, \frac{\partial}{\partial \tilde{x}^n}\Big|_p .$$

Thus the reference to the chart in the notation is done via the symbols for the coordinates. The relation between the two different bases follows from the following computation, which makes use of (2.5.4) (note Einstein's summation convention again):

$$\begin{aligned}\frac{\partial}{\partial x^i}\Big|_p &= \phi^{-1}_{*,p}(e_i) = \big(\tilde{\phi}^{-1}_{*,p} \circ \tilde{\phi}^{-1}_{*,p}\big)\big(\phi^{-1}_{*,p}(e_i)\big) = \tilde{\phi}^{-1}_{*,p}\big((\tilde{\phi}^{-1}_{*,p} \circ \phi^{-1}_{*,p})(e_i)\big) \\ &\overset{(2.5.4)}{=} \tilde{\phi}^{-1}_{*,p}\big(D_{\phi(p)}(\tilde{\phi} \circ \phi)(e_i)\big) = \tilde{\phi}^{-1}_{*,p}\Big(\frac{\partial(\tilde{\phi} \circ \phi)^j}{\partial x^i}(\phi(p)) \cdot e_j\Big) \\ &= \frac{\partial(\tilde{\phi} \circ \phi)^j}{\partial x^i}(\phi(p)) \cdot \tilde{\phi}^{-1}_{*,p}(e_j) = \frac{\partial(\tilde{\phi}^j \circ \phi)}{\partial x^i}(\phi(p)) \cdot \frac{\partial}{\partial \tilde{x}^j}\Big|_p .\end{aligned}$$

Recalling that the coordinates induced by a chart can be understood as the component functions of the chart, and dropping all reference to the points where one evaluates, this relation can be recast in the shorter and easier to memorize the "reduction rule"

$$\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \cdot \frac{\partial}{\partial \tilde{x}^j} ,$$

which might be familiar from the use of "curvilinear coordinates" in vector calculus (which are of course nothing else but different local coordinate systems in $\mathbb{R}^n$).

So far, we only considered real-valued functions, i.e. maps $f : \mathscr{M} \to \mathbb{R}$. For a differentiable map $F : \mathscr{M} \to \mathscr{N}$ between two differentiable manifolds $\mathscr{M}$ and $\mathscr{N}$, the derivative of F in a point $p \in \mathscr{M}$ is defined as the map

$$D_pF : T_p\mathscr{M} \to T_{F(p)}\mathscr{N} , \quad D_pF([c]) := [F \circ c] .$$

One can check that this is map is well-defined and linear. Further, if (ϕ, U) is a chart around $p \in \mathscr{M}$ with coordinates $x^1, \dots, x^n$, and (ψ, V) a chart around $F(p) \in \mathscr{N}$ with coordinates $y^1, \dots, y^m$, then one easily checks that

$$D_pF\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \frac{\partial(\psi \circ F \circ \phi^{-1})^j}{\partial x^i}(\phi(p)) \cdot \frac{\partial}{\partial y^j}\Big|_{F(p)} . \tag{2.5.7}$$

This shows that, with respect to the bases induced by the two charts ϕ and ψ, the linear map D_pF is represented by the Jacobian of the coordinate representation $\psi \circ F \circ \phi^{-1}$. Note that for a coordinate chart $\phi : U \to \mathbb{R}^n$ we have $D_p\phi = \phi_{*,p}$. For this reason, some authors denote the derivative of a smooth map F also by F_* instead of DF.

We remark that there are different but equivalent ways to define tangent vectors and thus also derivatives of smooth maps between smooth manifolds. One commonly used approach is via so-called *derivations*, which is an axiomatization of the product rule for (directional) derivatives (for details, see for example [**118**, Chapter 3]).

The collection of all tangent spaces to a differentiable manifold $\mathscr{M}$ as (disjoint) union $T\mathscr{M} := \bigcup_{p \in \mathscr{M}} T_p\mathscr{M}$ is called the *tangent bundle* of $\mathscr{M}$. It turns out that the tangent bundle is again a smooth manifold with certain additional structures. First note that we can define a map $\pi : T\mathscr{M} \to \mathscr{M}$ by mapping any tangent vector to its *base point*,

i.e. to the point of $\mathscr{M}$ to whose tangent space it belongs. Next, given a chart (ϕ, U) on $\mathscr{M}$, we can consider the map

$$T\phi : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}, \quad v \mapsto (\phi(\pi(v)), \phi_{*,\pi(v)}(v)).$$

This map is bijective onto the open subset $\phi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. One verifies immediately that all sets of the form $T\phi^{-1}(V)$, where ϕ is a chart on $\mathscr{M}$ and $V \subset \mathbb{R}^{2n}$ is an open subset, form what is called a basis for a topology on $T\mathscr{M}$, which is Hausdorff and σ-compact. In particular, all these sets are open subsets of $T\mathscr{M}$, so the maps $T\phi$ are local charts. (This is indeed the unique topology on $T\mathscr{M}$ with these properties; see [**118**, Chapter 3] for details.) Thus $T\mathscr{M}$ is a topological manifold. Furthermore, if (ϕ, U) and (ψ, U) are two charts on $\mathscr{M}$, then by (2.5.4) the transition map is between the two charts $T\phi$ and $T\psi$ on $T\mathscr{M}$ is given by

$$T\psi \circ T\phi^{-1}(x, v) = \big(x, D_x(\psi \circ \phi^{-1})(v)\big).$$

If $\mathscr{M}$ is a smooth manifold, then this is again a smooth map, showing that $T\mathscr{M}$ is also a smooth manifold.

Locally, the tangent bundle has a particular product structure which is already visible from the construction of the charts above. More precisely, note that any chart (ϕ, U) on $\mathscr{M}$ induces a local diffeomorphism

$$\widehat{\phi} : \pi^{-1}(U) \to U \times \mathbb{R}^n, \quad v \mapsto (\pi(v), \phi_{*,\pi(v)}(v)),$$

called a *local trivialization* of the tangent bundle. These local trivializations are compatible with the base point projection $\pi : T\mathscr{M} \to \mathscr{M}$ in the sense that $\mathrm{pr}_1 \circ \widehat{\phi} = \pi$, where $\mathrm{pr}_1 : U \times \mathbb{R}^n \to U$ denotes the projection to the first component. Moreover, they are also compatible with the vector space structures on the tangent spaces in the sense that for another chart (ψ, V) on $\mathscr{M}$ with $U \cap V \neq \varnothing$, we have

$$\widehat{\psi} \circ \widehat{\phi}^{-1}(p, v) = (p, D_{\phi(p)}(\psi \circ \phi^{-1})(v)),$$

and in the second component we have a smooth map

$$U \cap V \ni p \mapsto D_{\phi(p)}(\psi \circ \phi^{-1}) \in \mathrm{GL}(\mathbb{R}^n).$$

So the transition between two local trivialization simply amounts to a (pointwise fixed) linear transformation on $\mathbb{R}^n$.

The local structure of the tangent bundle just described can be generalized to the notion of a *vector bundle*. This notion is helpful for the understanding of causal fermion systems because, under suitable regularity assumptions, a causal fermion system will give rise to a vector bundle over spacetime with the spin spaces as fibers (see Section 11.1). We begin with the notion of a topological vector bundle and explain the differentiable structure afterward.

Definition 2.5.2. *Let $\mathcal{B}$ and $\mathscr{M}$ be topological spaces and $\pi : \mathcal{B} \to \mathscr{M}$ a continuous surjective map. Moreover, let Y be a (real or complex) vector space and $G \subset \mathrm{GL}(Y)$ a group acting on Y. Then $\mathcal{B}$ is a* **topological vector bundle** *with* **fiber** *Y and* **structure group** *G if there exists an open covering $\mathcal{U}$ of $\mathscr{M}$ and for every $U \in \mathcal{U}$ a homeomorphism $\phi_U : \pi^{-1}(U) \to U \times Y$, called a* **local trivialization** *or* **bundle chart**, *such that the following two properties are satisfied:*

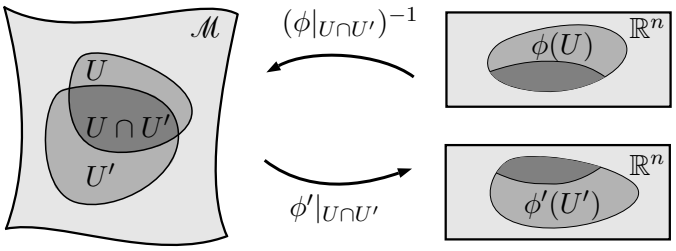


FIGURE 2.4. A transition map.

(i) *For any $U \in \mathcal{U}$, the diagram*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times Y \\ & \searrow^{\pi} & \downarrow{\scriptstyle \mathrm{pr}_1} \\ & & U \end{array} \tag{2.5.8}$$

commutes, where $\mathrm{pr}_1 : U \times Y \to U$ denotes the projection onto the first component.

(ii) *For all $U, V \in \mathcal{U}$ with $U \cap V \neq \varnothing$ we have*

$$\phi_U \circ \phi_V^{-1}(x, v) = (x, g_{UV}(x)v) \qquad \text{for all } x \in U \cap V, v \in Y\,, \tag{2.5.9}$$

where $g_{UV} : U \cap V \to G$ is a continuous transition function.

In the special case $G = \mathrm{GL}(Y)$, one simply speaks of a vector bundle (i.e., without explicitly mentioning the structure group).

Similar as for the tangent bundle, by the condition (2.5.9) the local trivializations allow to pull back the vector space structure of Y to the so-called fibers $\mathcal{B}_x := \pi^{-1}(\{x\}) \subseteq \mathcal{B}$ of a vector bundle. So one may think of a vector bundle $\pi : \mathcal{B} \to \mathscr{M}$ as a collection of vector spaces which are parametrized by $\mathscr{M}$.

A simple example of a vector bundle is the Cartesian product

$$\mathcal{B} = \mathscr{M} \times Y\,.$$

This vector bundle is globally trivialized, i.e. of product form. According to (2.5.8), a vector bundle has this product structure "locally" in $\pi^{-1}(U)$, but in general not globally. As we have already seen, the tangent bundle $T\mathscr{M}$ of a smooth manifold $\mathscr{M}$ is another example. Here the structure group will in general be $\mathrm{GL}(\mathbb{R}^n)$, but could possibly be taken a smaller group in special circumstances.

A *differentiable* (or *smooth*) vector bundle is a topological vector bundle where the base $\mathscr{M}$ is a differentiable (or smooth) manifold together with an atlas of bundle charts such that all transition maps are differentiable (or smooth, respectively). In this case, also the total space $\mathcal{B}$ is a differentiable (smooth) manifold and the projection map π is differentiable (smooth), as the local trivializations provide a differentiable atlas.

2.6. Exercises

EXERCISE 2.1. *(Closed sets)* Show that the closed sets of a topological space E (defined as the complements of the open sets) have the following properties:

(i) The sets $\varnothing$ and E are closed.

(ii) Closedness under finite unions: For any $A_1, \ldots, A_n \subset E$,

$$A_1, \ldots, A_n \text{ closed} \quad \Longrightarrow \quad A_1 \cup \cdots \cup A_n \text{ closed}\,.$$

(iii) Closedness under arbitrary intersections: For any family $\{A_\lambda\}_{\lambda\in\Lambda}$ of subsets of E,

$$A_\lambda \text{ closed for all } \lambda \in \Lambda \quad \Longrightarrow \quad \bigcap_{\lambda\in\Lambda} A_\lambda \text{ closed}\,.$$

EXERCISE 2.2. *(Notions of continuity)* Let $E = F = \mathbb{R}$ with the standard topology inherited from the metric $d(x, y) = |x-y|$. Show that a real-valued function $f : E \to F$ is continuous in the topological sense (2.1.2) if and only if for every $x \in \mathbb{R}$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\big|f(x) - f(y)\big| < \varepsilon \qquad \text{for all } y \in \mathbb{R} \text{ with } |y - x| < \delta\,.$$

EXERCISE 2.3. *(Extreme value theorem)* Let $(E, \mathcal{O})$ be a non-empty compact topological space and $f : E \to \mathbb{R}$ a continuous, real-valued function on E. Show that f attains its maximum, i.e. there is a point $x \in E$ with

$$f(x) \geq \inf_E f\,.$$

Hint: First show using basic definitions that a continuous function maps compact sets to compact sets. Then use what you know about compact subsets of $\mathbb{R}$.

Alternatively, one can use that a compact space is *sequentially compact* in the sense that every sequence has a convergent subsequence. If you are not familiar with these connections, try to prove them starting from the basic definitions.

EXERCISE 2.4. *(Examples of norms)*

(a) Show that the examples (2.2.1) and similarly (2.2.2) satisfy all properties of a norm. *Hint:* In the case $p < \infty$, the triangle inequality is also referred to as the *Minkowski inequality*. Its proof can be found for example in [**136**, Theorem 3.5].

(b) We let X be the vector space of all complex-valued finite sequences $(x_n)_{n\in\mathbb{N}}$ (by a finite sequence we mean a sequences for which only a finite number of members are non-zero). Show that

$$\big\|(x_n)_{n\in\mathbb{N}}\big\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} \quad \text{for } p < \infty \qquad \text{and} \qquad \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$$

defines a norm X. Show that the resulting normed space is infinite-dimensional.

EXERCISE 2.5. *(Examples of Banach spaces)*

(a) Show that $\mathbb{C}^n$ with the norm (2.2.1) is complete.

(b) Show that the space $(X, \|.\|_p)$ considered in Exercise 2.4 (b) is *not* complete. Let l_p be the space of all complex sequences (not necessarily finite) for which the norm $\|.\|_p$ is finite. Show that l_p is a Banach space.

(c) Show that the space $C_0^0(\mathbb{R}^n)$ with the norm (2.2.2) is *not* complete. *Hint*: construct a Cauchy sequences which has not limit in $C_0^0(\mathbb{R}^n)$. We remark that the completion of these spaces gives the Banach spaces $L^p(\mathbb{R}^n)$; details can be found for example in [**136**, Chapter 3].

EXERCISE 2.6. *(Completion of a metric space)* Let (E, d) be a metric space.

(a) Show that for two Cauchy sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in E, the limit

$$d'\Big(\big(x_n\big)_{n\in\mathbb{N}}, \big(y_n\big)_{n\in\mathbb{N}}\Big) := \lim_{n\to\infty} d(x_n, y_n)$$

exists. Show further that the function d' defined on the set of all Cauchy sequences of E in this way has the properties (ii) and (iii) in Definition 2.1.1, but will never satisfy property (i) unless E has only one element.

(b) Verify that

$$\big(x_n\big)_{n\in\mathbb{N}} \simeq \big(y_n\big)_{n\in\mathbb{N}} \qquad \text{if} \qquad d'\Big(\big(x_n\big)_{n\in\mathbb{N}}, \big(y_n\big)_{n\in\mathbb{N}}\Big) = 0$$

defines an equivalence relation on the set of all Cauchy sequences of E. Show further that the function d' induces a well-defined function d'' on the set $\tilde{E}$ of all equivalence classes. Verify that d'' has property (i) in Definition 2.1.1 and still satisfies (ii) and (iii).

(c) Show that $(\tilde{E}, d'')$ is a complete metric space. Show furthermore that the map $E \to \tilde{E}$, sending each $x \in E$ to the constant sequence $(x)_{n\in\mathbb{N}}$, is distance-preserving (and therefore injective).

(d) Modify the construction in order to form the completion of a scalar product space.

EXERCISE 2.7. *(Norm of a scalar product space)* Given a scalar product space $(V, \langle .|. \rangle)$, show that $\|u\| := \sqrt{\langle u|u\rangle}$ defines a norm (see Definitions 2.2.2 and 2.2.1).

EXERCISE 2.8. *(Boundedness and continuity of linear operators)* Let $A : V \to W$ be a linear operator between normed spaces V and W. Show that A is bounded if and only if it is continuous. *Hint:* Write down the usual condition for continuity of A and try to simplify it using linearity.

EXERCISE 2.9. *(Completeness of* $\mathrm{L}(V, W)$*)*

(a) Show that the operator norm on $\mathrm{L}(V, W)$ is indeed a norm, i.e. verify that it has all the properties in Definition 2.2.1.

(b) Show that $\mathrm{L}(V, W)$ is complete if and only if W is complete.

EXERCISE 2.10. *(Orthogonal projection to closed subspaces of a Hilbert space)* Let $\mathcal{H}$ be a Hilbert space and $V \subset \mathcal{H}$ a closed subspace.

(a) Show the parallelogram identity: For all $u, v \in \mathcal{H}$,

$$\|u + v\|^2 + \|u - v\|^2 = 2\,\|u\|^2 + 2\,\|v\|^2 \,.$$

(b) Given $u \in \mathcal{H}$, let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V which is a minimizing sequence of the distance to u, i.e.

$$\|u - v_n\| \to \inf_{v\in V} \|u - v\| \,.$$

Prove that the sequence $(v_n)_{n\in\mathbb{N}}$ converges. *Hint:* Apply the parallelogram identity to show that the sequence is Cauchy. Then make use of the completeness of the Hilbert space.

(c) Show that the limit vector $v := \lim_{n\to\infty} v_n$ has the property

$$\langle u - v, w\rangle = 0 \qquad \text{for all } w \in V \,.$$

In view of this equation, the vector v is also referred to as the orthogonal projection of u to V.

In the finite-dimensional setting, the orthogonal projection can be given more explicitly as explained in Exercise 2.12.

EXERCISE 2.11. *(Proof of the Fréchet-Riesz theorem)* Let $\mathcal{H}$ be a Hilbert space and let $\phi \in \mathcal{H}^*$ be non-zero.

(a) Show that the kernel of ϕ is a closed subspace of $\mathcal{H}$.

(b) Apply the result of Exercise 2.10 to construct a nonzero vector v which is orthogonal to $\ker \phi$. Show that this vector is unique up to scaling.
(c) Show that, after a suitable scaling, the vector v satisfies (2.2.3).
(d) Show that the vector v satisfying (2.2.3) is unique.

EXERCISE 2.12. *(Orthogonal complement of a finite-dimensional subspace)*
(a) Let I be a finite-dimensional subspace of a Hilbert space $(\mathcal{H}, \langle .|. \rangle)$. Show that its orthogonal complement $I^\perp$ defined by (2.2.6) is again a complex vector space.
(b) Show that restricting the scalar product to I, one again gets a Hilbert space. In particular, why is it complete again?
(c) Show that every vector $u \in \mathcal{H}$ has a unique decomposition of the form (2.2.7).
Hint: Choosing an orthonormal basis $e_1, \ldots, e_n$ of I, a good ansatz for $u^{||}$ is

$$u^{||} = \sum_{k=1}^{n} \langle e_k | u \rangle \, e_k \,.$$

EXERCISE 2.13. *(Multiplication operators)* Let $f \in C^0(\mathbb{R}, \mathbb{C})$ be a continuous, complex-valued function. Assume that it is bounded, i.e. that $\sup_{\mathbb{R}} |f| < \infty$. We consider the multiplication operator T_f on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, i.e.

$$T_f \,:\, \mathcal{H} \to \mathcal{H} \,, \qquad (T_f \,\phi)(x) = f(x)\, \phi(x) \,.$$

(a) Show that T_f is a (well-defined) bounded operator whose operator norm is given by

$$\|T_f\| = \sup_{\mathbb{R}} |f| \,.$$

(b) Show that T_f is symmetric if and only if f is real-valued. Under which assumptions on f is T_f unitary?

EXERCISE 2.14. *(Examples of measures)* Let $\mathcal{F}$ be a set and $\mathfrak{M}$ a σ-algebra on $\mathcal{F}$.
(a) Let $f : \mathcal{F} \to [0, \infty)$ be a function which is nonzero only at finitely many points of $\mathcal{F}$. Show that $\rho : \mathfrak{M} \to [0, \infty)$, $A \mapsto \sum_{x \in A} f(x)$ defines a measure.
(b) Show that any linear combination of measures on $\mathcal{F}$ with nonnegative coefficients is again a measure on $\mathcal{F}$.

EXERCISE 2.15. *(Basic properties of measures)* Let $(\mathcal{F}, \mathfrak{M}, \rho)$ be a measure space.
(a) Show that $\rho(A \cup B) = \rho(A) + \rho(B) - \rho(A \cap B)$ holds for any $A, B \in \mathfrak{M}$.
(b) Show that for any $A, B \in \mathfrak{M}$ with $A \subset B$, the inequality $\rho(A) \leq \rho(B)$ holds.
(c) Show that for any sequence $(A_n)_{n \in \mathbb{N}}$ of measurable sets (not necessarily pairwise disjoint), the inequality $\rho(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \rho(A_n)$ holds.

EXERCISE 2.16. *(Borel algebra)* Let $\mathcal{F}$ be a set.
(a) Show that the power set of $\mathcal{F}$ (i.e. the set of all subsets) forms a σ-algebra.
(b) Show that the intersection of σ-algebras is again a σ-algebra.
(c) Assume in addition that $\mathcal{F}$ is a topological space. Combine (a) and (b) to conclude that there is a smallest σ-algebra which contains all open subsets of $\mathcal{F}$.

EXERCISE 2.17. *(Completion of a measure)* Let $(\mathcal{F}, \mathfrak{M}, \rho)$ be a measure space. We introduce the family of sets

$$\tilde{\mathfrak{M}} := \{ A \subset \mathcal{F} \,|\, \text{there exist } B, N \in \mathfrak{M} \text{ with } (A \setminus B) \cup (B \setminus A) \subset N \text{ and } \rho(N) = 0 \} \,.$$

(a) Show that $\tilde{\mathfrak{M}}$ is again a σ-algebra.

(b) Show that the prescription $\tilde{\rho}(A) := \rho(B)$ defines a measure on $\tilde{\mathfrak{M}}$. (Where for $A \in \tilde{\mathfrak{M}}$ the set $B \in \mathfrak{M}$ is a set chosen as in the defining property of $\tilde{\mathfrak{M}}$.)
(c) Show that the measure $\tilde{\rho}$ is complete.

EXERCISE 2.18. *(Understanding the push-forward measure)* The purpose of this exercise is to introduce the so-called push-forward measure, which will be used later for the construction of causal fermion systems. Let $\mathscr{M} \subset \mathbb{R}^3$ be a smooth surface described by a parametrization Φ. More precisely, given an open subset $\Omega \subset \mathbb{R}^2$, we consider a smooth injective map

$$\Phi \,:\, \Omega \to \mathbb{R}^3$$

with the property that $D\Phi|_p : \mathbb{R}^2 \to \mathbb{R}^3$ has rank two for all $p \in \Omega$. Then the surface $\mathscr{M}$ is defined as the image $\Phi(\Omega) \subset \mathbb{R}^3$. We now introduce a measure ρ on $\mathbb{R}^3$ as the *push-forward measure* of the Lebesgue measure on $\mathbb{R}^2$ through Φ: Let μ be the Lebesgue measure on $\mathbb{R}^2$. We define a set $U \subset \mathbb{R}^3$ to be ρ-measurable if and only if its pre-image $\Phi^{-1}(U) \subset \mathbb{R}^2$ is μ-measurable. On the ρ-measurable sets we define the measure ρ by

$$\rho(U) = \mu\big(\Phi^{-1}(U)\big)\,.$$

Verify that the ρ-measurable sets form a σ-algebra, and that ρ is a measure. What are the sets of ρ-measure zero? What is the support of the measure ρ?

Suppose that Φ is no longer assumed to be injective. Is ρ still a well-defined measure? Is ρ well-defined if Φ is only assumed to be continuous? What are the minimal regularity assumptions on Φ needed for the push-forward measure to be well-defined? What is the support of ρ in this general setting?

EXERCISE 2.19. (a) Let $f \in C^0(\mathbb{R}^n, \mathbb{R})$ be continuous. Show that

$$\lim_{r \searrow 0} \int_{B_r(0)} f(x)\, \mathrm{d}^n x = 0\,. \tag{2.6.1}$$

Hint: Use that a continuous function is locally bounded.
(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue-integrable function. Show that (2.6.1) again holds. *Hint:* Use Lebesgue's monotone convergence theorem.

EXERCISE 2.20. *(Topology on the Schwartz space)*
(a) Show that the topology on $\mathcal{S}(\mathbb{R}^n)$ defined by (2.4.5) gives rise to the notion of convergence (2.4.6).
(b) Show that a linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ is continuous if and only if there are $p, q \in \mathbb{N}_0$ and $c > 0$ such that the inequality (2.4.7) holds.
(c) Show that for any $i \in \{1, \ldots, n\}$, the partial derivative $D_i : f \mapsto \partial_i f$ is a continuous linear mapping from $\mathcal{S}(\mathbb{R}^n)$ to itself.

EXERCISE 2.21. Prove the Fourier inversion formula for tempered distributions

$$\mathcal{F} \circ \mathcal{F}^* = \mathcal{F}^* \circ \mathcal{F} = \mathbb{1}_{\mathcal{S}'(\mathbb{R}^n)} \,:\, \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,.$$

Hint: Use Lemma 2.4.6 together with the definition of the Fourier transform of a tempered distribution.

EXERCISE 2.22. *(Dirac sequence)* Given $\varepsilon > 0$, consider the Gaussian $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ with

$$\eta_\varepsilon(x) := \frac{1}{\sqrt{4\pi\varepsilon}}\, \mathrm{e}^{-x^2/(4\varepsilon)}\,.$$

(a) Show that η_ε is a Schwartz function.

(b) Show that in the limit $\varepsilon \searrow 0$, the corresponding regular distribution converges to the δ distribution in the sense that for all $f \in \mathcal{S}(\mathbb{R})$,

$$\lim_{\varepsilon \searrow 0} T_{\eta_\varepsilon}(f) = \delta(f)\,.$$

(We recall that the δ distribution as introduced informally in Example 2.4.1 is defined by $\delta(f) = f(0)$ for all $f \in \mathcal{S}(\mathbb{R})$).

EXERCISE 2.23. This exercise is devoted to a clean proof of the distributional relation (16.5.2) in one dimension. More precisely, we want to prove the slightly more general statement that for any function $\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$,

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \eta(x) \left(\frac{1}{x - \mathrm{i}\,\varepsilon} - \frac{1}{x + \mathrm{i}\,\varepsilon} \right) \mathrm{d}x = 2\pi \mathrm{i}\; \eta(0)\,. \tag{2.6.2}$$

(a) Let $\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\eta(0) = 0$. Use Lebesgue's dominated convergence theorem to show that (2.6.2) holds.
(b) Use the residue theorem to show that (2.6.2) holds for the function $\eta(x) = 1/(x^2+1)$.
(c) Combine the results of (a) and (b) to prove (2.6.2) for general $\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$.

EXERCISE 2.24. Let $\Theta := \chi_{(0,\infty)} : \mathbb{R} \to \mathbb{R}$ be the *Heaviside function* defined by

$$\Theta(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0\,. \end{cases}$$

Show that $\Theta' = \delta$ holds in $\mathcal{S}'(\mathbb{R})$.

EXERCISE 2.25. *(Fourier transform of δ distribution)* Let $T \in \mathcal{S}'(\mathbb{R})$ be the regular distribution corresponding to the constant function with value one, i.e.

$$T(f) := \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x\,.$$

(a) Show that $T(f) = 2\pi(\mathcal{F}^* f)(0)$.
(b) Use the Fourier inversion theorem for tempered distributions to conclude that $\mathcal{F}T = 2\pi\,\delta$ (where δ is again the δ distribution).
(c) Apply the Fourier inversion theorem again to compute the Fourier transform of the δ distribution.
(d) Alternatively, one can compute these Fourier transforms directly working with convergence-generating factors in the style of (2.4.13). Do this carefully step by step, making sure that every computation step is mathematically well-defined.

EXERCISE 2.26. *(Schwartz space)*

(a) Show that Schwartz functions decay faster than polynomially at infinity together with all their partial derivatives. More precisely, show that for every multi-index α and for every $N \in \mathbb{N}$ there exists $C_{\alpha,N} \geq 0$ such that

$$|D^\alpha f(x)| \leq \frac{C_{\alpha,N}}{1 + |x|^N} \quad \text{for all } x \in \mathbb{R}^n.$$

(b) Which of the following functions belongs to $\mathcal{S}(\mathbb{R})$? Motivate your answers!

$$f_1(x) := \mathrm{e}^{-x^2}, \qquad f_2(x) := \frac{1}{1+x^4}, \qquad f_3(x) := \frac{\mathrm{e}^{-x^2}}{2 + \sin(\mathrm{e}^{x^2})}.$$

EXERCISE 2.27. *(The principal value integral)* For every $f \in \mathcal{S}(\mathbb{R})$ we define

$$A(f) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{f(x)}{x}\,\mathrm{d}x\,.$$

Does the limit exist? In fact, A is a tempered distribution. Show it.

EXERCISE 2.28. *(Multiplication operators)* Let $n \in \mathbb{N}$.

(a) Let g be a (not necessarily continuous) function $g : \mathbb{R}^n \to \mathbb{R}$ with the property that the mapping $x \mapsto g(x)/(1+|x|^r)$ belongs to $L^1(\mathbb{R}^n)$ for some $r > 0$. Show that the map $f \mapsto T_g(f) := \int_{\mathbb{R}^n} f(x)\,g(x)\,\mathrm{d}x$ defines a tempered distribution, $T_g \in \mathcal{S}(\mathbb{R}^n)'$.

(b) Find a smooth function $g \in L^1(\mathbb{R}^n)$ which is not pointwise bounded by any polynomial, i.e. there is no polynomial p such that

$$|g(x)| \le p(|x|) \quad \text{for all } x \in \mathbb{R}^n\,.$$

But the corresponding functional

$$T_g : \mathcal{S}(\mathbb{R}^n) \ni f \mapsto \int_{\mathbb{R}^n} g(x) f(x)\,\mathrm{d}^n x \tag{2.6.3}$$

still yields a well-defined tempered distribution.

EXERCISE 2.29. *(Approximating the δ distribution)* Not all distributions can be written as in (2.6.3). Nevertheless, it can be shown that every tempered distribution can be approximated by such functionals. Let us verify this statement in the concrete example of the δ distribution. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ fulfill $\varphi \ge 0$, $\operatorname{supp}\varphi \subset B_1(0)$ and $\|\varphi\|_{L^1} = 1$. Define, for every $\varepsilon > 0$,

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\,\varphi\left(\frac{x}{\varepsilon}\right) \in \mathcal{C}_0^\infty(\mathbb{R}^n) \quad \text{and} \quad \delta_\varepsilon := T_{\varphi_\varepsilon} \text{ (defined as in (2.6.3))}$$

Each functional δ_ε is a tempered distribution (why?). Show that, for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\delta_\varepsilon(f) \to \delta(f) = f(0) \quad \text{as } \varepsilon \searrow 0.$$

Remark: The more general statement that any tempered distribution can be approximated by a sequence of regular distributions can be shown using the method of convolution. We refer to [**89**, Chapter 5] for details.

EXERCISE 2.30. *(Another regular tempered distribution)* Let $n, k \in \mathbb{N}$ with $n > k$. Show that the mapping

$$\mathcal{S}(\mathbb{R}^n) \ni f \mapsto \int_{\mathbb{R}^n} \frac{f(x)}{|x|^k}\,\mathrm{d}^n x \in \mathbb{C}$$

is a well-defined tempered distribution.

EXERCISE 2.31. *(Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$)* Compute the Fourier transform of the following functions and tempered distributions.

(a) $f \in \mathcal{S}(\mathbb{R}^n)$ defined by $f(x) := \mathrm{e}^{-\lambda x^2}$ for $\lambda \ge 0$;

(b) $T_g \in \mathcal{S}'(\mathbb{R}^n)$ for the functions $g \in L^1(\mathbb{R})$ defined by

$$g(x) = \mathrm{e}^{-|x|} \quad \text{and} \quad g(x) = \frac{1}{1+x^2}.$$

EXERCISE 2.32. *(Fourier transform on $L^1(\mathbb{R}^n)$)* The functions in $L^1(\mathbb{R}^n)$ define tempered distributions by means of the identification $g \mapsto T_g$. As distributions they admit Fourier transform. However, for these functions the Fourier transform can also be defined

directly via the usual integral form. The goal of this exercise is to show that this is indeed true and that the two definitions coincide.

(a) Show that the Fourier transform

$$(\mathcal{F}g)(p) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}px} g(x)\, \mathrm{d}^n x$$

defines a map

$$\mathcal{F} : L^1(\mathbb{R}^n) \to C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Moreover, show that there exits a constant C_n such that

$$\|\mathcal{F}g\|_\infty \leq C_n \|g\|_{L^1} \quad \text{for all } g \in L^1(\mathbb{R}^n).$$

(b) Let $g \in L^1(\mathbb{R}^n)$. Show that the Fourier transform of the distribution $T_g \in \mathcal{S}'(\mathbb{R}^n)$ satisfies the relation

$$\mathcal{F}(T_g) = T_{\mathcal{F}g}.$$

Hint: You may use that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

Exercise 2.33. *(On the topology of $\mathcal{S}(\mathbb{R}^n)$)* Consider the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ equipped with the family of norms $\|\cdot\|_{p,q}$, where $p, q \in \mathbb{N}_0$. Show the following statements.

(a) For any $f, g \in \mathcal{S}(\mathbb{R}^n)$ the following series converges:

$$d(f,g) := \sum_{p,q,\in\mathbb{N}_0} \frac{1}{2^{p+q}} \frac{\|f-g\|_{p,q}}{1+\|f-g\|_{p,q}}.$$

Moreover it defines a metric on $\mathcal{S}(\mathbb{R}^n)$.

(b) The metric space $(\mathcal{S}(\mathbb{R}^n), d)$ is complete.

(c) Show that if a topology on a vector space is induced by a norm, then there is a neighborhood U of 0 such that for every other neighborhood V of 0, there is a positive number r such that $U \subset r \cdot V$. Can the topology of $\mathcal{S}(\mathbb{R}^n)$ be induced by a norm?

Exercise 2.34. *(Convolution of Schwartz functions)*

(a) Show that for two Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^n)$, the integral (2.4.15) defines a smooth function $(f * g)(x)$.

(b) Show that this function has rapid decay.

Exercise 2.35. *(Convolution and Fourier transform)* Prove the relation (2.4.16) on the Fourier transform of a convolution.
Hint: Use the definition of the Fourier transform (2.4.10) and rewrite the resulting double integral.

Exercise 2.36. *(Convolution with the δ distribution)* Let $\delta \in \mathcal{S}'(\mathbb{R}^n)$ be the δ distribution (at the origin). Show that $f * \delta = f$ holds for any $f \in \mathcal{S}(\mathbb{R}^n)$.

Exercise 2.37. *(Smoothing a distribution by convolution)* Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(R)^n)$.

(a) Show that the convolution $f * T$ as defined in (2.4.17) defines a smooth function.

(b) Show by a counter example that the function $f * T$ in general does *not* decay at infinity.

(c) Let $n \geq 1$, $\alpha \in \mathbb{N}^n$, $f \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$. Prove that $D^\alpha(f * T) = f * (D^\alpha T)$.

EXERCISE 2.38. *(A simple manifold)* Let $\mathscr{M} = S^1 \subset \mathbb{C}$ be the unit circle (considered as a subset of the complex plane). We choose the charts (ϕ_1, U_1) and (ϕ_2, U_2) with

$$U_1 = \left\{ \mathrm{e}^{\mathrm{i}\alpha} \,\middle|\, \alpha \in \Big(-\frac{3\pi}{2}, \frac{3\pi}{2} \Big) \right\}, \qquad \phi_1\big(\mathrm{e}^{\mathrm{i}\alpha}\big) = \alpha$$
$$U_2 = \left\{ -\mathrm{e}^{\mathrm{i}\alpha} \,\middle|\, \alpha \in \Big(-\frac{3\pi}{2}, \frac{3\pi}{2} \Big) \right\}, \qquad \phi_1\big(-\mathrm{e}^{\mathrm{i}\alpha}\big) = \alpha\,.$$

Show that these two charts define a smooth atlas of $\mathscr{M}$.

EXERCISE 2.39. Let $\mathscr{M}, \mathscr{N}$ be smooth manifolds and $F : \mathscr{M} \to \mathscr{N}$ a map. Show that F is smooth if and only if for any $p \in \mathscr{M}$ there exist charts (ϕ, U) around p and (ψ, V) around $F(p)$ with $F(U) \subset V$ and such that $\psi \circ F \circ \phi^{-1}$ is smooth (as map between open subsets of Euclidean spaces).

EXERCISE 2.40. Unwind the definitions to verify formula (2.5.7) for the derivative of a smooth map $F : \mathscr{M} \to \mathscr{N}$ between two smooth manifolds $\mathscr{M}, \mathscr{N}$ in local coordinates.

EXERCISE 2.41. *(A simple vector bundle)* Let $\mathcal{B} = \mathbb{R} \times S^1$ be the two-dimensional cylinder, $\mathscr{M} = S^1$ and

$$\pi \,:\, \mathcal{B} \to \mathscr{M}\,, \qquad (t, \mathrm{e}^{\mathrm{i}\alpha}) \mapsto \mathrm{e}^{\mathrm{i}\alpha}\,.$$

Show that $\mathcal{B}$ is a smooth vector bundle with fiber $Y = \mathbb{R}$.

EXERCISE 2.42. *(Another vector bundle)* Use the mapping

$$F \,:\, \mathbb{R} \times S^2 \to \mathbb{R}^3 \setminus \{0\}\,, \qquad (t, x) \mapsto \mathrm{e}^t\, x$$

(where we consider S^2 as the unit sphere embedded in $\mathbb{R}^3$) in order to give $\mathcal{B} = \mathbb{R}^3 \setminus \{0\}$ the structure of a vector bundle on S^2 with fiber $Y = \mathbb{R}$.

EXERCISE 2.43. *(The Möbius bundle)* Represent the circle S^1 as $S^1 = [0,1]/(0 \sim 1)$, i.e. as the unit interval with its end points being identified. Moreover, consider the space $X := ([0,1] \times \mathbb{R})/\sim$, where $\sim$ is the equivalence relation generated by $(0, v) \sim (1, -v)$ for all $v \in \mathbb{R}$. Show that the map $\pi : X \to S^1$, $[(x, v)] \mapsto [x]$ is well-defined and defines a vector bundle.

CHAPTER 3

Elements of Operator Theory

In this chapter we introduce some material from functional analysis which will be needed later in this book. More precisely, in Section 3.1 we explain the concept that linear operators with certain properties form a submanifold of the space of all linear operators. This concept will be useful when generalizing the causal action principle to causal variational principles in Chapter 6. In Section 3.2 we recall the spectral calculus for selfadjoint operators. Although this material is covered in most functional analysis lectures, in this book we do not expect that the reader is already familiar with this topic. The spectral theorem will be used only when developing functional analytic methods in spacetime in Chapter 15.

3.1. Manifolds of Operators

In this book, it is sometimes useful to observe that certain sets of operators on a Hilbert space form a smooth manifold. For the purposes in this book, it suffices to work out this concept in the case of a finite-dimensional Hilbert space (the generalization to the infinite-dimensional case is a bit more technical; for details see [**76**]).

We begin with a simple example which illustrates the basic concept.

EXAMPLE 3.1.1. **(Grassmann manifold)** Given $f \in \mathbb{N}$, we consider $\mathbb{C}^f$ with the canonical scalar product $\langle .,. \rangle_{\mathbb{C}^f}$. Let $\mathcal{G}$ be the set of all orthogonal projection operators to one-dimensional subspaces of $\mathbb{C}^f$,

$$\mathcal{G} := \left\{ \pi_V \text{ orthogonal projection to a one-dimensional subspace } V \subset \mathbb{C}^f \right\}.$$

Let us verify that $\mathcal{G}$ is a smooth manifold of dimension

$$\dim \mathcal{G} = 2f - 2 \,.$$

To this end, let π_V be such a projection operator. We choose a unit vector v which spans V. Next we let $V^\perp$ be the orthogonal complement of V and $W = B_1(0) \cap V^\perp$ its unit ball. We consider the mapping

$$F \,:\, W \to \mathcal{G} \,, \qquad u \mapsto \pi_{\text{span}(v+u)} \,.$$

It is verified by direct computation that F is injective and that its image is an open neighborhood of π_V in $\mathcal{G}$. Also, one easily verifies that it is a local homeomorphism and thus defines a chart

$$\phi = F|^{-1}_{F(W)} \,:\, F(W) \to W \subset V^\perp \simeq \mathbb{C}^{f-1} \simeq \mathbb{R}^{2f-2} \,.$$

Moreover, one verifies directly that all the charts obtained in this way form a smooth atlas (a more explicit method for constructing charts is explained in Exercise 3.1). This manifold is called *Grassmann manifold*. ◊

This concept can be generalized to so-called *flag manifolds* (see for example [**104**, page 142]):

DEFINITION 3.1.2. *Given $r \in \mathbb{N}$, we choose r integers*

$$0 < d_1 < \cdots < d_r < f \,.$$

Consider a sequence $(L_1, \ldots, L_r)$ of nested subspaces

$$L_1 \subset \cdots \subset L_r \subset \mathbb{C}^f \qquad \text{with} \qquad \dim L_i = d_i$$

for $i = 1, \ldots, r$. Then the set of such sequences $(L_1, \ldots, L_r)$ is referred to as the **flag manifold** $\mathcal{F}^{d_1,\ldots,d_r}$.

Here we do not need to verify that a flag manifold is indeed a manifold. Instead, it suffices to consider a specific set of operators which is related to a flag manifold. Namely, choosing again $\mathcal{H} = \mathbb{C}^f$ as well as integers p, q with $p + q \leq f$, we let $\mathcal{F}^{p,q}$ be the set

$$\mathcal{F}^{p,q} = \{A \in \mathrm{L}(\mathcal{H}) \,|\, A \text{ is symmetric and has } p \text{ positive and } q \text{ negative eigenvalues}\} \,,$$

where we count the eigenvalues with multiplicities. Taking L_1 as the subspace spanned by the eigenvectors corresponding to the positive eigenvalues and L_2 as the image of A, one gets a corresponding flag manifold with $r = 2$ and $d_1 = p$, $d_2 = p + q$. But the operators in $\mathcal{F}^{p,q}$ contain more information, namely the eigenvalues and the corresponding eigenspaces. Therefore the set $\mathcal{F}^{p,q}$ can be regarded as a flag manifold with additional structures. We now prove that this set is again a smooth manifold, following the method in [**69**, Section 3].

PROPOSITION 3.1.3. *The set $\mathcal{F}^{p,q}$ is a smooth manifold of dimension*

$$\dim \mathcal{F}^{p,q} = 2f\,(p+q) - (p+q)^2 \,.$$

PROOF. Let $x \in \mathcal{F}^{p,q}$. We denote its image by I and set $J = I^\perp$. Thus, using a block matrix representation in $\mathbb{C}^f = I \oplus J$, the matrix x has the representation

$$x = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \,. \tag{3.1.1}$$

The goal is to find a parametrization of operators of $\mathcal{F}^{p,q}$ in a small neighborhood of x. We first note that varying X by a sufficiently small symmetric matrix A, the resulting matrix $X + A$ has again p positive and q negative eigenvalues. In order to also vary the off-diagonal entries in (3.1.1), we make the ansatz

$$M = (\mathbb{1} + C) \begin{pmatrix} X + A & 0 \\ 0 & 0 \end{pmatrix} (\mathbb{1} + C)^*$$

with an $f \times f$-matrix C. This ansatz has the advantage that M is obviously symmetric and, for C sufficiently small, has again p positive and q negative eigenvalues (for details see Exercise 3.2). We want to choose C such that the upper right block matrix entry of M has a particularly simple form. This leads us to the parametrization matrix

$$M := \begin{pmatrix} \mathbb{1} & 0 \\ B^*(X+A)^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} X + A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & (X+A)^{-1}B \\ 0 & \mathbb{1} \end{pmatrix} \tag{3.1.2}$$

$$= \begin{pmatrix} X + A & B \\ B^* & B^*(X+A)^{-1}B \end{pmatrix} \tag{3.1.3}$$

with a linear operator $B : J \to I$. We also see that for $(A, B) = (0, 0)$ the parametrization matrix equals x, which is necessary for building a chart around x.

Thus, for sufficiently small ε, we obtain the mapping

$$\Lambda \,:\, \big(\mathrm{Symm}(I) \oplus L(I, J)\big) \cap B_\varepsilon(0) \to \mathcal{F}^{p,q} \,, \qquad (A, B) \mapsto M \tag{3.1.4}$$

(where $\mathrm{Symm}(I)$ denotes the symmetric linear operators). Let us verify that (again for sufficiently small ε) this mapping is a homeomorphism to an open neighborhood of $x \in \mathcal{F}^{p,q}$. It is obvious from (3.1.3) that Λ is injective. In order to verify that it maps to an open neighborhood of x, we let $y \in F^{p,q}$ with $\|x - y\| < \delta$ (with $\delta > 0$ to be specified below). Diagonalizing y with a unitary matrix U, we obtain the block matrix representation

$$y = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} X + C & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{pmatrix},$$

where C is a symmetric linear operator on I. In the limit $y \to x$, the image of y converges to the image of x, implying that the matrix U_{11} becomes unitary. Therefore, for sufficiently small $\delta > 0$, the matrix U_{11} is invertible, giving rise to the representation

$$y = \begin{pmatrix} \mathbb{1} & 0 \\ U_{21}\, U_{11}^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} U_{11}\,(X + C)\, U_{11}^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & (U_{11}^*)^{-1}\, U_{21}^* \\ 0 & \mathbb{1} \end{pmatrix}.$$

This is of the form (3.1.2), and one can even read off A and B,

$$\begin{aligned} A &= U_{11}\,(X + C)\, U_{11}^* - X \\ B &= \big(U_{11}\,(X + C)\, U_{11}^*\big)\, \big(U_{11}^*\big)^{-1}\, U_{21}^* \big). \end{aligned}$$

We conclude that Λ is a bijection to an open neighborhood of $x \in \mathcal{F}^{p,q}$. The continuity of Λ and of its inverse are obvious. We have thus constructed a chart around x.

Performing the above construction around every point $x \in \mathcal{F}^{p,q}$ gives an atlas. By direct computation one verifies that the transition maps are smooth. We conclude that, with the above atlas, $\mathcal{F}^{p,q}$ is indeed a smooth manifold.

We finally determine its dimension. The linear operator B is represented by a $(p+q) \times (f-p-q)$-matrix, giving rise to $2(p+q)(f-p-q)$ real degrees of freedom. The symmetric linear operator A, on the other hand, is represented by a Hermitian $(p+q)\times(p+q)$-matrix, described by $(p+q)^2$ real parameters. Adding these dimensions gives the result. □

We finally remark that the mapping Λ in (3.1.4) gives rise to distinguished charts, referred to as *symmetric wave charts*. They have the nice property that they are Gaussian normal coordinates with respect to the Riemannian metric induced by the Hilbert-Schmidt norm (for details see [**69**]).

3.2. The Spectral Theorem for Selfadjoint Operators

In this book, we will mainly encounter operators of *finite rank* (see Definition 2.2.5). In this case, a symmetric operator A on a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ (see again Definition 2.2.5) has real eigenvalues, and there is an orthonormal basis of eigenvectors. Given an eigenvalue λ, we refer to the dimension of the eigenspace as its *multiplicity*. Choosing an orthogonal basis $e_1, \ldots, e_k$ of this eigenspace, we can form the *orthogonal projection operator* E_λ to this eigenspace as

$$E_\lambda : \mathcal{H} \to \mathcal{H}, \qquad E_\lambda\, u := \sum_{i=1}^{k} e_i\, \langle e_i | u \rangle.$$

Denoting the set of eigenvalues by $\sigma(A) \subset \mathbb{R}$, the operator A has the *spectral decomposition*

$$A = \sum_{\lambda \in \sigma(A)} \lambda\, E_\lambda. \tag{3.2.1}$$

We now briefly outline how the spectral decomposition can be generalized to operators of infinite rank and unbounded operators. In this book, these results will be needed only in Chapters 15–18 and a few times implicitly in order to justify that for example the unitary time evolution in (4.6.2) is well-defined. With this in mind, the reader may skip the remainder of this section in a first reading. More details on the spectral theorem can be found in textbooks on functional analysis like for example [**131, 137, 116**]. We begin with the case that A is a *bounded* and *symmetric* linear operator (see Definitions 2.2.3 and 2.2.5). The *resolvent set* $\rho(A) \subset \mathbb{C}$ is defined as the set of all λ for which the operator $A - \lambda$ has a bounded inverse, the so-called *resolvent* $R_\lambda := (A-\lambda)^{-1} \in \mathrm{L}(\mathcal{H})$. The spectrum $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is defined as the complement of the resolvent set. The spectrum of a symmetric operator is always real, $\sigma(A) \subset \mathbb{R}$ (see Exercise 3.4). Given a complex polynomial $p(\lambda)$, we can form the operator $p(A)$ by replacing λ by the operator A (and monomials by powers of the operator). Clearly, this operation is compatible with taking adjoints and multiplications, i.e.

$$p(A)^* = \overline{p}(A) \qquad \text{and} \qquad p(A)\, q(A) = (pq)(A) \tag{3.2.2}$$

for any polynomials p and q. Moreover, the spectrum and norm of the operator $p(A)$ can be expressed easily in terms of the polynomial. Namely, the *spectral mapping theorem* states that (for details see Exercise 3.5)

$$\sigma\big(p(A)\big) = p\big(\sigma(A)\big)\,. \tag{3.2.3}$$

Moreover, the norm of $p(A)$ is given by the C^0-norm of the polynomial on the spectrum of A,

$$\big\|p(A)\big\| = \sup_{\lambda \in \sigma(A)} \big|p(\lambda)\big|\,. \tag{3.2.4}$$

These two properties make it possible to make sense of $p(A)$ for more general functions p. Namely, using the Stone-Weierstraß approximation theorem, the *continuous functional calculus* makes it possible to define $p(A)$ for continuous functions $p \in C^0(\mathbb{R})$. Next, measure-theoretic methods (more precisely, the Riesz representation theorem to be introduced in Section 12.2) make it possible to define $p(A)$ for any bounded Borel function (a function is called Borel if the pre-image of every open set is a Borel set). The relations (3.2.2), (3.2.3) remain valid, whereas (3.2.4) becomes an inequality, $\|p(A)\| \leq \sup_{\lambda \in \sigma(A)} |p(\lambda)|$.

The functional calculus for bounded Borel functions makes it possible to choose the function p in particular as the characteristic function of a Borel set $\Omega \subset \mathbb{R}$. We use the notation

$$E_\Omega := \chi_\Omega(A)\,.$$

The mapping which to a Borel set Ω associates the operator E_Ω is a *projection-valued measure*. This means that for every Borel set Ω, the operator E_Ω is an *orthogonal projection operator* in the sense that $E_\Omega^* = E_\Omega = E_\Omega^2$. Moreover, for any Borel sets U and V, these operators have the properties that $E_U E_V = E_{U \cap V}$. Finally, similar to Definition 2.3.2, a spectral measure has the property that $E_\varnothing = 0$ and $E_{\mathbb{R}} = \mathbb{1}$, whereas σ-additivity means that for any sequence of pairwise disjoint Borel sets $(\Omega_n)_{n \in \mathbb{N}}$ and every vector $u \in \mathcal{H}$,

$$E_{\bigcup_n \Omega_n} u = \sum_{n=1}^\infty \big(E_{\Omega_n} u\big)$$

(where the series converges in the Hilbert space $\mathcal{H}$; in other words, the series of operators converges strongly in $\mathrm{L}(\mathcal{H})$). Being in the measure-theoretic setting has the major

advantage that one can express the functional calculus as an integral, i.e.

$$f(A) = \int_{\mathbb{R}} f(\lambda)\ \mathrm{d}E_\lambda \tag{3.2.5}$$

for any real-valued bounded Borel function f. In particular, the operator A has the spectral decomposition

$$A = \int_{\mathbb{R}} \lambda\ \mathrm{d}E_\lambda\,. \tag{3.2.6}$$

In the case that the spectral measure is supported at a finite number of points, the integral in (3.2.6) reduces to a sum, giving us back the spectral decomposition 3.2.1. However, in general the integral (3.2.6) does not reduce to a sum or series. Instead, the support of the spectral measure coincides with the spectrum,

$$\sigma(A) = \operatorname{supp} E\,, \tag{3.2.7}$$

but the spectrum may contain open sets, giving rise to the so-called *continuous spectrum*. We remark that the equality in (3.2.4) again holds for bounded Borel functions if the supremum is replaced by the essential supremum (with respect to the spectral measure). In Exercise 3.7, the spectral theorem is illustrated in the example of a multiplication operator.

The above spectral theorem for bounded symmetric operators can be generalized to *bounded normal* operators. An operator $A \in \mathrm{L}(\mathcal{H})$ is called normal if it commutes with its adjoint,

$$[A, A^*] = 0\,.$$

The spectrum of a normal operator is in general complex. The spectral calculus reads

$$f(A) = \int_{\mathbb{C}} f(\lambda)\ \mathrm{d}E_\lambda\,, \tag{3.2.8}$$

where f is a complex-valued bounded Borel function on $\mathbb{C}$ and E a spectral measure supported in the complex plane. The formulas (3.2.3), (3.2.4) and (3.2.7) continue to hold. A typical example of a normal operator is a unitary operator (see Definition 2.2.5), in which case the spectrum lies on the unit circle (see Exercise 3.6).

Finally, the spectral theorem also applies to *unbounded selfadjoint* operators, as we now recall. An unbounded operator A is not defined on the whole Hilbert space, but only on a dense subspace $\mathcal{D}(A) \subset \mathcal{H}$. Thus it is a linear mapping

$$A\ :\ \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}\,.$$

The notion of a *symmetric* operator from Definition 2.2.5 extends to unbounded operators by imposing it only for vectors u and v in the domain, i.e.

$$\langle Au \,|\, v\rangle = \langle u \,|\, Av\rangle \qquad \text{for all } u, v \in \mathcal{D}(A)\,.$$

The operator A is *selfadjoint* if the following implication holds,

$$\langle Au \,|\, v\rangle = \langle u \,|\, w\rangle \quad \text{for all } u \in \mathcal{D}(A) \qquad \Longrightarrow \qquad v \in \mathcal{D}(A) \quad \text{and} \quad Av = w\,. \tag{3.2.9}$$

Clearly, a selfadjoint operator is symmetric. However, the converse is in general not true. Indeed, in order to obtain a selfadjoint operator one must construct a dense domain which must be balanced (i.e. not too large and not too small) such that the condition on the left of (3.2.9) implies that v lies in this domain. In this book, we shall not enter the methods for the construction of selfadjoint domains. Instead, we assume that a selfadjoint

operator A is given. Then the spectral theorem yields a spectral measure E on $\mathbb{R}$ such that (3.2.6) again holds, but with pointwise convergence on the domain, i.e.

$$Au = \int_{\mathbb{R}} \lambda \, \mathrm{d}\big(E_\lambda u\big) \qquad \text{for all } u \in \mathcal{D}(A) \,.$$

Given a bounded Borel function f on $\mathbb{R}$, the spectral calculus is again defined by (3.2.5), giving a bounded operator. If f is an unbounded Borel function, the spectral calculus (3.2.5) gives an in general unbounded function with dense domain

$$\mathcal{D}\big(f(A)\big) = \Big\{ u \in \mathcal{H} \,\Big|\, \int_{\mathbb{R}} |f(\lambda)|^2 \, \mathrm{d}\langle u | E_\lambda u\rangle < \infty \Big\} \,.$$

If f is real-valued, then the operator $f(A)$ is again selfadjoint.

EXAMPLE 3.2.1. **(An unbounded multiplication operator)** Generalizing the setting of Exercises 2.13 and 3.7, we let $g \in C^0(\mathbb{R}, \mathbb{R})$ be a possibly unbounded, real-valued function. We consider the corresponding multiplication operator A on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, i.e.

$$A := T_g \,:\, \mathcal{H} \to \mathcal{H} \,, \qquad (A\,\phi)(x) := g(x)\,\phi(x) \,.$$

In order to make mathematical sense of this operator, we need to specify a domain. In the first step, we choose $\mathcal{D}(A) = C_0^\infty(\mathbb{R})$ as all smooth test functions. With this domain, the operator is clearly symmetric, but it is not selfadjoint. In order to make A into a selfadjoint operator, we need to choose the domain as

$$\mathcal{D}(A) := \{\phi \in L^2(\mathbb{R}) \,|\, g\phi \in L^2(\mathbb{R})\} \,. \tag{3.2.10}$$

Namely, with this choice, the condition on the left of (3.2.9) implies that $\langle u | gv\rangle = \langle u | w\rangle$ for all $u \in \mathcal{D}(A)$. Using that the domain is dense, it follows that $w = gv$, so that the implication in (3.2.9) holds.

Using the relation $A^2 = T_{g^2}$ inductively, one sees that $p(A) = T_{p\circ g}$. Consequently, the functional calculus for the operator A can be written as $f(A) = T_{f\circ g}$. Choosing characteristic functions, one sees that the spectral measure is given by

$$E_\Omega = T_h \qquad \text{with} \qquad h(x) := \begin{cases} 1 & \text{if } f(x) \in \Omega \\ 0 & \text{otherwise} \,. \end{cases}$$

The support of this spectral measure coincides with the spectrum,

$$\sigma(A) = \operatorname{supp} E = g(\mathbb{R}) \,.$$

If g is equal to λ on a set of positive Lebesgue-measure, i.e. if

$$\mu\big(g^{-1}(\lambda)\big) > 0 \,,$$

then λ is an eigenvalue, and the corresponding eigenspace is given by

$$\ker(A - \lambda) = \{\phi \in L^2(R) \,|\, \phi|_{\mathbb{R}\setminus g^{-1}(\lambda)} = 0\} \,.$$

If λ is in the image of g, but the value λ is attained only on a set of measure zero, then λ lies in the continuous spectrum of A. More precisely, in this case the operator $A - \lambda$ has no kernel (i.e. λ is not an eigenvalue, because there are no corresponding eigenvectors in $\mathcal{H}$), but nevertheless the operator $A - \lambda$ has no bounded inverse. $\lozenge$

EXAMPLE 3.2.2. **(The Laplacian in $\mathbb{R}^3$)** We conclude with an example of a differential operator, namely the Laplacian Δ on the Hilbert space $L^2(\mathbb{R}^3)$. We begin with the simplest domain $\mathcal{D}(\Delta) = C_0^\infty(\mathbb{R}^3)$ of smooth test functions. With this domain, the Laplacian is symmetric, but it is not selfadjoint. One method for obtaining a domain for which Δ is selfadjoint is to take the Fourier transform and use the results for multiplication operators of the previous Example 3.2.1. To this end, it is preferable to choose the domain as the Schwartz functions, $\mathcal{D}(\Delta) = \mathcal{S}(\mathbb{R}^3)$. Taking the Fourier transform, the Laplacian becomes a multiplication operator, i.e. using the notation in Definition 2.4.4,

$$\mathcal{F}\Delta\mathcal{F}^* = T_g \qquad \text{with} \qquad g(p) = -|p|^2 \,.$$

Similar to (3.2.10), the multiplication operator T_g is selfadjoint with domain

$$\mathcal{D}\big(T_g\big) = \{\phi \in L^2(\mathbb{R}^3) \,|\, g\phi \in L^2(\mathbb{R}^3)\} \,.$$

Since the Fourier transform is a unitary transformation by Plancherel's theorem (see Theorem 2.4.8), we obtain a selfadjoint domain of the Laplacian simply by transforming the domain of T_g back to position space,

$$\mathcal{D}(\Delta) = \mathcal{F}^*\, \mathcal{D}\big(T_g\big) \,. \tag{3.2.11}$$

For the reader familiar with weak derivatives and Sobolev spaces we remark that this domain can be expressed more directly as $\mathcal{D}(\Delta) = L^2(\mathbb{R}^3) \cap W^{2,2}(\mathbb{R}^3)$. But for the purposes of this book, it suffices to write the domain according to (3.2.11). ◊

3.3. Exercises

EXERCISE 3.1. *(Charts of Grassmann manifold)* In this exercise we want to construct a chart of the Grassmann manifold of Example 3.1.1. To this end, we consider unit vectors $u \in \mathbb{C}^f$ with components

$$u(v) = \big(\sqrt{1 - \|v\|^2}, v\big) \qquad \text{with } v \in \mathbb{C}^{f-1} \text{ with } \|v\| < 1 \,.$$

We consider the mapping

$$F \,:\, B_1 \subset \mathbb{C}^{f-1} \to \mathcal{G} \,, \qquad F(v) := |u(v)\rangle\langle u(v)| \tag{3.3.1}$$

(where we used bra/ket-notation; in other words, $F(v)$ is the orthogonal projection to the span of $u(v)$). Show that F is differentiable at the origin and that $DF|_0$ has maximal rank. Use this method to construct a differentiable atlas of $\mathcal{G}$. Compare the mapping $\mathcal{F}$ in (3.3.1) with the mapping Λ in (3.1.4) in the case $p = 1$ and $q = 0$. What are the similarities and differences?

EXERCISE 3.2. Let x be a Hermitian $f \times f$-matrix of rank $p + q$ which (counting multiplicities) has p positive and q negative eigenvalues. Let A be another $f \times f$-matrix (not necessarily Hermitian). Prove the following statements:

(a) The matrix A^*xA has at most p positive and at most q negative eigenvalues.
Hint: Consider the maximal positive and negative definite subspaces of the bilinear forms $\langle ., x. \rangle_{\mathbb{C}^f}$ and $\langle ., A^*xA\, . \rangle_{\mathbb{C}^f}$. Use that

$$\langle u, A^*xA\, u \rangle_{\mathbb{C}^f} = \langle (Au), x\, (Au) \rangle_{\mathbb{C}^f} \,.$$

(b) If A is invertible, then the matrix A^*xA has again p positive and q negative eigenvalues.

EXERCISE 3.3. Let $A \in \mathrm{L}(H)$. For $\lambda \in \rho(A)$ in the resolvent set, we define the *resolvent* R_λ by $R_\lambda = (A - \lambda)^{-1}$. Prove the so-called *resolvent identity*

$$R_\lambda \, R_{\lambda'} = \frac{1}{\lambda - \lambda'} \, (R_\lambda - R_{\lambda'}) \ ,$$

valid for any $\lambda, \lambda' \in \rho(A)$. *Hint:* Multiply by $A - \lambda$ from the left and by $A - \lambda'$ from the right.

EXERCISE 3.4. Let $A \in \mathrm{L}(H)$ be a bounded symmetric operator. Show that its spectrum is real, $\rho(A) \subset \mathbb{R}$. *Hint:* It might be helpful to prove and make use of the inequality $\|(A - \lambda)\, u\| \geq \operatorname{Im} \lambda \, \|u\|$.

EXERCISE 3.5. Prove the spectral mapping theorem (3.2.3) for a polynomial p. *Hint:* Use that a complex polynomial can be factorized into a product of linear functions.

EXERCISE 3.6. Let $A \in \mathrm{L}(\mathcal{H})$ be unitary.

(a) Show that A is normal.
(b) Use the spectral calculus (3.2.8) to show that

$$\int_{\mathbb{C}} |\lambda|^2 \ \mathrm{d}E_\lambda = A^* A = \mathbb{1} \ .$$

(c) Conclude that the spectrum of A is contained in the unit circle.

EXERCISE 3.7. *(Spectrum and functional calculus for multiplication operators)* We return to the multiplication operators from Exercise 2.13. Let $f \in C^0(\mathbb{R}, \mathbb{C})$ be a continuous, complex-valued function. Assume that it is bounded, i.e. that $\sup_{\mathbb{R}} |f| < \infty$. We consider the multiplication operator T_f on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, i.e.

$$T_f \ : \ \mathcal{H} \to \mathcal{H} \, , \qquad (T_f \, \phi)(x) = f(x) \, \phi(x) \, .$$

(a) Show that, if the resolvent exists, it has the form

$$R_\lambda = (T_f - \lambda)^{-1} = T_g \qquad \text{with} \qquad g(x) = \frac{1}{f(x) - \lambda} \, .$$

For which values of λ has the operator $T_f - \lambda$ a bounded inverse? What is the spectrum of T_f?
(b) Are powers of T_f again multiplication operators? Use your findings to guess a formula for the continuous spectral calculus $f(T_a)$. What is the projection-valued spectral measure E?
(c) Work out the projection-valued spectral measure explicitly in the example

$$f(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}] \\ -x & \text{if } x \in (\frac{1}{2}, 1] \\ 0 & \text{otherwise} \, . \end{cases}$$

CHAPTER 4

Spinors in Curved Spacetime

This chapter provides a brief introduction to spinors in curved spacetime. In order to make this book accessible to a broader readership, we mainly restrict attention to systems in Minkowski space. Nevertheless, many constructions and results carry over to curved spacetime in a straightforward way. The present section is intended for providing the necessary background for these generalizations. The reader not interested in gravity and Lorentzian geometry may skip this section. More specifically, the results of this chapter will be used only in Sections 11.3, 13.6 and 19.3. We follow the approach in [**38**]; for other introductions to spinors on manifolds see for example [**115, 90**].

4.1. Curved Spacetime and Lorentzian Manifolds

The starting point for general relativity is the observation that a physical process involving gravity can be understood in different ways. Consider for example an observer at rest on earth looking at a freely falling person (e.g. a diver who just jumped from a diving board). The observer at rest argues that the earth's gravitational force, which he can feel himself, also acts on the freely falling person and accelerates him. The person in free fall, on the other hand, does not feel gravity. He can take the point of view that he himself is at rest, whereas the earth is accelerated toward him. He concludes that there are no gravitational fields, and that the observer on earth merely feels the force of inertia corresponding to his acceleration. Einstein postulated that these two points of view should be equivalent descriptions of the physical process. More generally, it depends on the observer whether one has a gravitational force or an inertial force. In other words,

equivalence principle: no physical experiment can distinguish between gravitational and inertial forces.

In mathematical language, observers correspond to coordinate systems, and so the equivalence principle states that the physical equations should be formulated in general (i.e. "curvilinear") coordinate systems, and should in all these coordinate systems have the same mathematical structure. This means that the physical equations should be invariant under diffeomorphisms, and thus spacetime is to be modeled by a *Lorentzian manifold* $(\mathscr{M}, g)$.

A Lorentzian manifold is "locally Minkowski space" in the sense that at every spacetime point $p \in \mathscr{M}$, the corresponding *tangent space* $T_p\mathscr{M}$ is a vector space endowed with a scalar product $\langle .,. \rangle_p$ of signature $(+ - - -)$. Therefore, we can distinguish between spacelike, timelike and null tangent vectors. Defining a non-spacelike curve $q(\tau)$ by the condition that its tangent vector $u(\tau) := \frac{\mathrm{d}}{\mathrm{d}\tau} q(\tau) \in T_{q(\tau)}\mathscr{M}$ be everywhere non-spacelike, our definition of the light cone and the notion of causality given in Section 1.2 in Minkowski space immediately carry over to a Lorentzian manifold. In a coordinate chart, the scalar product $\langle .,. \rangle_p$ can be represented in the form (1.2.1) where g_{jk} are the

components of the so-called *metric tensor* (and ξ^j are η^k are the components of two tangent vectors at p, again in this coordinate chart). In contrast to Minkowski space, the metric tensor is not a constant matrix but depends on the spacetime point, $g_{jk} = g_{jk}(p)$. Its ten components can be regarded as the relativistic analogue of Newton's gravitational potential. For every $p \in \mathscr{M}$ there are coordinate systems in which the metric tensor coincides with the Minkowski metric up to second order in a Taylor expansion about the point p,

$$g_{jk}(p) = \text{diag}(1, -1, -1, -1) \,, \qquad \partial_j g_{kl}(p) = 0 \,. \tag{4.1.1}$$

Such *Gaussian normal coordinates* correspond to the reference frame of a "freely falling observer" who feels no gravitational forces. However, it is in general impossible to arrange that also $\partial_{jk} g_{lm}(p) = 0$. This means that by going into a suitable reference frame, the gravitational field can be transformed away locally (=in one point), but not globally. With this in mind, a reference frame corresponding to Gaussian normal coordinates is also called a *local inertial frame*.

Physical equations can be carried over to a Lorentzian manifold by the prescription that they should in a local inertial frame have the same form as in Minkowski space; this is referred to as the *strong equivalence principle*. In practice, it amounts to replacing all partial derivatives by the corresponding *covariant derivatives* ∇ of the Levi-Civita connection of the Lorentzian manifold; we write symbolically (basics on the covariant derivative can be found for example in [**29, 117, 112**])

$$\textit{strong equivalence principle}: \quad \partial \;\longrightarrow\; \nabla \,.$$

The gravitational field is described via the curvature of spacetime. More precisely, the Riemannian *curvature tensor* is defined by the relations

$$R^i_{jkl}\, u^l = \nabla_j \nabla_k u^i - \nabla_k \nabla_j u^i \,. \tag{4.1.2}$$

Contracting indices, one obtains the *Ricci tensor* $R_{jk} = R^i_{jik}$ and *scalar curvature* $R = R^j_j$. The relativistic generalization of Newton's gravitational law are the Einstein equations

$$R_{jk} - \frac{1}{2}\, R\, g_{jk} = 8\pi\kappa\, T_{jk} \,,$$

where κ is the gravitational constant. Here the *energy-momentum tensor* T_{jk} describes the distribution of matter and energy in spacetime.

4.2. The Dirac Equation in Curved Spacetime

Dirac spinors on a manifold are often formulated using frame bundles, either an orthonormal frame [**9, 90**] or a Newman-Penrose null frame [**127, 20**]. We here outline an equivalent formulation of spinors in curved spacetime in the framework of a $\mathrm{U}(2,2)$ gauge theory (for details see [**38**]). We begin with constructions in local coordinates, whereas global issues like topological obstructions to the existence of spin structures will be discussed in Section 4.4 below. We let $\mathscr{M}$ be a 4-dimensional manifold (without Lorentz metric) and define the *spinor bundle* $S\mathscr{M}$ as a vector bundle over $\mathscr{M}$ with fiber $\mathbb{C}^4$. The fibers are endowed with an inner product $\prec .|. \succ$ of signature $(2,2)$, referred to as the *spin inner product*. Sections in the spinor bundle are called *spinors* or wave functions. Choosing local coordinates on $\mathscr{M}$ and local bases $\{e_\alpha\}_{\alpha=1,\ldots,4}$ of the spin spaces, a wave function is represented by a 4-component complex function in spacetime, usually denoted by $\psi(x)$. Choosing at every spacetime point a pseudo-orthonormal basis $(e_\alpha)_{\alpha=1,\ldots,4}$ in the fibers,

$$\prec e_\alpha | e_\beta \succ = s_\alpha\, \delta_{\alpha\beta} \,, \qquad s_1 = s_2 = 1,\; s_3 = s_4 = -1 \,. \tag{4.2.1}$$

the spin inner product takes the form (1.3.7). Clearly, such a frame $\{e_\alpha\}$ is not unique, as one can always perform a transformation according to

$$e_\alpha \;\longrightarrow\; (U^{-1})^\beta_\alpha\, e_\beta\,,$$

where U is an isometry of the spin inner product, $U \in \mathrm{U}(2,2)$. This basis transformation may depend on the spacetime point. Under this basis transformation the spinors behave as follows,

$$\psi^\alpha(x) \;\longrightarrow\; U^\alpha_\beta(x)\, \psi^\beta(x)\,. \tag{4.2.2}$$

In order to simplify the notation in the following computations, we omit the tensor indices and write this equation with a matrix multiplication,

$$\psi(x) \;\longrightarrow\; U(x)\, \psi(x) \tag{4.2.3}$$

(here we implicitly identify the spinor with its concrete realization in a spinor basis). In view of the analogy to gauge theories, we interpret the transformation law (4.2.3) of the wave functions as a local gauge transformation with gauge group $\mathcal{G} = \mathrm{U}(2,2)$. We refer to a choice of the spinor basis (e_α) as a *gauge*.

Our next goal is to formulate classical Dirac theory in such a way that the above $\mathrm{U}(2,2)$ gauge transformations correspond to a physical symmetry, the $\mathrm{U}(2,2)$ *gauge symmetry*. To this end, we shall consider a Dirac-type operator as the basic object on $\mathscr{M}$, from which we will then deduce the Lorentz metric and the gauge potentials. First we define a *differential operator of first order* as a linear operator $\mathcal{D}$ on wave functions which in any local chart and gauge takes the form

$$\mathcal{D} = \mathrm{i} G^j(x) \frac{\partial}{\partial x^j} + B(x) \tag{4.2.4}$$

with suitable (4×4)-matrices $G^j(x)$ and $B(x)$. This definition does not depend on a choice of coordinates and gauge, although the specific form of the matrices $G^j(x)$ and $B(x)$ clearly does. More precisely, if we change to other coordinates $\tilde{x}^i$ while keeping the gauge fixed, a short computation shows that the operator (4.2.4) transforms to

$$\mathrm{i} \left(G^k(x)\, \frac{\partial \tilde{x}^j}{\partial x^k} \right) \frac{\partial}{\partial \tilde{x}^j} + B(\tilde{x})\,. \tag{4.2.5}$$

On the other hand, if we perform a gauge transformation $\psi \to U\psi$ but keep the coordinates x^i fixed, the Dirac operator transforms according to

$$\mathcal{D}\psi \;\longrightarrow\; U\mathcal{D}\psi = U\mathcal{D}U^{-1}(U\psi)\,,$$

and a straightforward computation gives

$$U\mathcal{D}U^{-1} = \mathrm{i}\left(U G^j U^{-1}\right) \frac{\partial}{\partial x^j} + \left(U B U^{-1} + \mathrm{i} U G^j (\partial_j U^{-1})\right). \tag{4.2.6}$$

We now define an operator of Dirac type by the requirement that by choosing suitable coordinates and gauge, one can arrange that the coefficient matrices G^j of the partial derivatives "coincide locally" with the Dirac matrices of Minkowski space.

Definition 4.2.1. *A differential operator $\mathcal{D}$ of first order on a spinor bundle is called* **operator of Dirac type** *(or Dirac-type operator) if for every $p \in \mathscr{M}$ there is a chart (x^i, U) around p and a gauge $(e_\alpha)_{\alpha=1,\ldots,4}$ on U such that $\mathcal{D}$ is of the form* (4.2.4) *with*

$$G^j(p) = \gamma^j\,, \tag{4.2.7}$$

where the γ^j are the Dirac matrices of Minkowski space in the Dirac representation (1.3.3).

It may seem unconventional that we defined Dirac-type operators without having a covariant derivative on the spinor bundle yet. An advantage of our approach is that the condition (4.2.7) is natural and can be understood physically as an extension of the equivalence principle which incorporates gauge transformations (not just coordinate transformations). As we shall see, the above Dirac operator includes both gravitational fields and electromagnetic potentials, giving a unified description of electrodynamics and general relativity as a gauge theory. More concretely, from a Dirac-type operator we shall construct a gauge-covariant derivative D, also referred to as the corresponding *spin derivative* or *gauge-covariant derivative*. In preparation, we write the transformation law (4.2.2) in the shorter form

$$\psi(x) \;\longrightarrow\; U(x)\, \psi(x) \tag{4.2.8}$$

with $U \in \mathrm{U}(2,2)$. Clearly, partial derivatives of ψ do not have a nice behavior under gauge transformations because we pick up derivatives of U. This problem disappears if instead of partial derivatives we consider *gauge-covariant derivatives*

$$D_j = \partial_j - \mathrm{i} A_j \,, \tag{4.2.9}$$

where the gauge potential A_j is pointwise linear operator on spinor (i.e. a (4×4)-matrix valued function if represented in a gauge and coordinate system). Provided that these gauge potentials A_j transform under a gauge transformation U as

$$A_j \;\longrightarrow\; U A_j U^{-1} + \mathrm{i} U\, (\partial_j U^{-1}) \,, \tag{4.2.10}$$

a short calculation shows that then the gauge-covariant derivative behaves under gauge transformations according to

$$D_j \;\longrightarrow\; U\, D_j\, U^{-1} \,, \tag{4.2.11}$$

and thus the gauge-covariant derivatives of ψ obey the simple transformation rule

$$D_j \psi \;\longrightarrow\; U\, D_j \psi \,.$$

Thus our task is to find a way to construct matrices A_j which transform under local gauge transformations according to (4.2.10). This construction will also reveal the structure of the matrix B, and this will finally lead us to the definition of the Dirac operator, which involves precisely the gravitational and electromagnetic fields.

Before we come to this construction, we first explain how a Dirac type operator induces a Lorentzian metric. In a chart and gauge where (4.2.7) holds in a point p, it is obvious from (1.3.1) that the anti-commutator of the matrices $G^j(p)$ gives the Minkowski metric. Transforming from this chart to a general coordinate system and gauge using the transformation rules (4.2.5) and (4.2.6), one sees that in a general coordinate system and gauge, the anti-commutator of these matrices defines a Lorentzian metric,

$$g^{jk}(x)\, \mathbb{1} = \frac{1}{2}\, \{G^j(x),\, G^k(x)\} \,. \tag{4.2.12}$$

Moreover, using that the Dirac matrices in Minkowski space are symmetric with respect to the spin inner product in Minkowski space (see (1.3.8)), one sees (again from the transformation laws (4.2.5) and (4.2.6)) that the same is true for the matrices G^j,

$$\prec G^l \psi \,|\, \phi \succ \;=\; \prec \psi \,|\, G^l \phi \succ \qquad \text{for all } \psi, \phi \,. \tag{4.2.13}$$

Thus we see that, via (4.2.12), a Dirac-type operator induces on the manifold a Lorentzian structure. We refer to the matrices G^j as the Dirac matrices in curved spacetime. Since we can arrange by a choice of coordinate system and gauge that these matrices coincide at any given spacetime point with the Dirac matrices of Minkowski space, all the familiar

relations involving Dirac matrices generalize in an obvious way to curved space time. In particular, the pseudo-scalar operator $\Gamma(x)$ can be introduced in curved spacetime by writing (1.3.21) in a coordinate and gauge-covariant form as

$$\Gamma(x) = \frac{1}{4!}\, \varepsilon_{jklm}\, G^j(x)\, G^k(x)\, G^l(x)\, G^m(x)\,, \tag{4.2.14}$$

where the anti-symmetric tensor $\varepsilon_{jklm} = \sqrt{|\det g|}\, \epsilon_{jklm}$ differs from the anti-symmetric symbol ϵ_{jklm} by the volume density. The pseudo-scalar operator gives us again the notion of even and odd matrices and of chirality (1.3.22). Furthermore, we introduce the *bilinear matrices* σ^{jk} by

$$\sigma^{jk}(x) = \frac{\mathrm{i}}{2}\, \big[G^j(x),\, G^k(x)\big]\,.$$

As in Minkowski space, at any spacetime point the matrices

$$G^j\,, \quad \Gamma G^j\,, \quad \mathbb{1}\,, \quad \mathrm{i}\Gamma\,, \quad \sigma^{jk} \tag{4.2.15}$$

form a basis of the 16-dimensional (real) vector space of symmetric matrices (symmetric with respect to the spin inner product $\prec .|. \succ$). The matrices G^j and ΓG^j are odd, whereas $\mathbb{1}$, $\mathrm{i}\Gamma$ and σ^{jk} are even.

We are now ready for the construction of the spin connection from a Dirac type operator. The Lorentzian metric (4.2.12) induces the Levi-Civita connection ∇ on the tangent bundle, which further induces the Levi-Civita covariant derivative on arbitrary tensor fields (for basics on the Levi-Civita connection see again [**29, 117**]). Taking covariant derivatives of the Dirac matrices by the formula $\nabla_k G^j = \partial_k G^j + \Gamma^j_{kl}\, G^l$, where Γ^j_{kl} are the Christoffel symbols of the Levi-Civita connection, we obtain an expression which behaves under coordinate transformations like a tensor. However, it is not gauge covariant, because a gauge transformation (4.2.8) yields contributions involving first derivatives of U. More precisely, according to (4.2.6),

$$\begin{aligned} \nabla_k G^j \;\longrightarrow\; \nabla_k (U G^j U^{-1}) &= U(\nabla_k G^j) U^{-1} + (\partial_k U) G^j U^{-1} + U G^j (\partial_k U^{-1}) \\ &= U(\nabla_k G^j) U^{-1} - \big[U(\partial_k U^{-1}), U G^j U^{-1}\big] \end{aligned} \tag{4.2.16}$$

(in the last line we used the relation $\partial_j U = -U(\partial_j U^{-1})U$; for details see Exercise 4.2).

We can use the second summand in (4.2.16) to partially fix the gauge.

LEMMA 4.2.2. *For every spacetime point $p \in \mathscr{M}$ there is a gauge such that*

$$\nabla_k G^j(p) = 0 \qquad \textit{for all indices } j, k\,. \tag{4.2.17}$$

PROOF. We start with an arbitrary gauge and construct the desired gauge with two subsequent gauge transformations:

(i) The matrix $\partial_j \Gamma$ is odd, because

$$0 = \partial_j \mathbb{1} = \partial_j (\Gamma\Gamma) = (\partial_j \Gamma)\Gamma + \Gamma(\partial_j \Gamma)\,.$$

As a consequence, the matrix $i\Gamma(\partial_j \Gamma)$ is symmetric (with respect to the spin inner product). Note that for any symmetric matrix A the family of operators $U(q) = \exp(-iA(q-p))$ is unitary and $U(p) = 1$, $\partial_j U(p) = -iA$. Therefore, we can perform a gauge transformation U with $U(p) = \mathbb{1}$ and $\partial_j U(p) = \frac{1}{2}\Gamma(\partial_j \Gamma)$. In the new gauge the matrix $\partial_j \Gamma(p)$ vanishes, as one checks by the computation

$$\partial_j \Gamma_{|p} \;\longrightarrow\; \partial_j (U\Gamma U^{-1})_{|p} = \partial_j \Gamma_{|p} + \frac{1}{2}\, [\Gamma(\partial_j \Gamma),\, \Gamma]_{|p} = \partial_j \Gamma_{|p} - \Gamma^2 (\partial_j \Gamma)_{|p} = 0\,.$$

Differentiating the relation $\{\Gamma, G^j\} = 0$, one sees that the matrix $\nabla_k G^j_{|p}$ is odd (in the sense that it anti-commutes with Γ, exactly as defined in Minkowski space after (1.3.23)). We can thus represent it in the form

$$\nabla_k G^j_{|p} = \Lambda^j_{km}\, G^m_{|p} + \Theta^j_{km}\, \Gamma_{|p} G^m_{|p} \tag{4.2.18}$$

with suitable coefficients Λ^j_{km} and Θ^j_{km}.

This representation can be further simplified: According to Ricci's Lemma, $\nabla_n g^{jk} = 0$. Expressing the metric via the anti-commutation relation (4.2.12) and using the Leibniz rule, we get

$$\begin{aligned} 0 &= \{\nabla_n G^j,\, G^k\} + \{G^j,\, \nabla_n G^k\} \\ &= 2\Lambda^j_{nm}\, g^{mk} - \Theta^j_{nm}\, 2\mathrm{i}\Gamma\sigma^{mk} + 2\Lambda^k_{nm}\, g^{mj} - \Theta^k_{nm}\, 2\mathrm{i}\Gamma\sigma^{mj} \end{aligned} \tag{4.2.19}$$

and thus

$$\Lambda^j_{nm}\, g^{mk}_{|p} = -\Lambda^k_{nm}\, g^{mj}_{|p}\,. \tag{4.2.20}$$

Further, in the case $j = k \neq m$, the relation (4.2.19) yields that $\Theta^j_{nm} = 0$. For $j \neq k$, we obtain $\Theta^j_{nj}\, \sigma^{jk} + \Theta^k_{nk}\, \sigma^{kj} = 0$ and thus $\Theta^j_{nj} = \Theta^k_{nk}$ (j and k denote fixed indices, no summation is performed). We conclude that there are coefficients Θ_n with

$$\Theta^j_{nk} = \Theta_n\, \delta^j_k\,. \tag{4.2.21}$$

(ii) We perform a gauge transformation U with $U(p) = \mathbb{1}$ and

$$\partial_k U = -\frac{1}{2}\, \Theta_k\, \Gamma - \frac{\mathrm{i}}{4}\, \Lambda^m_{kn}\, g^{nl}\, \sigma_{ml} \tag{4.2.22}$$

(the existence of such a gauge transformation follows exactly as explained at the beginning of (i), using that the matrix on the right side of (4.2.22) is anti-symmetric with respect to the spin inner product). Using the representation (4.2.18) together with (4.2.20) and (4.2.21), the matrix $\nabla_k G^j$ transforms into

$$\begin{aligned} \nabla_k G^j \longrightarrow & \nabla_k G^j + [\partial_k U,\, G^j] \\ &= \Lambda^j_{km}\, G^m + \Theta_k\, \Gamma G^j - \Theta_k\, \Gamma G^j - \frac{\mathrm{i}}{4}\, \Lambda^m_{kn}\, g^{nl}\, [\sigma_{ml},\, G^j] \\ &= \Lambda^j_{km}\, G^m - \frac{\mathrm{i}}{4}\, \Lambda^m_{kn}\, g^{nl}\, 2\mathrm{i}\, (G_m\, \delta^j_l - G_l\, \delta^j_m) \\ &= \Lambda^j_{km}\, G^m + \frac{1}{2}\, \Lambda^m_{kn}\, g^{nj}\, G_m - \frac{1}{2}\, \Lambda^j_{km}\, G^m = 0\,. \end{aligned}$$

□

We call a gauge satisfying condition (4.2.17) a *normal gauge* around p. In order to analyze the remaining gauge freedom, we let U be a transformation between two normal gauges. Then according to (4.2.16) and (4.2.17), the commutator $[U(\partial_k U^{-1}),\, U G^j U^{-1}]$ vanishes at p or, equivalently,

$$[\mathrm{i}(\partial_k U^{-1})\, U,\, G^j]_{|p} = 0\,.$$

As is easily verified in the basis (4.2.15) using the anti-commutation relations, a matrix which commutes with all Dirac matrices is a multiple of the identity matrix. Moreover, the matrix $\mathrm{i}(\partial_j U^{-1})\, U$ is symmetric because

$$(\mathrm{i}(\partial_j U^{-1})\, U)^* = -\mathrm{i} U^{-1}\, (\partial_j U) = -\mathrm{i}\partial_j (U^{-1} U) + \mathrm{i}(\partial_j U^{-1})\, U = \mathrm{i}(\partial_j U^{-1})\, U\,.$$

We conclude that the matrix $\mathrm{i}(\partial_j U^{-1})\, U$ is a real multiple of the identity matrix. Transforming it unitarily with U, we see that it also coincides with the matrix $\mathrm{i} U(\partial_j U^{-1})$.

Restricting attention to normal gauges, it is easy to find expressions with the required behavior (4.2.10) under gauge transformations. Namely, setting

$$a_j = \frac{1}{4}\, \mathrm{Re}\, \mathrm{Tr}\left(G_j\, B\right) \mathbb{1}\,, \tag{4.2.23}$$

where "Tr" denotes the trace of a 4×4-matrix, one sees from (4.2.6) that

$$a_j \;\longrightarrow\; a_j + \frac{1}{4}\, \mathrm{Re}\;\, \mathrm{Tr}\left(G_j G^k\, i(\partial_k U^{-1})\, U\right) \mathbb{1} = a_j + \mathrm{i} U(\partial_j U^{-1})\,.$$

We can identify the a_j with the gauge potentials A_j and use (4.2.9) as the definition of the spin connection.

Definition 4.2.3. *The* **spin derivative** D *is defined by the condition that it behaves under gauge transformations* (4.2.8) *according to* (4.2.11)*, and that in normal gauges around* p *it has the form*

$$D_j(p) = \frac{\partial}{\partial x^j} - \mathrm{i} a_j \tag{4.2.24}$$

with the potentials a_j *according to* (4.2.23)*.*

In general gauges, the spin derivative can be written as

$$D_j = \frac{\partial}{\partial x^j} - \mathrm{i} E_j - \mathrm{i} a_j \tag{4.2.25}$$

with additional matrices $E_j(x)$, which involve the Dirac matrices and their first derivatives. The components of E_j are sometimes referred to as *spin coefficients*. The spin coefficients can be regarded in analogy to the Christoffel symbols of the Levi-Civita connection, but now they act on spinors, not on vector fields. A short calculation shows that the trace of the matrix E_j does not change under gauge transformations, and since it vanishes in normal gauges, we conclude that the matrices E_j are trace-free. It is straightforward to verify that they are given explicitly by (for details see Appendix A)

$$E_j = \frac{\mathrm{i}}{2}\, \Gamma\, (\partial_j \Gamma) - \frac{\mathrm{i}}{16}\; \mathrm{Tr}(G^m\, \nabla_j G^n)\, G_m G_n + \frac{\mathrm{i}}{8}\; \mathrm{Tr}(\Gamma G_j\, \nabla_m G^m)\, \Gamma\,. \tag{4.2.26}$$

In the next two theorems we collect the basic properties of the spin connection.

Theorem 4.2.4. *The spin derivative satisfies for all wave functions* ψ, ϕ *the equations*

$$\left[D_k, G^j\right] + \Gamma^j_{kl}\, G^l = 0 \tag{4.2.27}$$

$$\partial_j \prec \psi | \phi \succ = \prec D_j \psi \,|\, \phi \succ + \prec \psi \,|\, D_j \phi \succ\,. \tag{4.2.28}$$

Proof. The left side of (4.2.27) behaves under gauge transformations according to the adjoint representation $. \to U\, .\, U^{-1}$ of the gauge group. Therefore, it suffices to check (4.2.27) in a normal gauge, where

$$[D_k, G^j] + \Gamma^j_{kl}\, G^l = \nabla_k G^j - \frac{\mathrm{i}}{4}\; \mathrm{Re}\, \mathrm{Tr}\left(G_j B\right) [\mathbb{1}, G^j] = 0\,.$$

Since both sides of (4.2.28) are gauge invariant, it again suffices to consider a normal gauge. The statement is then an immediate consequence of the Leibniz rule for partial derivatives and the fact that the spin derivative differs from the partial derivative by an imaginary multiple of the identity matrix (4.2.24). □

The identity (4.2.27) means that the coordinate and gauge invariant derivative of the Dirac matrices vanishes. The relation (4.2.28) shows that the spin connection is compatible with the spin inner product.

We define the *curvature* $\mathcal{R}$ of the spin connection as the following two-form,

$$\mathcal{R}_{jk} = \frac{\mathrm{i}}{2} [D_j, D_k] .$$

THEOREM 4.2.5. *The spin connection satisfies the relation*

$$[D_j, G_k] = [D_k, G_j] .$$

Moreover, curvature has the form

$$\mathcal{R}_{jk} = \frac{1}{8} R_{mnjk}\, \sigma^{mn} + \frac{1}{2} (\partial_j a_k - \partial_k a_j) ,$$

where R_{mnjk} is the Riemannian curvature tensor and the a_j are given by (4.2.23).

PROOF. The identity (4.2.27) yields

$$\big[D_j, G_k\big] = \big[D_j, g_{kl}\, G^l\big] = (\partial_j g_{kl})\, G^l - g_{kl}\, \Gamma^l_{jm}\, G^m = \Gamma^m_{jk}\, G_m .$$

Thus, using that the Levi-Civita connection is torsion-free, we obtain

$$[D_j, G_k] = [D_k, G_j] = \big(\Gamma^m_{jk} - \Gamma^m_{kj}\big)\, G_m = 0 .$$

Next, again using (4.2.27), we can rewrite the covariant derivative as a spin derivative,

$$G_l\, \nabla_k u^l = \big[D_k, G_l u^l\big] .$$

Iterating this relation, we can express the Riemann tensor (4.1.2) by

$$\begin{aligned} G_i\, R^i_{jkl}\, u^l &= \big[D_j, [D_k, G_l u^l]\big] - \big[D_k, [D_j, G_l u^l]\big] \\ &= \big[[D_j, D_k], G_l u^l\big] = -2i \,\big[\mathcal{R}_{jk}, G_l u^l\big] . \end{aligned}$$

This equation determines curvature up to a multiple of the identity matrix, i.e.

$$\mathcal{R}_{jk}(x) = \frac{1}{8} R_{mnjk}\, \sigma^{mn} + \lambda_{jk} \mathbb{1}$$

with unknown parameters λ_{jk}. These parameters can be determined by computing the trace of curvature. Since the matrices E_j in (4.2.26) and their partial derivatives are trace-free, a direct computation starting from (4.2.25) gives

$$\lambda_{jk} = \frac{1}{4}\, \mathrm{Tr}(\mathcal{R}_{jk})\, \mathbb{1} = \frac{1}{8}\, \mathrm{Tr}\big(\partial_j A_k - \partial_k A_j\big)\, \mathbb{1} = \frac{1}{2}\, \big(\partial_j a_k - \partial_k a_j\big) ,$$

concluding the proof. □

We come to the physical interpretation of the above construction. According to Lemma 4.2.2 we can choose a gauge around p such that the covariant derivatives of the Dirac matrices vanish at p. Moreover, choosing normal coordinates and making a global (=constant) gauge transformation, we can arrange that $G(p) = \gamma^j$ and $\partial_j g_{kl}(p) = 0$. Then the covariant derivatives at p reduce to partial derivatives, and we conclude that

$$G^j(p) = \gamma^j , \qquad \partial_k G^j(p) = 0 . \tag{4.2.29}$$

These equations resemble the conditions for normal coordinates (4.1.1), except that the role of the metric is now played by the Dirac matrices. Indeed, by differentiating (4.2.12) one sees that (4.2.29) implies (4.1.1). Therefore, (4.2.29) is a stronger condition which not only poses a condition for the coordinates, but also for the gauge. We call a coordinate system and gauge where (4.2.29) is satisfied a *normal reference frame* around p.

In a normal reference frame around p, the Dirac matrices, and via (4.2.12) also the metric, are the same as in Minkowski space up to the order $\sim (x-p)^2$. Combining the equivalence principle with the usual minimal coupling procedure in physics, it seems a sensible physical assumption that the Dirac equation at p should coincide with that in Minkowski space. This implies that there should be a normal gauge such that all gauge potentials vanish at p, and thus the Dirac operator at p should coincide with the vacuum Dirac operator $i\partial\!\!\!/$. This physical argument makes it possible to specify the zero order term in (4.2.4).

DEFINITION 4.2.6. *A Dirac-type operator $\mathcal{D}$ is called* **Dirac operator** *if for any $p \in \mathscr{M}$ there is a normal reference frame around p such that $B(p) = 0$.*

Equivalently, the Dirac operator could be defined as a differential operator of first order (4.2.4) with the additional structure that for any $p \in \mathscr{M}$ there is a coordinate chart and gauge such that the following three conditions are satisfied,

$$G^j(p) = \gamma^j \,, \qquad \partial_k G^j(p) = 0 \,, \qquad B(p) = 0 \,.$$

This alternative definition has the disadvantage that it is a-priori not clear whether the second condition $\partial_k G^j(p) = 0$ can be satisfied for a general metric. This is the reason why we preferred to begin with only the first condition (Definition 4.2.1), then showed that the second condition can be arranged by choosing suitable coordinates and gauge, and satisfied the third condition at the end (Definition 4.2.6).

In general coordinates and gauge, the Dirac operator can be written as

$$\mathcal{D} = \mathrm{i} G^j D_j = \mathrm{i} G^j \left(\partial_j - \mathrm{i} E_j - \mathrm{i} a_j \right) , \tag{4.2.30}$$

where D is the spin connection of Definition 4.2.3. The matrices E_j take into account the gravitational field and are called *spin coefficients*, whereas the a_j can be identified with the *electromagnetic potential* (compare (1.3.6)). We point out that the gravitational field cannot be introduced into the Dirac equation by the simple replacement rule $\partial \to D$, because gravity has an effect on both the Dirac matrices and the spin coefficients. But factorizing the gauge group as $\mathrm{U}(2,2) = \mathrm{U}(1) \times \mathrm{SU}(2,2)$, the $\mathrm{SU}(2,2)$-gauge transformations are linked to the gravitational field because they influence G^j and E_j, whereas the $\mathrm{U}(1)$ can be identified with the gauge group of electrodynamics. In this sense, we obtain a unified description of electrodynamics and general relativity as a $\mathrm{U}(2,2)$ gauge theory. The Dirac equation

$$(\mathcal{D} - m)\, \psi = 0 \tag{4.2.31}$$

describes a Dirac particle in the presence of a gravitational and electromagnetic field. According to Theorem 4.2.5, the curvature of the spin connection involves both the Riemann tensor and the electromagnetic field tensor. One can express the classical action in terms of these tensor fields, so that the corresponding Euler-Lagrange equations give rise to the classical Einstein-Dirac-Maxwell equations.

For the probabilistic interpretation of the Dirac equation in curved spacetime, we choose a spacelike hypersurface $\mathcal{N}$ (corresponding to "space" for an observer) and consider in generalization of (1.3.12) on solutions of the Dirac equation the scalar product

$$(\psi|\phi)_{\mathcal{N}} = \int_{\mathcal{N}} \prec \psi \,|\, G^j \nu_j \, \phi \succ \, \mathrm{d}\mu_{\mathcal{N}} \,, \tag{4.2.32}$$

where ν is the future-directed normal on $\mathcal{N}$ and $d\mu_{\mathcal{N}}$ is the invariant measure on the Riemannian manifold $\mathcal{N}$ (in order to ensure that the integral exists, one can for example restrict attention to solutions of spatially compact support, as will be introduced after

Theorem 4.5.2 below). Then $(\psi|\psi)_{\mathcal{N}}$ is the *normalization integral*, which we again normalize to one. Its integrand has the interpretation as the *probability density*. In analogy to (1.3.9) the *Dirac current* is introduced by $J^k = \prec\psi \,|\, G^k\psi\succ$. Using Theorem 4.2.4 one sees similar as in Minkowski space that the Dirac current is divergence-free, $\nabla_k J^k = 0$. From the Gauß divergence theorem one obtains that the scalar product (4.2.32) does not depend on the choice of the hypersurface $\mathcal{N}$ (this follows similar as explained in Minkowski space after (1.3.11)).

In analogy to (1.3.16), we can introduce the *spacetime inner product*

$$<\psi|\phi> := \int_{\mathcal{M}} \prec\psi|\phi\succ_x \, \mathrm{d}\mu_{\mathcal{M}} \,. \tag{4.2.33}$$

in which the wave functions (which need not satisfy the Dirac equation but must have a suitable decay at infinity) are integrated over all of spacetime. We finally remark that, using Theorem 4.2.4 together with Gauß' divergence theorem, one easily verifies that the Dirac operator is symmetric with respect to this inner product.

4.3. Computation of the Dirac Operator

We now explain how the Dirac operator can be computed in an efficient way in a given spacetime. Thus suppose that the Lorentzian metric g_{ij} is given in a chosen chart. The general procedure is to first choose matrices $G^j(x)$ which are symmetric with respect to the spin inner product (4.2.13) (where in our formulation, the spin inner product is always given by (4.2.1)) and which satisfy the anti-commutation relations (4.2.12). Then the spin coefficients as given by (4.2.26) are obtained by a straightforward computation. Then the spin derivative is given by (4.2.25) (where a_j are the components of the electromagnetic potential; they are set to zero if no electromagnetic field is present). The Dirac operator is given by (4.2.30), i.e.

$$\mathcal{D} = \mathrm{i}G^j D_j = \mathrm{i}G^j\partial_j + G^j E_j + G^j a_j \,. \tag{4.3.1}$$

In this construction, one has a lot of freedom to choose the Dirac matrices $G^j(x)$ (as described systematically by the $\mathrm{U}(2,2)$-gauge transformations (4.2.8) and (4.2.11)). It is a promising strategy to use this gauge freedom such as to choose Dirac matrices for which the formulas for the spin coefficients (4.2.26) become as simple as possible. Moreover, one should keep in mind that for the computation of the Dirac operator, one does not need to know all the matrices E_j, but it suffices to compute the combination $G^j E_j$ in (4.3.1). Indeed, in many spacetimes of physical interest, making use of the gauge freedom, the combination $G^j E_j$ can be computed easily (for details see the computations in black hole geometries in [**85, 66**] or various examples in [**60**, Section 9]). We here illustrate this method by the example of a diagonal metric, in which case it is even unnecessary to compute the Christoffel symbols:

Proposition 4.3.1. *Assume that there is a local chart (x^i) in which the metric is diagonal, i.e.*

$$ds^2 = \sum_{i=0}^{3} g_{ii}(x)\, \mathrm{d}x_i^2 \,. \tag{4.3.2}$$

Then there is a gauge in which the Dirac operator (without electromagnetic field) takes the form

$$\mathcal{D} = \mathrm{i}G^j \frac{\partial}{\partial x^j} + B \,, \tag{4.3.3}$$

where

$$G^j(x) = g_{jj}(x)^{-\frac{1}{2}}\, \gamma^j \tag{4.3.4}$$

$$B(x) = \frac{\mathrm{i}}{2\sqrt{|\det g|}}\, \partial_j \left(\sqrt{|\det g|}\, G^j \right). \tag{4.3.5}$$

(here γ^j are again the Dirac matrices in Minkowski space).

PROOF. With (4.3.4) we have satisfied the anti-commutation relations

$$\{G^j, G^k\} = 2\, g^{jk}\, \mathbb{1}\,.$$

Moreover, the choice (4.3.4) ensures that the pseudo-scalar operator is constant, and that all derivatives of the G^j are in the span of $\gamma^0, \ldots, \sigma^k$. Therefore, the formula for the zero order term in the Dirac operator (4.3.1) simplifies to

$$B = -\frac{\mathrm{i}}{16}\, \mathrm{Tr}\left(G_m \left(\nabla_j G_n \right) \right) G^j G^m G^n\,, \tag{4.3.6}$$

where $\nabla_j G_n \equiv \partial_j G_n - \Gamma^k_{jn} G_k$ is the covariant derivative acting on the components of the spinorial matrix. Using the algebra of the Dirac matrices, one finds that (4.3.6) has a vectorial component (obtained by using the anti-commutation relations), and an axial component which is totally antisymmetric in the indices j, m, and n. This totally antisymmetric term vanishes for the following reasons: First, since the Levi-Civita connection is torsion-free, we may replace the covariant derivative by a partial derivative. Second, it follows from (4.3.4) that the matrix $\partial_j G_n$ is a multiple of G_n, implying that the trace $\mathrm{Tr}(G_m(\partial_j G_n))$ is symmetric in the indices m and n.

It remains to compute the vectorial component of (4.3.6). A short computation yields

$$B = \frac{\mathrm{i}}{2}\, \nabla_j G^j\,,$$

and the usual formula for the covariant divergence of a vector field gives the result. □

The result of this proposition is very useful for applications, as is illustrated in Exercise 4.4.

4.4. Formulation with Vector Bundles, the Spinor Bundle

So far, the Dirac operator was introduced in a local chart. We intentionally left a large local gauge freedom, having the advantage that this freedom can be used to simplify the form of the Dirac operator. The remaining question is whether our constructions in local charts can be made global to obtain a Dirac operator $\mathcal{D}$ acting on the sections of the so-called spinor bundle $S\mathscr{M}$. To this end, we shall consider the Dirac operator in different charts and patch the Dirac operators in the overlapping regions.

In preparation, we recall the structures introduced so far, using a more abstract notation which clarifies the dependence on gauge and coordinates. In our local construction at the beginning of Section 4.2, the spinor space at a point $x \in \mathscr{M}$ is simply $\mathbb{C}^4$ with the inner product (4.2.1). Using the same notation as in Section 1.4 in Minkowski space, we now denote the spinor space by $(S_x\mathscr{M}, \prec.|.\succ_x)$. Moreover, we denote the linear operators on $S_x\mathscr{M}$ which are symmetric with respect to the spin inner product by $\mathrm{Symm}(S_x\mathscr{M})$. It is a 16-dimensional real vector space spanned by the operators in (4.2.15). Given a Dirac-type operator $\mathcal{D}$, the Dirac matrices $G^j(x)$ span a four-dimensional subspace K_x of $\mathrm{Symm}(S_x\mathscr{M})$,

$$K_x := \mathrm{span}\big\{G^0(x), \ldots, G^3(x)\big\} \subset \mathrm{Symm}(S_x\mathscr{M})\,,$$

referred to as a *Clifford subspace* at x. Contracting a tangent vector u with the Dirac matrices gives rise to a mapping

$$\gamma : T_x\mathscr{M} \to K_x , \qquad u \mapsto u^j G_j .$$

Multiplying a spinor by $\gamma(u)$ is referred to as *Clifford multiplication*. The anti-commutation relations (4.2.12) can be written as

$$\frac{1}{2} \{\gamma(u), \gamma(v)\} = g_x(u,v)\, \mathbb{1}_{S_x\mathscr{M}} ,$$

showing that Clifford multiplication encodes the Lorentzian metric.

In view of the transformation law (4.2.5), the Clifford subspace does not depend on the choice of coordinates. But it clearly depends on the gauge. Indeed, in view of (4.2.6), it transforms according to

$$K_x \to U\, K_x\, U^{-1} \qquad \text{with } U \in \mathrm{U}(S_x) . \tag{4.4.1}$$

In order to simplify our problem, it is a good idea to arrange by a gauge transformation that the Clifford subspace agrees at every spacetime point with the standard Clifford subspace:

LEMMA 4.4.1. *By a gauge transformation* (4.4.1) *we can arrange that*

$$K_x = \mathrm{span}\{\gamma^0, \ldots, \gamma^3\}$$

(where γ^j are again the Dirac matrices in the Dirac representation).

PROOF. We consider the pseudo-scalar operator $\Gamma(x)$ as defined by (4.2.14). Working in a coordinate system and gauge where the Dirac matrices coincide at x with the usual Dirac matrices in Minkowski space, one sees immediately that the pseudo-scalar matrix satisfies also in curved spacetime the relations

$$\Gamma(x)^* = -\Gamma(x) \qquad \text{and} \qquad \Gamma(x)^2 = \mathbb{1} .$$

The first relation implies that $\Gamma(x)$ maps positive definite spinors to negative definite spinors and vice versa. Therefore, there is a pseudo-orthonormal basis of the spinor space in which $\Gamma(x)$ takes the same form as in Minkowski space,

$$\Gamma(x) = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{4.4.2}$$

Rewriting the change of basis as a gauge transformation, we have arranged by a transformation of the form (4.4.1) that the pseudo-scalar operator has the same form as in Minkowski space.

It follows from (4.2.14) and the anti-commutation relations that every vector in K anti-commutes with Γ. Therefore,

$$K \subset \mathrm{span}\big\{\gamma^0, \ldots, \gamma^3, \Gamma\gamma^0, \ldots, \Gamma\gamma^3\big\} .$$

We next show that the vector space $K \cap \mathrm{span}\{\gamma^0, \Gamma\gamma^0\}$ is one-dimensional. To this end, let $u, v \in T_x\mathscr{M}$ with $\gamma(u) = (a+b\Gamma)\,\gamma^0$ and $\gamma(v) = (c+d\Gamma)\,\gamma^0$ with real coefficients a, b, c, d. Then their anti-commutator is computed by

$$\{\gamma(u), \gamma(v)\} = 2\big(ac - bd\big)\, \mathbb{1} + 2\big(bc - ad\big)\, \Gamma ,$$

implying that $bc - ad = 0$. This implies that $\gamma(u)$ and $\gamma(v)$ are linearly dependent, giving the claim.

Repeating the last argument for $\gamma^1, \ldots, \gamma^3$, we conclude that there is a (not necessarily pseudo-orthonormal) basis $u_0, \ldots, u_3$ of $T_x\mathscr{M}$ such that

$$\gamma(u_j) = (a_j + b_j \Gamma)\, \gamma_j \,.$$

Then for any $j \neq k$,

$$\{\gamma(u_j), \gamma(u_k)\} = \big(b_j a_k - a_j b_k\big)\, \Gamma \left[\gamma_j, \gamma_k\right] ,$$

implying that the four vectors $(a_j, b_j) \in \mathbb{R}^2$ with $j = 0, \ldots, 3$ are all linearly independent. Therefore, by rescaling the basis vectors u_j we can arrange that

$$\gamma(u_j) = (a + b\Gamma)\, \gamma_j$$

for real parameters a and b.

The signature $(1,3)$ of the Lorentzian metric implies that $|a| > |b|$. Moreover, by flipping the sign of the vectors u_j if necessary we can arrange that $a > 0$. Therefore, we may represent K as

$$K = \text{span}\big\{(\mathrm{e}^{\alpha\Gamma}\, \gamma^0, \ldots \mathrm{e}^{\alpha\Gamma}\, \gamma^3\big\}$$

for some $\alpha \in \mathbb{R}$ (note that $\mathrm{e}^{\alpha\Gamma} = \cosh\alpha + \Gamma \sinh\alpha$). Performing the gauge transformation (4.4.1) with U according to

$$U = \exp\Big(-\frac{\alpha}{2}\, \Gamma \Big)$$

gives the result. □

After these preparations, we are ready to enter the patching construction. Thus let (x, U) and $(\tilde{x}, \tilde{U})$ be two local charts on $(\mathscr{M}, g)$ with non-empty overlap $U \cap \tilde{U}$. For technical simplicity, we always restrict attention to the case that spacetime is *time-oriented* in the sense that the transition maps between any two charts preserve the time direction. Moreover, for simplicity we restrict attention to spacetimes which are *oriented* in the sense that all transition maps also preserve the orientation (i.e., in more technical terms, we assume that the determinant of the Jacobian of every transition map is everywhere positive). We choose the charts such that x^0 and $\tilde{x}^0$ are time functions which increase to the future. Then we can write the Dirac operator in each chart according to (4.2.30), where for clarity we denote the objects in the chart $\tilde{x}$ with an additional tilde. We first consider the case without electromagnetic field where the potentials a_j vanish. According to Lemma 4.4.1, there is no loss of generality to restrict attention to gauges where the Dirac matrices are linear combinations of the Dirac matrices in Minkowski space, i.e.

$$G^j(x) = h^j_k(x)\, \gamma^k \qquad \text{and} \qquad \tilde{G}^j(\tilde{x}) = \tilde{h}^j_k(\tilde{x})\, \gamma^k \,.$$

Since x^0 is a time coordinate, the bilinear form $\prec . | G^0(x) . \succ_x$ is definite at very spacetime point x, and similar for the tilde coordinates. We choose the signs of the Dirac matrices such that the bilinear forms $\prec . | G^0(x) . \succ_x$ and $\prec . | \tilde{G}^0(\tilde{x}) . \succ_y$ are all *positive* definite. Moreover, as explained in Lemma 4.4.1, we always choose the gauge such that the pseudo-scalar operator (4.2.14) has the same form as in Minkowski space (4.4.2). Using that spacetime is oriented, this can be done consistently for all charts (meaning that if (4.4.2) holds in one chart and gauge, then it also hold in all other charts for the same gauge).

The transformation from the chart (x, U) to $(\tilde{x}, \tilde{U})$ involves the coordinate transformation as described by (4.2.5). After this transformation, the Dirac matrices

$$\tilde{G}^j(\tilde{x}) \qquad \text{and} \qquad G^k(x)\, \frac{\partial \tilde{x}^j}{\partial x^k}$$

will in general not coincide. But since the matrices are all formed as linear combinations of the Dirac matrices in Minkowski space, satisfy the same anti-commutation relations, and have the same time and spatial orientations, they can be obtained from each other by an orthochronous and proper Lorentz transformation, i.e.

$$\tilde{G}^j(\tilde{x}) = \Lambda^j_l\, G^k(x)\, \frac{\partial \tilde{x}^l}{\partial x^k}\,.$$

Now we can proceed just as in the proof of Lorentz invariance of the Dirac equation in Minkowski space (see Lemma 1.3.1) to conclude that there is a unitary transformation $U(x) \in \mathrm{U}(S_x)$ of the form

$$U := \exp\left(\frac{1}{4}\,\lambda_{lk}\,\gamma^l\,\gamma^k\right) \tag{4.4.3}$$

(with an anti-symmetric tensor λ_{lk}) such that the Dirac matrices agree after the gauge transformation, i.e.

$$U(\tilde{x})\,\tilde{G}^j(\tilde{x})\,U(\tilde{x})^{-1} = G^j(x)\,\frac{\partial \tilde{x}^l}{\partial x^k}\,.$$

Since the spin coefficients E_j in (4.2.25) are given explicitly in terms of the Dirac matrices and their derivatives (see (4.2.26)), the lower order terms in the resulting Dirac operators (4.2.30) also agree. Moreover, using that the only matrices which commute with all Dirac matrices are multiples of the identity, one sees that the gauge transformation $U(\tilde{x})$ of the form (4.4.3) is uniquely determined up to a sign. In this way, to every coordinate transformation, we have found a gauge transformation, unique up to a sign, such that the Dirac operators agree.

With the above construction, we have found a procedure for matching the Dirac operators in two overlapping charts. The involved gauge transformations of the form (4.4.3) are unique up to signs. Therefore, once we have decided on the signs, there is a unique way of identifying the Dirac wave functions in the overlapping region of two charts, such as to obtain Dirac wave functions in a larger patch of the manifold $\mathscr{M}$. Proceeding inductively, one can hope to obtain Dirac wave functions on all of $\mathscr{M}$. The subtle point is whether the signs of the transformations can be chosen in a compatible way for all charts. In more mathematical terms, one must satisfy the so-called cocycle conditions, and it turns out that these conditions can be fulfilled if and only if $\mathscr{M}$ satisfies a topological condition, the so-called *spin condition* (for details see for example [**115**, § II.1 and § II.2]). If the spin condition is satisfied, one can identify the spinor spaces via the mappings which patch the charts together. In this way, one obtains a vector bundle over $\mathscr{M}$, referred to as the *spinor bundle* $S\mathscr{M}$. The fibers of the spinor bundle are the spinor spaces $S_x\mathscr{M}$, which are four-dimensional complex vector spaces endowed with an inner product $\prec .|.\succ_x$ of signature $(2,2)$. The transformations of the form (4.4.3) generate a group, the so-called *spin group* denoted by

$$\mathrm{Spin}^{\uparrow}_x \subset \mathrm{U}(S_x) \tag{4.4.4}$$

(the reason why we write "generated by" is that the operators of the form (4.4.3) do *not* form a group; see Exercise 4.3). Elements of the spin group act on vectors of the Clifford subspace by the adjoint representation,

$$\gamma(v) \to U\,\gamma(v)\,U^{-1}\,,$$

we obtain another vector of the Clifford subspace, i.e.

$$U\,\gamma(v)\,U^{-1} = \gamma\big(\mathcal{O}_U(u)\big)\,.$$

Since the anti-commutation relations remain unchanged, the resulting transformation of the tangent space is an isometry. Indeed, by Lemma 1.3.1 it is a proper orthochronal Lorentz transformation,

$$\mathcal{O}_U \in \mathrm{SO}^\uparrow(T_x\mathscr{M})\,. \tag{4.4.5}$$

The indices $\uparrow$ in (4.4.4) and (4.4.5) indicate that we restrict attention to orthochronous transformations. We thus obtain the usual commutative diagram

$$\mathbb{Z}_2 \longrightarrow \mathrm{Spin}_x^\uparrow \xrightarrow{\mathcal{O}} \mathrm{SO}^\uparrow(T_x\mathscr{M}) \longrightarrow 0\,.$$

The connection to the usual spin group is obtained as follows. We say that a tangent vector $u \in T_x\mathscr{M}$ is a *unit vector* if $\langle u, u\rangle = \pm 1$. The spin group is defined by (see for example [**9, 115**], the concise summary in [**5**, Section 2] or similarly [**90**] in the Riemannian setting)

$$\mathrm{Spin}_x := \big\{\text{group generated by } \gamma(u)\,\gamma(v) \text{ with unit vectors } u, v \in T_x\mathscr{M}\big\}\,. \tag{4.4.6}$$

By expanding the exponential in (4.4.3), one sees that this matrix is generated by even products of Dirac matrices, showing that the group $\mathrm{Spin}_x^\uparrow$ in (4.4.4) is a subgroup of Spin_x. The group Spin_x also includes operators which are not unitary but satisfy instead the relation $U^*U = -\mathbb{1}$. These transformations describe reversals of the time orientation. Working with the general spin group (4.4.6) is of advantage in general dimension or signature. In four-dimensional time-oriented and orientable spacetimes, however, we can just as well restrict attention to orthochronous proper Lorentz transformations and the gauge transformations in (4.4.4).

We finally mention how to treat an electromagnetic field. Then the starting point is a time-oriented Lorentzian spin manifold $(\mathscr{M}, g)$ together with an anti-symmetric two-tensor F (the field tensor). In this situation, after the above coordinate and gauge transformations, the electromagnetic potentials a_j and $\tilde{a}_j$ in the two charts will in general not coincide. But, since the field tensor is prescribed, they coincide after a local $U(1)$-gauge transformation. Identifying the spinor spaces after this gauge transformation defines the Dirac operator as acting on the spinor bundle $S\mathscr{M}$. The resulting effective gauge group is $\mathrm{U}(1) \times \mathrm{Spin}_x^\uparrow$. We point out that this effective gauge group is obtained under the condition that the Clifford subspace is fixed at each spacetime point according to Lemma 4.4.1. Dropping this condition gives rise to the larger local gauge group $\mathrm{U}(2,2)$.

4.5. The Dirac Solution Space in Globally Hyperbolic Spacetimes

We now turn attention to solutions of the Dirac equation. In Minkowski space, a convenient method for constructing solutions is the Fourier transformation (see Section 2.4). However, this method can be used only for PDEs with constant coefficients, and therefore it does not apply to the Dirac equation in curved spacetime. Instead, a general method is to solve the Cauchy problem for given initial data, making it possible to identify the solution space of the Dirac equation with the space of suitable initial data. Clearly, for this method to be applicable, one needs to decompose spacetime into "space" and "time," because otherwise it would not be clear how to prescribe initial data at some initial time. In order to describe the Dirac solutions in all of spacetime by initial data, this splitting of spacetime into space and time must be performed globally. Moreover, there must be distinguished notions of the future and past of a spacelike hypersurface. Intuitively speaking, these requirements are necessary in order to make sense of the initial value problem in which, given initial data on a spacelike hypersurface, one seeks global solutions of the Dirac equation to both the future and the past.

In mathematical terms, the necessary assumptions on spacetime needed for a sensible formulation of the initial value problem, also referred to as the *Cauchy problem*, are subsumed in the notion of **global hyperbolicity**. We first give the formal definition and then explain its consequences. Let $(\mathscr{M}, g)$ be a Lorentzian manifold. We again assume that $\mathscr{M}$ is time-oriented (meaning that all transition maps preserve the time direction). We consider a parametrized piecewise C^1-curve $\gamma(\tau)$ in M which is *regular* in the sense that its tangent vector $\dot{\gamma}(\tau)$ is non-zero for all τ where γ is differentiable. This spacetime curve is said to be *causal* if its tangent vector is causal (i.e. timelike or lightlike) for all τ where γ is differentiable. Moreover, it is *future-directed* and *past-directed* its tangent vectors are future- and past-directed, respectively. The manifold $\mathscr{M}$ is said to satisfy the *strong causality condition* if there are *no almost closed causal curves* in the sense that for all $x \in M$ and for each open neighborhood U of x there is an open neighborhood $V \subset U$ of x such that every causal curve in $\mathscr{M}$ which is starting and ending in V is entirely contained in U. Moreover, in straightforward generalization of the corresponding notions in Minkowski space as introduced after (1.2.2), we let $J_x^\vee$ (and $J_x^\wedge$) be the set of all points $y \in \mathscr{M}$ which can be joined from x by a future-directed (respectively past-directed) causal curve. The manifold $\mathscr{M}$ is said to be *globally hyperbolic* if the strong causality condition holds and if the set $J_x^\vee \cap J_y^\wedge$ is compact for all $x, y \in \mathscr{M}$. For more details on the abstract definitions we refer to [**102**, Section 6.6], [**6**, Section 1.3], [**11**, Section 3.2] or [**125**, Chapter 14]).

A globally hyperbolic Lorentzian manifold $(\mathscr{M}, g)$ has remarkable properties, as we now explain. First, global hyperbolicity implies that $(\mathscr{M}, g)$ is diffeomorphic to a product,

$$\mathscr{M} \simeq \mathbb{R} \times \mathcal{N}\,, \tag{4.5.1}$$

where $\simeq$ means that there is a smooth diffeomorphism from $\mathscr{M}$ to $\mathbb{R} \times \mathcal{N}$ (with $\mathcal{N}$ a three-dimensional manifold). Thus every point $p \in \mathscr{M}$ can be written as $p = (t, x)$ with $t \in \mathbb{R}$ and $x \in \mathcal{N}$. One also refers to the above property that $\mathscr{M}$ admits a *smooth foliation* $\mathscr{M} = (\mathcal{N}_t)_{t \in \mathbb{R}}$, where $\mathcal{N}_t := \{t\} \times \mathcal{N}$. Moreover, the foliation can be chosen such as to have the following properties:

(i) Every surface $\mathcal{N}_t$ is *spacelike* (meaning that the metric induced by g on $\mathcal{N}_t$ is negative definite).

(ii) Every surface $\mathcal{N}_t$ is a *Cauchy surface*, meaning that every inextendible timelike curve in $\mathscr{M}$ intersects $\mathcal{N}_t$ exactly once. Here the timelike curve is said to be *inextendible* if it cannot be extended as a continuous curve.

The function t is also referred to as a *global time function*. These above properties of globally hyperbolic manifolds were proven in [**12**] (for more details and more references see again [**6**, Section 1.3]).

The property of $\mathcal{N}_t$ of being a Cauchy surface implies that the Cauchy problem for the Dirac equation is well-posed, as we now explain. To this end, let $(\mathscr{M}, g)$ be a four-dimensional globally hyperbolic spacetime. Then the topological splitting (4.5.1) implies that the spin condition mentioned before (4.4.4) is satisfied. Therefore, there is a spinor bundle $(S\mathscr{M}, \prec.|.\succ)$, being a vector bundle with fibers $S_x\mathscr{M} \simeq \mathbb{C}^4$ (there may be different spin structures, but we shall not go into this here). Moreover, the Dirac operator $\mathcal{D}$ is well-defined; in local coordinates and local spinor bases it takes the form (4.2.30). In the Cauchy problem, one seeks for solutions of the Dirac equation of mass m for prescribed initial data at time t_0, i.e.

$$(\mathcal{D} - m)\psi = \phi \qquad \text{with} \qquad \psi|_{\mathcal{N}_{t_0}} = \psi_0\,. \tag{4.5.2}$$

Then the following result holds:

THEOREM 4.5.1. *For smooth initial data $\psi_0 \in C^\infty(\mathcal{N}_{t_0}, S\mathcal{M})$ and a smooth inhomogeneity $\phi \in C^\infty(\mathcal{M}, S\mathcal{M})$ the Cauchy problem* (4.5.2) *has a unique global solution $\psi \in C^\infty(\mathcal{M}, S\mathcal{M})$.*

The proof of this theorem uses methods of hyperbolic partial differential equations and will be given in Section 13.6 later in this book.

Having a Cauchy surface is also very useful because we can then define a scalar product on the solution space as the spatial integral (4.2.32), where $\mathcal{N}$ is chosen as a Cauchy surface. However, for the integral in (4.2.32) to be well-defined, working with smooth solutions is not suitable. Instead, similar as explained in Minkowski space in Section 1.4, we better assume that the solution has compact support on the Cauchy surface. Due to finite propagation speed for solutions of hyperbolic partial differential equations (as will be made precise in Section 13.2), the following result holds:

THEOREM 4.5.2. *If the initial data and the inhomogeneity have compact support,*

$$\psi_0 \in C_0^\infty(\mathcal{N}_{t_0}, S\mathcal{M}) \qquad \text{and} \qquad \phi \in C_0^\infty(\mathcal{M}, S\mathcal{M})\,,$$

then the solution ψ of the Cauchy problem (4.5.2) *also has compact support on any other Cauchy surface $\mathcal{N}_t$.*

The proof of this theorem will again be given in Section 13.6 below.

Using the same notion as in Section 1.4, we refer to smooth solutions as in the above theorem as having *spatially compact support*. Smooth and spatially compact sections of the spinor bundle are again denoted by $C^\infty_{\text{sc}}(\mathcal{M}, S\mathcal{M})$. For Dirac solutions in this class, the scalar product (4.2.32) is well-defined. Moreover, due to current conservation, this scalar product does not depend on the choice of the Cauchy surface (as explained after (4.2.32)). Exactly as explained in Section 1.4, taking the completion gives the Hilbert space $(\mathcal{H}_m, (.|.))$ of weak solutions of the Dirac equation with the property that their restriction to any Cauchy surface is square integrable (where "square integrable" is defined via (4.2.32)).

4.6. Hamiltonian Formulation in Stationary Spacetimes

Given a foliation $(\mathcal{N}_t)_{t\in\mathbb{R}}$ of the globally hyperbolic spacetime $(\mathcal{M}, g)$ by Cauchy surfaces, one can rewrite the Dirac equation in the Hamiltonian form

$$\mathrm{i}\partial_t \psi = H\psi$$

with a Hamiltonian H. In order to compute H in a local chart, one chooses a coordinate system (x^i) such that $x^0 = t$ coincides with the time function. Then, writing the Dirac operator in (4.2.31) in the form (4.2.30) and solving for the time derivatives, one obtains in generalization of (1.3.15)

$$H = -\big(G^0\big)^{-1}\left(\sum\nolimits_{\alpha=1}^3 \mathrm{i}G^\alpha\left(\partial_\alpha - \mathrm{i}E_\alpha - \mathrm{i}a_\alpha\right) - m\right) - E_0 - a_0\,.$$

When analyzing the Dirac equation in the Hamiltonian form, one must be be careful because the Hamiltonian in general is *not symmetric* with respect to the Hilbert space scalar product (4.2.32). This can be seen as follows. For the Dirac equation in Minkowski space, the symmetry of the Hamiltonian is obtained just as for the Schrödinger equation by using that the scalar product is conserved in time (1.1.6). In curved spacetime, the scalar product is still conserved (due to current conservation). But when taking the time

derivative, one must take into account that the scalar product itself is time dependent. More precisely, assuming for notational simplicity that the Cauchy surfaces admit global charts,

$$\begin{aligned} 0 = \partial_t(\phi|\psi) &= \frac{\partial}{\partial t} \int_{\mathcal{N}_t} \prec\!\phi \,|\, G^j \nu_j \, \psi\!\succ \, \mathrm{d}\mu_{\mathcal{N}_t} \\ &= (\partial_t \phi \,|\, \psi) + (\phi \,|\, \partial_t \psi) + \int_{\mathbb{R}^3} \prec\!\psi \,|\, \Big(\partial_t \big(G^j \nu_j \sqrt{\det g_{\mathcal{N}_t}} \big) \Big)\, \phi\!\succ \mathrm{d}^3x \\ &= -\mathrm{i} \Big((H\phi|\psi) - (\phi|H\psi) \Big) + \int_{\mathbb{R}^3} \prec\!\psi \,|\, \Big(\partial_t \big(G^j \nu_j \sqrt{\det g_{\mathcal{N}_t}} \big) \Big)\, \phi\!\succ \mathrm{d}^3x \end{aligned} \tag{4.6.1}$$

(where $g_{\mathcal{N}_t}$ denotes the induced Riemannian metric on the Cauchy surface $\mathcal{N}_t$). The integral in the last line is in general non-zero. In this case, the Hamiltonian is obviously not symmetric. At first sight, this might seem surprising because it seems to contradict the axioms of quantum mechanics (for a detailed account on this issue see [**2, 3**]). However, one should keep in mind that the non-symmetric contributions to the Hamiltonian are needed in order to compensate for the fact that the scalar product itself is time-dependent.

Our interpretation of the above problem is that the Hamiltonian formulation of the Dirac equation is useful only in situations when the integral in (4.6.1) vanishes. This can be arranged if the all the coefficients of the metric are time-independent. In other words, spacetime should be *stationary* with corresponding Killing field given by ∂_t. Under these assumptions, the Hamiltonian H is also time-independent. Moreover, the computation (4.6.1) shows that the operator is symmetric. Using that the time evolution maps smooth and compactly supported initial data on the Cauchy surface at time t_0 to a smooth and compactly supported solution at an arbitrary time t, one can use abstract methods to construct a selfadjoint extension of H (see for example [**82**] for a general situation involving additional boundary conditions). Then the Cauchy problem can be solved immediately using the spectral theorem for selfadjoint operators,

$$\psi(t,x) = \big(\mathrm{e}^{-\mathrm{i}tH}\, \psi_0 \big)(x)\,. \tag{4.6.2}$$

This formulation is particularly useful for analyzing the long-time behavior of the solutions (see for example the analysis in the Kerr geometry in [**68, 67**]).

4.7. Exercises

EXERCISE 4.1. Verify by elementary integration by parts in a chart that for a diagonal metric (4.3.2), the Dirac operator (4.3.3) is symmetric with respect to the inner product (4.2.33).

EXERCISE 4.2. Let $U(x)$ with $x \in \mathbb{R}^4$ be a smooth function of invertible matrices. Show that

$$\partial_j U = -U (\partial_j U^{-1}) U\,.$$

Hint: Differentiate the relation $U(x)\, U(x)^{-1} = \mathbb{1}$.

EXERCISE 4.3. The goal of this exercise is to show that the unitary operators of the form (4.4.3) do not form a group (in more mathematical language, the spin group is not exponential; for details see [**30**] and the references therein). We proceed in several steps:

(a) Let λ_{jk} be an anti-symmetric tensor. Show using the anti-commutation relations that

$$\left(\frac{1}{4}\, \lambda_{lk}\, \gamma^l\, \gamma^k \right)^2 = -\frac{1}{16}\, \lambda_{lk}\, \lambda^{lk}\, \mathbb{1} + \frac{\mathrm{i}}{16 \cdot 4!}\, \Gamma\, \varepsilon^{ijkl}\, \lambda_{ij} \lambda_{kl}\,. \tag{4.7.1}$$

(b) Deduce from (a) that the corresponding unitary transformation (4.4.3) is a linear combination of the matrices

$$\mathbb{1}\,, \qquad \Gamma\,, \qquad \lambda_{lk}\,\gamma^l\,\gamma^k \qquad \text{and} \qquad \lambda_{lk}\,\Gamma\,\gamma^l\,\gamma^k\,.$$

(c) Show under the additional assumption $\varepsilon^{ijkl}\,\lambda_{ij}\lambda_{kl} = 0$ that

$$\exp\left(\frac{1}{4}\,\lambda_{lk}\,\gamma^l\gamma^k\right) = \begin{cases} \mathbb{1}\cos\alpha + \dfrac{1}{4}\,\lambda_{lk}\,\gamma^l\gamma^k\,\dfrac{\sin\alpha}{\alpha} & \text{if } \lambda_{lk}\,\lambda^{lk} > 0 \\ \mathbb{1}\cosh\alpha + \dfrac{1}{4}\,\lambda_{lk}\,\gamma^l\gamma^k\,\dfrac{\sinh\alpha}{\alpha} & \text{if } \lambda_{lk}\,\lambda^{lk} < 0\,, \end{cases} \tag{4.7.2}$$

where $\alpha := \sqrt{|\lambda_{lk}\,\lambda^{lk}|}/4$. *Hint:* Use (4.7.1) in the power series of the exponential.

(d) Choose a specific tensor λ_{lk} for which the matrix in (4.7.2) is equal to minus the identity.

(e) We now restrict attention to tensors λ for which the corresponding unitary transformation (4.4.3) is a linear combination of the matrices $\mathbb{1}$ and $\gamma^0\gamma^1$. Infer from (b) that in this case, there are real numbers α, β such that

$$\frac{1}{4}\,\lambda_{lk}\,\gamma^l\,\gamma^k = (\alpha + \mathrm{i}\Gamma\beta)\,\gamma^0\gamma^1\,.$$

Deduce that

$$\exp\left(\frac{1}{4}\,\lambda_{lk}\,\gamma^l\gamma^k\right) = \cosh\big(\alpha + \mathrm{i}\Gamma\beta\big) + \gamma^0\gamma^1\,\sinh\big(\alpha + \mathrm{i}\Gamma\beta\big)\,.$$

(f) Show that the last expression involves no contribution $\sim \Gamma$ only if either $\alpha = 0$ or $\beta = 0$ (*Hint:* It might be convenient to work in an eigenvector basis of $i\Gamma$). Infer that in the case $\alpha = 0$, this expression is a linear combination of the matrices $\mathbb{1}$ and $\gamma^0\gamma^1$ only if $\beta \in \pi\mathbb{Z}$. Conclude that

$$\exp\left(\frac{1}{4}\,\lambda_{lk}\,\gamma^l\gamma^k\right) = a\,\mathbb{1} + b\,\gamma^0\gamma^1 \qquad \text{with } a > 0\,.$$

(g) Deduce from (c) and (e) that the matrices of the form (4.4.3) do not form a group.

Exercise 4.4. The *Schwarzschild geometry* is the simplest mathematical model of a black hole (for the physical background see for example [**112**]). In so-called Boyer-Lindquist coordinates $t \in \mathbb{R}$ (the time coordinate), $r \in (2M, \infty)$ (the radial coordinate) and $\vartheta \in (0, \pi)$, $\varphi \in (0, 2\pi)$ (the angular coordinates), the line element takes the form

$$\mathrm{d}s^2 = g_{jk}\;\mathrm{d}x^j\;\mathrm{d}x^k = \Big(1 - \frac{2M}{r}\Big)\;\mathrm{d}t^2 - \Big(1 - \frac{2M}{r}\Big)^{-1}\;\mathrm{d}r^2 - r^2\;\mathrm{d}\vartheta^2 - r^2\,\sin^2\vartheta\;\mathrm{d}\varphi^2\,,$$

where $M > 0$ is the mass of the black hole. Compute the Dirac operator in this geometry. *Hint:* Use the formulas in Proposition 4.3.1. The results can be found in the literature for example in [**85**] or [**83**, Section 2.2].

Part 2

Causal Fermion Systems: Fundamental Structures

CHAPTER 5

A Brief Introduction to Causal Fermion Systems

In this chapter we define and explain the basic objects and structures of a causal fermion system. Since causal fermion systems introduce a new language to describe our physical world, we begin with preliminary considerations which explain how the basic objects of the theory come about and how to think about them. In order to provide different perspectives, the preliminary considerations motivate causal fermion systems in two somewhat different ways. In Section 5.1 the motivating question is whether and how spacetime structures can be encoded in quantum mechanical wave functions. In Section 5.2, on the other hand, we begin with the example of a two-dimensional lattice system and ask the question how one can formulate physical equations in this discrete spacetime without making use of specific lattice structures like the nearest neighbor relations and the lattice spacing. By extending the setting from the motivating examples (Section 5.3) we are led to the general definition of a causal fermion system (Section 5.4). Next, as a further example, we explain how the Minkowski vacuum can be described by a causal fermion system (Section 5.5). In order to formulate equations describing the dynamics of a causal fermion system, we introduce a variational principle, the so-called causal action principle (Section 5.6). We proceed by explaining how to obtain a spacetime as well as structures therein (Section 5.7). We conclude by discussing the form of the causal action principle (Section 5.8) and by explaining the underlying physical concepts (Section 5.9).

5.1. Motivation: Encoding Spacetime Structures in Wave Functions

For the introductory considerations, following [**58**, Section 2.1.1] we begin with a quantum particle described by a quantum mechanical wave function ψ satisfying the Klein-Gordon equation (1.2.5) in Minkowski space or in a curved spacetime. Suppose that we have access only to the information contained in the absolute square $|\psi(x)|^2$ of this wave function. We ask the question: Given this information, what can we infer on the structure of spacetime? First, let the wave function ψ be a solution evolved from compactly supported initial data ψ_0 as illustrated in Figure 5.1. Then finite speed of propagation guarantees that the absolute square $|\psi(x)|^2$ vanishes outside the causal future of the support of the initial data. In this way, the support of $|\psi(x)|^2$ gives us some information on the causal structure of our spacetime. But, of course, there is only a limited amount of information which can be extracted from a single wave function. However, if instead we probe with many wave functions, as illustrated in Figure 5.2, we gain more information. If we aggregate the information contained in all wave functions evolved from compactly supported initial data, then we can extract the complete causal structure of our spacetime. We remark that this determines the metric up to a conformal factor [**103, 119**].

We next consider the situation if an electromagnetic background field is present. The coupling of the scalar field to the electromagnetic field is described by the Klein-Gordon equation (1.2.6). Now the wave functions are deflected by the electromagnetic force.

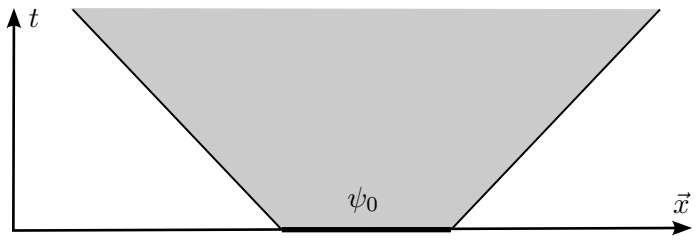


FIGURE 5.1. Causal propagation of a wave function.

Therefore, their absolute square also encodes information on the electromagnetic field. In order to retrieve this information, one can use the following procedure. Suppose that we have access to two wave functions ψ and ϕ and that we can also measure the absolute value of superpositions, i.e.

$$\big|\alpha\psi(x) + \beta\phi(x)\big|^2 = \big|\alpha\psi(x)\big|^2 + 2\,\mathrm{Re}\left(\overline{\alpha}\,\beta\,\overline{\psi(x)}\,\phi(x)\right) + \big|\beta\phi(x)\big|^2$$

for arbitrary complex coefficients α and β. By varying these coefficients, we can determine the quantity

$$\overline{\psi(x)}\phi(x)\,,$$

which tells us about the correlation of the two wave function ψ and ϕ at the spacetime point x. This allows us to probe the electromagnetic field, as shown schematically in Figure 5.3. Here we do not need to be specific on what "probing" exactly means (for example, one could determine deflection angles, recover the Aharanov-Bohm phase shifts of the wave function, etc.). All that counts is that we can get information also on the electromagnetic field. Generally speaking, the more wave functions we have to our disposal, the more information on the electromagnetic field can be retrieved. It seems sensible to expect that, after suitably increasing the number of wave functions, we can recover both the spacetime structures and the matter fields therein from the knowledge of the absolute squares of all these wave functions alone.

Now we go one step further and formulate the idea of encoding spacetime structures in a family of wave functions in mathematical terms. To this end, we consider a (for simplicity finite) number f of linearly independent wave functions $\psi_1, \ldots, \psi_f : \mathscr{M} \to \mathbb{C}$, mapping from a classical spacetime $\mathscr{M}$ to the complex numbers. On the complex vector space $\mathcal{H}$ spanned by these wave functions we introduce a scalar product $\langle .|. \rangle_{\mathcal{H}}$ by demanding that the wave functions $\psi_1, \ldots, \psi_f$ are orthonormal, i.e.

$$\langle \psi_k | \psi_l \rangle_{\mathcal{H}} = \delta_{kl}\,.$$

We thus obtain an f-dimensional Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. At any spacetime point $x \in \mathscr{M}$ we can now introduce the *local correlation operator* $F(x) : \mathcal{H} \to \mathcal{H}$ as the linear operator whose matrix representation in the basis $\psi_1, \ldots, \psi_f$ is given by

$$(F(x))^j{}_k = \overline{\psi_j(x)}\psi_k(x)\,. \tag{5.1.1}$$

The diagonal entries of this matrix are the absolute squares of the wave functions, whereas the off-diagonal entries tell us about the correlation of two different wave functions at the spacetime point x. This is why we refer to $F(x)$ as the local correlation operator. Alternatively, the local correlation operator can be characterized in a basis-invariant form by the identity

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} = \overline{\psi(x)}\phi(x) \qquad \text{for all } \psi, \phi \in \mathcal{H}\,.$$

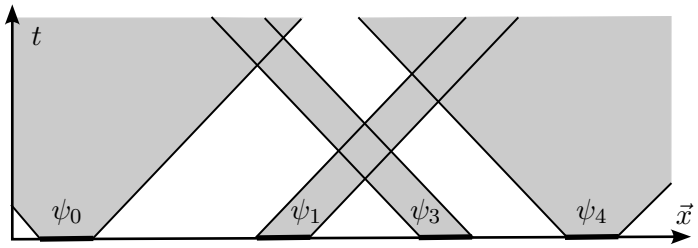


FIGURE 5.2. Probing with many wave functions.

By construction, the operator $F(x)$ is positive semi-definite and has rank at most one (in order not to distract from the main construction, this will be explained in more detail after (5.2.2) in Section 5.2 below). By varying the point x, we obtain a map $F : \mathscr{M} \to \mathcal{F}$ from the classical spacetime $\mathscr{M}$ to the set $\mathcal{F}$ of positive semi-definite linear operators of rank at most one,

$$\mathcal{F} := \big\{ y \in \mathrm{L}(\mathcal{H}) \,\big|\, y \text{ positive semi-definite of rank at most one} \big\} .$$

This map encodes all the physical information[1] contained in the wave functions of $\mathcal{H}$.

Next, we need to formalize the idea that we want to restrict attention to the information encoded in the wave functions. This entails that we want to disregard all the information contained in the usual structures of Minkowski space or a curved spacetime (like the causal structure, the metric, the spinor bundle, and all that). In order to do so mathematically, we focus on the family of all local correlation operators. Thus, instead of considering F as a mapping from our classical spacetime to $\mathcal{F}$, we restrict attention to its image $M := F(\mathcal{F})$ as a subset of $\mathcal{F}$,

$$M \subset \mathcal{F} . \tag{5.1.2}$$

In this way, Minkowski space and the corresponding classical spacetime structures no longer enter our description. Instead, spacetime and all structures therein are encoded in and must be recovered from the information contained in the family of wave functions. This point of view of recovering all spacetime structures from the wave functions will be taken seriously in this book, and we will unravel its consequences step by step.

It turns out that working as in (5.1.2) merely with a subset of $\mathcal{F}$ is not quite sufficient. In order to get into the position to formulate physical equations, we need one more structure: a measure ρ on spacetime. Here by a "measure on spacetime" we mean a mapping which to a subset $\Omega \subset M$ associates a non-negative number, which can be thought of as the "volume" of the spacetime region corresponding to Ω. In non-technical terms, this measure can be obtained by combining the volume measure in Minkowski space with the map F. More precisely, we take the pre-image $F^{-1}(\Omega) \subset \mathcal{F}$ and integrate over it,

$$\rho(\Omega) := \int_{F^{-1}(\Omega)} \mathrm{d}\mu \,,$$

where $d\mu = \mathrm{d}^4x$ is the volume measure on Minkowski space $\mathscr{M}$ (and similarly $\mathrm{d}\mu = \sqrt{|\det g|}\; \mathrm{d}^4x$ in curved spacetime). In more mathematical terms, the measure ρ is the

[1]Here by "physical" we mean the information up to local gauge phases, which drop out in (5.1.1). Local gauge freedom and local gauge transformations will be discussed in Section 5.9.

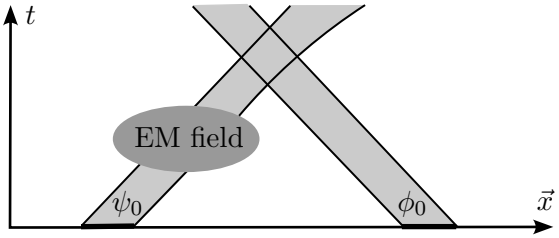


FIGURE 5.3. Probing an electromagnetic field.

push-forward of μ under F (for basics on measure theory and the push-forward measure see Section 2.3).

This construction leads us to consider a measure ρ on a set of linear operators on a Hilbert space as the basic structure describing a physical system in spacetime. These are indeed all the basic ingredients to define a causal fermion system. The only modification to be made later is that, instead of complex wave functions, we will work with sections of a spinor bundle. One consequence of that is that the local correlation operators will no longer be positive semi-definite. Instead, they will be of finite rank with a fixed upper bound on the number of positive and negative eigenvalues.

Before coming to these generalizations (Section 5.3), we next explain why encoding information in the wave functions also has benefits if one wants to formulate physical equations in a setting which goes beyond a classical continuous spacetime.

5.2. Motivating Example: Formulating Equations in Discrete Spacetimes

It is generally believed that for distances as small as the Planck length, spacetime can no longer be described by Minkowski space or a Lorentzian manifold, but that it should have a different, possibly discrete structure. There are different approaches to model such spacetimes. The simplest approach is to replace Minkowski space by a discrete lattice. Indeed, causal fermion systems provide another, more general approach. In any such approach one faces the challenge of how to formulate physical equations if one gives up the continuous structure of spacetime and thus can no longer work with partial differential equations like the Klein-Gordon equation or the Dirac equation.

In order to explain the underlying problem more concretely, we now have a closer look at the simple example of a spacetime lattice (this example was first given in [**48**, Section 1]). For simplicity, we consider a two-dimensional lattice (one space and one time dimension), but higher-dimensional lattices could be described similarly. Thus let $\mathscr{M} \subset \mathbb{R}^{1,1}$ be a rectangular lattice in two-dimensional Minkowski space. We denote the spacing in time direction by Δt and in spatial direction by Δx (see Figure 5.4). The usual procedure for setting up equations on a lattice is to replace derivatives by difference quotients, giving rise to an evolution equation which can be solved time step by time step according to deterministic rules. A simple example is the discretization of the two-dimensional wave equation for a function $\phi : \mathscr{M} \to \mathbb{C}$ on the lattice,

$$\begin{aligned} 0 = \Box\phi(t,x) &:= \frac{1}{(\Delta t)^2}\Big(\phi(t+\Delta t, x) - 2\phi(t,x) + \phi(t-\Delta t, x)\Big) \\ &\quad - \frac{1}{(\Delta x)^2}\Big(\phi(t, x+\Delta x) - 2\phi(t,x) + \phi(t, x-\Delta x)\Big)\,. \end{aligned}$$

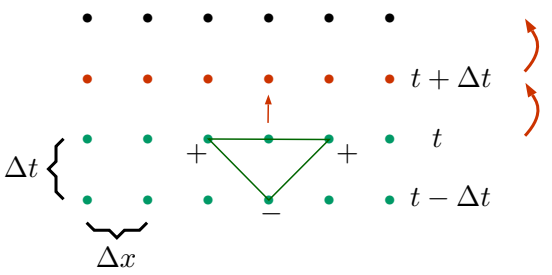


FIGURE 5.4. Time evolution of a lattice system $\mathscr{M} \subset \mathbb{R}^{1,1}$.

Solving this equation for $\phi(t+\Delta t, x)$ gives a deterministic rule for computing $\phi(t+\Delta t, x)$ from the values of ϕ at earlier times t and $t - \Delta t$ (see again Figure 5.4).

While this method for setting up equations in a discrete spacetime is very simple and yields well-defined evolution equations, it also has several drawbacks:

- The above method of discretizing the continuum equations is very *ad hoc*. Why do we choose a regular lattice, why do we work with difference quotients? There are many other ways of discretizing the wave equation.
- The method is *not background-free*. In order to speak of the "lattice spacing," the lattice must be thought of as being embedded in a two-dimensional ambient spacetime.
- The concept of a spacetime lattice is not invariant under general coordinate transformations. In other words, the assumption of a spacetime lattice is *not compatible with the equivalence principle*.

In view of these shortcomings, the following basic question arises:

> Can one formulate equations without referring to the nearest neighbor relation and the lattice spacing?

The answer to this question is yes, and we will now see how this can be done in the example of our two-dimensional lattice system. Although our example is somewhat oversimplified, this consideration will lead us quite naturally to the setting of causal fermion systems.

In order to formulate the equations, we consider on our lattice a family of complex-valued wave functions $\psi_1, \ldots, \psi_f : \mathscr{M} \to \mathbb{C}$ (for simplicity a finite number, i.e. $f < \infty$). At this stage, these wave functions do not need to satisfy any wave equation. On the complex vector space $\mathcal{H}$ spanned by these wave functions we introduce a scalar product $\langle .|. \rangle_{\mathcal{H}}$ by demanding that the wave functions $\psi_1, \ldots, \psi_f$ are orthonormal, i.e.

$$\langle \psi_k | \psi_l \rangle_{\mathcal{H}} = \delta_{kl} \,.$$

We thus obtain an f-dimensional Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. Note that the scalar product is given abstractly (meaning that it has no representation in terms of the wave functions as a sum over lattice points). Next, for any lattice point $(t, x) \in \mathscr{M}$ we introduce the so-called *local correlation operator* $F(t, x) : \mathcal{H} \to \mathcal{H}$ as the linear operator whose matrix representation in the basis $\psi_1, \ldots, \psi_f$ is given by

$$(F(t,x))^j_k = \overline{\psi_j(t,x)} \psi_k(t,x) \,. \tag{5.2.1}$$

The diagonal elements of this matrix are the absolute squares $|\psi_k(t,x)|^2$ of the corresponding wave functions. The off-diagonal elements, on the other hand, tell us about the correlation of the j^{th} and k^{th} wave function at the lattice point (t, x). This is the reason

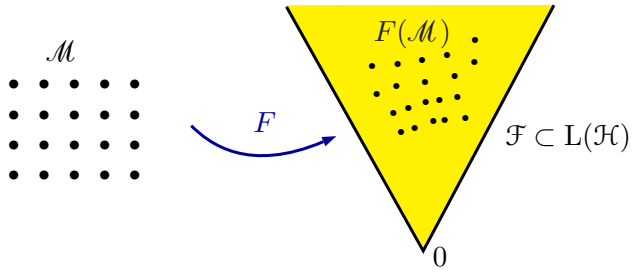


FIGURE 5.5. Embedding in $\mathcal{F}$.

for the name "local correlation operator." This operator can also be characterized in a basis-invariant way by the relations

$$\langle \psi \,|\, F(t,x)\, \phi \rangle_{\mathcal{H}} = \overline{\psi(t,x)} \phi(t,x)\,, \tag{5.2.2}$$

to be satisfied for all $\psi, \phi \in \mathcal{H}$.

We now analyze some properties of the local correlation operators. Taking the complex conjugate, one sees immediately that the matrix defined by (5.2.1) is Hermitian. Stated equivalently independent of bases, the local correlation operator is a *symmetric* linear operator on $\mathcal{H}$ (see Definition 2.2.5 in the preliminaries). Moreover, a local correlation operator has *rank at most one* and is *positive semi-definite*. This can be seen in detail by expressing it in terms of the operator

$$e(t,x) : \mathcal{H} \to \mathbb{C}\,, \qquad \psi \mapsto \psi(t,x)\,,$$

which to every vector associates the corresponding wave function evaluated at the spacetime point (t,x) (this mapping is sometimes referred to as the *evaluation map*). Indeed, rewriting the right side of (5.2.2) as

$$\overline{\psi(t,x)} \phi(t,x) = \overline{\big(e(t,x)\psi\big)} \big(e(t,x)\phi\big) = \langle \psi \,|\, e(t,x)^*\, e(t,x)\, \phi \rangle_{\mathcal{H}}\,,$$

(where $e(t,x)^* : \mathbb{C} \to \mathcal{H}$ is the adjoint of the operator $e(t,x)$ as defined by (2.2.5)), we can compare with the left side of (5.2.2) to conclude that

$$F(t,x) = e(t,x)^*\, e(t,x)\,. \tag{5.2.3}$$

This shows that $F(t,x)$ is positive semi-definite. Moreover, being a mapping to $\mathcal{H}$, the operator $e(t,x)$ has rank at most one. As a consequence, also $F(t,x)$ has rank at most one.

It is useful to denote the set of all operators with the above properties by

$$\begin{aligned} \mathcal{F} := \big\{ F \in \mathrm{L}(\mathcal{H}) \,\big|\, & F \text{ is symmetric,} \\ & \text{positive semi-definite and has rank at most one} \big\}\,. \end{aligned} \tag{5.2.4}$$

Varying the lattice point, we obtain a mapping (see Figure 5.5)

$$F : \mathscr{M} \to \mathcal{F}\,, \qquad (t,x) \mapsto F(t,x)\,.$$

For clarity, we note that the set $\mathcal{F}$ is *not* a vector space, because a linear combination of operators in $\mathcal{F}$ in general has rank greater than one. But it is a *conical* set in the sense that a positive multiple of any operator in $\mathcal{F}$ is again in $\mathcal{F}$ (this is why in Figure 5.5 the set $\mathcal{F}$ is depicted as a cone).

We point out that the local correlation operators do not involve the lattice spacing or the nearest neighbor relation (as a matter of fact, we did not even use that $\mathscr{M}$ is

a lattice); instead they contain information only on the local correlations of the wave functions at each lattice point. With this in mind, our strategy for formulating equations which do not involve the specific structures of the lattice is to work exclusively with the local correlation operators, i.e. with the subset $F(\mathscr{M}) \subset \mathcal{F}$. In other words, in Figure 5.5 we want to disregard the lattice on the left and work exclusively with the objects on the right.

How can one set up equations purely in terms of the local correlation operators? In order to explain the general procedure, we consider a finite number of operators $F_1, \ldots, F_L \in \mathcal{F}$. Each of these operators can be thought of as encoding information on the local correlations of the wave functions at a corresponding spacetime point. However, this "spacetime point" is no longer a lattice point, because the notions of lattice spacing and nearest lattice point have been dropped. At this stage, spacetime is merely a point set, where each point is an operator on the Hilbert space. In order to obtain a "spacetime" in the usual sense (like Minkowski space, a Lorentzian manifold or a generalization thereof), one needs additional structures and relations between the spacetime points. Such relations can be obtained by multiplying the operators. Indeed, the operator product $F_i F_j$ tells us about correlations of the wave functions at different spacetime points. Taking the trace of this operator product gives a real number. Our method for formulating physical equations is to use the operators F_i and their products to set up a variational principle. This variational formulation has the advantage that symmetries give rise to conservation laws by Noether's theorem (as will be explained in Chapter 9). Therefore, we want to minimize an action $\mathcal{S}$ defined in terms of the operators $F_1, \ldots, F_L$. A simple example is to

$$\text{minimize} \qquad \mathcal{S}(F_1, \ldots, F_L) := \sum_{i,j=1}^{L} \text{Tr}(F_i F_j)^2 \tag{5.2.5}$$

under variations of the points $F_1, \ldots, F_L \in \mathcal{F}$. In order to obtain a mathematically sensible variational principle, one needs to impose certain constraints. Here we do not enter the details, because the present example is a bit too simple (see however Exercise 5.1). Instead, we merely use it as a motivation for the general setting of causal fermion systems, which we now introduce.

5.3. Toward the General Definition of a Causal Fermion System

In order to get from the previous motivating examples to the general setting of causal fermion systems, we extend the above constructions in several steps:

(a) The previous example works similarly in higher dimensions, in particular for a lattice $\mathscr{M} \subset \mathbb{R}^{1,3}$ in four-dimensional Minkowski space. This has no effect on the resulting structure of a finite number of distinguished operators $F_1, \ldots, F_L \in \mathcal{F}$.

(b) Suppose that we consider multi-component wave functions $\psi : \mathscr{M} \to \mathbb{C}^N$. Then, clearly, we cannot directly multiply two such wave functions pointwise as was done on the right side of (5.2.1). However, assuming that we are given an inner product on $\mathbb{C}^N$, which we denote by $\prec .|. \succ$ (in mathematical terms, this inner product is a non-degenerate sesquilinear form; we always use the convention that the wave function in the first argument is complex conjugated), we can adapt the definition of the local correlation operator (5.2.1) to

$$(F(t,x))^j_k = -\prec \psi_j(t,x) | \psi_k(t,x) \succ$$

(the minus sign compared to (5.2.1) merely is a useful convention). The resulting local correlation operator is no longer an operator of rank at most one, but it has rank at most N (as can be seen for example by writing it similar to (5.2.3) in the form $F(t,x) = -e(t,x)^* e(t,x)$ with the evaluation map $e(t,x) : \mathcal{H} \to \mathbb{C}^N, \psi \mapsto \psi(t,x)$). If the inner product $\prec.|.\succ$ on $\mathbb{C}^N$ is positive definite, then the operator $F(t,x)$ is negative semi-definite. However, in the physical applications in mind, this inner product will *not* be positive definite. Indeed, a typical example in mind is that of four-component Dirac spinors. The Lorentz invariant inner product $\overline{\psi}\phi$ on Dirac spinors in Minkowski space (with the usual adjoint spinor $\overline{\psi} := \psi^\dagger \gamma^0$) is indefinite of signature $(2,2)$. In order to describe systems involving leptons and quarks, one must take direct sums of Dirac spinors, giving the signature (n,n) with $n \in 2\mathbb{N}$. With this in mind, we assume more generally that

$$\prec.|.\succ \quad \text{has signature } (n,n) \text{ with } n \in \mathbb{N}\,.$$

Then the resulting local correlation operators are symmetric operators of rank at most $2n$, which (counting multiplicities) have at most n positive and at most n negative eigenvalues.

(c) Finally, it is useful to generalize the setting such as to allow for continuous spacetimes and for spacetimes which may have both continuous and discrete components. In preparation, we note that the sums over the operators $F_1, \ldots, F_L$ in (5.2.5) can be written as integrals,

$$\mathcal{S}(\rho) = \int_{\mathcal{F}} \mathrm{d}\rho(x) \int_{\mathcal{F}} \mathrm{d}\rho(y)\ \mathrm{Tr}(xy)^2\,, \tag{5.3.1}$$

if the measure ρ on $\mathcal{F}$ is chosen as the sum of Dirac measures supported at these operators,

$$\rho = \sum_{i=1}^{L} \delta_{F_i}\,. \tag{5.3.2}$$

Note that, in this formulation, the measure plays a double role: First, it distinguishes the points $F_1, \ldots, F_L$ as those points where the measure is non-zero, as is made mathematically precise by the notion of the *support* of the measure (for details see Definition 2.3.4), i.e.

$$\operatorname{supp}\rho = \{F_1, \ldots, F_L\}\,. \tag{5.3.3}$$

Second, a measure makes it possible to integrate over its support, an operation which for the measure (5.3.2) reduces to the sum over $F_1, \ldots, F_L$.

Now one can extend the setting simply by considering (5.3.1) for more general measures on $\mathcal{F}$ (like for example regular Borel measures). The main advantage of working with measures is that we get into a mathematical framework in which variational principles like (5.2.5) can be studied with powerful analytic methods.

5.4. Basic Definition of a Causal Fermion System

Motivated by the previous considerations we now give the basic definition of a causal fermion system. This definition evolved over several years. Based on preparations in [**41**], the present formulation was first given in [**57**].

DEFINITION 5.4.1. *Given a separable complex Hilbert space $\mathcal{H}$ with scalar product $\langle .|. \rangle_{\mathcal{H}}$ and a parameter $n \in \mathbb{N}$ (the* **spin dimension***), we let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all symmetric operators on $\mathcal{H}$ of finite rank, which (counting multiplicities) have at most n positive*

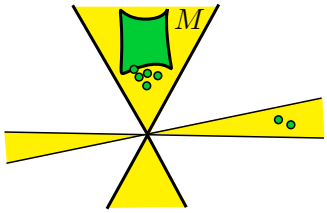


FIGURE 5.6. A causal fermion system.

and at most n negative eigenvalues. Moreover, let ρ be a positive measure on $\mathcal{F}$ (defined on a σ-algebra of subsets of $\mathcal{F}$). We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a **causal fermion system**.

The definition of a causal fermion system is illustrated in Figure 5.6. The set $\mathcal{F}$ is invariant under the transformation where an operator is multiplied by a real number, as is indicated in the figure by the double cones. The support of the measure, denoted by

$$M := \operatorname{supp} \rho \,, \tag{5.4.1}$$

is referred to as *spacetime* (intuitively speaking, the support of a measure consists of all points where the measure is non-zero; for mathematical details see Definition 2.3.4). In contrast to the example of the lattice system, where spacetime consisted of discrete points (5.3.3), in general the measure ρ can also have continuous components. For example, M could be a subset of $\mathcal{F}$ having the additional structure of being a four-dimensional manifold. The space $\mathcal{F}$ should be thought of as a space of very large dimension[2], so that M typically is a low-dimensional subset of $\mathcal{F}$. The measure $\rho(\Omega)$ of a measurable subset $\Omega \subset M$ can be regarded as the volume of the spacetime region Ω. In the example of the lattice system, this volume is simply the number of spacetime points in Ω, whereas for a continuous spacetime it is the four-dimensional Lebesgue measure of Ω. It is a specific feature of a causal fermion system that a spacetime point $x \in M$ is a linear operator on the Hilbert space $\mathcal{H}$. This endows spacetime with a lot of additional structure. In particular, as will be explained in Section 5.7, the spacetime point operators give rise to a family of spinorial wave functions and to causal and geometric structures. The general idea is that a causal fermion system describes a spacetime together with all structures therein. Before entering these structures in more detail, we illustrate the general definition by the simple and concrete example of Dirac wave functions in Minkowski space.

5.5. Example: Dirac Wave Functions in Minkowski Space

As a further example, we now explain how to construct a causal fermion system in Minkowski space. Recall that in Section 1.4 (and similarly in curved spacetime in Section 4.5), for a given parameter $m \in \mathbb{R}$ we introduced the Hilbert space $(\mathcal{H}_m, (.|.))$ of all solutions of the Dirac equation with mass m (recall that the scalar product is defined as the spatial integral (1.3.12)). We now choose a closed subspace $\mathcal{H}$ of this Hilbert space and denote the scalar product $(.|.)$ restricted to this subspace by $\langle .|. \rangle_{\mathcal{H}}$ (changing the notation from round to pointed brackets clarifies that we consider $\langle .|. \rangle_{\mathcal{H}}$ as an abstract

[2]This statement is made precise in [**69, 76**] as follows. The operators of $\mathcal{F}$ of maximal rank $2n$ form a Banach manifold. If the Hilbert space $\mathcal{H}$ is finite-dimensional, then this manifold also has finite dimension given by $4n \dim \mathcal{H} - 4n^2$; see also Proposition 3.1.3 in the Preliminaries.

scalar product, without referring to its representation as a spatial integral (1.3.12)). We thus obtain the

$$\text{Hilbert space} \qquad \big(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}}\big)\,.$$

By construction, the vectors in this Hilbert space are solutions of the Dirac equation. They can be thought of as the "occupied states" of the system. We prefer the notion of *physical wave functions*, where "physical" means intuitively that these wave functions are realized in our physical system (whatever this means; we shall not enter philosophical issues here). The choice of the subspace $\mathcal{H} \subset \mathcal{H}_m$ is part of the input which characterizes the physical system. For example, in order to describe the vacuum, one chooses $\mathcal{H}$ as the subspace of all negative-energy solutions of the Dirac equation (see Section 1.5). In order to model a system involving electrons, however, the subspace $\mathcal{H}$ must be chosen to include the electronic wave functions of positive frequency. At this stage, we do not need to specify $\mathcal{H}$, and in order to clarify the concepts, it seems preferable to keep our considerations on a general abstract level. Specific choices and explicit computations can be found in [**45**, Section 1.2] and in later chapters of this book (Chapters 15-19).

We point out that the functions in $\mathcal{H}$ do not need to be continuous (instead, as mentioned at the end of Section 1.4, their restriction to any Cauchy surface merely is an L^2-function). Therefore, we cannot evaluate the wave functions pointwise at a spacetime point $x \in \mathscr{M}$. However, for the following constructions it is crucial to do so. The way out is to introduce so-called *regularization operators* $(\mathfrak{R}_\varepsilon)$ with $0 < \varepsilon < \varepsilon_{\max}$ as linear operators which map $\mathcal{H}$ to the continuous wave functions,

$$\mathfrak{R}_\varepsilon \;:\; \mathcal{H} \to C^0(\mathscr{M}, S\mathscr{M}) \qquad \text{linear}\,. \tag{5.5.1}$$

In the limit $\varepsilon \searrow 0$, these operators should go over to the identity (in a suitable sense which we do not specify here as it will not be needed). The physical picture is that on a small length scale, which can be thought of as the Planck length scale $\varepsilon \approx 10^{-35}$ meters, the structure of spacetime must be modified. The regularization operators specify this microscopic structure of spacetime. Different choices of regularization operators are possible. A simple example of a regularization operator is obtained by mollifying with a test function. Thus we let $h \in C_0^\infty(\mathscr{M}, \mathbb{R})$ be a non-negative test function with

$$\int_{\mathscr{M}} h(x)\, \mathrm{d}^4x = 1\,.$$

We define the operators $\mathfrak{R}_\varepsilon$ for $\varepsilon > 0$ as the convolution operators (for basics on the convolution see the paragraph after (2.4.15) in Section 2.4)

$$(\mathfrak{R}_\varepsilon u)(x) := \frac{1}{\varepsilon^4} \int_{\mathscr{M}} h\Big(\frac{x-y}{\varepsilon}\Big)\, u(y)\; \mathrm{d}^4y\,.$$

Another method is to work in Fourier space (for preliminaries see Sections 1.5 and 2.4) by setting

$$u(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \hat{u}(k)\, \mathrm{e}^{-\mathrm{i}kx}\,,$$

and to regularize by multiplication with an exponentially decaying cutoff function, i.e.

$$\big(\mathfrak{R}_\varepsilon u\big)(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \hat{u}(k)\, \mathrm{e}^{-\varepsilon\,|\omega|}\, \mathrm{e}^{-\mathrm{i}kx} \qquad \text{with} \qquad \omega = k^0\,. \tag{5.5.2}$$

This so-called *iε-regularization* is most convenient for explicit computations (for more details see [**45**, §2.4.1]). Clearly, these methods of regularizing Dirac solutions are very

special and should be thought of merely as a mathematical tool for constructing simple and explicit examples of causal fermion systems.

Before going on, we briefly remark for the reader familiar with quantum field theory (QFT) how the above regularization is related to the ultraviolet regularization procedures used in relativistic QFT. Both in QFT and the setting of causal fermion systems, regularizations are needed in order to make the theory mathematically well-defined. In the renormalization program in QFT, one shows that the UV regularization can be taken out if other parameters of the theory (like masses and coupling constants) are suitably rescaled. Then the regularization can be understood merely as a computational tool. In the causal fermion systems, however, the physical picture behind the regularization is quite different. Namely, in our setting the *regularized* objects are to be considered as the fundamental physical objects. The regularization models the microscopic structure of spacetime and has therefore a physical significance.

Next, for any $x \in \mathscr{M}$ we consider the bilinear form

$$b_x^\varepsilon \;:\; \mathcal{H} \times \mathcal{H} \to \mathbb{C}\,, \quad b_x^\varepsilon(u,v) = -\prec (\mathfrak{R}_\varepsilon\, u)(x) \,|\, (\mathfrak{R}_\varepsilon\, v)(x) \succ .$$

This bilinear form is well-defined and bounded because $\mathfrak{R}_\varepsilon$ is defined pointwise and because evaluation at x gives a linear operator of finite rank (see Exercise 5.3). Thus for any $v \in \mathcal{H}$, the anti-linear form $b_x^\varepsilon(.,v) : \mathcal{H} \to \mathbb{C}$ is continuous. By the Fréchet-Riesz theorem (Theorem 2.2.4), there is a unique vector $w^\varepsilon \in \mathcal{H}$ such that $b_x^\varepsilon(u,v) = \langle u | w^\varepsilon \rangle_{\mathcal{H}}$ for all $u \in \mathcal{H}$. The mapping $v \mapsto w^\varepsilon$ is linear and bounded. We thus obtain a bounded linear operator $F^\varepsilon(x)$ on $\mathcal{H}$ such that

$$b_x^\varepsilon(u,v) = \langle u \,|\, F^\varepsilon(x)\, v \rangle_{\mathcal{H}} \qquad \text{for all } u, v \in \mathcal{H}\,, \tag{5.5.3}$$

referred to as the *local correlation operator*. Taking into account that the inner product on the Dirac spinors at x has signature $(2,2)$, the local correlation operator $F^\varepsilon(x)$ is a symmetric operator on $\mathcal{H}$ of rank at most four, which has at most two positive and at most two negative eigenvalues.

Varying the point $x \in \mathscr{M}$, for any ε we obtain a mapping

$$F^\varepsilon \;:\; \mathscr{M} \to \mathcal{F}\,, \tag{5.5.4}$$

where $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ is the set of all symmetric operators on $\mathcal{H}$ of finite rank which (counting multiplicities) have at most two positive and at most two negative eigenvalues. We sometimes refer to F^ε as the *local correlation map*. The last step is to drop all other structures (like the metric and causal structures of Minkowski space, the spinorial structures, etc.). As mentioned earlier, the basic concept behind causal fermion systems is to work exclusively with the local correlation operators corresponding to the physical wave functions. In order to formalize this concept, we introduce the measure ρ^ε on $\mathcal{F}$ as the push-forward of the volume measure on $\mathscr{M}$ (for details see Section 2.3 or Exercise 2.18),

$$\rho^\varepsilon := F^\varepsilon_* \mu\,. \tag{5.5.5}$$

We thus obtain a causal fermion system of spin dimension $n = 2$ (see Definition 5.4.1). The local correlation operators are encoded in ρ as the support M of this measure. Working exclusively with the structures of a causal fermion system, we no longer have the usual spacetime structures (particles, fields, causal structure, geometry, ...). The underlying idea is that all these spacetime structures are encoded in the local correlation operators. At this point, it is not obvious that this concept is sensible. But, as we shall see in the later sections in this book, it is indeed possible to reconstruct all spacetime

structures from the local correlation operators. In this sense, the structures of a causal fermion system give a complete description of the physical system.

5.6. The Causal Action Principle

Having given the general definition of a causal fermion system (see Definition 5.4.1), the question arises how physical equations can be formulated in this setting. To this end, we now introduce a variational principle, the so-called *causal action principle*. In this variational principle, we minimize a functional, the so-called *causal action*, under variations of the measure ρ. The minimality property will then impose strong conditions on the possible form of this measure. The mathematical structure of the causal action is similar to the action (5.2.5) given in our example of the lattice system. Its detailed form, however, is the result of many computations and longer considerations, as will be outlined in Section 5.8 below.

For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. However, in general it is no longer a symmetric operator because $(xy)^* = yx$, and this is different from xy unless x and y commute. As a consequence, the eigenvalues of the operator xy are in general complex. We denote these eigenvalues counting algebraic multiplicities by $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$ (more specifically, denoting the rank of xy by $k \leq 2n$, we choose $\lambda_1^{xy}, \ldots, \lambda_k^{xy}$ as all the non-zero eigenvalues and set $\lambda_{k+1}^{xy}, \ldots, \lambda_{2n}^{xy} = 0$). We introduce the Lagrangian and the causal action by

$$\text{causal Lagrangian:} \qquad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \Big(|\lambda_i^{xy}| - |\lambda_j^{xy}| \Big)^2 \tag{5.6.1}$$

$$\text{causal action:} \qquad \mathcal{S}(\rho) = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(x)\, \mathrm{d}\rho(y)\,. \tag{5.6.2}$$

The *causal action principle* is to minimize $\mathcal{S}$ by varying the measure ρ under the following constraints:

$$\text{volume constraint:} \qquad \rho(\mathcal{F}) = \text{const} \tag{5.6.3}$$

$$\text{trace constraint:} \qquad \int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho(x) = \text{const} \tag{5.6.4}$$

$$\text{boundedness constraint:} \qquad \mathcal{T}(\rho) := \iint_{\mathcal{F}\times\mathcal{F}} \Big(\sum_{j=1}^{2n} |\lambda_j^{xy}| \Big)^2 \mathrm{d}\rho(x)\, \mathrm{d}\rho(y) \leq C\,, \tag{5.6.5}$$

where C is a given parameter (and tr denotes the trace of a linear operator on $\mathcal{H}$ of finite rank). As already mentioned, we postpone the physical explanation of the detailed form of the Lagrangian to Section 5.8. The constraints can be understood mathematically as being needed in order to get a well-posed variational principle with non-trivial minimizers. This will be explained in Chapter 12 (see in particular Section 12.4; also the Exercise 5.4 is related).

Before going on, for clarity we point out that the mathematical structure of the causal action principle is quite different from other variational principles considered in physics and mathematics. There does not seem to be a direct way of deriving or even motivating the causal action principle from other known action principles or Lagrangians. The only way to get the connection to the known physical equation is by studying suitable limiting cases of the causal action principle and the corresponding Euler-Lagrange equations (it

will be outlined in Sections 21 and 22 how to get a connection to classical field theory and quantum field theory, respectively).

In order to make the causal action principle mathematically well-defined, one needs to specify the class of measures in which to vary ρ. To this end, on $\mathcal{F}$ we consider the topology induced by the operator norm

$$\|A\| := \sup \big\{ \|Au\|_{\mathcal{H}} \text{ with } \|u\|_{\mathcal{H}} = 1 \big\} \tag{5.6.6}$$

(for basics see the preliminaries in Sections 2.1 and 2.2). In this topology, the Lagrangian as well as the integrands in (5.6.4) and (5.6.5) are continuous. The σ-algebra generated by the open sets of $\mathcal{F}$ consists of the so-called Borel sets. A *regular Borel measure* is a measure on the Borel sets with the property that it is continuous under approximations by compact sets from inside and by open sets from outside (for basics see the preliminaries in Section 2.3). The right prescription is to vary ρ within the class of regular Borel measures on $\mathcal{F}$.

One must distinguish two settings:

(a) The *finite-dimensional setting*: $\dim \mathcal{H} < \infty$ and $\rho(\mathcal{F}) < \infty$.
In this case, we will prove the existence of minimizing measures in Chapter 12. This will also clarify the significance of the constraints (see in particular the examples in Section 12.4).

(b) The *infinite-dimensional setting*: $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) = \infty$.
An obvious complication in this setting is that the volume constraint (5.6.3) is infinite. Likewise, the other constraints as well as the causal action may diverge. These divergences can be avoided by restricting attention to variations which change the measure only on a set of finite volume. By doing so, the differences of the action and the constraints are well-defined and finite (this method will be introduced in Sections 6.3 and 12.8).

With this in mind, the remaining problem is to deal with infinite-dimensional Hilbert spaces. The question whether physics is to be described on the fundamental level by finite- or infinite-dimensional Hilbert spaces seems of more philosophical nature, and we shall not enter this question here. One way of getting along with the finite-dimensional setting is to take the point of view that, on a fundamental physical level, the total volume is finite and the Hilbert space $\mathcal{H}$ is finite-dimensional, whereas the infinite-dimensional setting merely is a mathematical idealization needed in order to describe systems in infinite volume involving an infinite number of quantum particles. Even if this point of view is taken, the infinite-dimensional case is of independent mathematical interest and should also be the appropriate effective description in many physical situations. This case also seems to be mathematically sensible. However, the existence theory has not yet been developed. But at least, it is known that the Euler-Lagrange equations corresponding to the causal action principle still have a mathematical meaning in the infinite-dimensional setting (for details see [**45**]).

We now explain how the spacetime of a causal fermion system is endowed with a topological and causal structure. Recall that, given a minimizing measure ρ, *spacetime* $M \subset \mathcal{F}$ is defined as the support of ρ (see (5.4.1); this is illustrated in Exercise 2.18). Thus the spacetime points are symmetric linear operators on $\mathcal{H}$. On M we consider the topology induced by $\mathcal{F}$ (generated by the sup-norm (5.6.6) on $\mathrm{L}(\mathcal{H})$). Moreover, the measure $\rho|_M$ restricted to M can be regarded as a volume measure on spacetime. This turns spacetime into a *topological measure space*. Furthermore, one has the following notion of causality:

DEFINITION 5.6.1. **(causal structure)** *For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. We denote its non-trivial eigenvalues (counting algebraic multiplicities) by $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$. The points x and y are called* **spacelike** *separated if all the λ_j^{xy} have the same absolute value. They are said to be* **timelike** *separated if the λ_j^{xy} are all real and do not all have the same absolute value. In all other cases (i.e. if the λ_j^{xy} are not all real and do not all have the same absolute value), the points x and y are said to be* **lightlike** *separated.*

Restricting the causal structure of $\mathcal{F}$ to M, we get causal relations in spacetime.

Before going on, we point out that it is not obvious whether and in which sense this definition of causality agrees with the usual notion of causality in Minkowski space (or, more generally, in a Lorentzian spacetime). In order to get the connection, one can consider the causal fermion system constructed in Section 5.5 with the Hilbert space $\mathcal{H} \subset \mathcal{H}_m$ chosen as the subspace of all negative-energy solutions of the Dirac equation (thereby realizing the concept of the Dirac sea as explained in Section 1.5). Then the above "spectral definition" of causality goes over to the causal structure of Minkowski space in the limiting case $\varepsilon \searrow 0$. Since the detailed computations for getting this correspondence are a bit lengthy, we do not present them here but refer the interested reader instead to [**45**, Section 1.2].

The Lagrangian (5.6.1) is compatible with the above notion of causality in the following sense. Suppose that two points $x, y \in \mathcal{F}$ are spacelike separated. Then the eigenvalues λ_i^{xy} all have the same absolute value. As a consequence, the Lagrangian (5.6.1) vanishes. Thus pairs of points with spacelike separation do not enter the action. This can be seen in analogy to the usual notion of causality where points with spacelike separation cannot influence each other. This analogy is the reason for the notion "causal" in "causal fermion system" and "causal action principle."

A causal fermion also system distinguishes a *direction of time*. In order to see this, for $x \in \mathcal{F}$ we let π_x be the orthogonal projection in $\mathcal{H}$ on the subspace $x(\mathcal{H}) \subset \mathcal{H}$ and introduce the functional

$$\mathcal{C} : M \times M \to \mathbb{R}, \qquad \mathcal{C}(x,y) := i \operatorname{tr}\big(y\, x\, \pi_y\, \pi_x - x\, y\, \pi_x\, \pi_y\big). \tag{5.6.7}$$

Obviously, this functional is anti-symmetric in its two arguments, making it possible to introduce the notions

$$\begin{cases} y \text{ lies in the } \textit{future} \text{ of } x & \text{if } \mathcal{C}(x,y) > 0 \\ y \text{ lies in the } \textit{past} \text{ of } x & \text{if } \mathcal{C}(x,y) < 0\,. \end{cases}$$

We remark that the detailed form of the functional (5.6.7) is not obvious; it must be justified by working out that it gives back the time direction of Minkowski space in a suitable limiting case (for details see Exercise 5.9 and [**45**, §1.2.5]).

By distinguishing a direction of time, we get a structure similar to a causal set (see for example [**17**]). However, in contrast to a causal set, our notion of "lies in the future of" is not necessarily transitive.

5.7. Basic Inherent Structures

It is the general concept that a causal fermion system describes spacetime as well as all structures therein (like the causal and metric structures, particles, fields, etc.). Thus all these structures must be constructed from the basic objects of the theory alone, using the information already encoded in the causal fermion system. We refer to these constructed structures as being *inherent* in the causal fermion system. We now introduce

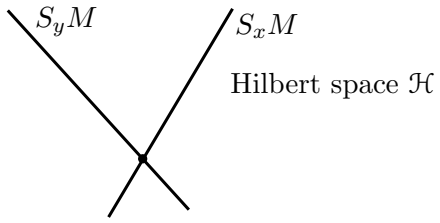


FIGURE 5.7. The spin spaces

and explain the most important of these structures: the *spin spaces*, the *physical wave functions* and the *kernel of the fermionic projector*. Other inherent structures will be introduced later in this book (see Chapters 9–11); for a more complete account we also refer to [**45**, Chapter 1].

The causal action principle depends crucially on the eigenvalues of the operator product xy with $x, y \in \mathcal{F}$. For computing these eigenvalues, it is convenient not to consider this operator product on the (possibly infinite-dimensional) Hilbert space $\mathcal{H}$, but instead to restrict attention to a finite-dimensional subspace of $\mathcal{H}$, chosen such that the operator product vanishes on the orthogonal complement of this subspace. This construction leads us to the spin spaces and to the kernel of the fermionic projector, which we now introduce. For every $x \in \mathcal{F}$ we define the *spin space* S_x as the image of the operator x, i.e.

$$S_x := x(\mathcal{H})\,; \tag{5.7.1}$$

it is a subspace of $\mathcal{H}$ of dimension at most $2n$ (see Figure 5.7). Moreover, we let

$$\pi_x \;:\; \mathcal{H} \to S_x \tag{5.7.2}$$

be the orthogonal projection in $\mathcal{H}$ on the subspace $S_x \subset \mathcal{H}$. For any $x, y \in M$ we define the *kernel of the fermionic projector* $P(x,y)$ by (see Figure 5.8).

$$P(x,y) = \pi_x\, y|_{S_y} \;:\; S_y \to S_x \tag{5.7.3}$$

(where π_x is again the orthogonal projection on the subspace $x(\mathcal{H}) \subset \mathcal{H}$). Taking the trace of (5.7.3) in the case $x = y$, one finds that $\mathrm{tr}(x) = \mathrm{Tr}_{S_x}(P(x,x))$, making it possible to express the integrand of the trace constraint (5.6.4) in terms of the kernel of the fermionic projector. In order to also express the eigenvalues of the operator xy in terms of the kernel of the fermionic projector, we introduce the *closed chain* A_{xy} as the product

$$A_{xy} = P(x,y)\, P(y,x) \;:\; S_x \to S_x\,. \tag{5.7.4}$$

The closed chain can be computed in more detail using the formula (5.7.3). In preparation, we note that, from the definition of π_x as the orthogonal projection to the image of x (5.7.2), it follows immediately that $\pi_x x = x$. Taking the adjoint of this relation, we conclude that

$$\pi_x\, x = x = x\, \pi_x\,. \tag{5.7.5}$$

Using these identities, we can compute the closed chain by

$$A_{xy} = (\pi_x y)(\pi_y x)|_{S_x} = \pi_x\, yx|_{S_x}\,.$$

Applying this equation iteratively and using again (5.7.5), we obtain for the p^{th} power of the closed chain

$$(A_{xy})^p = \pi_x\, (yx)^p|_{S_x}\,.$$

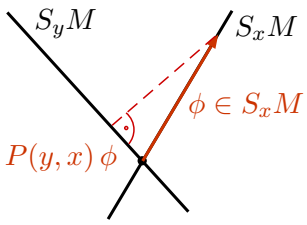


FIGURE 5.8. The kernel of the fermionic projector

Taking the trace, one sees in particular that

$$\mathrm{Tr}_{S_x}(A_{xy}^p) = \mathrm{tr}\big((yx)^p\big) = \mathrm{tr}\big((xy)^p\big) \tag{5.7.6}$$

(where the last identity simply is the invariance of the trace under cyclic permutations). Since all our operators have finite rank, for any $x, y \in \mathcal{F}$ there is a finite-dimensional subspace I of $\mathcal{H}$ such that xy maps I to itself and vanishes on the orthogonal complement of I. For example, one can choose I as the span of the image of xy and the orthogonal complement of the kernel of xy,

$$I = \mathrm{span}\big\{(xy)(\mathcal{H}), \ker(xy)^\perp\big\}\,.$$

Then the non-trivial eigenvalues of the operator product xy are the non-zero roots of the characteristic polynomial of the restriction $xy|_I : I \to I$. The coefficients of this characteristic polynomial (like the trace, the determinant, etc.) are symmetric polynomials in the eigenvalues and can therefore be expressed in terms of traces of powers of the operator $xy|_I : I \to I$ (for details see Exercise 5.10). Using this result similarly for the characteristic polynomial of A_{xy} and using (5.7.6), we conclude that the eigenvalues of the closed chain coincide with the non-trivial eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ of the operator xy in Definition 5.6.1 (including multiplicities). In particular, one sees that kernel of the fermionic projector encodes the causal structure of M. The above argument also implies that the operator products xy and yx are isospectral. This shows that the causal structure is symmetric in x and y. The main advantage of working with the kernel of the fermionic projector is that the closed chain (5.7.4) is a linear operator on a vector space of dimension at most $2n$, making it possible to compute the $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ as the eigenvalues of a matrix (in finite dimensions).

Next, it is very convenient to choose inner products on the spin spaces in such a way that the kernel of the fermionic projector is symmetric in the sense that

$$P(x,y)^* = P(y,x)\,, \tag{5.7.7}$$

where the star denotes the adjoint with respect to yet to be specified inner products on the spin spaces. This identity indeed holds if on the spin space S_x (and similarly on S_y) one chooses the *spin inner product* $\prec .|. \succ_x$ defined by

$$\prec u|v\succ_x = -\langle u|xv\rangle_{\mathcal{H}} \qquad (\text{for all } u, v \in S_x)\,. \tag{5.7.8}$$

Due to the factor x on the right, this definition really makes the kernel of the fermionic projector symmetric, as is verified by the computation

$$\begin{aligned} \prec u \,|\, P(x,y)\, v\succ_x &= -\langle u \,|\, x\, P(x,y)\, v\rangle_{\mathcal{H}} = -\langle u \,|\, xy\, v\rangle_{\mathcal{H}} \\ &= -\langle \pi_y\, x\, u \,|\, y\, v\rangle_{\mathcal{H}} = \prec P(y,x)\, u \,|\, v\succ_y\,, \end{aligned}$$

where we again used (5.7.5) (and $u \in S_x$, $v \in S_y$). The spin space $(S_x, \prec .|. \succ_x)$ is an *indefinite* inner product of signature (p, q) with $p, q \leq n$ (for textbooks on indefinite inner

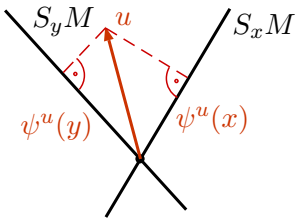


FIGURE 5.9. The physical wave function

product spaces see [**16, 94**]). In this way, indefinite inner product spaces arise naturally when analyzing the mathematical structure of the causal action principle.

The kernel of the fermionic projector plays a central role in the analysis for several reasons:

- The Lagrangian can be expressed in terms of $P(x,y)$ (via the closed chain (5.7.4) and its eigenvalues).
- Being a mapping from one spin space to another, $P(x,y)$ gives relations between different spacetime points. In this way, it carries geometric information. This will be explained in Chapter 11 (see also [**56**] or the introductory survey paper [**47**]).
- The kernel of the fermionic projector also encodes all the wave functions of the system. In order to see the connection, for a vector $u \in \mathcal{H}$ one introduces the corresponding *physical wave function* ψ^u as (see Figure 5.9)

$$\psi^u \;:\; M \to \mathcal{H}\,, \qquad \psi^u(x) = \pi_x u \in S_x\,. \tag{5.7.9}$$

Then, choosing an orthonormal basis (e_i) of $\mathcal{H}$ and using the completeness relation as well as (5.7.8), one obtains for any $\phi \in S_y$

$$P(x,y)\,\phi = \pi_x y|_{S_y}\,\phi = \sum_i \pi_x e_i\, \langle e_i | y\,\phi\rangle_{\mathcal{H}} = -\sum_i \psi^{e_i}(x)\, \prec\!\psi^{e_i}(y)\,|\,\phi\!\succ_y\,,$$

showing that $P(x,y)$ is indeed composed of all the physical wave functions, i.e. in bra/ket notation

$$P(x,y) = -\sum_i |\psi^{e_i}(x)\!\succ \prec\!\psi^{e_i}(y)|\,. \tag{5.7.10}$$

We remark that knowing the kernel of the fermionic projector in spacetime makes it possible to reconstruct the causal fermion system (the detailed construction can be found in [**57**, Section 1.1.2]). We also note that the representation of the kernel of the fermionic projector (5.7.10) also opens the door to the detailed study of causal fermion systems in Minkowski space as carried out in [**45**]; see also the Exercises 5.15–5.18.

Taking a slightly different perspective, one can say that all structures of the causal fermion system are encoded in the *family of physical wave functions* ψ^u with $u \in \mathcal{H}$ as defined in (5.7.9). In order to make this statement precise, it is most convenient to introduce the *wave evaluation operator* $\Psi(x)$ at the spacetime point $x \in M$ by

$$\Psi(x) \;:\; \mathcal{H} \to S_x\,, \qquad u \mapsto \psi^u(x) = \pi_x u\,. \tag{5.7.11}$$

Clearly, using (5.7.9), the wave evaluation operator can be written simply as

$$\Psi(x) = \pi_x\,. \tag{5.7.12}$$

The wave evaluation operator describes the family of all physical wave functions. Indeed, applying the wave evaluation operator to a vector u and varying the point x, we get back

the corresponding physical wave function ψ^u. We next compute the adjoint of $\Psi(x)$,

$$\Psi(x)^* : S_x \to \mathcal{H} .$$

Taking into account the corresponding inner products, we obtain for any $\phi \in S_x$ and $u \in \mathcal{H}$,

$$\langle \Psi(x)^* \phi | u \rangle_{\mathcal{H}} = \prec \phi | \Psi(x)\, u \succ_x \overset{(5.7.8)}{=} -\langle \phi | x\, \Psi(x)\, u \rangle_{\mathcal{H}} .$$

This shows that

$$\Psi(x)^* = -x|_{S_x} . \tag{5.7.13}$$

Combining (5.7.12) and (5.7.13) and comparing with (5.7.3), one sees that

$$x = -\Psi(x)^*\, \Psi(x) \qquad \text{and} \qquad P(x,y) = -\Psi(x)\, \Psi(y)^* . \tag{5.7.14}$$

In this way, all the spacetime point operators and the kernel of the fermionic projector can be constructed from the wave evaluation operator. Moreover, the conclusion after (5.7.6) that the eigenvalues of the closed chain coincide with the nontrivial eigenvalues of the operator product xy can be seen more directly from the computation

$$\begin{aligned} A_{xy} &= P(x,y)\, P(y,x) = \Psi(x) \Psi(y)^* \Psi(y) \Psi(x)^* \\ &= -\Psi(x) \big(y\, \Psi(x)^* \big) \simeq -\Psi(x)^* \Psi(x)\, y = xy , \end{aligned}$$

where by $\simeq$ we mean that the operators are isospectral (in the sense that they have the same non-zero eigenvalues with the same algebraic multiplicities). Here we used that for any two matrices $A \in \mathbb{C}^{p \times q}$ and $B \in \mathbb{C}^{q \times p}$, the matrix product AB is isospectral to BA (for details see Exercise 5.5).

5.8. How Did the Causal Action Principle Come About?

Causal fermion systems and the causal action principle came to light as a result of many considerations and computations carried out over several years. We now give an outline of these developments, also explaining the specific form of the causal action principle.

The starting point for the considerations leading to causal fermion systems was the belief that in order to overcome the conceptual problems of quantum field theory, the structure of spacetime should be modified. Moreover, instead of starting from differential equations in a spacetime continuum, one should formulate the physical equations using the new structures of spacetime, which might be non-smooth or discrete. A more concrete idea in this direction was that the spacetime structures should be encoded in the family of wave functions which is usually associated to the Dirac sea (for basics see Section 1.5). Thus, instead of disregarding the sea states, one should take all these wave functions into account. The mutual interaction of all these wave functions should give rise to the structures of spacetime as we experience them.

The first attempts toward making this idea more precise go back to the early 1990s. The method was to consider families of Dirac solutions (the formalism of quantum fields was avoided in order to keep the setting as simple and non-technical as possible). In order to describe such a family mathematically, the corresponding two-point kernel $P(x,y)$ was formed

$$P(x,y) := -\sum_{l=1}^{f} |\psi_l(x) \succ \prec \psi_l(y)|$$

(where $\psi_1, \ldots, \psi_f$ are suitably normalized solutions of the Dirac equation; for preliminaries see Section 1.3). The kernel $P(x,y)$ is also referred to as the *kernel of the fermionic*

projector. In the Minkowski vacuum, this kernel is formed of all the states of the Dirac sea. Then the sum goes over to an integral over the lower mass shell

$$P^{\text{vac}}(x,y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \,(\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k_0)\, \mathrm{e}^{-\mathrm{i}k(x-y)} \tag{5.8.1}$$

(this integral is well-defined as the Fourier transform of a tempered distribution; see the preliminaries in Section 2.4). Likewise, a system involving particles and anti-particles is described by "occupying additional states of positive energy" and by "creating holes in the Dirac sea," respectively. Thus, more technically, one sets

$$P(x,y) = P^{\text{vac}}(x,y) - \sum_a |\psi_a(x) \succ\!\prec \psi_a(y)| + \sum_b |\phi_b(x) \succ\!\prec \phi_b(y)|\,, \tag{5.8.2}$$

where ψ_a and ϕ_b are suitably normalized Dirac solutions of positive and negative energy, respectively. In case a bosonic interaction is present, the kernel of the fermionic projector should no longer satisfy the vacuum Dirac equation, but the Dirac equation in the presence of a, say, external potential $\mathcal{B}$. Clarifying the dependence on the bosonic potential with an additional tilde, we write the resulting Dirac equation as

$$(\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{P}(x,y) = 0\,. \tag{5.8.3}$$

Analyzing the distribution $\tilde{P}(x,y)$ in Minkowski space reveals the following facts:

(a) The kernel $\tilde{P}(x,y)$ contains all the information on the wave functions of the particles and anti-particles of the system. This statement can be understood from the representation (5.8.2) in which all these wave functions appear. Alternatively, the wave functions can be reconstructed from $\tilde{P}(x,y)$ as being the image of the corresponding integral operator on $C_0^\infty(\mathscr{M}, S\mathscr{M})$

$$\phi \mapsto \int_{\mathscr{M}} \tilde{P}(.,y)\, \phi(y)\; \mathrm{d}^4 x\,.$$

(b) The kernel $\tilde{P}(x,y)$ has singularities on the light cone. The detailed form of the singularities involves integrals of the potential $\mathcal{B}$ and its derivatives along the light cone. In particular, knowing the kernel $\tilde{P}(x,y)$ makes it possible to reconstruct the potential $\mathcal{B}$ at every spacetime point. These statements follow immediately by looking at the so-called light-cone expansion of $\tilde{P}(x,y)$ (see Chapter 19 in this book or [**45**, Section 2.2 and Appendix B]).

(c) The singularity structure of $\tilde{P}(x,y)$ encodes the causal structure of Minkowski space. This can be seen again from the light-cone expansion of $\tilde{P}(x,y)$ (see again Chapter 19 in this book or [**45**, Section 2.2 and Appendix B]).

These findings show that, at least for Dirac systems in the presence of classical bosonic potentials, the kernel $\tilde{P}(x,y)$ contains all the information on the physical system. This led to the concept to regard $\tilde{P}(x,y)$ as the basic physical object in spacetime. The more familiar structures and objects like Minkowski space with its causal structure, the Dirac equation, the classical field equations for the bosonic fields (like the Maxwell or Einstein equations), however, should no longer be considered as being fundamental. Consequently, the physical equations should be formulated directly in terms of the kernel of the fermionic projector.

Formalizing this idea in a clean way also made it necessary to disregard or to prescind from the usual spacetime structures. This led to the *principle of the fermionic projector* as formulated around 1990 (see the unpublished preprint [**35**] and the monograph [**41**]).

We here present a slightly different but equivalent formulation which is somewhat closer to the setting of causal fermion systems. Let M be a discrete set (i.e. a point set without additional structures), the *discrete spacetime*. Moreover, for every $x \in M$ one chooses an indefinite inner product space $(S_x, \prec.|.\succ_x)$, referred to as the *spin space* at x (usually, one chooses the dimensions and signatures of all spin spaces to be the same, but this is not crucial for the construction). Next, we consider a collection of *wave functions* $(\psi_a)_a$, each being a mapping which to every discrete spacetime point $x \in M$ associates a vector $\psi_a(x) \in S_x$ of the corresponding spin space. Out of these wave functions, one can form the kernel of the fermionic projector

$$P(x,y) := -\sum_a |\psi_a(x)\succ\prec\psi_a(y)| \;:\; S_y \to S_x\,.$$

The principle of the fermionic projector asserts that the physical equations should be formulated purely in terms of the kernel of the fermionic projector in discrete spacetime.

The next question was how precisely these physical equations should look like. This was a difficult question which took many years to be answered. Apart from the structural requirements coming from the principle of the fermionic projector, the following considerations served as guiding principles[3]:

(i) In analogy to classical field theory, a *variational approach* should be used. One main advantage is the resulting connection between symmetries and conservation laws (corresponding to the classical Noether theorem), which seems of central importance in physical applications.
(ii) Classical field theory should be obtained in a certain limiting case. More specifically, the Euler-Lagrange equations coming from our variational principle should reproduce the *Maxwell* and *Einstein equations*.
(iii) Also the *Dirac equation* should be recovered in a certain limiting case.

More mathematically, the strategy was to form composite expressions of the kernel of the fermionic projector. Choosing n points $x_1, \ldots, x_n \in M$, one can form the *closed chain*

$$A_{x_1,\ldots,x_n} := P(x_1,x_2)\,P(x_2,x_3)\cdots P(x_{n-1},x_n)\,P(x_n,x_1) \;:\; S_{x_1} \to S_{x_1}\,. \tag{5.8.4}$$

Being an endomorphism of the spin space, one can compute the eigenvalues of the closed chain and form a Lagrangian $\mathcal{L}[A_{x_1,\ldots,x_n}]$ as a symmetric function of these eigenvalues. Summing over the spacetime points gives an ansatz for the

$$n\text{-point action} \qquad \mathcal{S} = \sum_{x_1,\ldots,x_n \in M} \mathcal{L}[A_{x_1,\ldots,x_n}]\,.$$

This general ansatz can be made more specific and concrete by considering *gauge phases*. This consideration was motivated by the fact that the kernel of the fermionic projector $\tilde{P}(x,y)$ formed of Dirac solutions involves gauge phases. More specifically, choosing the potential in the Dirac equation (5.8.3) as an electromagnetic potential, i.e. $\mathcal{B} = \A$, then the leading contribution to the kernel are gauge phases described by line integrals over the electromagnetic potential,

$$\tilde{P}(x,y) = \mathrm{e}^{-\mathrm{i}\int_x^y A_j \xi^j}\; P^{\text{vac}}(x,y) + \cdots\,, \tag{5.8.5}$$

[3]Of course, it is also an important requirement that our variational principle should give agreement with quantum field theory. But this connection was not used for finding the causal action principle. It was worked out more recently; for more details see Chapter 22.

where

$$\int_x^y A_j \xi^j = \int_0^1 A_j(\alpha y + (1-\alpha)x)\, (y-x)^j \, \mathrm{d}\alpha$$

(this can again be seen from the light-cone expansion; more specifically, see [**45**, §2.2.4]). Here $\cdots$ stands for many other contributions to $\tilde{P}(x,y)$ which involve derivatives of the potential (like the field tensor, the Maxwell current, etc.). All these additional contributions are small in the sense that they are less singular on the light cone. These findings will be made precise by the Hadamard and light-cone expansions of the kernel of the fermionic projector in Chapter 21 of this book. At this stage, we do not need to be specific. All we need is that gauge phases come into play, which involves integrals of the potential along the line segment joining the points x and y.

Let us analyze the effect of the gauge phases on the closed chain (5.8.4). First of all, the closed chain is *gauge invariant*. Indeed, if one considers a pure gauge potential $A_j = \partial_j \Lambda$, then the gauge phases in (5.8.5) simplify to

$$\tilde{P}(x,y) = \mathrm{e}^{-\mathrm{i}\Lambda(y)+\mathrm{i}\Lambda(x)}\, P^{\text{vac}}(x,y)\,, \tag{5.8.6}$$

and the phase factors of neighboring factors cancel in (5.8.4). This consideration also gives a relation between local gauge invariance and the fact that the adjacent factors in (5.8.4) must coincide. In the case $n = 1$, the kernel of the fermionic projector is evaluated only on the diagonal $P(x,x)$. This turns out to be too simple for formulating physical equations, as can be understood from the fact that no relations between spacetime points are taken into account. If $n \geq 3$, the gauge phases in (5.8.4) can be rewritten using Stokes' theorem as *flux integrals* of the electromagnetic field through the two-dimensional polygon with vertices $x_1, \ldots, x_n$. Analyzing the situation in some more detail, one finds that the resulting Euler-Lagrange equations will be satisfied only if all fluxes vanish. This implies that the electromagnetic potential must be a pure gauge potential. In other words, the case $n \geq 3$ does not allow for an interaction via gauge potentials. This is the reason why this case was disregarded (for some more details on this argument see [**41**, Remark 6.2.5]).

After these considerations, we are left with the

$$\text{two-point action} \qquad \mathcal{S} = \sum_{x,y \in M} \mathcal{L}[A_{xy}]\,,$$

where A_{xy} is the closed chain formed of two points,

$$A_{xy} := P(x,y)\, P(y,x)\,. \tag{5.8.7}$$

In this case, the polygon with vertices x and y degenerates to a straight line, implying that the flux through this polygon vanishes as desired. The starting point for a more quantitative analysis was to choose the Lagrangian formed by taking products and sums of traces of powers of the closed chain. A typical example is the Lagrangian

$$\mathcal{L}[A_{xy}] := \mathrm{Tr}_{S_x}\left(A_{xy}^2\right) - c\left(\mathrm{Tr}_{S_x}(A_{xy})\right)^2 \tag{5.8.8}$$

with a real parameter c. In such examples, the Lagrangian is a symmetric polynomial in the eigenvalues of the closed chain. The methods and results of this early analysis can be found in the unpublished preprints [**35, 36**].

Generally speaking, the study of such polynomial Lagrangians seemed a promising strategy toward formulating physically sensible equations. However, the more detailed analysis revealed the basic problem that *chiral gauge phases* come into play: As just

explained after (5.8.6), the closed chain and therefore also the Lagrangian are gauge invariant for the electromagnetic potential. However, the situation changes if chiral gauge potentials are considered. Here chiral gauge potentials are left- or right-handed potentials A_L and A_R which can be inserted into the Dirac equation by generalizing (1.3.6) to

$$\big(\mathrm{i}\partial\!\!\!/ + \chi_R A\!\!\!/_L + \chi_L A\!\!\!/_R - m\big)\psi = 0\,,$$

where $\chi_{L/R}$ are the chiral projection operators (1.3.22) (for details see for example [**45**, §2.2.3]). In physics, the electroweak interaction involves left-handed gauge potentials. In this case, the left- and right-handed components of $P(x,y)$ involve phase transformations by the left- and right-handed gauge potentials, respectively. When forming the closed chain (5.8.7), the left- and right-handed components of $P(x,y)$ are multiplied together. As a consequence, the closed chain involves relative phases of the left- and right-handed gauge potentials, i.e. phase factors of the form

$$\mathrm{e}^{\pm \mathrm{i} \int_x^y (A_L - A_R)_j \xi^j}\,,$$

where A_L and A_R are the left- and right-handed gauge potentials (here for simplicity again Abelian). As a consequence, also the eigenvalues of the closed chain are multiplied by these relative phases. The traces of powers of the closed chain as in (5.8.8) are still real-valued (this is because the phase factors always come as complex conjugate pairs), but they do not have fixed signs. Working out the Euler-Lagrange equations, one sees that they also involve the relative gauge phases, making it difficult to allow for chiral gauge fields. In order to bypass these difficulties, from around 1999 on Lagrangians were considered which involved *absolute values* of the eigenvalues of the closed chain. This had two major advantages:

(a) The *chiral gauge phases drop out* of the Lagrangian.
(b) It became natural to formulate non-negative Lagrangians. As a consequence, in the variational principle one *minimize the action* instead of merely looking for critical points.
(c) A connection to *causality* was obtained. In order to see how this comes about, we give a simple computation in the Minkowski vacuum. Suppose that the points x and y are either timelike or spacelike separated. Then $P(x,y)$ is well-defined and finite even without regularization and, due to Lorentz symmetry, it has the form

$$P(x,y) = \alpha\, \xi_j \gamma^j + \beta\, \mathbb{1}$$

with two complex-valued functions α and β (where again $\xi = y - x$, and γ^j are the Dirac matrices). Taking the adjoint with respect to the spin inner product, we see that

$$P(y,x) = \overline{\alpha}\, \xi_j \gamma^j + \overline{\beta}\, \mathbb{1}\,.$$

As a consequence,

$$A_{xy} = P(x,y)\, P(y,x) = a\, \xi_j \gamma^j + b\, \mathbb{1}$$

with two real parameters a and b given by

$$a = \alpha\overline{\beta} + \beta\overline{\alpha}\,, \qquad b = |\alpha|^2\, \xi^2 + |\beta|^2 \tag{5.8.9}$$

(here $\xi^2 = \xi^i \xi_i$ denotes the Minkowski inner product, which may be negative). Applying the formula $(A_{xy} - b\mathbb{1})^2 = a^2\, \xi^2\, \mathbb{1}$, the roots of the characteristic polynomial of A_{xy} are computed by

$$b \pm \sqrt{a^2\, \xi^2}\,. \tag{5.8.10}$$

Therefore, the eigenvalues of the closed chain are either real, or else they form a complex conjugate pair. Moreover, one gets a connection to causality: By explicit computation in Minkowski space one sees that a is non-zero (the details can be found in [**56**, proof of Lemma 4.3]). Therefore, if ξ is timelike (i.e. $\xi^2 > 0$) then the relations (5.8.9) and (5.8.10) show that the eigenvalues are distinct, both real and have the same sign. If ξ is spacelike, on the other hand, the eigenvalues are complex and have the same absolute value. In this way, one gets agreement with the spectral definition of causality in Definition 5.6.1. Moreover, choosing a Lagrangian which depends only on differences of absolute values of the eigenvalues vanishes for spacelike separation, making it possible to build causality into the action principle.

The further analysis led to the class of Lagrangians

$$\mathcal{L} = \sum_{i,j} \left(|\lambda_i^{xy}|^p - |\lambda_j^{xy}|^p \right)^2$$

with a parameter $p \in \mathbb{N}$, where the λ_i^{xy} are the eigenvalues of A_{xy} (again counted with algebraic multiplicities). The case $p = 1$ gives the causal Lagrangian (5.6.1) (albeit with the difference of working instead of the local correlation operators with the kernel of the fermionic projector; the connection will be explained below). The decision for $p = 1$ was taken based on the so-called *state stability analysis*, which revealed that the vacuum Dirac sea configuration (5.8.1) is a local minimizer of the causal action only if $p = 1$ (for details see [**41**, Section 5.5]). Now that the form of the causal action was fixed, the monograph [**41**] was completed and published. The causal action principle is given in this book as an example of a variational principle in discrete spacetime (see [**41**, Section 3.5]). The boundedness constraint (5.6.5) already appears, and the causal Lagrangian (5.6.1) arises when combining the Lagrangian with the Lagrange multiplier term corresponding to the boundedness constraint. The volume constraint (5.6.3) is also implemented, however in discrete spacetime simply as the condition that the number of spacetime points be fixed (and ρ-integrals are replaced by sums over the spacetime points). The trace constraint, however, was not yet recognized as being necessary and important.

After the publication of the monograph [**41**], the causal action principle was analyzed in more detail and more systematically, starting from simple systems and proceeding to more realistic physical models, concluding with systems showing all the interactions of the standard model and gravity (see [**45**, Chapters 3–5]). This detailed study also led to the causal action principle in the form given in Section 5.6 above. The path from the monograph [**41**] to the present formulation in [**45**] is outlined in [**41**, Preface to second online edition]. We now mention a few points needed for the basic understanding.

One major conceptual change compared to the setting in indefinite inner product spaces was to recognize that an underlying Hilbert space structure is needed in order for the causal variational principle to be mathematically well-defined. This became clear when working on the existence theory in discrete spacetime [**42**]. This Hilbert space structure is built in most conveniently by working instead of the kernel of the fermionic projector with the local correlation operators which relate the Hilbert space scalar product to the spin inner product by

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} = - \prec \psi(x) | \phi(x) \succ_x . \tag{5.8.11}$$

Using that the operator product $F(x)F(y)$ has the same non-trivial eigenvalues as the closed chain A_{xy} given by (5.8.7) (as we already observed in Section 5.7 after (5.7.4)),

the causal action principle can also be formulated in terms of the local correlation operators $F(x)$ with $x \in M$. Moreover, it turned out that measure-theoretic methods can be used to generalize the setting such as to allow for the description of not only discrete, but also continuous spacetimes. In this formulation, the sums over the discrete spacetime points are replaced by integrals with respect to a measure μ on M. This setting was first introduced in [**43**] when working out the existence theory. In this formulation, the only a-priori structure of spacetime is that of a measure space (M, μ). The local correlation operators give rise to a mapping

$$F \,:\, M \to \mathcal{F}\,, \quad x \mapsto F(x)\,,$$

where $\mathcal{F}$ is the subset of finite rank operators on $\mathcal{H}$ which are symmetric and (counting multiplicities) have at most n positive and at most n negative eigenvalues (where n is introduced via the signature (n, n) of the indefinite inner product in (5.8.11)). This analysis also revealed the significance of the trace constraint. As the final step, instead of working with the measure μ, the causal action can be expressed in terms of the push-forward measure $\rho = F_*\mu$, being a measure on $\mathcal{F}$ (defined by $\rho(\Omega) = \mu(F^{-1}(\Omega))$). Therefore, it seems natural to leave out the measure space (M, μ) and to work instead directly with the measure ρ on $\mathcal{F}$.

These considerations led to the general definition of causal fermion systems in Section 5.4, where the physical system is described by a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ and the measure ρ on $\mathcal{F}$. The causal action principle takes the form as stated in Section 5.6.

5.9. Underlying Physical Concepts

We now briefly explain a few physical concepts behind causal fermion systems and the causal action principle. The aim is to convey the reader the correct physical picture in a non-technical way. Doing so already here makes it necessary to anticipate some ideas on a qualitative level which will be introduced more systematically and thoroughly later in this book.

It is a general feature of causal fermion systems that the usual distinction between the structure of spacetime itself (being modelled by Minkowski space or a Lorentzian manifold) and structures in spacetime (like wave functions and matter fields) ceases to exist. Instead, all these structures are described as a whole by *a single object*: the measure ρ on $\mathcal{F}$. Spacetime and all structures therein are different manifestations of this one object. The dynamics of spacetime and of all objects in spacetime are described in a unified and holistic manner by the causal action principle. Clearly, in order to get a connection to the conventional description of physics, one still needs to construct the familiar physical objects from a causal fermion system. Also, one needs to rewrite the dynamics as described by the causal action principle in terms of these familiar physical objects. This study is a main objective of this book. As already exemplified in Section 5.7 by the spin spaces and physical wave functions, the strategy is to identify suitable inherent structures in a causal fermion system, which then may be given suitable names. This must be done carefully in such a way that these names convey the correct physical picture. Ultimately, the inherent structures serve the purpose of getting a better understanding of the causal action principle. As we shall see, this will be achieved by reformulating the Euler-Lagrange equations of the causal action principle in terms of the inherent structures. When this is done, also the physical names of the inherent structures will be justified by showing that they agree with the familiar physical objects in specific limiting cases and generalize these objects in a sensible way.

In view of this unified description of all physical structures by a single mathematical object, it is difficult to describe the essence of causal fermion systems using the familiar notions from physics. One simple way of understanding the causal action principle is to focus on the structure of the *physical wave functions* and the kernel of the fermionic projector. Clearly, the resulting picture is a bit oversimplified, because it only captures part of the structures encoded in a causal fermion system. Nevertheless, it conveys a good and the correct intuition of what the causal action principle is about. We saw in Section 5.7 that a causal fermion system gives rise to the family of physical wave functions $(\psi^u)_{u \in \mathcal{H}}$ (see (5.7.9)). The kernel of the fermionic projector (5.7.10) is built up of all the physical wave functions and thus describes the whole family. It gives rise to the closed chain (5.7.4), which in turn determines the causal action and the constraints. In this way, the causal action principle becomes a variational principle for the family of physical wave functions. Thus the interaction described by the causal action principle can be understood as a direct mutual interaction of all the physical wave functions. In simple terms, the causal action principle aims at bringing the family of wave functions into an "optimal" configuration. For such optimal configurations, the family of wave functions gives rise to the spacetime structures as we know them: the causal and metric structure, the bosonic fields, and all that.

The last step can be understood more concretely starting from Dirac's hole theory and the picture of the *Dirac sea* (for basics see again Section 1.5). In our approach, the Dirac sea is taken literally. Thus all the states of the Dirac sea correspond to physical wave functions. All the information contained in these wave functions induces spacetime with the familiar structures. As a simple example, the bosonic potentials $\mathcal{B}$ are determined via the Dirac equation (5.8.3) from the family of wave functions as described by $\tilde{P}(x,y)$. Clearly, in order to make this picture precise, one needs to verify that, in a certain limiting case, the kernel of the fermionic projector corresponding to a minimizer of the causal action principle indeed satisfies a Dirac equation of the form (5.8.3) and thus gives rise to a potential $\mathcal{B}$. This will be one of the objectives of the later chapters in this book.

We now discuss which *physical principles* enter the approach, and how they were incorporated. Causal fermion systems evolved from an attempt to combine several physical principles in a coherent mathematical setting. As a result, these principles appear in a specific way:

- The **principle of causality**: A causal fermion system gives rise to a causal structure (see Definition 5.6.1). The causal action principle is compatible with this notion of causality in the sense that the pairs of points with spacelike separation do not enter the Euler-Lagrange equations. In simple terms, points with spacelike separation do not interact.
- The **local gauge principle**: Already in the above discussion of how the causal action principle came about, we mentioned that the Lagrangian is gauge invariant in the sense that gauge phases drop out of the Lagrangian (see the explanation after (5.8.6) in Section 5.8). When starting from a general causal fermion system, **local gauge invariance** becomes apparent when representing the physical wave functions in bases of the spin spaces. More precisely, choosing a pseudo-orthonormal basis $(\mathfrak{e}_\alpha(x))_{\alpha=1,\ldots,\dim S_x}$ of S_x, a physical wave function can be represented as

$$\psi(x) = \sum_{\alpha=1}^{\dim S_x} \psi^\alpha(x)\, \mathfrak{e}_\alpha(x) \tag{5.9.1}$$

with component functions $\psi^1, \ldots, \psi^{\dim S_x}$. The freedom in choosing the basis $(\mathfrak{e}_\alpha)$ is described by the group of unitary transformations with respect to the indefinite spin inner product. This gives rise to the transformations

$$\mathfrak{e}_\alpha(x) \to \sum_\beta U^{-1}(x)^\beta_\alpha \, \mathfrak{e}_\beta(x) \qquad \text{and} \qquad \psi^\alpha(x) \to \sum_\beta U(x)^\alpha_\beta \, \psi^\beta(x) \tag{5.9.2}$$

with $U \in \mathrm{U}(p,q)$. As the basis $(\mathfrak{e}_\alpha)$ can be chosen independently at each spacetime point, one obtains *local gauge transformations* of the wave functions, where the gauge group is determined to be the isometry group of the spin inner product. The causal action is *gauge invariant* in the sense that it does not depend on the choice of spinor bases.

- The **Pauli exclusion principle** is incorporated in a causal fermion system, as can be seen in various ways. One formulation of the Pauli exclusion principle states that every fermionic one-particle state can be occupied by at most one particle. In this formulation, the Pauli exclusion principle is respected because every wave function can either be represented in the form ψ^u (the state is occupied) with $u \in \mathcal{H}$ or it cannot be represented as a physical wave function (the state is not occupied). Via these two conditions, the fermionic projector encodes for every state the occupation numbers 1 and 0, respectively, but it is impossible to describe higher occupation numbers.

  More technically, one may obtain the connection to the fermionic Fock space formalism by choosing an orthonormal basis $u_1, \ldots, u_f$ of $\mathcal{H}$ and forming the f-particle Hartree-Fock state

  $$\Psi := \psi^{u_1} \wedge \cdots \wedge \psi^{u_f} .$$

  Clearly, the choice of the orthonormal basis is unique only up to the unitary transformations

  $$u_i \to \tilde{u}_i = \sum_{j=1}^f U_{ij} \, u_j \quad \text{with} \quad U \in \mathrm{U}(f) .$$

  Due to the anti-symmetrization, this transformation changes the corresponding Hartree-Fock state only by an irrelevant phase factor,

  $$\psi^{\tilde{u}_1} \wedge \cdots \wedge \psi^{\tilde{u}_f} = \det U \; \psi^{u_1} \wedge \cdots \wedge \psi^{u_f} .$$

  Thus the configuration of the physical wave functions can be described by a fermionic multi-particle wave function. The Pauli exclusion principle becomes apparent in the total anti-symmetrization of this wave function.

  Clearly, the above Hartree-Fock state does not account for quantum entanglement. Indeed, the description of entanglement requires more general Fock space constructions (this will be described in more detail in Chapter 22).

- The **equivalence principle**: Starting from a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$, spacetime M is given as the support of the measure ρ. Thus spacetime is a topological space (with the topology on M induced by the operator norm on $\mathrm{L}(\mathcal{H})$). In situations when spacetime has a smooth manifold structure, one can describe spacetime by choosing coordinates. However, there is no distinguished coordinate systems, giving rise to the freedom of performing general coordinate transformations. The causal action as well as all the constraints are invariant under such transformations. In this sense, the equivalence principle is implemented in the setting of causal fermion systems.

However, other physical principles are missing. For example, the principle of locality is not included. Indeed, the causal action principle is *non-local*, and locality is recovered

only in the continuum limit. Moreover, our concept of causality is quite different from *causation* (in the sense that the past determines the future) or *microlocality* (stating that the observables of spacelike separated regions must commute).

5.10. A Summary of the Basic Concepts and Objects

In this section we summarize all important concepts of the preceding sections. You may use this as a reference list for frequently used concepts and objects.

Basic concept	**Summary and Comments**
Causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$	A separable Hilbert space $\mathcal{H}$, a natural number $n \in \mathbb{N}$, the set $\mathcal{F}$ of symmetric linear operators on $\mathcal{H}$ with at most n positive and n negative eigenvalues as well as a measure ρ defined on a σ-algebra on $\mathcal{F}$ forms a **causal fermion system**.

Remarks:

- The structure of a causal fermion system provides a general framework for describing generalized spacetimes. Concrete physical systems correspond to specific choices of $\mathcal{H}$, n and the measure ρ.

- $\mathcal{H}$ should be considered as the Hilbert space spanned by all one-particle wave functions realized in our system (the *physical wave functions*).

- We are mainly interested in the case $n = 2$ (at most two positive and two negative eigenvalues). This case allows for the description of Dirac spinors in four-dimensional spacetimes.

Spacetime M	By definition, we describe spacetime by the support of the measure $M := \text{supp}(\rho)$.

Remarks:

- All points $x \in M$ are linear operators on $\mathcal{H}$. This fact implies that our spacetime is endowed with more structures and contains additional information.

- In order to describe systems in Minkowski space, we identify spacetime points $x \in M$ with corresponding points in Minkowski space $\mathscr{M}$ via a mapping $F^\varepsilon : \mathscr{M} \to M$ (for more details see (5.5.4)).

The measure ρ	The measure ρ in Definition 5.4.1 is the most important object of the theory. It describes spacetime as well as all objects therein.

Remarks:

- A lot of structure is encoded in the measure ρ. In particular, it describes the behavior of spacetime on microscopic scales (Planck scale).

- In the example of causal fermion systems describing Minkowski space, the measure is obtained as the push-forward of the Minkowski volume measure $\mathrm{d}\mu = \mathrm{d}^4x$ under the local correlation map F^ε, i.e. we set $\rho = F^\varepsilon_* \mu$.

The causal action	We define a Lagrangian $\mathcal{L}(x,y)$ for two spacetime points x and y using the eigenvalues $(\lambda_i^{xy})_{i=1,\ldots,2n}$ of the product xy, which is an operator of rank at most $2n$. The Lagrangian is given by $\mathcal{L}(x,y) := \frac{1}{4n} \sum_{i,j=1}^{2n} (|\lambda_i^{xy}| - |\lambda_j^{xy}|)^2$. Finally, the causal action is defined by taking the double integral $\mathcal{S}(\rho) := \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y) \mathrm{d}\rho(x) \mathrm{d}\rho(y)$.

Remarks:

- It may happen that the rank of the operator xy is smaller than $2n$. In this case, some of the eigenvalues $\lambda_1^{xy}, \ldots \lambda_{2n}^{xy}$ are zero.
- The action depends nonlinearly on the measure ρ. Since ρ describes spacetime and all objects therein, the action also depends on spacetime and on all these object.

The causal action principle	The causal action principle states that measures describing physical systems must be minimizers of the causal action under variations of ρ, respecting the constraints (5.6.3), (5.6.4) and (5.6.5).

Remarks:

- The Euler-Lagrange equations corresponding to the causal action principle are the physical equations of the theory.
- By varying the measure ρ, we also vary spacetime as well as all structures therein.

The physical wave functions	Every vector $u \in \mathcal{H}$ can represented in spacetime by the physical wave function ψ^u defined by $\psi^u(x) = \pi_x u \in S_x$, where π_x denotes the orthogonal projection in $\mathcal{H}$ onto the subspace $x(\mathcal{H}) \subset \mathcal{H}$.
The kernel of the fermionic projector	For any spacetime point operator $x \in M$ we define the spin space S_x as its image $S_x := x(\mathcal{H})$. This gives rise to a mapping between spin spaces at different spacetime points $x, y \in M$ by $P(x,y) := \pi_x y|_{S_x} : S_y \to S_x$, The mapping $P(x,y)$ is the kernel of the fermionic projector. It can be expressed in terms of all physical wave functions by $P(x,y) = -\sum_i |\psi^{e_i}(x) \succ\!\prec \psi^{e_i}(y)|$, where the (e_i) form an orthonormal basis of $\mathcal{H}$.

Remarks:

- The kernel of the fermionic projector gives relations between spacetime points. In particular, it encodes the causal structure and the geometry of spacetime.
- In order to compute the Lagrangian, it is useful to form the *closed chain* $A(x,y) := P(x,y)P(y,x)$.

5.11. Exercises

EXERCISE 5.1. This exercise is devoted to the study of the variational principle (5.2.5) of the motivating example.

(a) Assume that the operators $F_1, \ldots, F_L$ are a minimizer of the action (5.2.5) under variations of $F_i \in \mathcal{F}$ with $\mathcal{F}$ according to (5.2.4). Given $i \in \{1, \ldots f\}$, represent F_i as

$$F_i = |\psi_i\rangle\langle\psi_i| \qquad \text{with } \psi_i \in \mathcal{H}\,.$$

Vary the vector ψ_i to derive the following Euler-Lagrange (EL) equations,

$$\sum_{i,j=1}^{f} \mathrm{Tr}(F_j F_i)\, F_j\, \psi_i = 0\,.$$

(b) Deduce that all the matrices F_i must vanish. *Hint:* It is useful to first show that

$$\sum_{i,j=1}^{f} \big|\,\mathrm{Tr}(F_j F_i)\big|^2 = 0\,. \tag{5.11.1}$$

(c) In order to get non-trivial solutions, one can for example impose the constraint

$$\sum_{i=1}^{f} \mathrm{Tr}\left(F_i^2\right) = 1\,.$$

Derive the corresponding EL equations.

(d) The constraint (5.11.1) also makes it possible to prove existence of minimizer with a compactness argument. Work out this existence proof in detail.

EXERCISE 5.2. *(A causal fermion system on ℓ_2)* Let $\mathcal{H} = \ell_2$ the Hilbert space of square-summable complex-valued sequences, equipped with the scalar product

$$\langle u|v\rangle = \sum_{i=1}^{\infty} \bar{u}_i\, v_i, \quad u = (u_i)_{i\in\mathbb{N}},\ v = (v_i)_{i\in\mathbb{N}}.$$

For any $k \in \mathbb{N}$, let $x_k \in \mathrm{L}(\mathcal{H})$ be the operator defined by

$$(x_k u)_k := u_{k+1}, \quad (x_k u)_{k+1} := u_k, \quad (x_k u)_i = 0 \text{ for } i \notin \{k, k+1\}.$$

In other words,

$$x_k u = \big(\underbrace{0, \ldots, 0}_{k-1 \text{ entries}}, u_{k+1}, u_k, 0, \ldots\big)$$

Finally, let μ the counting measure on $\mathbb{N}$ (i.e. $\mu(X) = |X|$ equals the cardinality of $X \subset \mathbb{N}$.)

(a) Show that every operator x_k has rank two, is symmetric, and has one positive and one negative eigenvalue. Make yourself familiar with the concept that every operator is a point in $\mathcal{F}$ for spin dimension $n = 1$.

(b) Let $F : \mathbb{N} \to \mathcal{F}$ be the mapping which to every k associates the corresponding operator x_k. Show that the push-forward measure $\rho = F_*\,\mu$ defined by $\rho(\Omega) := \mu(F^{-1}(\Omega))$ defines a measure on $\mathcal{F}$. Show that this measure can also be characterized by

$$\rho(\Omega) = |\{k \in \mathbb{N} \,|\, x_k \in \Omega\}|.$$

(c) Show that $(\mathcal{H}, \mathcal{F}, \rho)$ is a causal fermion system of spin dimension one.

(d) Show that the support of ρ consists precisely of all the operators x_k. What is spacetime M? What is the causal structure on M? What is the resulting causal action?

EXERCISE 5.3. *(Boundedness of operators of finite rank)* Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be a Hilbert space and $(V, \|.\|)$ a normed space of finite dimension n. Moreover, let $A : \mathcal{H} \to V$ be a linear mapping.

(a) Show that the kernel of A is a closed subspace of $\mathcal{H}$. Show that its orthogonal complement $(\ker A)^\perp$ has dimension at most n.
(b) Derive a block matrix representation of A on $\mathcal{H} = (\ker A) \oplus (\ker A)^\perp$.
(c) Deduce that A is bounded, i.e. that there is a constant $c > 0$ with $\|Au\| \leq c \,\|u\|_{\mathcal{H}}$ for all $u \in \mathcal{H}$.

EXERCISE 5.4. *(On the trace constraint)* This exercise shows that the trace constraint ensures that the action is non-zero. Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system of spin dimension n.

(a) Assume that $\mathrm{tr}(x) \neq 0$. Show that $\mathcal{L}(x, x) > 0$. (For a quantitative statement of this fact in the setting of discrete spacetimes see [**42**, Proposition 4.3].)
(b) Assume that $\int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho \neq 0$. Show that $\mathcal{S}(\rho) > 0$.

EXERCISE 5.5. *(On the spectrum of the closed chain)* This exercise is devoted to analyzing general properties of the spectrum of the closed chain.

(a) We let x and y be symmetric operators of finite rank on a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. Show that there is a finite-dimensional subspace $I \subset \mathcal{H}$ on which both x and y are invariant. By choosing an orthonormal basis of I and restricting the operators to I, we may represent both x and y by Hermitian matrices. Therefore, the remainder of this exercise is formulated for simplicity in terms of Hermitian matrices.
(b) Show that for any matrix Z, the characteristic polynomials of Z and of its adjoint Z^* (being the transposed complex conjugate matrix) are related by complex conjugation, i.e. $\det(Z^* - \overline{\lambda}\, \mathbb{1}) = \overline{\det(Z - \lambda\, \mathbb{1})}$.
(c) Let X and Y be symmetric matrices. Show that the characteristic polynomials of the matrices XY and YX coincide.
(d) Combine (b) and (c) to conclude that the characteristic polynomial of XY has real coefficients, i.e. $\det(XY - \overline{\lambda}\, \mathbb{1}) = \overline{\det(XY - \lambda\, \mathbb{1})}$. Infer that the spectrum of the matrix product XY is symmetric about the real axis, i.e.

$$\det(XY - \lambda\, \mathbb{1}) = 0 \quad \Longrightarrow \quad \det(XY - \overline{\lambda}\, \mathbb{1}) = 0\,. \tag{5.11.2}$$

(e) For the closed chain (5.7.4), the mathematical setting is somewhat different, because A_{xy} is a symmetric operator on the indefinite inner product space $(S_x, \prec .|. \succ_x)$. On the other hand, we saw after (5.7.4) that A_{xy} is isospectral to xy. Indeed, the symmetry result (5.11.2) can be used to prove a corresponding statement for A_{xy},

$$\det(A_{xy} - \lambda\, \mathbb{1}) = 0 \quad \Longrightarrow \quad \det(A_{xy} - \overline{\lambda}\, \mathbb{1}) = 0\,.$$

This result is well-known in the theory of indefinite inner product spaces (see for example the textbooks [**16, 94**] or [**42**, Section 3]). In order to derive it from (5.11.2), one can proceed as follows: First, represent the indefinite inner product in the form $\prec .|. \succ = \langle .|S\, x \rangle$, where $\langle .|. \rangle$ is a scalar product and S is an invertible operator which is symmetric (with respect to this scalar product). Next, show that the operator $B := A_{xy} S$ is symmetric (again with respect to this scalar product). Finally, write the closed chain as $A_{xy} = BS^{-1}$ and apply (5.11.2).

EXERCISE 5.6. *(Regular spacetime points)* Let $x \in \mathcal{F}$ have $p(x) \leq n$ negative and $q(x) \leq n$ positive eigenvalues. The pair $\text{sign}(x) := (p(x), q(x))$ is referred to as the *signature* of x. The operator x is said to be *regular* if $\text{sign}(x) = (n, n)$. The goal of this exercise is to show that the set $\mathcal{F}^{\text{reg}}$ of regular points is open in $\mathcal{F}$. Let us define the *positive* and *negative components* of x as the operators

$$x_\pm := \frac{x \pm |x|}{2}, \quad |x| := \sqrt{x^2}.$$

From the functional calculus it follows that $x\,|x| = |x|\,x$. Prove the following statements.

(a) Let $\{e_i\,,\ i = 1, \ldots, m\}$ be an orthogonal set. Show that any vector set $\{h_i\,,\ i = 1, \ldots, m\}$ which fulfills the following condition is linearly independent,

$$\|e_i - h_i\| < \frac{\inf\{\|e_i\|\,,\ i = 1, \ldots, m\}}{m} \quad \text{for all } i = 1, \ldots, m.$$

(b) For every $x \in \mathcal{F}$,

$$x(\operatorname{im} x_\pm) \subset \operatorname{im} x_\pm \quad \text{and} \quad x_+\, x_- = 0.$$

Moreover, $x|_{\operatorname{im} x_-}$ and $x|_{\operatorname{im} x_+}$ are negative and positive definite, respectively.

(c) Let $x_0 \in \mathcal{F}$. Then there is an orthonormal set $\{e_i \,|\, i = 1 \ldots \dim S_{x_0}\}$ of eigenvectors of x_0 such that

$$\langle e_i |\, x_0\, e_i \rangle < 0 \text{ for all } i \leq p(x_0), \quad \langle e_i |\, x_0\, e_i \rangle > 0 \text{ for all } p(x_0) < i \leq q(x_0).$$

(d) The following functions are continuous,

$$f_i : B_r(x_0) \ni x \mapsto f_i(x) := \begin{cases} x_-\, e_i & i \leq p(x_0) \\ x_+\, e_i & p(x_0) < i \leq p(x_0) + q(x_0)\,. \end{cases}$$

Hint: You can use the general inequality $\big\| |A| - |B| \big\| \leq \|A^2 - B^2\|$

(e) There is a $r > 0$ such that $p(x) \geq p(x_0)$ and $q(x) \geq q(x_0)$ for every $x \in B_r(x_0)$. *Hint: Use the statements above.*

(f) Conclude that $\mathcal{F}^{\text{reg}}$ is an open subset of $\mathcal{F}$.

EXERCISE 5.7. *(On the spectrum of the closed chain - part 2)* Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system and $x, y \in \mathcal{F}$. For the closed chain

$$A_{xy} := P(x,y)P(y,x) : (S_x, \prec.|.\succ_x) \to (S_x, \prec.|.\succ_x),$$

the mathematical setting analyzed in Exercise 3.3 is somewhat different, because A_{xy} is a symmetric operator on an *indefinite inner product space*. On the other hand, we know that A_{xy} is isospectral to xy. Indeed, the symmetry result in Exercise 3.3-(iv) can be used to prove a corresponding statement for the closed chain:

$$\det(A_{xy} - \lambda\, \mathbb{I}) = 0 \iff \det(A_{xy} - \overline{\lambda}\, \mathbb{I}) = 0.$$

This result is well-known in the theory of indefinite inner product spaces. In order to derive it from Exercise 3.3-(iv), one can proceed as follows: First, represent the indefinite inner product in the form $\prec \cdot\,,\, \cdot \succ = \langle \cdot\,, S\, \cdot \rangle$, where $\langle \cdot\,,\, \cdot \rangle$ is a scalar product and S is an invertible operator which is symmetric (with respect to this scalar product). Next, show that the operator $B := A_{xy} S$ is symmetric (again with respect to this scalar product). Finally, write the closed chain as $A_{xy} = BS^{-1}$ and apply Exercise 3.3 (iv).

EXERCISE 5.8. *(A causal causal fermion system on ℓ_2 - part 2)* We return to the example of Exercise 5.2.. This time we equip it with a *Krein structure*.

(a) For any $k \in \mathbb{N}$, construct the spin space S_{x_k} and its spin scalar product.

(b) Given a vector $u \in \mathcal{H}$, what is the corresponding wave function ψ^u? What is the Krein inner product $< ., . >$?
(c) What is the topology on the Krein space $\mathcal{K}$? Does the wave evaluation operator $\Psi : u \mapsto \psi^u$ give rise to a well-defined and continuous mapping $\Psi : \mathcal{H} \to \mathcal{K}$? If yes, is it an embedding? Is it surjective?
(d) Repeat part (c) of this exercise for the causal fermion system obtained if the operators x_k are multiplied by k, i.e.

$$x_k u := (0, \ldots, 0, k\, u_{k+1}, k\, u_k\, 0, \ldots).$$

EXERCISE 5.9. *(Time direction)* The ability to distinguish between past and future can be described in mathematical terms by the existence of an antisymmetric functional $\mathcal{T} : M \times M \to \mathbb{R}$. One then says that

$$\begin{cases} y \text{ lies in the } \textit{future} \text{ of } x & \text{if } \mathcal{T}(x,y) > 0 \\ y \text{ lies in the } \textit{past} \text{ of } x & \text{if } \mathcal{T}(x,y) < 0\,. \end{cases}$$

Can you think of simple non-trivial examples of such a functional which involve only products and linear combinations of the spacetime operators and the orthogonal projections on the corresponding spin spaces? *Hint:* One possible functional is

$$\mathcal{T}(x,y) := \mathrm{tr}\left(y\,\pi_x - x\,\pi_y\right);$$

this is considered further in [**45**, Exercise 1.22].

EXERCISE 5.10. This exercise is devoted to clarifying the connection between the characteristic polynomial and traces of powers of a matrix. We let A be an $N \times N$-matrix (not necessarily Hermitian) and denote the zeros of its characteristic polynomials counting multiplicities by $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, i.e.

$$\det(\lambda\, \mathbb{1} - A) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_N)\,.$$

Moreover, we denote the coefficients of the characteristic polynomial by a_k, i.e.

$$\det(\lambda \mathbb{1} - A) = \lambda^N + a_1 \lambda^{N-1} + \cdots + a_N\,.$$

(a) Show that the coefficients are symmetric polynomials in the eigenvalues of the form

$$a_n = c_n \sum_{\substack{B \subset \{1,\ldots,N\} \\ \text{with } \#B = n}} \prod_{k \in B} \lambda_k\,,$$

where the sum goes over all subsets of $\{1, \ldots N\}$ with n elements, and c_n are combinatorial prefactors. Compute the c_n.
(b) Show that each coefficient a_n can be written in the form

$$\begin{aligned} a_n &= d_n\, \mathrm{Tr}(A^n) + d_{1,n-1}\, \mathrm{Tr}(A)\ \mathrm{Tr}(A^{n-1}) + d_{1,1,n-2}\, \mathrm{Tr}(A)\ \mathrm{Tr}(A)\ \mathrm{Tr}(A^{n-2}) + \cdots \\ &= \sum_{k=1}^{n} \sum_{\substack{1 \leq j_1 \leq \cdots \leq j_k \\ \text{with } j_1 + \cdots + j_k = n}} d_{j_1,\ldots,j_k}\ \mathrm{Tr}\left(A^{j_1}\right) \cdots \mathrm{Tr}\left(A^{j_j}\right) \end{aligned}$$

with suitable combinatorial factors $d_n, d_{1,n-1}, \ldots$. *Hint:* This formula can be derived in various ways. One method is to proceed inductively in n. Alternatively, one can use a dimensional argument.

EXERCISE 5.11. *(Embedding of $S_x\mathscr{M}$ into $S_{F(x)}$)* The goal of this exercise is to explain how the fibers of the spinor bundle $S\mathscr{M}$ are related to the spin spaces S_x of the corresponding causal fermion system. In order to keep the setting as simple as possible, we let $(\mathscr{M}, g)$ be Minkowski space and $\mathscr{H}$ a finite-dimensional subspace of the Dirac solution space $\mathscr{H}_m$, consisting of smooth wave functions of spatially compact support, i.e.

$$\mathscr{H} \subset C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathscr{H}_m \qquad \text{finite-dimensional}\,.$$

We again let $F(x)$ be the local correlation operators, i.e.

$$\langle \psi | F(x) \psi \rangle = -\prec \psi(x) | \phi(x) \succ \qquad \text{for all } \psi, \phi \in \mathscr{H}$$

(since $\mathscr{H}$ consists of smooth functions, we may leave out the regularization operators). Defining the measure again by $\mathrm{d}\rho = F_*(\mathrm{d}^4x)$, we again obtain a causal fermion system of spin dimension $n = 2$. We next introduce the *evaluation map* e_x by

$$e_x \,:\, \mathscr{H} \to S_x\mathscr{M}\,, \qquad e_x(\psi) = \psi(x)\,.$$

Restricting the evaluation mapping to the spin space $S_{F(x)}$ at the spacetime point $F(x)$ (defined as in (5.7.1) as the image of the operator $F(x)$), we obtain a mapping

$$e_x|_{S_{F(x)}} \,:\, S_{F(x)} \to S_x\mathscr{M}\,.$$

(a) Show that $e_x|_{S_{F(x)}}$ is an isometric embedding.
(b) Show that for all $u \in \mathscr{H}$ and $x \in \mathscr{M}$,

$$e_x|_{S_{F(x)}}\big(\psi^u(F(x))\big) = u(x)\,.$$

EXERCISE 5.12. *(Identification of $S\mathscr{M}$ with SM)* In the setting of the previous exercise, we now make two additional assumptions:

(i) The mapping $F : \mathscr{M} \to \mathcal{F}$ is injective and its image is closed in $\mathcal{F}$.
(ii) The resulting causal fermion system is *regular* in the sense that for all $x \in \mathscr{M}$, the operator $F(x)$ has rank $2n$.

Using the results of the previous exercise, explain how the following objects can be identified:

(a) x with $F(x)$
(b) $\mathscr{M}$ with M
(c) The spinor space $S_x\mathscr{M}$ with the corresponding spin space $S_{F(x)}$
(d) $u \in \mathscr{H}$ with its corresponding physical wave function ψ^u

EXERCISE 5.13. *(The space $C^0(M, SM)$)* A *wave function* ψ is defined a mapping from M to H such that $\psi(x) \in S_x$ for all $x \in M$. It is most convenient to define *continuity* of a wave function by the requirement that for all $x \in M$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\big\| \sqrt{|y|}\, \psi(y) - \sqrt{|x|}\, \psi(x) \big\|_{\mathscr{H}} < \varepsilon \qquad \text{for all } y \in M \text{ with } \|y - x\| \leq \delta\,.$$

Show that, using this definition, every physical wave function is continuous. Thus, denoting the space of continuous wave functions by $C^0(M, SM)$, we obtain an embedding

$$\mathscr{H} \hookrightarrow C^0(M, SM)\,.$$

Hint: You may use the inequality

$$\Big\| \sqrt{|y|} - \sqrt{|x|} \Big\| \leq \|y - x\|^{\frac{1}{4}}\, \|y + x\|^{\frac{1}{4}}\,.$$

EXERCISE 5.14. *(A causal fermion system in $\mathbb{R}^3$)* We choose $\mathcal{H} = \mathbb{C}^2$ with the canonical scalar product. Moreover, we choose let $\mathscr{M} = S^2 \subset \mathbb{R}^3$ and $\mathrm{d}\mu$ the Lebesgue measure on $\mathscr{M}$. Consider the mapping

$$F : \mathscr{M} \to \mathrm{L}(\mathcal{H}) , \qquad F(p) = 2 \sum_{\alpha=1}^{3} p^\alpha \sigma^\alpha + \mathbb{1} ,$$

where σ^α are the three Pauli matrices (1.3.4).

(a) Show that for every $p \in S^2$,

$$\mathrm{tr}\big(F(p)\big) = 2 , \qquad \mathrm{tr}\big(F(p)^2\big) = 10 .$$

Conclude that the eigenvalues of $F(p)$ are equal to 1 ± 2.

(b) We introduce the measure ρ as the push-forward measure $\rho = F_* \mu$ (i.e. $\rho(\Omega) := \mu(F^{-1}(\Omega))$). Show that $(\mathcal{H}, \mathcal{F}, \rho)$ is a causal fermion system of spin dimension one.

(b) Show that the support of ρ coincides with the image of F. Show that it is is homeomorphic to S^2.

We refer to this example as a *Dirac sphere*. This example is also referred to as the *Dirac sphere*; this and other similar examples can be found in [**43**, Examples 2.8 and 2.9] or [**60**, Example 2.2].

EXERCISE 5.15. *(The regularized fermionic projector in Minkowski space)* The goal of this exercise is to compute the kernel of the fermionic projector in the Minkowski vacuum for the simplest regularization, the $i\varepsilon$-regularization (5.5.2).

(a) Use the identifications of Exercises 5.11 and 5.12 to show that (5.7.10) holds in the example of Dirac wave functions in Minkowski space (as constructed in Section 5.5) but now with Dirac wave functions and the spin inner product thereon.

(b) More specifically, we now choose $\mathcal{H} = \mathcal{H}_m^-$ as the subspace of all negative-frequency solutions of the Dirac equation. Moreover, we choose the $i\varepsilon$-regularization (5.5.2). For clarity, we denote the corresponding kernel of the fermionic projector by $P^{2\varepsilon}(x,y)$. Show that

$$P^\varepsilon(x,y) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \, (\not{k} + m) \, \delta(k^2 - m^2) \, \Theta(-k^0) \, \mathrm{e}^{-\mathrm{i}k(x-y)} \, \mathrm{e}^{\varepsilon k^0} . \tag{5.11.3}$$

Hint: Work in a suitable orthonormal basis of the Hilbert space. Without regularization, the computation can be found in [**45**, Lemma 1.2.8].

(c) Show that $P^\varepsilon(x,y)$ can be written as $\not{v}^\varepsilon + \beta^\varepsilon$ with $v_j^\varepsilon, \beta^\varepsilon$ smooth functions of $\xi = y - x$.

(d) Compute $P^\varepsilon(x,x)$. Is this matrix invertible? How does it scale in ε? Why does this result show that the resulting causal fermion system is regular? *Hint:* The details can also be found in [**45**, Section 2.5]. For an alternative way of proving regularity see Exercise 5.18.

(e) For ξ spacelike or timelike, i.e. away from the lightcone, the limit $\varepsilon \searrow 0$ of (5.11.3) is well-defined. More precisely, it can be shown that $v_j^\varepsilon \to \alpha \, \xi_j$ and $\beta^\varepsilon \to \beta$ pointwise, for α, β smooth complex functions. Find smooth real functions a, b such that

$$\lim_{\varepsilon \searrow 0} A^\varepsilon_{xy} = a \not{\xi} + b. \tag{5.11.4}$$

EXERCISE 5.16. *(Correspondence of the causal structure in Minkowski space I)* Let $x, y \in \mathscr{M}$ be timelike separated vectors and assume that $\xi := y - x$ is normalized

to $\xi^2 = 1$. As explained in Exercise 5.15, the limit $\varepsilon \searrow 0$ of the closed chain A^ε_{xy} takes the form $A := a \,\xi\!\!\!/ + b$. Consider the matrices

$$F_\pm := \frac{1}{2} (\mathbb{1} \pm \xi\!\!\!/) \in \mathrm{L}(\mathbb{C}^4) .$$

Prove the following statements.

(a) The matrices $F_\pm$ have rank two and map to eigenspaces of A. What are the corresponding eigenvalues? Conclude that the points x and y are timelike separated in the sense of Definition 5.6.1.
(b) The matrices $F_\pm$ are idempotent and symmetric with respect to the spin inner product $\prec .|. \succ$.
(c) The image of the matrices $F_\pm$ is positive or negative definite (again with respect to the spin inner product).
(d) The image of F_+ is orthogonal to that of F_- (again with respect to the spin inner product).
(e) The eigenvalues of A are strictly positive. *Hint:* Use how the functions a and b came up in (5.11.4).

The result of (a)–(d) can be summarized by saying that the $F_\pm$ are the spectral projection operators of A. We remark that the findings also mean that the x and y are even *properly timelike separated* as introduced in [**45**, Definition 1.1.6].

EXERCISE 5.17. *(Correspondence of the causal structure in Minkowski space II)* We now let $x, y \in \mathscr{M}$ be spacelike separated vectors and assume that $\xi := y - x$ is normalized to $\xi^2 = -1$. Consider again the matrix $A := a \,\xi\!\!\!/ + b$ of Exercise 5.16 and set

$$F_\pm := \frac{1}{2} (\mathbb{1} \pm \mathrm{i}\, \xi\!\!\!/) \in \mathrm{L}(\mathbb{C}^4) .$$

(a) The matrices $F_\pm$ have rank two and map to eigenspaces of A. What are the corresponding eigenvalues? Conclude that the points x and y are spacelike separated in the sense of Definition 5.6.1.
(b) The matrices $F_\pm$ are idempotent and $F_+^* = F_-$.
(c) The image of the matrices $F_\pm$ is null (in other words, it is a lightlike subspace of the spinor space).

These findings illustrate the more general statement that symmetric operators on an indefinite inner product space may have complex eigenvalues, in which case they form complex conjugate pairs.

EXERCISE 5.18. *(Spin spaces for the regularized Dirac sea vacuum)* We consider the causal fermion system constructed in Section 5.5, where we choose $\mathcal{H} = \mathcal{H}_m^-$ as the space of all negative-energy solutions of the Dirac equation. Moreover, we choose the $i\varepsilon$-regularization (5.5.2). For clarity, we denote the corresponding kernel of the fermionic projector by $P^\varepsilon(x,y)$. This causal fermion system is also referred to as the *regularized Dirac sea vacuum*.

(a) Let Σ_0 denote the Cauchy surface at time $t = 0$. Show that, for any $x \in \mathscr{M}$ and $\chi \in \mathbb{C}^4$,

$$(\mathrm{i}\partial\!\!\!/ - m) P^\varepsilon(\,\cdot\,, x)\chi = 0 \quad \text{and} \quad P^\varepsilon(\,\cdot\,, x)\chi\big|_{\Sigma_0} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) .$$

Conclude that $P^\varepsilon(\,\cdot\,, x)\chi \in \mathcal{H}_m^- \cap C^\infty(\mathbb{R}^4, \mathbb{C}^4)$.

(b) Convince yourself that

$$\mathfrak{R}_\varepsilon(P^\varepsilon(\,\cdot\,,x)\chi) = P^{2\varepsilon}(\,\cdot\,,x)\chi\,.$$

(c) Let $\{\mathfrak{e}_1,\ldots,\mathfrak{e}_4\}$ denote the canonical basis of $\mathbb{C}^4$. Using Exercise 5.15 (b), show that the wave functions $P^\varepsilon(\,\cdot\,,x)\mathfrak{e}_\mu$ for $\mu=1,2,3,4$ are linearly independent.

(d) Let $S_x := F^\varepsilon(x)(\mathcal{H}_m^-)$ endowed with $\prec u,v\succ_x := -\langle u|F^\varepsilon(x)v\rangle$ be the *spin space* at $x\in\mathscr{M}$. Show that the following mapping is an isometry of indefinite inner products (i.e. injective and product preserving),

$$\Phi_x : S_x \ni u \mapsto \mathfrak{R}_\varepsilon u(x) \in \mathbb{C}^4.$$

Conclude that the causal fermion system is regular at $x\in\mathscr{M}$, i.e. $\dim S_x = 4$, if and only if there exist vectors $u_\mu\in\mathcal{H}_m^-$, for $\mu=1,2,3,4$, such that the $\mathfrak{R}_\varepsilon u_\mu(x)\in\mathbb{C}^4$ are linearly independent.

(e) Conclude that the causal fermion system is regular at every spacetime point.

CHAPTER 6

Causal Variational Principles

The causal action principle as introduced in Section 5.6 has quite a rich structure and is rather involved. Therefore, it is difficult to analyze it in full generality in one step. It is preferable to begin with special cases and simplified situations, and to proceed from there step by step. In fact, doing so leads to a whole class of variational principles, referred to as *causal variational principles*. These different variational principles capture different features and aspects of the causal action principle. Proceeding in this way also gives a better understanding of the physical meaning of the different structures of a causal fermion system and of the interaction as described by the causal action principle. We now give an overview of the different settings considered so far. This has the advantage that in the later chapters of this book, we can always work in the setting which is most suitable for the particular question in mind. Moreover, for pedagogical reasons, in this book we shall sometimes idealize the setting for example by assuming for technical simplicity that the Lagrangian is smooth.

6.1. The Causal Variational Principle on the Sphere

Clearly, the trace constraint (5.6.4) and the boundedness constraint (5.6.5) complicate the analysis. Therefore, it might be a good idea to consider a simplified setting where these constraints are not needed. This can be accomplished most easily by *prescribing the eigenvalues* of the operators in $\mathcal{F}$. This method was first proposed in [**43**, Section 2] in a slightly different formulation. We now explain the method in a way which fits best to our setting. Given $n \in \mathbb{N}$, we choose real numbers $\nu_1, \ldots, \nu_{2n}$ with

$$\nu_1 \leq \cdots \leq \nu_n \;\leq 0 \;\leq\; \nu_{n+1} \leq \ldots \leq \nu_{2n} \,. \tag{6.1.1}$$

We let $\mathcal{F}$ be the set of all symmetric operators on $\mathcal{H}$ of rank $2n$ whose eigenvalues (counted with multiplicities) coincide with $\nu_1, \ldots, \nu_{2n}$. If $\mathcal{H}$ is finite-dimensional, the set $\mathcal{F}$ is compact. This is the reason why it is sensible to minimize the causal action (5.6.2) keeping only the volume constraint (5.6.3), which for simplicity we implement by restricting attention to normalized measures,

$$\rho(\mathcal{F}) = 1 \,.$$

Note that, since $\mathcal{F}$ is compact and the Lagrangian $\mathcal{L}$ is continuous on $\mathcal{F} \times \mathcal{F}$, also the action $S(\rho)$ is finite for any normalized measure ρ.

The simplest interesting case is obtained by choosing the spin dimension $n = 1$ and the Hilbert space $\mathcal{H} = \mathbb{C}^2$. In this case, according to (6.1.1) we have one non-negative and one non-positive eigenvalue. If these eigenvalues have the same absolute value, all the operators have trace zero. This case is not of interest because there are trivial minimizers (for details see Example 12.4.1 in Section 12.4). With this in mind, it suffices to consider the case that $|\nu_1| \neq |\nu_2|$. Since scaling all the eigenvalues in (6.1.1) by a real constant does not change the essence of the variational principle, it is no loss of generality to assume

that the two eigenvalues ν_1, ν_2 satisfy the relation $\nu_1 + \nu_2 = 2$, making it possible to parametrize the eigenvalues by

$$\nu_{1/2} = 1 \mp \tau \qquad \text{with} \qquad \tau \geq 1\,.$$

Then $\mathcal{F}$ consists of all Hermitian 2×2-matrices F which have eigenvalues ν_1 and ν_2. These matrices can be represented using the Pauli matrices by

$$\mathcal{F} = \left\{ F = \tau\, \vec{x} \vec{\sigma} + \mathbb{1} \quad \text{with} \quad \vec{x} \in S^2 \subset \mathbb{R}^3 \right\}. \tag{6.1.2}$$

Thus the set $\mathcal{F}$ can be identified with the unit sphere S^2.

The volume constraint (5.6.3) can be implemented most easily by restricting attention to *normalized* regular Borel measures on $\mathcal{F}$ (i.e. measures with $\rho(\mathcal{F}) = 1$). Such a measure can be both continuous, discrete or a mixture. Examples of *continuous measures* are obtained by multiplying the Lebesgue measure on the sphere by a non-negative smooth function. By a *discrete measure*, on the other hand, we here mean a *weighted counting measure*, i.e. a measure obtained by inserting weight factors into (5.3.2),

$$\rho = \sum_{i=1}^{L} c_i\, \delta_{x_i} \qquad \text{with} \qquad x_i \in \mathcal{F}\,, \quad c_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{L} c_i = 1\,. \tag{6.1.3}$$

In this setting, a straightforward computation yields for the Lagrangian (5.6.1) (see Exercise 6.1)

$$\begin{aligned} &\mathcal{L}(x,y) = \max\big(0, \mathcal{D}(x,y)\big) \qquad \text{with} \\ &\mathcal{D}(x,y) = 2\tau^2 \left(1 + \langle x,y\rangle\right)\Big(2 - \tau^2 \left(1 - \langle x,y\rangle\right)\Big)\,, \end{aligned} \tag{6.1.4}$$

and $\langle x,y\rangle$ denotes the Euclidean scalar product of two unit vectors $x, y \in S^2 \subset \mathbb{R}^3$ (thus $\langle x,y\rangle = \cos\vartheta$, where ϑ is the angle between x and y).

The resulting so-called *causal variational principle on the sphere* was introduced in [**43**, Chapter 1] and analyzed in [**84**, Sections 2 and 5] and more recently in [**10**]. We now explain a few results from these papers.

First of all, the causal variational principle on the sphere is well-posed, meaning that the minimum is attained in the class of all normalized regular Borel measures; (the proof of this statement will be given in Chapter 12 using measure-theoretic methods to be developed later in this book). Minimizing numerically in the class of weighted counting measures for increasing number L of points and different values of the parameter τ, the resulting minimal value of the action has an interesting non-smooth structure shown in Figure 6.1. In particular, one finds that the minimizing measure is not unique; indeed, there are typically many minimizers. From the mathematical perspective, this non-uniqueness can be understood from the fact that the causal action principle is a *non-convex* variational principle where one cannot expect uniqueness. To give a concrete example for the non-uniqueness, we note that in the case $\tau = \tau_c = \sqrt{2}$, there is a minimizing measure which is supported on an octahedron (for details see [**84**, Section 2]). This minimizing measure is not unique, because every measure obtained from it by a rotation in SO(3) is again a minimizer. But the non-uniqueness goes even further in the sense that there are pairs of minimizing measures which cannot be obtained from each other by a rotation or reflection. For example, again in the case $\tau = \sqrt{2}$, the normalized Lebesgue measure on the sphere is also a minimizer.

Moreover, the study in [**84**, Section 2] gives the following

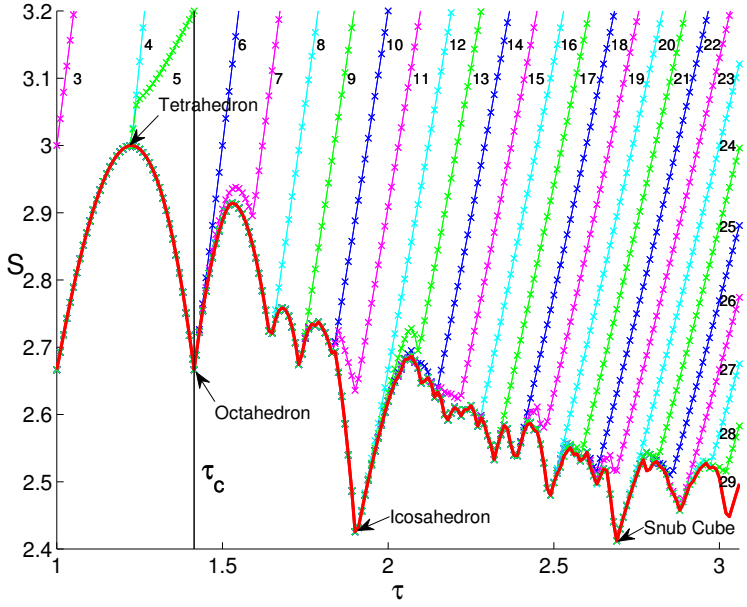


FIGURE 6.1. Numerical minima for the weighted counting measure on the sphere.

> *numerical result*: If $\tau > \sqrt{2}$, every minimizing measure is a weighted counting measure (6.1.3).

Thus, although we minimize over all regular Borel measures (i.e. measures which can have discrete and continuous components), a minimizing measure always describes a discrete spacetime consisting of a finite number of spacetime points. This result can be interpreted physically as an indication that the causal action principle should give rise to discrete spacetime structures. More details on the numerical findings and the physical interpretation can be found in the review [**57**, Section 4]. A more advanced numerical study of the causal action principle in low dimensions can be found in [**59**].

The above numerical findings can be underpinned by analytic results. We finally mention some of these results, although they will not be needed later on, and the methods of proof will not be covered in this book. First, it was proven in [**84**, Section 5.1] that the support has an empty interior:

THEOREM 6.1.1. *If $\tau > \sqrt{2}$, the support of any minimizing measure does not contain an open subset of S^2.*

Intuitively speaking, this result shows that the spacetime points are a subset of the sphere of dimension strictly smaller than two. More recently, it was shown in [**10**] that the dimension of the support is even strictly smaller than one:

THEOREM 6.1.2. *In the case $\tau > \sqrt{6}$, the support of any minimizing measure is totally disconnected and has Hausdorff dimension at most* $6/7$.

The proof of these theorems uses techniques which will not be covered in this book. Therefore, we refer the reader interested in more details to the papers cited above.

6.2. Causal Variational Principles in the Compact and Smooth Settings

Generalizing the causal variational principle on the sphere, one can replace $\mathcal{F}$ by a smooth compact manifold of dimension $m \geq 1$.

DEFINITION 6.2.1. *Let $\mathcal{F}$ be a smooth compact manifold of dimension $m \geq 1$ and $\mathcal{D} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R})$. Define the Lagrangian $\mathcal{L} \in C(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)$ by*

$$\mathcal{L}(x,y) := \max\big(0, \mathcal{D}(x,y)\big) \tag{6.2.1}$$

and assume that $\mathcal{L}$ has the following properties:

(i) *$\mathcal{L}$ is symmetric: $\mathcal{L}(x,y) = \mathcal{L}(y,x)$ for all $x, y \in \mathcal{F}$.*
(ii) *$\mathcal{L}$ is strictly positive on the diagonal: $\mathcal{L}(x,x) > 0$ for all $x \in \mathcal{F}$.*

The **causal variational principle in the compact setting** *is to minimize the causal action*

$$\mathcal{S} = \int_{\mathcal{F}} \mathrm{d}\rho(x) \int_{\mathcal{F}} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \tag{6.2.2}$$

under variations of measures ρ in the class of all regular Borel measures on $\mathcal{F}$ which are normalized, i.e.

$$\rho(\mathcal{F}) = 1\,. \tag{6.2.3}$$

This setting was introduced in [**84**, Section 1.2]. It is the preferable choice for studying phenomena for which the detailed form of the Lagrangian as well as the constraints of the causal action principle are irrelevant. Note that also in the compact setting the action $S(\rho)$ is finite for any normalized measure ρ, because $\mathcal{F}$ is compact and $\mathcal{L}$ is continuous.

Given a minimizing measure ρ, the Lagrangian induces on spacetime $M := \operatorname{supp} \rho$ a causal structure. Namely, two spacetime points $x, y \in M$ are said to be *timelike* and *spacelike* separated if $\mathcal{L}(x,y) > 0$ and $\mathcal{L}(x,y) = 0$, respectively. But, of course, compared to the causal action principle for causal fermion systems, spin spaces and physical wave functions (as defined in Section 5.7) are missing in this setting.

We point out that in (6.2.1) we merely assumed that the function $\mathcal{D}$ is smooth. The Lagrangian, however, is only Lipschitz continuous. It is in general non-differentiable on the boundary of the light cone as defined by the level set $\mathcal{D}(x,y) = 0$. In order to avoid differentiability issues, it is sometimes useful to simplify the setting even further by assuming that the Lagrangian itself is smooth,

$$\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)\,. \tag{6.2.4}$$

This is the so-called **smooth setting**. We point out that the Lagrangian of the causal action (5.6.1) is not smooth if some of the eigenvalues vanish or are degenerate (more precisely, the causal Lagrangian is only Hölder continuous, as is worked out in detail in [**76**, Section 5]). Indeed, this non-smoothness yields interesting effects like the results on the singular support in [**84, 10**]. In view of these results, the smoothness assumption (6.2.4) is a mathematical simplification which, depending on the application in mind, may or may not be justified. In this book, we choose the smooth setting mainly for pedagogical reasons, keeping in mind that the generalizations to non-smooth Lagrangians are rather straightforward. The reader who is interested in or needs these generalizations will find the details in the research papers.

Before going on, we point out that the assumptions that $\mathcal{F}$ is a smooth manifold and that the function $\mathcal{D}$ in (6.2.1) is smooth are convenient and avoid certain technicalities. But these assumptions are much more than what is needed for the analysis. More

generally, one can choose $\mathcal{L}$ as a non-negative continuous function,

$$\mathcal{L} \in C^0(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)\,. \tag{6.2.5}$$

Going one step further, one may relax the continuity of the Lagrangian by the condition that $\mathcal{L}$ be *lower semi-continuous*, i.e. that for all sequences $x_n \to x$ and $y_{n'} \to y$,

$$\mathcal{L}(x,y) \leq \liminf_{n,n' \to \infty} \mathcal{L}(x_n, y_{n'})\,.$$

Since the Lagrangian of the causal action (5.6.1) is continuous, lower semi-continuity is an unphysical generalization. Nevertheless, this setting is useful for two reasons: First, from the point of view of the calculus of variations, it is a natural generalization to which most methods still apply. And second, lower semi-continuous Lagrangians are convenient for formulating explicit examples (like the lattice model in [**71**, Section 5]).

We finally note that also the assumption of $\mathcal{F}$ being a smooth manifold can be relaxed. From the point of view of calculus of variations, the right setting is to assume that $\mathcal{F}$ is a compact topological Hausdorff space.

In this book, for pedagogical reasons we do not aim for the highest generality and minimal smoothness and regularity assumptions. An introduction to a more general and more abstract setting can be found in [**75**, Section 3].

6.3. Causal Variational Principles in the Non-Compact Setting

In the compact setting, spacetime M is a compact subset of $\mathcal{F}$. This is not suitable for describing situations when spacetime has an asymptotic future or past or when spacetime has singularities (like at the big bang or inside a black hole). For studying such situations, it is preferable to work in the so-called *non-compact setting* introduced in [**71**, Section 2.1] where $\mathcal{F}$ is chosen to be a non-compact manifold.

In the non-compact setting, it is not sensible to restrict attention to normalized measures. Instead, the total volume $\rho(\mathcal{F})$ is typically infinite. In this situation, the causal action (6.2.2) could also be infinite. Therefore, we need to define in which sense a measure is a minimizer of the action.

DEFINITION 6.3.1. **(Causal variational principles in the non-compact setting)** *Let $\mathcal{F}$ be a non-compact smooth manifold of dimension $m \geq 1$, and let $\mathcal{D} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R})$ be such that the Lagrangian $\mathcal{L} \in C^0(\mathcal{F} \times \mathcal{F}; \mathbb{R}_0^+)$ defined by* (6.2.1) *has the properties* (i) *and* (ii) *in Definition 6.2.1. Given a regular Borel measure ρ on $\mathcal{F}$, another regular Borel measure $\tilde{\rho}$ on $\mathcal{F}$ is a* **variation** *of ρ* **of finite volume** *if it satisfies the conditions*

$$\big|\tilde{\rho} - \rho\big|(\mathcal{F}) < \infty \qquad \text{and} \qquad \big(\tilde{\rho} - \rho\big)(\mathcal{F}) = 0\,, \tag{6.3.1}$$

where $|\tilde{\rho} - \rho|$ is the total variation measure (see Definition 2.3.5 in Section 2.3). For such a variation of finite volume we consider the (formal) difference of the actions defined by

$$\begin{aligned} \big(\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)\big) := &\int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\ &+ \int_{\mathcal{F}} \mathrm{d}\rho(x) \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(y)\, \mathcal{L}(x,y) \\ &+ \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(y)\, \mathcal{L}(x,y)\,. \end{aligned} \tag{6.3.2}$$

The measure ρ is said to be a **minimizer** *of the causal action with respect to variations of finite volume if this difference is non-negative for all $\tilde{\rho}$ satisfying* (6.3.1)*,*

$$\big(\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)\big) \geq 0\,. \tag{6.3.3}$$

We note for clarity that integrals with respect to $\tilde{\rho} - \rho$ are defined by

$$\int_{\mathcal{F}} f(x)\, \mathrm{d}(\tilde{\rho} - \rho)(x) := \int_{\mathcal{F}} f \;\mathrm{d}\mu^+ - \int_{\mathcal{F}} f \;\mathrm{d}\mu^-$$

with the finite measures $\mu_\pm$ as given in Definition 2.3.5. In particular, $(\tilde{\rho} - \rho)(\mathcal{F}) := \mu^+(\mathcal{F}) - \mu^-(\mathcal{F})$.

Exactly as mentioned at the end of the previous section, the assumption that $\mathcal{F}$ is a smooth manifold could be weakened. From the point of view of calculus of variations, the right setting is to assume that $\mathcal{F}$ is a σ-locally compact topological Hausdorff space (for details see again [**75**, Section 3]).

6.4. The Local Trace is Constant

Causal variational principles as introduced in the previous sections are of interest in their own right as a novel class of nonlinear variational principles. Nevertheless, since we are primarily interested in causal fermion systems, it is important to get a concise mathematical connection to the causal action principle. In preparation, we now analyze the trace constraint and derive a first general result on minimizing measures of the causal action principle. We present this result at such an early stage of this book because this result can be used to simplify the setup of causal fermion systems, getting the desired connection to causal variational principles (see Section 6.5 below). The following result was first obtained in [**13**] (albeit with a different method); see also [**45**, Proposition 1.4.1]. For technical simplicity, we restrict attention to the finite-dimensional setting (in the infinite-dimensional case, this problem has not yet been studied). Then the total volume of spacetime as well as the minimal action are finite.

PROPOSITION 6.4.1. *Consider the causal action principle in the finite-dimensional setting* $\dim \mathcal{H} < \infty$*. Let ρ be a minimizer of finite total volume, $\rho(\mathcal{F}) < \infty$. Then there is a real constant c such that*

$$\mathrm{tr}(x) = c \qquad \text{for all } x \in M := \mathrm{supp}\, \rho\,.$$

We often refer to $\mathrm{tr}(x)$ as the *local trace* at the point x.

PROOF OF PROPOSITION 6.4.1. We will prove the theorem by contradiction and therefore assume that the local trace is *not* constant. The idea is to use this assumption to construct a suitable variation

$$(\rho_\tau)_{\tau \in (-\delta, \delta)} \qquad \text{with} \qquad \rho_0 = \rho$$

which satisfies the constraints, but makes the action smaller, in contradiction to ρ being a minimizer.

For the construction of the variation we combine two different general methods. One method is to multiply the measure ρ by a positive measurable function $f_\tau : M \to \mathbb{R}^+$,

$$\rho_\tau = f_\tau\, \rho$$

(alternatively, one can also write this relation as $\mathrm{d}\rho_\tau(x) = f_\tau(x)\, \mathrm{d}\rho(x)$). Clearly, such a variation does not change the support of the measure. In order to change the support,

our second method is to consider a measurable function $F_\tau : M \to \mathcal{F}$ and take the push-forward measure,

$$\rho_\tau = (F_\tau)_* \rho \,.$$

Combining these two methods, we are led to considering variations of the form

$$\rho_\tau = (F_\tau)_* \big(f_\tau\, \rho\big) \,. \tag{6.4.1}$$

The condition $\rho_0 = \rho$ gives rise to the conditions

$$f_0 \equiv 1 \qquad \text{and} \qquad F_0 \equiv \mathbb{1} \,. \tag{6.4.2}$$

Finally, we assume that the functions f_τ and F_τ are smooth in τ. The ansatz (6.4.1) is particularly convenient for computations. Namely, by definition of the push-forward measure,

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho_\tau(y) = \int_M \mathcal{L}\big(x, F_\tau(y)\big)\, f_\tau(y)\, \mathrm{d}\rho(y) \,,$$

and similarly for all other integrals.

Next, for arbitrarily given f_τ, we want to choose F_τ in such a way that the last integral becomes independent of τ. To this end, we choose

$$F_\tau(x) := \frac{x}{\sqrt{f_\tau(x)}} \,.$$

Using that the causal Lagrangian $\mathcal{L}(x,y)$ is homogeneous of degree two in y (as is obvious from (5.6.1) and the fact that the eigenvalues λ_i^{xy} are homogeneous of degree one in y), it follows that

$$\begin{aligned} \int_M \mathcal{L}\big(x, F_\tau(y)\big)\, f_\tau(y)\, \mathrm{d}\rho(y) &= \int_M \mathcal{L}\left(x, \frac{y}{\sqrt{f_\tau(y)}}\right) f_\tau(y)\, \mathrm{d}\rho(y) \\ &= \int_M \mathcal{L}(x,y) \frac{1}{f_\tau(y)}\, f_\tau(y)\, \mathrm{d}\rho(y) = \int_M \mathcal{L}(x,y)\, \mathrm{d}\rho(y) \,. \end{aligned}$$

Arguing similarly in the variable x, one sees that our variation does not change the action. Since the integrand $|\lambda_j^{xy}|^2$ of the boundedness constraint (5.6.5) is again homogeneous of degree two in x and y, the above argument applies similarly to the functional $\mathcal{T}$, showing that the boundedness constraint is respected by our variations.

Let us analyze the volume and trace constraints. In order to satisfy the volume constraint, we make the ansatz

$$f_\tau = 1 + \tau g \,, \tag{6.4.3}$$

where g is a bounded function with zero mean,

$$\int_M g(x)\, \mathrm{d}\rho(x) = 0 \,. \tag{6.4.4}$$

This ensures that the volume constraint is satisfied. We finally consider the variation of the trace constraint,

$$\begin{aligned}
&\int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho_\tau(x) - \int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho(x) \\
&= \int_M \mathrm{tr}\big(F_\tau(x)\big)\, f_\tau(x)\, \mathrm{d}\rho(x) - \int_M \mathrm{tr}(x)\, \mathrm{d}\rho(x) \\
&= \int_M \mathrm{tr}\Big(\frac{x}{\sqrt{f_\tau(x)}}\Big)\, f_\tau(x)\, \mathrm{d}\rho(x) - \int_M \mathrm{tr}(x)\, \mathrm{d}\rho(x) \\
&= \int_M \mathrm{tr}(x)\, \Big(\sqrt{f_\tau(x)} - 1\Big)\, \mathrm{d}\rho(x)\,.
\end{aligned}$$

Employing again the ansatz (6.4.3) and differentiating with respect to τ, we obtain for the first variation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho_\tau \Big|_{\tau=0} = \frac{1}{2} \int_M \mathrm{tr}(x)\, g(x)\, \mathrm{d}\rho(x) \tag{6.4.5}$$

(here we may differentiate the integrand using Lebesgue's dominated convergence theorem). Now we use of our assumption that the local trace is *not* constant on M, making it possible to choose $x_1, x_2 \in M$ with $\mathrm{tr}(x_1) > \mathrm{tr}(x_2)$. Moreover, we choose a function g which supported in a small neighborhood of x_1 and x_2, has zero mean (6.4.4) and is positive at x_1 and negative at x_2. In this way, we can arrange that the right side of (6.4.5) is strictly positive. Hence, using (6.4.4), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\big(\rho_\tau - \rho\big)(x) \Big|_{\tau=0} > 0\,. \tag{6.4.6}$$

To summarize, we have found a variation which respects the boundedness and the volume constraint and preserves the causal action, but increases the integral of the trace (6.4.6).

The final step is to modify the variation in such a way that the trace and volume constraints are respected, whereas the action and the boundedness constraints become smaller. To this end, we transform the measures according to

$$\rho_\tau \to (G_\tau)_*(\rho_\tau)$$

with

$$G_\tau(x) = x \left(\int_M \mathrm{tr}(x)\, \mathrm{d}\rho \right) \Big/ \left(\int_M \mathrm{tr}(x)\, \mathrm{d}\rho_t \right). \tag{6.4.7}$$

A short computation shows that the trace constraint is respected, and so is the volume constraint. Moreover, in view of (6.4.6), for small positive τ the scaling factor in (6.4.7) is strictly smaller than one, implying that the first variations of the action and of the boundedness constraint are both strictly negative (here we use again the homogeneity of the Lagrangian and of the integrand of the boundedness constraint). This is a contradiction to the fact that ρ is a minimizer (here we make essential use of the fact that the boundedness constraint (5.6.5) is an *inequality* constraint, so that decreasing $\mathcal{T}$ in the variation is allowed). We conclude that the local trace must be constant. □

6.5. How the Causal Action Principle Fits into the Non-Compact Setting

Under mild technical assumptions on the minimizing measure, the causal action principle for causal fermion systems is a special case of the causal variational principle in the non-compact setting, as we now explain.

Since for minimizers of the causal action principle, all operators in M have the same trace (see Proposition 6.4.1), we can simplify the setting by restricting attention to linear operators in $\mathcal{F}$ which all have the same trace. Then the trace constraint can be disregarded, as it follows from the volume constraint. We now implement this simplification by modifying our setting. At the same time, we implement the boundedness constraint by a *Lagrange multiplier* term. Here we apply this method naively by modifying the Lagrangian to

$$\mathcal{L}_\kappa(x,y) := \frac{1}{4n} \sum_{i,j=1}^{2n} \Big(|\lambda_i^{xy}| - |\lambda_j^{xy}| \Big)^2 + \kappa \left(\sum_{i=1}^{2n} |\lambda_i^{xy}| \right)^2 , \tag{6.5.1}$$

where $\kappa > 0$ is the Lagrange multiplier. We refer to $\mathcal{L}_\kappa$ as the *κ-Lagrangian*. The justification for this procedure as given in [**13**] is a bit subtle, and for brevity we shall not enter these constructions here. It is important to note that, in contrast to the usual Lagrange multiplier, where a minimizer under constraints in general merely is a critical point of the Lagrangian including the Lagrange multipliers, here we obtain again a *minimizer* of the effective action (for details see [**13**, Theorem 3.13]).

Finally, we make a mild technical simplification. A spacetime point $x \in M$ is said to be *regular* if x has the maximal possible rank $2n$. Otherwise, the spacetime point is *singular*. In typical physical applications, all spacetimes points are regular, except maybe at singularities like the center of black holes. For example, the construction of a causal fermion system in Minkowski space from Dirac wave functions in Section 5.5 gives regular spacetime points if $\mathcal{H}$ is chosen sufficiently large (in particular for all negative-frequency solutions). More generally, the interacting systems considered in [**45**, Chapters 3-5] all have regular spacetime points. The same is true for the similar construction in globally hyperbolic spacetimes (for details see [**47**]). With this in mind, in this book we usually assume that the *causal fermion system* is *regular* in the sense that all spacetime points are regular. This assumption has the advantage that the set of all regular points of $\mathcal{F}$ is a smooth manifold (see Proposition 3.1.3 in Section 3.1). We remark that, in the case that $\mathcal{H}$ is infinite-dimensional, the set of regular points of $\mathcal{F}$ can be endowed with the structure of a Banach manifold (for details see [**76**, Section 3]). These consideration lead us to the following setting:

DEFINITION 6.5.1. *Let $\mathcal{H}$ be a complex Hilbert space. Moreover, suppose we are given parameters $n \in \mathbb{N}$ (the spin dimension), $c > 0$ (the constraint for the local trace) and $\kappa > 0$ (the Lagrange multiplier of the boundedness constraint). We then let $\mathcal{F}^{\text{reg}} \subset \mathrm{L}(\mathcal{H})$ be the set of all symmetric operators F on $\mathcal{H}$ with the following properties:*

(a) *F has finite rank and (counting multiplicities) has exactly n positive and n negative eigenvalues.*

(b) *The local trace is equal to c, i.e.*

$$\mathrm{tr}(F) = c\,.$$

On $\mathcal{F}^{\text{reg}}$ we again consider the topology induced by the sup-norm on $\mathrm{L}(\mathcal{H})$. The **reduced causal action principle for regular systems** *is to minimize the reduced causal action*

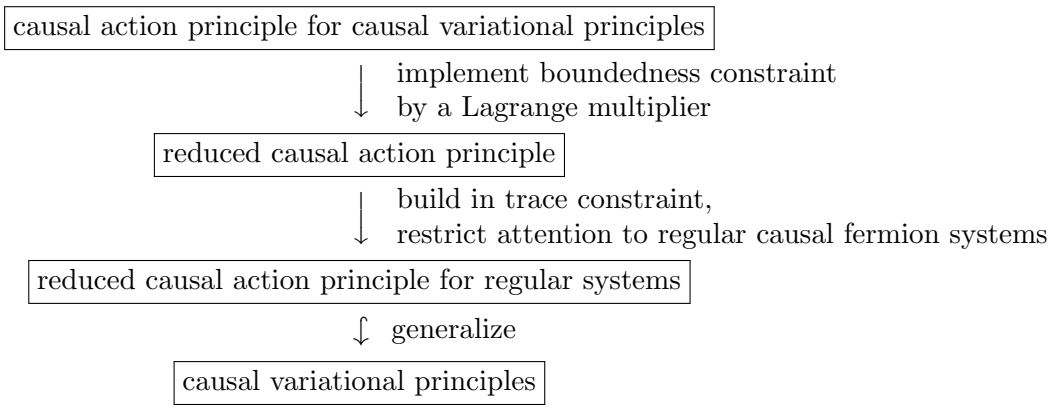
$$\mathcal{S}_\kappa(\rho) = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}_\kappa(x,y)\, \mathrm{d}\rho(x)\, \mathrm{d}\rho(y)$$

over all regular Borel measures under variations which preserve the total volume.

In this way, the causal action principle fits into the framework of causal variational principles in the non-compact setting as introduced in Section 6.3. In agreement with (6.2.5), the causal Lagrangian is continuous (in fact, it is even locally Hölder continuous; for details see [**76**, Section 5.1]). Moreover, it has the desired properties (i) and (ii) on page 124 (it is strictly positive because the Lagrangian can be estimated from below in terms of the local trace; see Exercise 5.4).

In order to avoid misunderstandings, we point out that the above description of causal fermion systems by measures on $\mathcal{F}^{\text{reg}}$ is not a suitable setting for the existence theory (as will be developed in Chapter 12). The reason is that $\mathcal{F}^{\text{reg}}$ is not closed in $\mathcal{F}$, because the boundary points in $\mathcal{F}$ are missing (in a converging sequence, some of the eigenvalues could tend to zero in the limit). As a consequence, considering a minimizing sequence $(\rho_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{F}^{\text{reg}}$, the limiting measure might well be supported also on $\mathcal{F} \setminus \mathcal{F}^{\text{reg}}$. For this reason, there is no existence theory for measures on $\mathcal{F}^{\text{reg}}$. But if a minimizing measure is given, it seems sensible to assume that the resulting causal fermion system is regular. Under this assumption, the analysis of the causal fermion system can be carried out exclusively in $\mathcal{F}^{\text{reg}}$, whereas $\mathcal{F}$ is no longer needed. For a convenient and compact notation, in such situations we shall even omit the superscript "reg", so that $\mathcal{F}$ denotes the set of all symmetric operators on $\mathcal{H}$ with the above properties (a) and (b). Moreover, we shall omit the subscript κ. Thus, with a slight abuse of notation, we shall denote the Lagrangian including the Lagrange multiplier term (6.5.1) again by $\mathcal{L}$.

In this way, assuming that the causal fermion systems under consideration are regular, we have recovered the causal action principle as a specific causal variational principle. The connection is summarized schematically as follows:



Whenever the specific form of the causal Lagrangian (6.5.1) is not needed, we will work in the more general setting of causal variational principles. Apart from the sake of greater generality, this has the advantage that it becomes clearer which structures are needed for which results. Moreover, it is often more convenient not to specify the form of the Lagrangian. Generally speaking, we can work with causal variational principles unless the physical wave functions and their induced geometric and analytic structures are invoked.

6.6. Exercises

EXERCISE 6.1. *(Derivation of the causal variational principle on the sphere)* We consider the causal fermion systems in the case $n = 1$ and $f = 2$. For a given parameter $\tau > 1$ we introduce the mapping $F : M \to \mathcal{F}$ by

$$F(\vec{x}) = \tau\, \vec{x}\vec{\sigma} + \mathbb{1}\,. \tag{6.6.1}$$

(a) Compute the eigenvalues of the matrix $F(\vec{x})$ and verify that it has one positive and one negative eigenvalue.

(b) Use the identities between Pauli matrices

$$\sigma^i \sigma^j = \delta^{ij} + \mathrm{i} \epsilon^{ijk}\, \sigma^k \,,$$

in order to compute the matrix product,

$$F(\vec{x})\, F(\vec{y}) = \big(1 + \tau^2\, \vec{x}\vec{y}\big)\, \mathbb{1} + \tau\, (\vec{x} + \vec{y}) \vec{\sigma} + \mathrm{i} \tau^2\, (\vec{x} \wedge \vec{y}) \vec{\sigma}.$$

(c) compute the eigenvalues of this matrix product to obtain

$$\lambda_{1/2} = 1 + \tau^2 \cos\vartheta \pm \tau \sqrt{1 + \cos\vartheta}\, \sqrt{2 - \tau^2\, (1 - \cos\vartheta)}\,,$$

where ϑ denotes the angle ϑ between $\vec{x}$ and $\vec{y}$.

(d) Verify that if $\vartheta \leq \vartheta_{\max}$ with

$$\vartheta_{\max} := \arccos\left(1 - \frac{2}{\tau^2}\right),$$

then the eigenvalues $\lambda_{1/2}$ are both real. Conversely, if $\vartheta > \vartheta_{\max}$, then the eigenvalues form a complex conjugate pair.

(e) Use the formula

$$\lambda_1 \lambda_2 = \det(F(\vec{x}) F(\vec{y})) = \det(F(\vec{x}))\, \det(F(\vec{y})) = (1+\tau)^2 (1-\tau)^2 > 0$$

to conclude that if the eigenvalues $\lambda_{1/2}$ are both real, then they have the same sign.

(f) Combine the above findings to conclude that the causal Lagrangian (5.6.1) can be simplified to (6.1.4).

EXERCISE 6.2. *(The action and boundedness constraint of the Lebesgue measure on the sphere)* We consider the causal variational principle on the sphere as introduced in Section 6.1. We let $\mathrm{d}\mu$ be the surface area measure, normalized such that $\mu(S^2) = 1$.

(a) Use the formula for the causal Lagrangian on the sphere (6.1.4) to compute the causal action (5.6.2). Verify that

$$\mathcal{S}[F] = \frac{1}{2} \int_0^{\vartheta_{\max}} \mathcal{L}(\cos\vartheta)\, \sin\vartheta\, \mathrm{d}\vartheta = 4 - \frac{4}{3\tau^2}\,. \tag{6.6.2}$$

(b) Show that the functional $\mathcal{T}$ is given by

$$\mathcal{T}[F] = 4\tau^2(\tau^2 - 2) + 12 - \frac{8}{3\tau^2}\,. \tag{6.6.3}$$

Hence the causal action (6.6.2) is bounded uniformly in τ, although the function F, (6.6.1), as well as the functional $\mathcal{T}$, (6.6.3), diverge as $\tau \to \infty$.

EXERCISE 6.3. *(A minimizer with singular support)* We again consider the causal variational principle on the sphere as introduced in Section 6.1. Verify by direct computation that in the case $\tau = \sqrt{2}$, the causal action of the normalized counting measure supported on the *octahedron* is smaller the action of μ. *Hint:* For $\tau = \sqrt{2}$ the opening angle of the light cone is given by $\vartheta = 90^\circ$, so that all distinct spacetime points are spacelike separated. Moreover, the causal action of the normalized Lebesgue measure is given in Exercise 6.2 (a).

It turns out that normalized counting measure supported on the octahedron is indeed a minimizer of the causal action. More details and related considerations can be found in [**43, 84, 57**].

EXERCISE 6.4. *(A causal variational principle on $\mathbb{R}$)* We let $\mathcal{F} = \mathbb{R}$ and consider the Lagrangians

$$\mathcal{L}_2(x,y) = (1+x^2)(1+y^2) \qquad \text{and} \qquad \mathcal{L}_4(x,y) = (1+x^4)(1+y^4)\,.$$

We minimize the corresponding causal actions (6.2.2) within the class of all normalized regular Borel measures on $\mathbb{R}$. Show with a direct estimate that the Dirac measure δ supported at the origin is the unique minimizer of these causal variational principles.

EXERCISE 6.5. *(A causal variational principle on S^1)* We let $\mathcal{F} = S^1$ be the unit circle parametrized as $\mathrm{e}^{\mathrm{i}\varphi}$ with $\varphi \in \mathbb{R} \bmod 2\pi$ and consider the Lagrangian

$$\mathcal{L}(\varphi, \varphi') = 1 + \sin^2\big(\varphi - \varphi' \bmod 2\pi\big)\,.$$

We minimize the corresponding causal action (6.2.2) within the class of all normalized regular Borel measures on S^1. Show by direct computation and estimates that every minimizer is of the form

$$\rho = \tau\, \delta\big(\varphi - \varphi' - \varphi_0 \bmod 2\pi\big) + (1-\tau)\, \delta\big(\varphi - \varphi' - \varphi_0 + \pi \bmod 2\pi\big) \tag{6.6.4}$$

for suitable values of the parameters $\tau \in [0,1]$ and $\varphi_0 \in \mathbb{R} \bmod 2\pi$.

CHAPTER 7

The Euler-Lagrange Equations

In classical field theory, the dynamics of the physical system is revealed by analyzing the Euler-Lagrange equations corresponding to the classical action principle. These Euler-Lagrange equations are the physical equations (like the Maxwell or Einstein equations). They have the mathematical structure of partial differential equations. Likewise, for the causal action principle and causal variational principles, the Euler-Lagrange equations describe the dynamics. However, they are no longer differential equations but have a quite different form. In this chapter we shall derive the *Euler-Lagrange equations* and discuss their general structure.

7.1. The Euler-Lagrange Equations

Let ρ be a minimizer of the causal variational principle in the non-compact setting (more precisely, a minimizer with respect to variations of finite volume; see Section 6.3). We now derive the Euler-Lagrange (EL) equations, following the method in the compact setting [**84**, Lemma 3.4]. We again define spacetime as the support of ρ,

$$M := \operatorname{supp} \rho \subset \mathcal{F} .$$

In words, the EL equations state that the causal action is minimal under first variations of the measure. In order to make mathematical sense of the variations, we need the following assumptions:

(i) The measure ρ is *locally finite* (meaning that any $x \in \mathcal{F}$ has an open neighborhood U with $\rho(U) < \infty$).

(ii) The function $\mathcal{L}(x, .)$ is ρ-integrable for any $x \in \mathcal{F}$, and the function

$$x \mapsto \int_{\mathcal{F}} \mathcal{L}(x, y)\, \mathrm{d}\rho(y)$$

is a bounded continuous function on $\mathcal{F}$.

These technical assumptions are satisfied in most applications and are sufficiently general for the purpose of this book (we note that the continuity assumption in (ii) could be relaxed to lower semi-continuity; the details are worked out in [**75**]). We introduce the function

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x, y)\, \mathrm{d}\rho(y) - \mathfrak{s} \;:\; \mathcal{F} \to \mathbb{R} , \tag{7.1.1}$$

where $\mathfrak{s} \in \mathbb{R}$ is a parameter whose value will be specified below.

THEOREM 7.1.1. *(The Euler-Lagrange equations) Let ρ be a minimizer of the causal action with respect to variations of finite volume and assume that ρ satisfies the conditions* (i) *and* (ii) *above. Then*

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell . \tag{7.1.2}$$

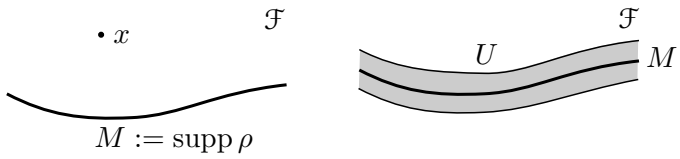


FIGURE 7.1. Evaluation of ℓ away from and near M.

PROOF. Given $x_0 \in \operatorname{supp}\rho$, we choose an open neighborhood U with $0 < \rho(U) < \infty$. For any $y \in \mathcal{F}$ we consider the family of measures $(\tilde{\rho}_\tau)_{\tau\in[0,1)}$ given by

$$\tilde{\rho}_\tau = \chi_{M\setminus U}\,\rho + (1-\tau)\,\chi_U\,\rho + \tau\,\rho(U)\,\delta_y \tag{7.1.3}$$

(where δ_y is the Dirac measure supported at y). Then

$$\tilde{\rho}_\tau - \rho = -\tau\,\chi_U\,\rho + \tau\,\rho(U)\,\delta_y = \tau\big(\rho(U)\,\delta_y - \chi_U\,\rho\big)\,. \tag{7.1.4}$$

Using this formula one readily verifies that $\tilde{\rho}_\tau$ is a variation of finite volume satisfying the volume constraint. Hence

$$\begin{aligned} 0 \leq \big(\mathcal{S}(\tilde{\rho}_\tau) - \mathcal{S}(\rho)\big) &= 2\tau\left(\rho(U)\left(\ell(y)+\mathfrak{s}\right) - \int_U \Big(\ell(x)+\mathfrak{s}\Big)\,\mathrm{d}\rho(x)\right) + \mathcal{O}\big(\tau^2\big) \\ &= 2\tau\left(\rho(U)\,\ell(y)) - \int_U \ell(x)\mathrm{d}\rho(x)\right) + \mathcal{O}\big(\tau^2\big) \end{aligned}$$

(here we may carry out the integrals in arbitrary order using Tonelli's theorem for non-negative integrands). Since this holds for any $\tau \in [0,1)$, the linear term must be non-negative, and thus

$$\ell(y) \geq \frac{1}{\rho(U)} \int_U \ell(x)\,\mathrm{d}\rho(x)\,. \tag{7.1.5}$$

Now assume that (7.1.2) is false. Then there is $x_0 \in \operatorname{supp}\rho$ and $y \in \mathcal{F}$ such that $\ell(x_0) > \ell(y)$. Continuity of ℓ implies that there is an open neighborhood U of x_0 such that $\ell(x) > \ell(y)$ for all $x \in U$. But this contradicts (7.1.5). □

It is indeed no loss of generality to restrict attention to first variations within the special class (7.1.4); for details see Exercise 7.1.

We always choose the parameter $\mathfrak{s}$ such that the infimum of ℓ in (7.1.2) is zero. Then the EL equations read

$$\ell|_{\operatorname{supp}\rho} \equiv \inf_{\mathcal{F}} \ell = 0\,. \tag{7.1.6}$$

The parameter $\mathfrak{s}$ can be understood as the "action per volume" (see Exercise 7.2). We finally point out that solutions of the EL equations do not need to be minimizers of the causal action principle. Similar to the situation for local maxima or saddle points in the finite-dimensional setting, there may be variations for which $\mathcal{S}$ is stationary, but whose second or higher variations are negative.

7.2. The Restricted Euler-Lagrange Equations in the Smooth Setting

The EL equations (7.1.6) make a statement on the function ℓ even at points $x \in \mathcal{F}$ which are far away from spacetime M (see the left of Figure 7.1). In this way, the EL equations contain much more information than conventional physical equations formulated in spacetime. At present, it is unclear how this additional information can be used or interpreted. One way of understanding this situation is to take the point of view

that all information on the physical system must be obtained by performing observations or measurements in spacetime, which means that the information contained in ℓ away from M is inaccessible for fundamental reasons. Here we shall not take sides or discuss whether or to which extent this point of view is correct. Instead, we simply note that it seems preferable and physically sensible to restrict attention to the function ℓ in an arbitrarily small neighborhood U of M in $\mathcal{F}$ (see the right of Figure 7.1). In practice, this means that we shall evaluate ℓ as well as its derivatives only on M. In this way, the causal action principle gives rise to an interaction described by equations in spacetime.

This concept leads us to the so-called *restricted EL equations*, which we now introduce. For technical simplicity, we again restrict attention to the *smooth setting* (for a more general derivation see [**71**, Section 4]). This means that we assume that the Lagrangian is smooth (see (6.2.4) and the discussion thereafter). To avoid confusion, we point out that this assumption does *not* entail that spacetime $M := \operatorname{supp} \rho$ is a smooth manifold. Nevertheless, we can speak of a smooth function or a smooth vector field on M, meaning that the function (or vector field) has a smooth extension to $\mathcal{F}$.[1] Moreover, for technical simplicity we assume that also the function ℓ defined by (7.1.1) is smooth on $\mathcal{F}$. Under these assumptions, the minimality of ℓ implies that the derivative of ℓ vanishes on M. We thus obtain the equations

$$\ell|_M \equiv 0 \qquad \text{and} \qquad D\ell|_M \equiv 0 \tag{7.2.1}$$

(where $D\ell(p) : T_p\mathcal{F} \to \mathbb{R}$ is the derivative). In order to combine these two equations in a compact form, it is convenient to consider a pair $\mathfrak{u} := (a, u)$ consisting of a real-valued function a on M and a vector field u on $T\mathcal{F}$ along M, and to denote the combination of multiplication and directional derivative by

$$\nabla_{\mathfrak{u}} \ell(x) := a(x)\, \ell(x) + \big(D_u \ell\big)(x)\,. \tag{7.2.2}$$

The pair $\mathfrak{u} = (a, u)$ is referred to as a *jet*. This jet is a vector in a corresponding *jet space* $\mathfrak{J}$ defined by

$$\mathfrak{u} = (a, u) \;\in\; \mathfrak{J} := C^\infty(M, \mathbb{R}) \oplus \Gamma^\infty(M, T\mathcal{F})\,, \tag{7.2.3}$$

where $C^\infty(M, \mathbb{R})$ and $\Gamma^\infty(M, T\mathcal{F})$ denote the space of real-valued functions and vector fields on M, respectively, which admit a smooth extension to $\mathcal{F}$. Then the equations (7.2.1) imply that $\nabla_{\mathfrak{u}} \ell(x)$ vanishes for all $x \in M$,

$$\nabla_{\mathfrak{u}} \ell|_M = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}\,. \tag{7.2.4}$$

These are the so-called *restricted EL equations*. For brevity, a solution of the restricted EL equations is also referred to as a *critical measure*. We remark that, in the literature, the restricted EL equations are sometimes also referred to as the *weak* EL equations. Here we prefer the notion "restricted" in order to avoid potential confusion with weak solutions of these equations (as constructed in [**22**]; see also Chapter 14 below).

[1] We remark that the question on whether a function or vector field on M can be extended smoothly to $\mathcal{F}$ is rather subtle. The needed conditions are made precise by Whitney's extension theorem (see for example the more recent account in [**34**]). Here we do not enter the details of these conditions, but use them as implicit assumptions entering our notion of smoothness. We remark that these conditions are fulfilled whenever $M := \operatorname{supp} \rho$ carries a manifold structure.

7.3. Symmetries and Symmetric Criticality

In many applications, variational principles have an underlying symmetry (for example spherical symmetry or time independence). Typically, it simplifies the variational problem to vary within the class of functions which respect this symmetry. Having found a minimizer within this restricted class, the question arises whether it is also a minimizer of the full variational problem. The general answer to this question is no, simply because the absolute minimizer does not necessarily respect the symmetry of the variational principle. For causal variational principles, the situation is similar, if we only replace "function" by "measure." As a simple example, we saw in Section 6.1 for the causal variational principle on the sphere that, although the variational principle is spherically symmetric, minimizing measures are typically weighted counting measures, thus breaking spherical symmetry.

Nevertheless, one can hope that minimizers within the class of symmetric functions are critical points of the full variational problem. This statement, referred to as the *principle of symmetric criticality*, has been formulated and proven under general assumptions in [**126**]. In this section we explain how the principle of symmetric criticality can be stated and proved in the setting of causal variational principles. As we shall see, the proof is quite simple and rather different from that in the classical calculus of variations. We begin by explaining the basic idea in the simplest possible situation, where we consider the *compact setting* and assume that also the *symmetry group is compact*. Afterward we explain how to treat a *non-compact symmetry group*.

As in Section 6.2 we let $\mathcal{F}$ be a compact manifold. Moreover, we again assume that the Lagrangian $\mathcal{L}$ is continuous (6.2.5), symmetric and strictly positive on the diagonal (see the assumptions (i) and (ii) on page 124). In order to describe the symmetry, we let $\mathcal{G}$ be a *compact Lie group*, which should act as a group of diffeomorphisms on $\mathcal{F}$ (for basics on Lie groups see for example [**118**, Chapter 7]). More precisely, we assume the group action $\Phi : \mathcal{G} \times \mathcal{F} \to \mathcal{F}$ to be a continuous mapping with the properties that $\Phi_g := \Phi(g, .)$ is a diffeomorphism of $\mathcal{F}$ for any $g \in \mathcal{G}$, and that

$$\Phi_g \circ \Phi_h = \Phi_{gh} \qquad \text{for all } g, h \in \mathcal{G}\,.$$

Moreover, the symmetry is expressed by the condition that the Lagrangian be invariant under the group action, i.e.

$$\mathcal{L}(\Phi_g x, \Phi_g y) = \mathcal{L}(x, y) \qquad \text{for all } x, y \in \mathcal{F} \text{ and } g \in \mathcal{G}\,. \tag{7.3.1}$$

We denote the set of normalized regular Borel measures on $\mathcal{F}$ by $\mathfrak{M}$. Taking the push-forward of Φ, we obtain a group action on $\mathfrak{M}$ (for the definition of the push-forward measure see again Section 2.3). We denote the measures which are invariant under this group action by $\mathfrak{M}^{\mathcal{G}}$, i.e.

$$\mathfrak{M}^{\mathcal{G}} := \big\{ \rho \in \mathfrak{M} \,\big|\, (\Phi_g)_* \rho = \rho \quad \text{for all } g \in \mathcal{G} \big\}\,. \tag{7.3.2}$$

We also refer to the measures in $\mathfrak{M}^{\mathcal{G}}$ as being *equivariant* (for more details on equivariant causal variational principles see [**13**, Section 4]).

THEOREM 7.3.1. *Let ρ be a minimizer of the causal action under variations within the class $\mathfrak{M}^{\mathcal{G}}$ of equivariant measures. Then ρ is a critical point of the full variational principle in the sense that the EL equations* (7.1.6) *hold.*

We point out for clarity that minimizers under variations in $\mathfrak{M}^{\mathcal{G}}$ will in general *not* be minimizers under variations in $\mathfrak{M}$. The reason is that the minimizers in $\mathfrak{M}$ are typically not invariant under the action of the group $\mathcal{G}$. A concrete example for this phenomenon is given in Exercise 7.4.

PROOF OF THEOREM 7.3.1. We denote the orbits of the group action by $\langle x \rangle := \Phi_{\mathcal{G}} x$ with $x \in \mathcal{F}$. Since $\mathcal{G}$ is compact, so are the orbits. On $\mathcal{G}$ there is a uniquely defined normalized measure which is invariant under the group action by left multiplication, the so-called *normalized Haar measure* μ (for details on the Haar measure see for example [**118**, Chapter 16]). A particular class of equivariant measures is obtained by taking the push-forward of μ by the mapping $\Phi(., x)$. More precisely, for given $x \in \mathcal{F}$ we define the (also normalized) Borel measure $\delta_{\langle x \rangle}$ on $\mathcal{F}$ by

$$\delta_{\langle x \rangle}(\Omega) := \mu\big(\{g \in \mathcal{G} \,|\, \Phi(g, x) \in \Omega\}\big) \tag{7.3.3}$$

for any Borel set $\Omega \subset \mathcal{F}$. The subscript $\langle x \rangle$ indicates that, being equivariant, this measure depends only on the orbit.

Given $y \in \mathcal{F}$, we now consider the variation $(\tilde{\rho}_\tau)_{\tau \in [0,1)}$ within the class of equivariant measures defined by

$$\tilde{\rho}_\tau = (1 - \tau)\, \rho + \tau\, \delta_{\langle y \rangle} \,. \tag{7.3.4}$$

Note that, as convex combination of two normalized measures, also $\tilde{\rho}_\tau$ is normalized. Using that ρ is a minimizer within this class, we can proceed similarly as in the proof of Theorem 7.1.1 to obtain

$$\int_{\mathcal{F}} \ell(x)\, \mathrm{d}\delta_{\langle y \rangle}(x) \geq \int_{\mathcal{F}} \ell(x)\, \mathrm{d}\rho(x) \,. \tag{7.3.5}$$

Moreover, it follows by symmetry that the function ℓ is constant on the orbits, because

$$\begin{aligned} \ell\big(\Phi_g y\big) &= \int_{\mathcal{F}} \mathcal{L}\big(\Phi_g y, x\big)\, \mathrm{d}\rho(x) - \mathfrak{s} = \int_{\mathcal{F}} \mathcal{L}\big(y, \Phi_{g^{-1}} x\big)\, \mathrm{d}\rho(x) - \mathfrak{s} \\ &= \int_{\mathcal{F}} \mathcal{L}\big(y, x\big)\, \mathrm{d}\rho(x) - \mathfrak{s} = \ell(y) \,, \end{aligned}$$

where in the first line we used symmetry of $\mathcal{L}$ and in the second line we used that ρ is equivariant. Hence, integrating over the orbit, we obtain

$$\ell(y) = \int_{\mathcal{F}} \ell(x)\; \mathrm{d}\delta_{\langle y \rangle} \,.$$

Combining this identity with (7.3.5), we conclude that

$$\ell(y) \geq \int_{\mathcal{F}} \ell(x)\, \mathrm{d}\rho(x) \qquad \text{for all } y \in \mathcal{F} \,.$$

Now we can argue exactly as in the proof of Theorem 7.1.1 to obtain the result. □

We next consider the case that the symmetry group $\mathcal{G}$ is a *noncompact Lie group*. A typical example is the translation group, giving rise to the homogeneous causal action principle as considered in [**54**]. We again assume that $\mathcal{G}$ acts on $\mathcal{F}$ as a group of diffeomorphisms $\Phi : \mathcal{G} \times \mathcal{F} \to \mathcal{F}$. We can again single out the equivariant measures $\mathfrak{M}^{\mathcal{G}}$ by (7.3.2). Moreover, on $\mathcal{G}$ one can introduce a left-invariant measure μ (again referred to as the Haar measure). However, in contrast to the case of a compact Lie group, now the measure μ has infinite total volume. As a consequence, it cannot be normalized, and moreover it is unique only up to a positive prefactor. It is a basic difficulty that for any non-zero equivariant measure ρ, the integrals in the causal action (6.2.2) diverge,

because the integral over the group elements g describing the symmetry (7.3.1) diverge. In simple terms, this group integral gives an infinite prefactor. This suggests that the problem could be cured simply by leaving out this integral. We now explain how this can be done. For simplicity, we restrict attention to the case that $\mathcal{G}$ *acts freely* (in the sense that $gx = x$ with $g \in \mathcal{G}$ implies that $g = e$ is the neutral element). Then, for any $x \in \mathcal{G}$, the mapping $g \mapsto \Phi(g, x)$ is a continuous injective mapping from $\mathcal{G}$ to $\mathcal{F}$. In other words, each orbit is homeomorphic to $\mathcal{G}$. Again denoting the space of orbits by $\mathcal{F}/\mathcal{G}$, we can thus identify $\mathcal{F} \simeq (\mathcal{F}/\mathcal{G}) \times \mathcal{G}$. Moreover, using this identification, the equivariant measure can be written as

$$\rho = \rho_{\mathcal{F}/\mathcal{G}} \times \mu \,,$$

where $\rho_{\mathcal{F}/\mathcal{G}}$ is a measure on the orbits. Now we replace the action (6.2.2) by

$$\mathcal{S}(\rho) = \int_{\mathcal{F}/\mathcal{G}} \mathrm{d}\rho_{\mathcal{F}/\mathcal{G}}(x) \int_{\mathcal{F}} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \qquad \text{with } \rho \in \mathfrak{M}^{\mathcal{G}} \,. \tag{7.3.6}$$

The *equivariant causal variational principle* is to minimize this action under variations in $\mathfrak{M}^{\mathcal{G}}$, leaving the total volume of $\rho_{\mathcal{F}/\mathcal{G}}$ fixed. If $\mathcal{F}/\mathcal{G}$ is compact, we can normalize this total volume by demanding that

$$\rho_{\mathcal{F}/\mathcal{G}}(\mathcal{F}/\mathcal{G}) = 1 \,. \tag{7.3.7}$$

If $\mathcal{F}/\mathcal{G}$ is non-compact, the volume constraint can be treated similar as explained for causal variational principles in the non-compact setting in Section 6.3. For more details on this procedure and the resulting existence theory we refer to [**13**, Section 4] and [**54**].

The justification for considering the equivariant causal variational principle (7.3.6) is that it gives a method for constructing critical points of the full variational principle. The basis is the following result which applies in the case that $\mathcal{F}/\mathcal{G}$ is compact.

THEOREM 7.3.2. **(Symmetric criticality for causal variational principles)** *Let $\mathcal{G}$ be a noncompact Lie group acting freely on $\mathcal{F}$ as a group of diffeomorphisms. Assume that $\mathcal{F}/\mathcal{G}$ is compact. Let ρ be a minimizer of the equivariant causal action principle which is normalized on the orbits* (7.3.7)*. Then ρ is a critical point of the full variational principle in the sense that the EL equations* (7.1.6) *hold.*

PROOF. The measure (7.3.3) is normalized on $\mathcal{F}/\mathcal{G}$. Therefore, the variation (7.3.4) satisfies the volume constraint (7.3.7). Computing the first variation of the action, in analogy to (7.3.4) we now obtain

$$\int_{\mathcal{F}/G} \ell(x)\; \mathrm{d}\delta_{\langle y \rangle}(x) \geq \int_{\mathcal{F}/\mathcal{G}} \ell(x)\, \mathrm{d}\rho_{\mathcal{F}/\mathcal{G}}(x)$$

(note that the integrands are constant on the orbits). Carrying out the integral on the left side, we conclude that

$$\ell(y) = \int_{\mathcal{F}/\mathcal{G}} \ell(x)\, \mathrm{d}\rho_{\mathcal{F}/\mathcal{G}}(x) \,,$$

giving the claim. □

In the case that $\mathcal{F}/\mathcal{G}$ is not compact, it is not clear if minimizers exist. One strategy for constructing minimizers is to exhaust $\mathcal{F}/\mathcal{G}$ by compact sets, similar as is done in [**75**] for causal variational principles in the non-compact setting (see also Section 12.8). If an equivariant minimizer ρ exists, we know by symmetry that ℓ is constant on the orbits, and moreover the corresponding EL equations imply that ℓ is minimal on the orbits in the support of ρ. Combining these facts, we immediately obtain the EL equations (7.1.2).

In this way, we conclude that symmetric criticality always holds for causal variational principles.

7.4. Exercises

EXERCISE 7.1. *(More general first variations)* In the proof of Theorem 7.1.1 we restricted attention to very specific variations (7.1.3). In this exercise we verify that the resulting EL equations (7.1.2) guarantee that the action is minimal also under more general variations. To this end, let μ be a normalized measure on $\mathcal{F}$, for technical simplicity with compact support. Consider variations of the form

$$\tilde{\rho}_\tau = \chi_{M\setminus U}\, \rho + (1-\tau)\, \chi_U\, \rho + \tau\, \rho(U)\, \mu$$

Show that (7.1.2) implies the inequality

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \mathcal{S}\big(\tilde{\rho}_\tau\big)\Big|_{\tau=0} \geq 0\,.$$

EXERCISE 7.2. Assume that ρ is a minimizer of a causal variational principle with finite total volume. Show that the parameter $\mathfrak{s}$ in (7.1.2) takes the value

$$\mathfrak{s} = \frac{\mathcal{S}(\rho)}{\rho(\mathcal{F})}\,.$$

EXERCISE 7.3. *(Non-smooth EL equations)* We return to the example of the counting measure on the octahedron as considered in Exercise 6.3.

(a) Compute the function $\ell(x)$. Show that the EL equations (7.1.2) are satisfied.
(b) Show that the function ℓ is *not* differentiable at any point x of the octahedron. Therefore, it is not possible to formulate the restricted EL equations (7.2.4)

This example illustrates why in the research papers [**70, 61**] one carefully keeps track of differentiability properties by introducing suitable jet spaces.

EXERCISE 7.4. *(Symmetric criticality on the sphere)* We consider the causal variational principle on the sphere as introduced in Section 6.1.

(a) Show that the symmetric measure on the sphere

$$\mathrm{d}\mu(\vartheta, \varphi) = \frac{1}{4\pi}\; \mathrm{d}\varphi\, \sin\vartheta\; \mathrm{d}\vartheta$$

is critical in the sense that it satisfies the EL equations (7.1.2).
(b) Use the minimizer with singular support constructed in Exercise 6.3 to argue that minimizers within the class of symmetric measure are in general *not* minimizers within the class of measures without symmetries. More details on this effect of symmetry breaking can be found in [**43, 84, 57**].

CHAPTER 8

The Linearized Field Equations

The EL equations as derived in the previous chapter (see Theorem 7.1.1 or the restricted EL equations in (7.2.4)) are *nonlinear* equations. This can be seen immediately from the fact that the measure ρ enters in a twofold way: It determines the function ℓ via the integration (7.1.1), and it also determines via its support M where the function ℓ is to be evaluated. As usual, such nonlinear equations are difficult to analyze. Therefore, it is a good idea to simplify these equations by linearization. This leads us to the so-called *linearized field equations* which describe linear perturbations of the system which respect the EL equations. This chapter is devoted to the derivation of the linearized field equations and to the construction of explicit examples. We remark that the linearized field equations are also a suitable starting point for the analysis of the full EL equations, for example by using perturbative methods (see [**51**] or Chapter 18).

8.1. Derivation of the Linearized Field Equations in the Smooth Setting

Linearizations are used frequently in physics and mathematical analysis. Typical physical examples are the anharmonic oscillator for small displacements or nonlinear waves of small amplitudes. In these examples, the nonlinearity of the underlying wave equation may be neglected, making it possible to describe the dynamics by a linear wave equation. In mathematical terms, the dynamics is usually described by nonlinear equations which are difficult to solve (like for example the Einstein equations). Linearizing gives linear equations (for example the equations of linearized gravity), which are much easier to analyze. In order to derive the linearized equations, one typically considers a family G_τ of solutions. The parameter τ can be thought of as the "amplitude" of the perturbation, and $G_\tau|_{\tau=0}$ describes the unperturbed system (for example an anharmonic oscillator at rest or the vacuum). Then the derivative

$$\frac{\mathrm{d}}{\mathrm{d}\tau} G_\tau \big|_{\tau=0}$$

is the linearized field; it satisfies a linear equation obtained by differentiating the nonlinear equation (like Hamilton's equation for the anharmonic oscillator or a nonlinear wave equation) with respect to τ.

The concept of linearization is also fruitful in the context of causal variational principles. Since the system is described by the measure ρ, the above family of solutions now corresponds to a family of measures $(\tilde{\rho}_\tau)_{\tau \in [0,\delta)}$ which are all critical points of the causal action principle. The basic question is how to vary the measure. Indeed, there are many ways of varying. We begin with a simple method, which we will generalize and discuss afterward. In order to keep the presentation as simple as possible, we again restrict attention to the *smooth setting* (as defined in Section 6.2; this means that the Lagrangian is smooth (6.2.4), but M does not need to have a smooth manifold structure). Moreover, for technical simplicity we assume that the Lagrangian has the following property.

DEFINITION 8.1.1. *The Lagrangian has* **compact range** *if for every compact $K \subset \mathcal{F}$ there is a compact set $K' \subset \mathcal{F}$ such that*

$$\mathcal{L}(x,y) = 0 \qquad \text{for all } x \in K \text{ and } y \notin K' .$$

We choose a family f_τ of positive weight functions and a family F_τ of mapping from M to $\mathcal{F}$. These functions should all be smooth, also in the parameter τ, i.e.

$$f \in C^\infty([0,\delta) \times M, \mathbb{R}^+) \qquad \text{and} \qquad F \in C^\infty([0,\delta) \times M, \mathcal{F})$$

(here, as explained before (7.2.1), smoothness on M is defined via the existence of a smooth extension to $\mathcal{F}$). We multiply ρ by f_τ and then take the push-forward under F_τ,

$$\tilde{\rho}_\tau := (F_\tau)_*\big(f_\tau\, \rho\big) . \tag{8.1.1}$$

We assume that for $\tau = 0$ the variation is trivial,

$$f_0 \equiv 1 \qquad \text{and} \qquad F_0 \equiv \mathbb{1} . \tag{8.1.2}$$

Since multiplying by a positive function leaves the support unchanged, the support of the measure is transformed only by F_τ; more precisely,

$$\operatorname{supp} \tilde{\rho}_\tau = \overline{F_\tau\big(\operatorname{supp} \rho\big)} \tag{8.1.3}$$

(for details see Exercise 8.1).

The assumption that all the measures $\tilde{\rho}_\tau$ are critical means that they all satisfy the restricted EL equations (7.2.4). Taking into account that the support of the measures changes according to (8.1.3), we know that for all $\mathfrak{u} \in \mathfrak{J}_0$ and all $x \in M$,

$$\begin{aligned} 0 &= \nabla_\mathfrak{u}\bigg(\int_\mathcal{F} \mathcal{L}\big(F_\tau(x), y\big)\; \mathrm{d}\tilde{\rho}_t(y) - \mathfrak{s} \bigg) \\ &= \nabla_\mathfrak{u}\bigg(\int_\mathcal{F} \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\; f_\tau(y)\, \mathrm{d}\rho(y) - \mathfrak{s} \bigg) , \end{aligned}$$

where in the last line we used the definition of the push-forward measure. It is convenient to multiply this equation by $f_\tau(x)$. We can write this factor inside the brackets,

$$0 = \nabla_\mathfrak{u}\bigg(\int_\mathcal{F} f_\tau(x)\, \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\; f_\tau(y)\, \mathrm{d}\rho(y) - f_\tau(x)\, \mathfrak{s} \bigg) ,$$

because the terms obtained when the derivative $\nabla_\mathfrak{u}$ acts on $f_\tau(x)$ vanish in view of the restricted EL equations (7.2.4). Since this equation holds for all $\tau \in [0,\delta)$, we can differentiate at $\tau = 0$,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\tau} \nabla_\mathfrak{u}\bigg(\int_\mathcal{F} f_\tau(x)\, \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\; f_\tau(y)\, \mathrm{d}\rho(y) - f_\tau(x)\, \mathfrak{s} \bigg)\bigg|_{\tau=0} \\ &= \nabla_{\mathfrak{u}(x)}\bigg(\int_\mathcal{F} \big(\dot{f}_0(x) + D_{1,\dot{F}_0(x)} + \dot{f}_0(y) + D_{2,\dot{F}_0(y)}\big)\mathcal{L}(x,y)\; \mathrm{d}\rho(y) - \dot{f}_0(x)\, \mathfrak{s} \bigg) , \end{aligned}$$

where D_1 and D_2 denote partial derivatives acting on the first and second argument of the Lagrangian, respectively. Here we were allowed to interchange the derivative with the integral because the integration range is compact by the assumption in Definition 8.1.1. We write this equation in the shorter form

$$\nabla_{\mathfrak{u}(x)}\bigg(\int_\mathcal{F} \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_\mathfrak{v}\, \mathfrak{s} \bigg) = 0 , \tag{8.1.4}$$

where $\mathfrak{v}$ is the jet generated by the functions f_τ and F_τ,

$$\mathfrak{v} := \frac{\mathrm{d}}{\mathrm{d}\tau}\big(f_\tau, F_\tau\big)\big|_{\tau=0} \in \mathfrak{J}\,. \tag{8.1.5}$$

Note that in (8.1.4) the $\mathfrak{u}$-derivative also acts on the jet $\mathfrak{v}(x)$, giving rise to the terms

$$\int_{\mathcal{F}} \nabla_{1,D_u\mathfrak{v}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{D_u\mathfrak{v}}\, \mathfrak{s}\,. \tag{8.1.6}$$

But these terms vanish in view of the restricted Euler-Lagrange equations (7.2.4). This observation makes it possible to simplify the formulation of the linearized field equations by adopting the following computational conventions for partial derivatives and jet derivatives:

(i) Partial and jet derivatives with an index $i \in \{1,2\}$ only act on the respective variable of the function $\mathcal{L}$. This implies, for example, that the derivatives commute,

$$\nabla_{1,\mathfrak{v}} \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) = \nabla_{1,\mathfrak{u}} \nabla_{1,\mathfrak{v}} \mathcal{L}(x,y)\,.$$

(ii) The partial or jet derivatives which do not carry an index act as partial derivatives on the corresponding argument of the Lagrangian. This implies, for example, that

$$\nabla_{\mathfrak{u}} \int_{\mathcal{F}} \nabla_{1,\mathfrak{v}}\, \mathcal{L}(x,y)\, \mathrm{d}\rho(y) = \int_{\mathcal{F}} \nabla_{1,\mathfrak{u}} \nabla_{1,\mathfrak{v}}\, \mathcal{L}(x,y)\, \mathrm{d}\rho(y)\,.$$

We will use these conventions from now on throughout this book. We point out that, following these conventions, *jets are never differentiated*. This is a very convenient convention. Clearly, we must always verify that this convention may really be used. As already mentioned above, in (8.1.4) this convention is justified because the additional terms obtained if the derivative $\nabla_{\mathfrak{u}}$ acted on the jet $\mathfrak{v}$ vanish as a consequence of the restricted EL equations.

We remark that, from a differential geometric perspective, defining higher derivatives on $\mathcal{F}$ would make it necessary to introduce a connection on $\mathcal{F}$. While this could be done, we here use the simpler method that higher derivatives on $\mathcal{F}$ are defined as partial derivatives carried out in *distinguished charts*. More precisely, around each point $x \in \mathcal{F}$ we choose a distinguished chart and carry out all derivatives as partial derivatives acting on each tensor component in this chart. We remark that, in the setting of causal fermion systems, an atlas of distinguished charts is provided by the so-called symmetric wave charts (for details see the remark after Proposition 3.1.3 and [**69**, Section 6.1] or [**76**, Section 3]).

DEFINITION 8.1.2. *Let ρ be a solution of the restricted EL equations* (7.2.4). *A jet $\mathfrak{v} \in \mathfrak{J}$ is referred to as a* **solution of the linearized field equations** *if*

$$\langle \mathfrak{u}, \Delta\mathfrak{v}\rangle(x) := \nabla_{\mathfrak{u}}\bigg(\int_{\mathcal{F}} \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}}\, \mathfrak{s} \bigg) = 0 \tag{8.1.7}$$

for all $\mathfrak{u} \in \mathfrak{J}$ and all $x \in M$. The vector space of all linearized solutions is denoted by $\mathfrak{J}^{\text{lin}} \subset \mathfrak{J}$.

We often write the linearized field equation in the short form $\Delta\mathfrak{v} = 0$. For the mathematical analysis of the linearized field equations, it is preferable to include an *inhomogeneity* $\mathfrak{w}$

$$\Delta\mathfrak{v} = \mathfrak{w}\,. \tag{8.1.8}$$

In view of the pairing with the jet $\mathfrak{u}$ in (8.1.7), the inhomogeneity is a dual jet, i.e. $\mathfrak{u}(x) \in (\mathfrak{J}_x)^*$, making it possible to add on the right side of (8.1.7) the dual pairing $\langle \mathfrak{u}(x), \mathfrak{w}(x)\rangle_x$.

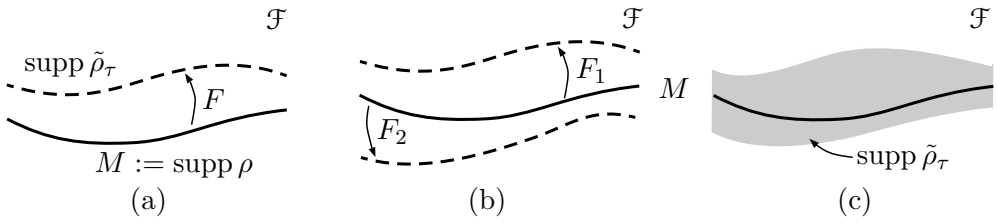


FIGURE 8.1. Fragmentation of the measure ρ.

We remark that, in the analysis of the linearized field equations in Chapter 14, the distinction between jets and dual jets will become unnecessary because we will identify them via a scalar product on the jets given at every spacetime point.

We conclude this section with a brief discussion of our ansatz (8.1.1). Intuitively speaking, this ansatz means that the support of the measure is changed smoothly as a whole. In particular, if M is a smooth four-dimensional submanifold of $\mathcal{F}$, then the varied measure M_τ will again have this property. In physical terms, measures where M has such a manifold structure describe *classical spacetimes*. Consequently, the ansatz (8.1.1) and the corresponding linearization (8.1.5) correspond to *classical fields* in a classical spacetime. In contrast, if the support $M := \operatorname{supp}\rho$ of the measure does *not* have the structure of a four-dimensional manifold, then we refer to M as a *quantum spacetime*. The notion "quantum spacetime" appears in the literature in different contexts with rather different meanings. Here we take the above notion as the definition. In this way, the notion "quantum spacetime" gets a precise mathematical meaning. Our notion is very general. In particular, it allows for the description of non-smooth spacetime structures. The name "*quantum* spacetime" is justified by the fact, in our setting, all spacetime structures are encoded in the family of all physical wave functions, being the fundamental quantum objects of a causal fermion system.

In order to give an idea for how such a quantum spacetime may look like, let us consider the example where the unperturbed measure ρ describes a classical spacetime M (for example Minkowski space $M \simeq \mathbb{R}^4$). As just explained, the ansatz (8.1.1) changes the support of the measure smoothly as a whole (see Figure 8.1 (a)). More generally, one can consider the situation where the measure ρ "disintegrates" into several "components" which are perturbed differently (see Figure 8.1 (b)). For the mathematical description, we choose a parameter $L \in \mathbb{N}$ (the "number of subsystems") and consider mappings

$$f_\mathfrak{a} \in C^\infty\big([0,\delta) \times M, \mathbb{R}^+\big)\,, \quad F_\mathfrak{a} \in C^\infty\big([0,\delta) \times M, \mathcal{F}\big) \qquad \text{with } \mathfrak{a} = 1, \ldots, L\,.$$

For the so-called *measure with fragmentation*, in generalization of (8.1.1) we make the ansatz

$$\tilde{\rho}_\tau = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} (F_{\mathfrak{a},\tau})_*\big(f_{\mathfrak{a},\tau}\, \rho\big)\,. \tag{8.1.9}$$

The larger L is chosen, the more freedom we have in perturbing the measure. We point out that we may choose L arbitrarily large. In the limit $L \to \infty$, one can even describe situations where the support of the measure ρ is "enlarged" by the perturbation, as is shown in Figure 8.1 (c). In this way, a classical spacetime point may correspond to many points of the quantum spacetime, making it possible to encode additional local degrees of freedom. Integrating with respect to the measure ρ also entails an integration

over these additional degrees of freedom, bearing some similarity to integrating over field configurations in a path integral.

Assuming that the family $(\tilde{\rho}_\tau)_{\tau\in[0,\delta)}$ satisfies the restricted EL equations for all τ, we can again linearize in τ to obtain the corresponding linearized field equations. They again have the form (8.1.7), but now with $\mathfrak{v}$ being the "averaged jet"

$$\mathfrak{v} = \frac{1}{L}\sum_{\mathfrak{a}=1}^{L} \mathfrak{v}_\mathfrak{a} \qquad \text{with} \qquad \mathfrak{v}_\mathfrak{a} = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(f_{\mathfrak{a},\tau}, F_{\mathfrak{a},\tau}\big)\big|_{\tau=0}\,.$$

Therefore, for linearized fields the fragmentation does not give anything essentially new. But on the nonlinear level, fragmentation yields additional effects. We refer the interested reader for more details to [**49**, Section 5] and [**51**, Section 5] as well as to the applications to quantum field theory in [**62, 23**] (see also Chapter 22).

In view of this consideration, the only restriction in describing linear perturbations of a measure ρ by a jet $\mathfrak{v}$ of the form (8.1.5) is that the support of the measure ρ is changed continuously in τ, in the sense that the support $\operatorname{supp}\tilde{\rho}_\tau$ lies in a small neighborhood of M (for details see Exercise 8.2). In particular, we do not cover variations of the form (7.1.3) where part of the measure is "transported" to a point $y \in \mathcal{F}$ which may be far away from M. The reason for disregarding such variations is that, similar as explained before introducing the restricted EL equations in Section 7.2 (see Figure 7.1), analyzing the EL equations outside a small neighborhood of M does not seem to be of physical relevance.

8.2. Commutator Jets in Causal Fermion Systems

In order to illustrate the linearized field equations, we conclude this chapter by deriving explicit classes of solutions. These solutions correspond to inherent symmetries of the system. In this section, we consider *commutator jets*, which describe the unitary invariance of a causal action principle. In the next section (Section 8.3), we shall derive *inner solutions* by using the invariance of the measure under diffeomorphisms of M combined with a suitable multiplication of ρ by a smooth weight function.

Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system. The causal action principle is *unitarily invariant* in the following sense. Let $\mathcal{U} \in \mathrm{U}(\mathcal{H})$ be a unitary transformation. Given a measure ρ on $\mathcal{F}$, we can unitarily transform the measure by setting

$$(\mathcal{U}\rho)(\Omega) = \rho\big(\mathcal{U}^{-1}\,\Omega\,\mathcal{U}\big) \qquad \text{for} \qquad \Omega \subset \mathcal{F}\,. \tag{8.2.1}$$

By construction of the integral, this also means that

$$\int_\mathcal{F} \phi(x)\, \mathrm{d}(\mathcal{U}\rho)(x) = \int_F \phi\big(\mathcal{U} x \mathcal{U}^{-1}\big)\, \mathrm{d}\rho(x)$$

for any integrable function ϕ. Since the eigenvalues of an operator are invariant under unitary transformations, a measure ρ is a minimizer or critical point of the causal action principle if and only if $\mathcal{U}\rho$ is.

Infinitesimally, this unitary invariance gives rise to a special class of solutions of the linearized field equations, as we now explain. Let ρ be a critical measure. We let $\mathcal{A}$ be a symmetric operator on $\mathcal{H}$, for technical simplicity of finite rank. By exponentiating, we obtain a family of unitary operators $(\mathcal{U}_\tau)_{\tau\in\mathbb{R}}$ with

$$\mathcal{U}_\tau := \exp(\mathrm{i}\tau\mathcal{A})\,. \tag{8.2.2}$$

According to (8.2.1), the support of the measures $\tilde{\rho}_\tau := \mathcal{U}_\tau\rho$ is given by

$$\tilde{M}_\tau := \operatorname{supp}\tilde{\rho}_\tau = \mathcal{U}_\tau\, M\, \mathcal{U}_\tau^{-1}\,.$$

Due to the unitary invariance of the Lagrangian, the measures $\tilde{\rho}_\tau$ all satisfy the EL equations. Infinitesimally, the unitary transformations are described by the jet $\mathfrak{v}$ given by

$$\mathfrak{v} := (0, v) \in \mathfrak{J}^{\text{lin}} \qquad \text{with} \qquad v(x) = \frac{\mathrm{d}}{\mathrm{d}\tau} \big(\mathcal{U}_\tau\, x\, \mathcal{U}_\tau^{-1}\big)\big|_{\tau=0} = \mathrm{i}\big[\mathcal{A}, x\big]\,. \tag{8.2.3}$$

Due to the commutator in the last equation, we refer to jets of this form as **commutator jets** (this notion was first introduced in [**64**, Section 3]). The fact that commutator jets generate families of critical measures implies that they satisfy the linearized field equations:

LEMMA 8.2.1. *The commutator jet* $\mathfrak{v}$ *in* (8.2.3) *satisfies the linearized field equations* (8.1.7).

PROOF. Due to the unitary invariance of the Lagrangian,

$$\mathcal{L}\big(\mathcal{U}_\tau x \mathcal{U}_\tau^{-1},\, \mathcal{U}_\tau y \mathcal{U}_\tau^{-1}\big) = \mathcal{L}(x, y)\,.$$

Differentiating with respect to τ and applying the product and chain rules gives

$$(D_{1,v} + D_{2,v})\, \mathcal{L}(x, y)\, \mathrm{d}\rho(y) = 0\,.$$

Hence the integrand in (8.1.7) vanishes for all $x, y \in \mathcal{F}$. As a consequence, the integral in (8.1.7) vanishes for all $x \in \mathcal{F}$. Consequently, also its derivative in the direction of u vanishes. Using our convention that the jet derivatives act only on the Lagrangian (see the end of Section 8.1), this directional derivative differs from the jet derivative in (8.1.7) by the term $D_{D_u \mathfrak{v}} \ell(x)$ (similar as explained for (8.1.6)). This term vanishes in view of the restricted EL equations (7.2.4). □

As we shall see in Section 9.4, commutator jets are very useful because they give rise to conserved quantities.

8.3. Inner Solutions in Smooth Spacetimes

We now return to causal variational principles in the smooth setting (thus we again assume that the Lagrangian is smooth (6.2.4)). We introduce an additional smoothness assumption for the measure ρ and explain why it is useful in some applications.

DEFINITION 8.3.1. *Spacetime* $M := \operatorname{supp} \rho$ *has a* **smooth manifold structure** *if the following conditions hold:*

(i) *M is diffeomorphic to a smooth oriented manifold $\mathscr{M}^k$ of dimension k.*
(ii) *In a chart (x, U) of $\mathscr{M}^k$, the measure ρ is absolutely continuous with respect to the Lebesgue measure with a smooth, strictly positive weight function, i.e.*

$$\mathrm{d}\rho = h(x)\, \mathrm{d}^k x \qquad \text{with} \quad h \in C^\infty(\mathscr{M}^k, \mathbb{R}^+)\,. \tag{8.3.1}$$

Even though there is no reason why physical spacetime should have a smooth manifold structure on the Planck scale, this assumption is clearly justified on the macroscopic scale of atomic and gravitational physics. With this in mind, the assumption of a smooth manifold structure seems admissible in all applications in which the microscopic structure of spacetime should be irrelevant. Before going on, we point out that one should carefully distinguish the assumption of a smooth manifold structure from the smooth setting introduced in Section 6.2. In particular, should keep in mind that the smoothness of $\mathcal{L}$ does not imply that M has a smooth manifold structure, nor vice versa.

The fact that ρ is defined independent of charts implies that the function h in (8.3.1) transform like a tensor density. More precisely, on the overlap of two charts (x, U) and $(\tilde{x}, \tilde{U})$, we know that

$$h(x)\, \mathrm{d}^k x = \tilde{h}(\tilde{x})\, \mathrm{d}^k \tilde{x}$$

and thus

$$h(x) = \det \left(\frac{\partial x^i}{\partial \tilde{x}^j} \right) \tilde{h}(\tilde{x})\,.$$

This transformation law makes it possible to define the covariant divergence of a vector field v on $M \simeq \mathscr{M}^k$ in a local chart by

$$\operatorname{div} v = \frac{1}{h}\, \partial_j \big(h\, v^j \big) \tag{8.3.2}$$

(where, following the Einstein summation convention, we sum over $j = 0, \ldots, k$). Alternatively, the divergence of a vector field $v \in \Gamma(M, TM)$ can be defined independent of charts by the relation

$$\int_M \operatorname{div} v\, \eta(x)\, \mathrm{d}\rho = -\int_M D_v \eta(x)\, \mathrm{d}\rho(x)\,, \tag{8.3.3}$$

to be satisfied for all test functions $\eta \in C_0^\infty(M, \mathbb{R})$. Indeed, integrating partial derivatives by parts and using (8.3.1), we obtain

$$\begin{aligned} -\int_M D_v \eta(x)\, \mathrm{d}\rho(x) &= -\int_M v^j(x)\, \frac{\partial \eta(x)}{\partial x^j}\, h(x)\, \mathrm{d}^k x \\ &= \int_M \eta(x)\, \frac{1}{h(x)} \left(\frac{\partial}{\partial x^j} \Big(h(x)\, v^j(x) \Big) \right) h(x)\, \mathrm{d}^k x\,, \end{aligned}$$

and using (8.3.2) and again (8.3.1) gives back the left side of (8.3.3). We remark that the right side of (8.3.3) can be understood as a weak formulation of the divergence. Such a formulation has the advantage that it can be used even in cases where v is not differentiable. In what follows, we will always restrict attention to smooth vector fields, so that the weak and pointwise formulations (8.3.3) and (8.3.2) are equivalent. We usually prefer to work with (8.3.2), but the weak formulation (8.3.3) still has its value in being manifestly coordinate independent.

Having vector fields and the divergence to our disposal, we can now introduce a specific class of linearized solutions. We first define them and explain their significance afterward.

DEFINITION 8.3.2. *An* **inner solution** *is a jet $\mathfrak{v} \in \mathfrak{J}$ of the form*

$$\mathfrak{v} = (\operatorname{div} v, v) \qquad \textit{with} \qquad v \in \Gamma(M, TM)\,.$$

The vector space of all inner solution is denoted by $\mathfrak{J}^{\text{in}} \subset \mathfrak{J}^1$.

This notion was first introduced in [**61**, Section 3]. The name "inner *solution*" is justified by the following lemma:

LEMMA 8.3.3. *Every inner solution $\mathfrak{v} \in \mathfrak{J}^{\text{in}}$ of compact support is a solution of the linearized field equations, i.e.*

$$\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle_M = 0 \qquad \textit{for all } \mathfrak{u} \in \mathfrak{J}\,.$$

PROOF. Applying the Gauß divergence theorem, one finds that for every function $f \in C^1_0(M, \mathbb{R})$,

$$\int_M \nabla_{\mathfrak{v}} f \, \mathrm{d}\rho = \int_M \big(\operatorname{div} v \, f + D_v f\big) \, \mathrm{d}\rho = \int_M \operatorname{div}\big(f v\big) \, \mathrm{d}\rho = 0 \,.$$

Likewise, in the linearized field equations we may integrate by parts in y,

$$\begin{aligned} \langle \mathfrak{u}, \mathfrak{v} \rangle_M &= \nabla_{\mathfrak{u}} \bigg(\int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathcal{L}(x,y) \, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}} \mathfrak{s} \bigg) \\ &= \nabla_{\mathfrak{u}} \bigg(\int_M \nabla_{1,\mathfrak{v}} \mathcal{L}(x,y) \, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}} \mathfrak{s} \bigg) \\ &= \nabla_{\mathfrak{u}} \nabla_{\mathfrak{v}} \ell(x) = \nabla_{\mathfrak{v}(x)} \big(\nabla_{\mathfrak{u}} \ell(x)\big) - \nabla_{D_v \mathfrak{u}} \ell(x) = 0 \,. \end{aligned}$$

The last equality comes about as follows. The summand $\nabla_{D_v \mathfrak{u}} \ell(x)$ vanishes by the restricted EL equations. Moreover, the restricted EL equations yield that the function $\nabla_{\mathfrak{u}} \ell$ vanishes identically on M. Therefore, this function is differentiable in the direction of the vector field v on M, and this directional derivative is zero. Therefore,

$$\nabla_{\mathfrak{v}(x)} \big(\nabla_{\mathfrak{u}} \ell(x)\big) = D_{v(x)} \big(\nabla_{\mathfrak{u}} \ell(x)\big) + \operatorname{div} v(x) \, \nabla_{\mathfrak{u}} \ell(x) = 0 \,,$$

giving the result. □

This result also holds for inner solutions $\mathfrak{v}$ of non-compact support, provided that the vector field v has suitable decay properties at infinity. For details we refer to [**61**, Section 3].

We now explain the significance of inner solutions. To this end, we let $\Phi_t : M \to M$ with $t \in (-\delta, \delta)$ and some $\delta > 0$ be the local flow generated by the vector field v, i.e.

$$\Phi_0 = \mathrm{id}_M \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \Phi_t(x) = v\big|_{\Phi_t(x)} \quad \text{for all } t \in (-\delta, \delta) \,.$$

We consider the corresponding flow of the measure as described by the push-forward measures $\rho_t := (\Phi_t)_* \rho$. These measures will in general be different from ρ (more precisely, $\rho_t = \rho$ for all t if and only if the vector field is divergence-free). But one can arrange the measure to remain unchanged by modifying the weight of the measure with a function f_t, i.e.

$$\rho = (\Phi_t)_* \big(f_t \, \rho\big) \qquad \text{for all } t \in (-\delta, \delta) \tag{8.3.4}$$

for a suitable function $f_t \in C^\infty(M, \mathbb{R}^+)$ (here we again make use of our assumption that M has a smooth manifold structure; see Definition 8.3.1). One verifies by direct computation that the function f_t agrees infinitesimally with the divergence of v, i.e.

$$\dot{f}_0(x) = \operatorname{div} v(x) \,.$$

Therefore, the inner solution $\mathfrak{v} = (\operatorname{div} v, v)$ is the infinitesimal generator of the transformation on the right side of (8.3.4). In other words, inner solutions are infinitesimal generators of transformations of M which leave the measure ρ unchanged. Since the causal fermion system is defined in terms of the measure ρ, inner solutions do not change the causal fermion system, but they merely describe symmetry transformations of the measure. For the reader familiar with general relativity, it may be helpful to see inner solutions as the analog of infinitesimal coordinate transformations, in which case the infinitesimal change of the metric satisfies the Einstein equations (simply because curvature remains unchanged).

One application of inner solution is that they can be used to simplify the scalar components of jets. We now explain the general procedure (explicit examples will be worked out in Chapter 20). A preparatory question is which scalar components can be realized by inner solutions. This question can be answered in great generality by applying Moser's theorem (see for example [**113**, Section XVIII, §2] or straightforward generalizations to non-compact manifolds). For simplicity, we here make additional assumptions which make it possible to use hyperbolic methods.

THEOREM 8.3.4. *Assume that $M \simeq \mathscr{M}^k$ has a smooth manifold structure. Moreover, assume that $\mathscr{M}$ is topologically of the form $\mathscr{M} = \mathbb{R} \times \mathcal{N}$ with a manifold $\mathcal{N}$. Let $a \in C^\infty_{\text{sc}}(\mathscr{M}, \mathbb{R})$ be a smooth function with spatially compact support (meaning that, for all $t \in \mathbb{R}$, the function $a(t,.)$ is compactly supported in $\mathcal{N}$). Then there is a vector field $v \in C^\infty_{\text{sc}}(\mathscr{M}, T\mathscr{M})$, again with spatially compact support, such that the jet $\mathfrak{v} := (a, v)$ is an inner solution.*

PROOF. Our task is to solve the equation $\operatorname{div} v = a$, which can be written equivalently as

$$\partial_j \big(h\, v^j\big) = ha\,. \tag{8.3.5}$$

We first consider the case that a has compact support. In order to solve the partial differential equation (8.3.5), on $\mathcal{N}$ we choose a complete Riemannian metric $g_{\mathcal{N}}$ (such a metric exists according to [**123**]). Moreover, on $\mathscr{M}$ we choose the Lorentzian metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - g_{\mathcal{N}}\,. \tag{8.3.6}$$

Here the choice of the Riemannian metric $g_{\mathcal{N}}$ is irrelevant, and the arbitrariness in choosing this metric corresponds to the fact that (8.3.5) is an under-determined equation which admits many different solutions.

Assume that $\phi \in C^\infty_{\text{sc}}(\mathscr{M}, \mathbb{R})$ is a spatially compact solution of the inhomogeneous wave equation

$$\left(\frac{ha}{\sqrt{|\det g|}}\right)(x) = \Box\phi(x) := \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x^j} \left(\sqrt{|\det g|}\, g^{jk}\, \partial_k \phi(x) \right) \tag{8.3.7}$$

(the existence of such a solution will be shown in the next paragraph). Then a direct computation shows that the vector field

$$v^j := \frac{\sqrt{|\det g|}}{h}\, g^{jk}\, \partial_k \phi \tag{8.3.8}$$

indeed satisfies (8.3.5) (note that, in view of (8.3.1), we may divide by h to again obtain a smooth vector field with spatially compact support).

It remains to show that the inhomogeneous wave equation (8.3.7) has solutions of spatially compact support. Here we must anticipate results for hyperbolic partial differential equations which will be treated in Chapter 13 later in this book. A simple method of obtaining the desired solutions uses the existence of advanced and retarded Green's operators denoted by $S^\vee$ and $S^\wedge$. In the case that the inhomogeneity a has compact support, we can simply set $\phi = S^\wedge(ha/\sqrt{|\det g|})$. In the case that a merely has spatially compact support, we decompose it as

$$a = a_+ + a_-\,,$$

where a_+ is supported in the set $\{t > 0\}$ and a_- is supported in the set $\{t < 1\}$. Denoting the advanced and retarded Green's operators of the scalar wave equation corresponding

to the Lorentzian metric (8.3.6) by $S^\vee$ and $S^\wedge$, respectively, the function

$$\phi := S^\wedge\left(\frac{h}{\sqrt{|\det g|}}\, a_+\right) + S^\vee\left(\frac{h}{\sqrt{|\det g|}}\, a_-\right)$$

is a well-defined solution of the equation $\Box\phi = ha$ which is smooth and has spatially compact support. Therefore, we can again define the vector field v by (8.3.8). This gives the result. □

The result of this proposition can be used to change the scalar component of a linearized solution arbitrarily. As a concrete example, let us consider a causal fermion system describing an interacting system in Minkowski space which near spatial infinity is the vacuum. In this case, all the jets describing the interaction have spatially compact support. Therefore, we can compensate the scalar components by corresponding inner solutions. After doing so, the interacting system is described purely in terms of jets without scalar components. We denote the corresponding jet space similar to (7.2.3) by

$$\Gamma := \{0\} \oplus \Gamma^\infty(M, T\mathcal{F})$$

(where $\Gamma^\infty(M, T\mathcal{F})$ again denotes the space of vector fields on M which admit a smooth extension to $\mathcal{F}$). For clarity, we again point out that this simplification can be made only if spacetime has a smooth manifold structure.

We finally note that, in the above setting, the scalar components of the jets may be disregarded also for testing. Thus the restricted EL equations (7.2.4) and the linearized field equations (8.1.7) can be written equivalently as

$$D_u \ell|_M = 0 \qquad \text{for all } u \in \Gamma_0 \tag{8.3.9}$$

$$D_u \int_{\mathcal{F}} \big(D_{1,v} + D_{2,v}\big)\mathcal{L}(x,y)\, \mathrm{d}\rho(y) = 0 \qquad \text{for all } u \in \Gamma_0\,. \tag{8.3.10}$$

In this way, the scalar components of jets can be left out completely for spacetimes which have a smooth manifold structure and also satisfy the other assumptions of Theorem 8.3.4. We now sketch how to reduce to (8.3.9) (for the linearized field equation (8.3.10) one argues similarly; the details of this this reduction can be found in [**52**, Section 3]). Clearly, (7.2.4) implies (8.3.9). In order to show that also (8.3.9) implies (7.2.4), we assume that (8.3.9) holds. Since the vector field u can be chosen arbitrarily at any given point x, it follows that the function ℓ is constant on M. After changing the parameter $\mathfrak{s}$ if necessary, the function ℓ vanishes identically, implying (7.2.4).

8.4. Summary of the Linearized Field Equations

Similar as in Section 5.10 we now summarize the important objects of Chapters 7 and 8. One may think of this as a helpful bookmark for the commonly used structures.

Basic concept	**Summary and Comments**
The Euler-Lagrange equations	For technical simplicity we here assume that (1) the measure ρ is locally finite in the sense that any $x \in \mathcal{F}$ has a neighborhood U with $\rho(U) < \infty$. (2) the function $\mathcal{L}(x,.)$ is ρ-integrable for all $x \in \mathcal{F}$. Under these assumptions, we introduce the function $\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \mathfrak{s}$. The EL equations are formulated in our causal fermion system by $\ell|_{supp\ \rho} \equiv \inf_{\mathcal{F}} \ell$.

Remarks:

- The parameter $\mathfrak{s}$ can be chosen arbitrarily. For convenience, we always choose it such that $\inf_{\mathcal{F}} \ell = 0$. For this choice, the EL equations simplify to $\ell|_{\mathrm{supp}\rho} \equiv 0$
- Every minimizing measure ρ is a solution of the EL equations.
- One should keep in mind that not every solution of the EL equations is also a minimizer of the causal action. It is only a critical point of the variational principle.

Jet space $\mathfrak{J}$	It is useful to introduce a pair $\mathfrak{u} := (a, u)$ with a real-valued function $a \in C^\infty(M, \mathbb{R})$ and a vector field $u \in C^\infty(M, T\mathcal{F})$ as a so-called *jet*. The set of all jets is called the jet space $\mathfrak{J} = C^\infty(M, \mathbb{R}) \oplus C^\infty(M, T\mathcal{F})$.

Remarks:

- We often restrict attention to variations of the measure ρ in the form $\rho_\tau = F_\tau^*(f_\tau \rho)$ with families of smooth functions $(F_\tau) : M \to \mathcal{F}$ and $(f_\tau) : M \to \mathbb{R}^+$. Infinitesimally, these variations are described by the jets $\mathfrak{v} := \frac{\mathrm{d}}{\mathrm{d}\tau}(f_\tau, F_\tau)|_{\tau=0}$.

The restricted EL equations

The above EL equations also contains information for points on $\mathcal{F}$ which are far away from our spacetime M. In order to get the connection to usual physical equations formulated in spacetime, it suffices to restricting attention to a small neighborhood of points in M. In the smooth setting considered in this book, this leads to the following two equations:

(1) $\ell|_M \equiv 0$
(2) $D\ell|_M \equiv 0$

These the two equations can be combined in a compact form by using a jet $\mathfrak{u} = (a, u)$ to define $\nabla_{\mathfrak{u}}\ell(x) := a(x)\ell(x) + (D_u\ell)(x)$. Therefore, the *restricted EL equations* can be written as $\nabla_{\mathfrak{u}}\ell(x)|_M = 0$ for all $\mathfrak{u} \in \mathfrak{J}$.

Remarks:

- We also refer to solutions of the restricted EL equations as *critical measures*.

The linearized field equations

The linearized field equations reads

$$\langle \mathfrak{u}, \Delta\mathfrak{v} \rangle(x) := \nabla_{\mathfrak{u}} \left(\int_{\mathcal{F}} \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\right) \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}}\, \mathfrak{s} \right) = 0$$

for all $\mathfrak{u} \in \mathfrak{J}$ an $x \in M$. All $\mathfrak{v} \in \mathfrak{J}$ which satisfy this equation are called *linearized solutions*. The vector space of all linearized solutions is denoted by $\mathfrak{J}^{\text{lin}}$.

Remarks:

- For simplicity of presentation, in this book we only consider the *smooth setting* where we assume that the Lagrangian is smooth on $\mathcal{F} \times \mathcal{F}$.

- The EL equations are nonlinear equations for ρ. They are simplified by linearization, giving the linearized field equations.

- Often we use the shorthand notation $\Delta\mathfrak{v} = 0$ for the linearized field equation. One also can include an inhomogeneity $\mathfrak{w}$ by writing $\Delta\mathfrak{v} = \mathfrak{w}$.

- In order to avoid questions of differentiability of the jets, we use a formalism where the jets are never differentiated.

- Solutions of the linearized field equations can be used to obtain solutions for the full EL equations by means of a perturbation expansion.

The commutator jet	Let $\mathcal{A}$ be a symmetric operator on $\mathcal{H}$. We get a family of unitary transformations by $\mathcal{U}_\tau := \exp(\mathrm{i}\tau\mathcal{A})$. Jets of the form $\mathfrak{v} := (0, v) \in \mathfrak{J}^{\text{lin}}$ with $v(x) = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\mathcal{U}_\tau\, x\, \mathcal{U}_\tau^{-1}\big)\big|_{\tau=0} = \mathrm{i}\big[\mathcal{A}, x\big]$ are referred to as commutator jets. They are solutions the linearized field equations.

Remarks:

- This is the first special class of solutions of the linearized field equations which we consider.

The inner solutions	If our spacetime has a smooth manifold structure, the measure can be written in terms of the Lebesgue measure by $\mathrm{d}\rho = h(x)\, \mathrm{d}^k x$ with $h \in C^\infty(\mathscr{M}^k, \mathbb{R}^+)$. Now we can formulate the covariant divergence of a vector field v by $\operatorname{div} v = \frac{1}{h}\, \partial_j\big(h\, v^j\big)$. We refer to the jets of the form $\mathfrak{v} = (\operatorname{div} v, v)$ with $v \in \Gamma(M, TM)$ as **inner solutions**. All inner solutions with compact support are solutions of the linearized field equations.

Remarks:

- This is the second class of special solutions.
- The space of all inner solutions is denoted by $\mathfrak{J}^{\text{in}}$.

8.5. Exercises

EXERCISE 8.1. Let $F : \mathcal{F} \to \mathcal{F}$ be continuous and ρ a measure on $\mathcal{F}$. Show that

$$\operatorname{supp} F_* \rho = \overline{F(\operatorname{supp} \rho)} .$$

Hint: Recall the definition of the support of a measure (2.3.4) and use that the preimage of an open set under a continuous mapping is open.

EXERCISE 8.2. (a) Assume that $\mathcal{F}$ is locally compact. Moreover, assume that $F \in C^0([0,\delta) \times M, \mathcal{F})$ is continuous and that its preimage of any compact set is compact. Then for any $y \notin M$ there is $\tau_0 \in (0,\delta)$ such that

$$y \notin \operatorname{supp} \tilde{\rho}_\tau \qquad \text{for all } \tau \in [0, \tau_0]$$

(where $\tilde{\rho}_\tau$ are again the measures (8.1.1)) *Hint:* Use the result of Exercise 8.1.

(b) Show that this result remains valid for the variation (8.1.9) with a finite number of subsystems.

(c) What happens for an infinite number of subsystems? Also, is the assumption necessary that the preimage of a compact set under F is compact?

EXERCISE 8.3. *(Linearization of nonlinear partial differential equations)* In this exercise you are given two *non-linear* partial differential equations with corresponding (soliton) solutions. Check that the functions ϕ do indeed solve the equations. Then try to figure out what it means to *linearize* the equations around the given solutions and do it.

(a) The *sine-Gordon equation* of velocity $v \in (-1, 1)$:

$$\partial_{tt}\phi - \partial_{xx}\phi + \sin\phi = 0, \quad \phi(t,x) = 4 \arctan\left(\mathrm{e}^{\frac{x-vt}{\sqrt{1-v^2}}} \right).$$

(b) The *Korteweg-de-Vries equation* of unit speed:

$$\partial_t \phi + 6\,\phi\,\partial_x \phi + \partial_{xxx}\phi = 0, \quad \phi(t,x) = \frac{1}{2} \operatorname{sech}^2\left(\frac{x-vt}{2} \right).$$

Hint: You may use the following identities,

$$\sin(4 \arctan(x)) = -4 \frac{x^3 - x}{(1+x^2)^2}, \quad \tanh(x) - \tanh^3(x) = \operatorname{sech}^2(x) \tanh(x).$$

EXERCISE 8.4. *(Linearized fields on the sphere)* Let ρ be a minimizing measure of the causal variational principle of the sphere as introduced in Section 6.1 (for example the octahedron in Exercise 6.3 (b)).

(a) Let v be the vector field $\partial/\partial\varphi$ (where φ is the azimuth angle). Show that $\mathfrak{v} = (0, v)$ is a solution of the linearized field equations. *Hint:* One can use the fact that the causal variational principle is rotationally symmetric.

(b) Show that $\mathfrak{v}$ can be written as a commutator jet, i.e. in analogy to (8.2.3),

$$v(x) = \mathrm{i} \left[c\sigma^3, F(x) \right],$$

where $F : S^2 \subset \mathbb{R}^3 \to \mathcal{F}$ is the mapping in (6.1.2). Compute the constant c.

EXERCISE 8.5. *(Linearized fields for the causal variational principle on $\mathbb{R}$)* We return to the causal variational principles on $\mathbb{R}$ introduced in Exercise 6.4. Let $\rho = \delta$ be the unique minimizer.

(a) Show that the jet $\mathfrak{v} = (0, v)$ with the vector field $v = \partial_x$ is a solution of the linearized field equations for the causal variational principle corresponding to $\mathcal{L}_4$.

(b) Show that the jet $\mathfrak{v} = (0, v)$ from (a) does *not* satisfy the linearized field equations for the causal variational principle corresponding to $\mathcal{L}_2$.

EXERCISE 8.6. *(Linearized fields for the causal variational principle on S^1)* We return to the causal variational principle on $\mathbb{R}$ introduced in Exercise 6.5. Let ρ be a minimizing measure (6.6.4) for $0 < \tau < 1$.

(a) Show that the jet $\mathfrak{v} = (0, v)$ with the vector field $v = \partial_\varphi$ satisfies the linearized field equations. *Hint:* One can use the fact that the variational principle is rotationally symmetric.

(b) Show that the jet $\mathfrak{v} = (b, 0)$ with $b(\phi_0) = -b(\phi_0 + \pi)$ is a solution of the linearized field equations. *Hint:* Use that the causal action is independent of the parameter τ.

(c) Show that every solution of the linearized field equations is a linear combination of the linearized fields in (a) and (b).

EXERCISE 8.7. *(The commutator of commutator jets)* For any symmetric operator $\mathcal{A}$ of finite rank, the commutator vector field $\mathcal{C}(\mathcal{A})$ is defined by $\mathcal{C}(\mathcal{A})(x) = \mathrm{i}\big[\mathcal{A}, x\big]$ (see (8.2.3)). It is a vector field on M. For two vector fields u, v on M, their commutator is defined by $[u, v](f) = u(v(f)) - v(u(f))$ (with f any smooth function on M). Show that the commutator of two commutator vector fields is again a commutator vector field and

$$\big[\mathcal{C}(\mathcal{A}), \mathcal{C}(\mathcal{B})\big] = -\mathcal{C}\big(\mathrm{i}[\mathcal{A}, \mathcal{B}]\big)\,.$$

Hint: The proof can be found in [**64**, Lemma A.2].

CHAPTER 9

Surface Layer Integrals and Conservation Laws

In this chapter we introduce *surface layer integrals* as an adaptation of surface integrals to causal fermion systems and causal variational principles. The mathematical structure of a surface layer integral fits nicely to the analytic structures (namely, the EL equations and the linearized field equations as introduced in Chapters 7 and 8). This will become apparent in *conservation laws* which generalize Noether's theorem and the symplectic form to the setting of causal variational principles. Moreover, we shall introduce a so-called *nonlinear surface layer integral* which makes it possible to compare two measures ρ and $\tilde{\rho}$ at a given time. Finally, we will explain how *two-dimensional* surface integrals can be described by surface layer integrals.

9.1. The Concept of a Surface Layer Integral

In daily life we experience space and objects therein. These objects are usually described by densities, and integrating these densities over space gives particle numbers, charges, the total energy, etc. In mathematical terms, the densities are typically described as the normal components of vector fields on a Cauchy surface, and conservation laws express that the values of these integrals do not depend on the choice of the Cauchy surface, i.e.

$$\int_{\mathcal{N}_1} J^k \nu_k \, \mathrm{d}\mu_{\mathcal{N}_1}(x) = \int_{\mathcal{N}_2} J^k \nu_k \, \mathrm{d}\mu_{\mathcal{N}_2}(x) \,, \tag{9.1.1}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are two Cauchy surfaces, ν is the future-directed normal, and $\mathrm{d}\mu_{\mathcal{N}_{1/2}}$ is the induced volume measure.

In the setting of causal variational principles, surface integrals like (9.1.1) are undefined. Instead, one considers so-called *surface layer integrals*, which we now introduce. In general terms, a surface layer integral is a double integral of the form

$$\int_\Omega \left(\int_{M \setminus \Omega} (\cdots)\, \mathcal{L}(x,y) \, \mathrm{d}\rho(y) \right) \mathrm{d}\rho(x) \,, \tag{9.1.2}$$

where one variable is integrated over a subset $\Omega \subset M$, and the other variable is integrated over the complement of Ω. Here $(\cdots)$ stands for a differential operator acting on the Lagrangian to be specified below. In order to explain the basic idea, we begin with the additional assumption that the Lagrangian is of *short range* in the following sense. We let $d \in C^0(M \times M, \mathbb{R}_0^+)$ be a suitably chosen distance function on M. Then the assumption of short range can be quantified by demanding that $\mathcal{L}$ should vanish on distances larger than δ, i.e.

$$d(x,y) > \delta \quad \Longrightarrow \quad \mathcal{L}(x,y) = 0 \,. \tag{9.1.3}$$

Under this assumption, the surface layer integral (9.1.2) only involves pairs (x,y) of distance at most δ, with x lying in Ω and y lying in the complement $M \setminus \Omega$. As a

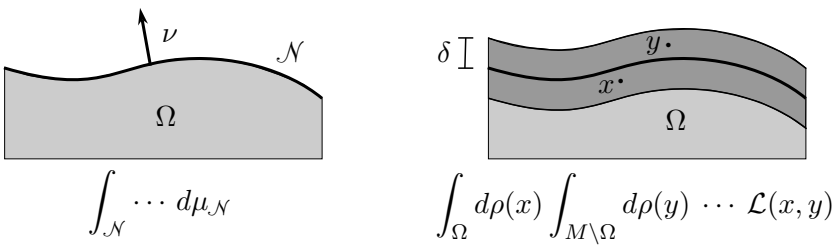


FIGURE 9.1. A surface integral and a corresponding surface layer integral.

consequence, the integral only involves points in a layer around the boundary of Ω of width δ, i.e.

$$x, y \in B_\delta\big(\partial\Omega\big) \,.$$

Therefore, a double integral of the form (9.1.2) can be regarded as an approximation of a surface integral on the length scale δ, as shown in Figure 9.1. In the setting of causal variational principles, such surface layer integrals take the role of surface integrals.

We point out that the causal Lagrangian is *not* of short range in the sense (9.1.3). But it decays on a length scale which typically coincides with the Compton scale $1/m$ (where m denotes the rest mass of the Dirac particles). With this in mind, the above consideration and the qualitative picture of a surface layer integral in Figure 9.1 apply to the causal action principle as well.

9.2. A Noether-Like Theorem

In modern physics, the connection between symmetries and conservation laws is of central importance. For continuous symmetries, this connection is made mathematically precise by Noether's theorem (see [**122**] or the textbooks [**95**, Section 13.7], [**7**, Chapter III]). As shown in [**70**], the connection between symmetries and conservation laws can be extended to the setting of causal variational principles. As we shall see, both the statement and the proof are quite different from the classical Noether theorem; this is why we refer to our result as a *Noether-like theorem*.

The first step is to formulate a symmetry condition for the Lagrangian $\mathcal{L}(x,y)$ of a causal variational principle. Similar to the procedure in Section 7.3, one could describe the symmetry by a group of diffeomorphisms. For our purposes, the correct setting would be to consider a one-parameter group of diffeomorphisms Φ_τ on $\mathcal{F}$, i.e.

$$\Phi : \mathbb{R} \times \mathcal{F} \to \mathcal{F} \qquad \text{with} \qquad \Phi_\tau \Phi_{\tau'} = \Phi_{\tau+\tau'} \tag{9.2.1}$$

(we usually write the first argument as a subscript, i.e. $\Phi_\tau(x) \equiv \Phi(\tau, x)$). The symmetry condition could be imposed by demanding that the Lagrangian be invariant under this one-parameter group in the sense that

$$\mathcal{L}(x,y) = \mathcal{L}\big(\Phi_\tau(x), \Phi_\tau(y)\big) \qquad \text{for all } \tau \in \mathbb{R} \text{ and } x, y \in \mathcal{F} \,. \tag{9.2.2}$$

It turns out that this condition is unnecessarily strong for two reasons. First, it suffices to consider families which are defined locally for $\tau \in (-\tau_{\max}, \tau_{\max})$. Second, the mapping Φ does not need to be defined on all of $\mathcal{F}$. Instead, it is more appropriate to impose the symmetry condition only on spacetime $M \subset \mathcal{F}$. This leads us to consider instead of (9.2.1) a mapping

$$\Phi : (-\tau_{\max}, \tau_{\max}) \times M \to \mathcal{F} \qquad \text{with} \qquad \Phi_0 = \mathrm{id}_M \,. \tag{9.2.3}$$

We refer to Φ_τ as a **variation** of M in $\mathcal{F}$. Next, we need to specify what we mean by "smoothness" of this variation. This is a subtle point because, as explained in the example of the causal variational principle on the sphere in Section 6.1, the support of a minimizing measure will in general be singular. Moreover, the function ℓ defined by (7.1.1) in general will only be Lipschitz continuous. Our Noether-like theorems only require that this function be differentiable in the direction of the variations:

DEFINITION 9.2.1. *A variation Φ_τ of the form* (9.2.3) *is* **continuously differentiable** *if the composition*

$$\ell \circ \Phi \;:\; (-\tau_{\max}, \tau_{\max}) \times M \to \mathbb{R}$$

is continuous and if its partial derivative $\partial_\tau(\ell \circ \Phi)$ exists and is continuous.

The next question is how to adapt the symmetry condition (9.2.2) to the mapping Φ defined only on $(-\tau_{\max}, \tau_{\max}) \times M$. This is not obvious because setting $\tilde{x} = \Phi_\tau(x)$ and using the group property, the condition (9.2.2) can be written equivalently as

$$\mathcal{L}\big(\Phi_{-\tau}(\tilde{x}), y\big) = \mathcal{L}\big(\tilde{x}, \Phi_\tau(y)\big) \qquad \text{for all } \tau \in \mathbb{R} \text{ and } \tilde{x}, y \in \mathcal{F}\,. \tag{9.2.4}$$

But if we restrict attention to pairs $x, y \in M$, the equations in (9.2.2) and (9.2.4) are different. For the general mathematical formulation, it is preferable to weaken the condition (9.2.2) starting from the expression in (9.2.4).

DEFINITION 9.2.2. *A variation Φ_τ of the form* (9.2.3) *is a* **symmetry of the Lagrangian** *if*

$$\mathcal{L}\big(x, \Phi_\tau(y)\big) = \mathcal{L}\big(\Phi_{-\tau}(x), y\big) \qquad \textit{for all } \tau \in (-\tau_{\max}, \tau_{\max}) \textit{ and } x, y \in M\,. \tag{9.2.5}$$

We now now state and prove our Noether-like theorem.

THEOREM 9.2.3. *Let ρ be a critical measure and Φ_τ a continuously differentiable symmetry of the Lagrangian. Then for any compact subset $\Omega \subset M$,*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big) \Big)\Big|_{\tau=0} = 0\,. \tag{9.2.6}$$

PROOF. Integrating (9.2.5) over $\Omega \times \Omega$ gives

$$\begin{aligned}
0 &= \iint_{\Omega \times \Omega} \Big(\mathcal{L}\big(x, \Phi_\tau(y)\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big) \Big)\, \mathrm{d}\rho(x)\, \mathrm{d}\rho(y) \\
&= \iint_{\Omega \times \Omega} \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big)\big) \Big)\, \mathrm{d}\rho(x)\, \mathrm{d}\rho(y) \\
&= \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \chi_\Omega(y)\, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big)\big) \Big)\,,
\end{aligned}$$

where in the first step we used the Lagrangian is symmetric in its two arguments and that the integration range is symmetric in x and y. We rewrite the characteristic function $\chi_\Omega(y)$ as $1 - (1 - \chi_\Omega(y))$, multiply out and use the definition of ℓ, (7.1.1). We thus obtain

$$\begin{aligned}
0 &= \int_\Omega \Big(\ell\big(\Phi_\tau(x)\big) - \ell\big(\Phi_{-\tau}(x)\big) \Big)\, \mathrm{d}\rho(x) \\
&\quad - \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \chi_{M \setminus \Omega}(y)\, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big) \Big)\,.
\end{aligned}$$

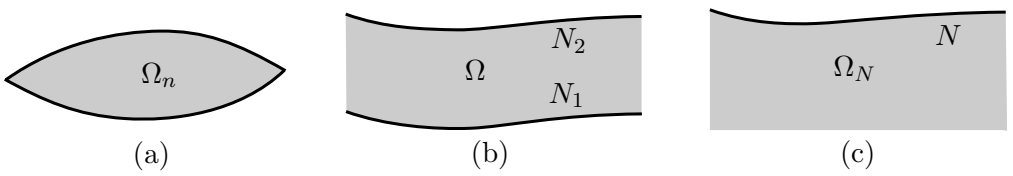


FIGURE 9.2. Choices of spacetime regions.

We thus obtain the identity

$$\begin{aligned}\int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y) \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big)\Big) \\ = \int_\Omega \Big(\ell\big(\Phi_\tau(x)\big) - \ell\big(\Phi_{-\tau}(x)\big)\Big)\, \mathrm{d}\rho(x)\,.\end{aligned} \tag{9.2.7}$$

Using that $\ell(\Phi_\tau(x))$ is continuously differentiable (see Definition 9.2.1) and that Ω is compact, we conclude that the right side of this equation is differentiable at $\tau = 0$. Moreover, we are allowed to interchange the τ-differentiation with integration. The EL equations (7.1.6) imply that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ell\big(\Phi_\tau(x)\big)\Big|_{\tau=0} = 0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\ell\big(\Phi_{-\tau}(x)\big)\Big|_{\tau=0}\,.$$

Hence the right side of (9.2.7) is differentiable at $\tau = 0$, and the derivative vanishes. This gives the result. □

This theorem requires a detailed explanation. We first clarify the connection to surface layer integrals. To this end, let us assume for technical simplicity that Φ_τ and the Lagrangian are differentiable in the sense that the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\tau(x)\big|_{\tau=0} =: u(x) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{L}\big(\Phi_\tau(x), y\big)\big|_{\tau=0} \tag{9.2.8}$$

exist for all $x, y \in M$ and are continuous on M respectively $M \times M$. Then one may exchange differentiation and integration in (9.2.6) and apply the chain rule to obtain

$$\int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, D_{1,u}\mathcal{L}(x,y) = 0\,,$$

where $D_{1,u}$ is the partial derivative at x in the direction of the vector field $u(x)$. This expression is a surface layer integral as in (9.1.2). In general, the derivatives in (9.2.8) need *not* exist, because we merely imposed the weaker differentiability assumption of Definition 9.2.1. In this case, the statement of the theorem implies that the derivative of the integral in (9.2.6) exists and vanishes.

We next explain the connection to conservation laws. Let us assume that M admits a sensible notion of "spatial infinity" and that the vector field $\partial_\tau\Phi \in \Gamma(M, T\mathcal{F})$ has suitable decay properties at spatial infinity. Then one can chose a sequence $\Omega_n \subset M$ of compact sets which form an exhaustion of a set Ω which extends up to spatial infinity (see Figure 9.2 (a) and (b)). Considering the surface layer integrals for Ω_n and passing to limit, one concludes that also the surface layer integral corresponding to Ω vanishes. Let us assume that the boundary $\partial\Omega$ has two components N_1 and N_2 (as in Figure 9.2 (b)). Then the above theorem implies that the surface layer integrals over N_1 and N_2 coincide (where

the surface layer integral over N is defined as the surface layer integral corresponding to a set Ω_N with $\partial\Omega_N = N$ as shown in Figure 9.2 (c)). In other words, the quantity

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\Omega_N} \mathrm{d}\rho(x) \int_{M\setminus\Omega_N} \mathrm{d}\rho(y)\, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big)\Big)\Big|_{\tau=0}$$

is well-defined and independent of the choice of N. In this setting, the surfaces N can be interpreted as Cauchy surfaces, and the conservation law of Theorem 9.2.3 means that the surface layer integral is preserved under the time evolution.

As a concrete example, the unitary invariance of the causal action principle gives rise to a conservation law, which corresponds to current conservation. This example will be considered in detail in Section 9.4. We finally remark that the conservation laws for *energy-momentum* can also be obtained from Theorem 9.2.3, assuming that the causal fermion system has symmetries as described by generalized Killing symmetries. We refer the interested reader to [**70**, Section 6].

9.3. A Class of Conservation Laws in the Smooth Setting

In the previous section we saw that surface layer integrals can be used to formulate a Noether-like theorem which relates symmetries to conservation laws. In this section we shall derive conservation laws even in the absence of symmetries. Instead, these conservation laws are closely tied to the structure of the linearized field equations as derived in Section 8.1. In order to focus on the essence of the construction, we again restrict attention to the smooth setting (6.2.4). The basic idea of the construction is explained in the following proposition:

PROPOSITION 9.3.1. *Let ρ be a critical measure and $\Omega \subset M$ be compact. Then for any solution $\mathfrak{v} \in \mathfrak{J}^{\text{lin}}$ of the linearized field equations* (8.1.7)*,*

$$\gamma_\rho^\Omega(\mathfrak{v}) := \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) = \int_\Omega \nabla_\mathfrak{v}\, \mathfrak{s}\; \mathrm{d}\rho\,. \tag{9.3.1}$$

PROOF. In view of the anti-symmetry of the integrand,

$$\int_\Omega \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) = 0\,.$$

Adding this equation to the left side of (9.3.1), we obtain

$$\begin{aligned}
\gamma_\rho^\Omega(\mathfrak{v}) &= \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) \\
&= 2 \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \big(\nabla_{1,\mathfrak{v}}\big)\mathcal{L}(x,y) - \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) \\
&= \int_\Omega \mathrm{d}\rho(x) \Big(2\, \nabla_\mathfrak{v}\Big(\ell(x) + \mathfrak{s}\Big) - \Big(\big(\Delta\mathfrak{v}\big)(x) + \nabla_\mathfrak{v}\, \mathfrak{s}\Big)\Big)\,,
\end{aligned}$$

where in the last line we used the definitions of ℓ and Δ (see (7.1.1) and (8.1.7)). Applying the restricted EL equations (7.2.4) and the linearized field equations (8.1.7) gives the result. □

Viewing γ_ρ^Ω as a linear functional on the linearized solutions, we also refer to $\gamma_\rho^\Omega(\mathfrak{v})$ as the *conserved one-form*. We remark that the identity (9.3.1) has a similar structure as the conservation law in the Noether-like theorem (9.2.6). In order to make the connection

precise, one describes the symmetry Φ_τ infinitesimally by a jet $\mathfrak{v}$ with vanishing scalar component,

$$\mathfrak{v}(x) := \frac{\mathrm{d}}{\mathrm{d}\tau}\big(0, \Phi_\tau(x)\big)\Big|_{\tau=0} .$$

Using the symmetry property (9.2.5), one verifies similar as in the proof of Lemma 8.2.1 that this jet satisfies the linearized field equations (8.1.7). Therefore, Proposition 9.3.1 applies, and the right side vanishes because $\mathfrak{v}$ has no scalar component. We thus recover the identity obtained by carrying out the τ-derivative in (9.2.6).

We conclude that Proposition 9.3.1 is a generalization of Theorem 9.2.3. Instead of imposing symmetries, the identity (9.3.1) is a consequence of the linearized field equations. Again choosing Ω as the region between two Cauchy surfaces (see Figure 9.2), one obtains a relation between the surface layer integrals around N_1 and N_2. If the scalar component of $\mathfrak{v}$ vanishes, we obtain a conservation law. Otherwise, the right side of (9.3.1) tells us how the surface layer integral changes in time.

We now generalize Proposition 9.3.1. The basic idea is to integrate anti-symmetric expressions in x and y which involve higher derivatives of the Lagrangian. We again restrict attention to the smooth setting (for the general proof see [**72**]). Let $\tilde\rho_{s,t}$ with $s,t \in (-\delta,\delta)$ be a two-parameter family of measures which are solutions of the restricted EL equations. We assume that these measures are of the form

$$\tilde\rho_{s,t} = (F_{s,t})_*\big(f_{s,t}\,\rho\big)\,, \tag{9.3.2}$$

where $f_{s,t}$ and $F_{s,t}$ are smooth,

$$f \in C^\infty\big((-\delta,\delta)^2 \times \mathcal{F}, \mathbb{R}^+\big) \qquad \text{and} \qquad F \in C^\infty\big((-\delta,\delta)^2 \times \mathcal{F}, \mathcal{F}\big)\,, \tag{9.3.3}$$

and are trivial in the case $s = t = 0$ (6.4.2). Moreover, we need the following technical assumption:

(ta) For all $x \in M$, $p, q \geq 0$ and $r \in \{0, 1\}$, the following partial derivatives exist and may be interchanged with integration,

$$\begin{aligned}&\int_M \partial_{s'}^r \partial_s^p \partial_t^q \mathcal{L}\big(F_{s+s',t}(x), F_{s,t}(y)\big)\Big|_{s'=s=t=0}\, \mathrm{d}\rho(y) \\ &\qquad = \partial_{s'}^r \partial_s^p \partial_t^q \int_M \mathcal{L}\big(F_{s+s',t}(x), F_{s,t}(y)\big)\, \mathrm{d}\rho(y)\Big|_{s'=s=t=0} .\end{aligned}$$

We now state a general identity between a surface layer integral and a volume integral which was first obtained in [**72**]. It generalizes the result of Proposition 9.3.1 and gives rise to additional conservation laws for surface layer integrals, which will be analyzed subsequently (in Section 9.5). The proof of the following theorem also works out the mathematical essence of our conservation laws.

THEOREM 9.3.2. *Let f and F be as in* (9.3.3) *and* (6.4.2) *which satisfy the above assumption* (ta). *Moreover, assume that the measures $\tilde\rho_{s,t}$ given by* (9.3.2) *satisfy the restricted EL equations for all s and t. Then for every compact $\Omega \subset M$ and every $k \in \mathbb{N}$,*

$$\begin{aligned} I^\Omega_{k+1} &:= \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y) \\ &\qquad \times \big(\partial_{1,s} - \partial_{2,s}\big)\big(\partial_{1,t} + \partial_{2,t}\big)^k f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y)\Big|_{s=t=0} \\ &= \mathfrak{s} \int_\Omega \partial_s \partial_t^k f_{s,t}(x)\Big|_{s=t=0}\, \mathrm{d}\rho(x)\,. \end{aligned} \tag{9.3.4}$$

PROOF. Introducing the short notation

$$L\big(x_{s,t}, y_{s,t}\big) = f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y)\,,$$

the restricted EL equations (8.1.7) read

$$\nabla_{\mathfrak{u}}\bigg(\int_M L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y) - \mathfrak{s}\, f_{s,t}(x) \bigg) = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}\,.$$

In particular for any $k \geq 0$ and any vector $v = v^s \partial_s + v^t \partial_s$, we obtain

$$\int_M \partial_{1,s} \big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y)\Big|_{s=t=0} = \mathfrak{s}\, \partial_s \partial_v^k f_{s,t}(x)\big|_{s=t=0} \tag{9.3.5}$$

$$\int_M \big(\partial_{1,v} + \partial_{2,v}\big)^{k+1} L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y)\Big|_{s=t=0} = \mathfrak{s}\, \partial_v^{k+1} f_{s,t}(x)\big|_{s=t=0} \tag{9.3.6}$$

(the derivatives exist and can be exchanged with the integration according to the above assumption (ta)). Differentiating the last equation with respect to v^s and dividing by $k+1$, we obtain

$$\int_M \big(\partial_{1,s} + \partial_{2,s}\big) \big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y) = \mathfrak{s}\, \partial_s \partial_v^k f_{s,t}(x)\,.$$

Subtracting twice the identity (9.3.5), we obtain for any $k \geq 0$ the equation

$$\int_M \big(\partial_{1,s} - \partial_{2,s}\big) \big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y) = \mathfrak{s}\, \partial_s \partial_v^k f_{s,t}(x)\,.$$

Integrating the last equation over Ω gives

$$\begin{aligned} \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \big(\partial_{1,s} - \partial_{2,s}\big) \big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big) \\ = \mathfrak{s} \int_\Omega \partial_s \partial_v^k f_{s,t}(x)\; \mathrm{d}\rho(x)\,. \end{aligned} \tag{9.3.7}$$

On the other hand, since the integrand is anti-symmetric in its arguments x and y, we also know that

$$\int_\Omega \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\, \big(\partial_{1,s} - \partial_{2,s}\big) \big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big) = 0\,.$$

Subtracting this equation from (9.3.7) and evaluating at $s = t = 0$ gives the result. □

Specializing the statement of this theorem to the case $k = 0$ and setting

$$\mathfrak{v} = \frac{\mathrm{d}}{\mathrm{d}s} \big(f_{s,t}, F_{s,t}\big)\Big|_{s=t=0}\,,$$

we recover the statement of Proposition 9.3.1. The case $k = 1$ will be studied in more detail in Section 9.5.

We conclude this section by discussing the conservation law of Proposition 9.3.1 for *inner solutions* as considered in Section 8.3 (commutator jets will be considered afterward in Section 9.4). To this end, we need to assume again that spacetime has a smooth manifold structure. We first define an integration measure on the boundary of Ω.

DEFINITION 9.3.3. *Let $\mathfrak{v} = (\operatorname{div} v, v) \in \mathfrak{J}_\rho^{\text{in}}$ be an inner solution and $\Omega \subset M$ closed with smooth boundary $\partial\Omega$. On the boundary, we define the measure $d\mu(\mathfrak{v}, x)$ as the contraction of the volume form on M with v, i.e. in local charts*

$$\mathrm{d}\mu(\mathfrak{v}, x) = h\, \epsilon_{ijkl}\, v^i\; \mathrm{d}x^j\, \mathrm{d}x^k\, \mathrm{d}x^l\,,$$

where ϵ_{ijkl} is the totally anti-symmetric symbol (normalized by $\epsilon_{0123} = 1$).

We now let $\mathfrak{v} = (\operatorname{div} v, v)$ be an inner solution. Then the integral on the right side of (9.3.1) reduces the integral over the divergence of the vector field v,

$$\int_\Omega \nabla_{\mathfrak{v}}\, \mathfrak{s}\; \mathrm{d}\rho = \mathfrak{s} \int_\Omega \operatorname{div} v\; \mathrm{d}\rho\,. \tag{9.3.8}$$

On the left side of (9.3.1), on the other hand, similar as in Lemma 8.3.3 we can integrate by parts. But now boundary terms remain,

$$\begin{aligned} \gamma^\Omega_\rho(\mathfrak{v}) &= \int_{\partial\Omega} \mathrm{d}\mu(\mathfrak{v}, x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) + \int_\Omega \mathrm{d}\rho(x) \int_{\partial\Omega} \mathrm{d}\mu(\mathfrak{v}, y)\, \mathcal{L}(x,y) \\ &= \int_{\partial\Omega} \mathrm{d}\mu(\mathfrak{v}, x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) = \mathfrak{s} \int_{\partial\Omega} \mathrm{d}\mu(\mathfrak{v}, x)\,, \end{aligned} \tag{9.3.9}$$

where in the last line we used the symmetry of $\mathcal{L}$ and employed the EL equations. In this way, the surface layer integral in (9.3.1) reduces to a usual surface integral over the hypersurface $\partial\Omega$. Moreover, combining (9.3.1) with (9.3.9) and (9.3.8), we get back the Gauß divergence theorem

$$\mathfrak{s} \int_{\partial\Omega} \mathrm{d}\mu(\mathfrak{v}, x) = \mathfrak{s} \int_\Omega \operatorname{div} v\; \mathrm{d}\rho\,.$$

This illustrates that Proposition 9.3.1 is a generalization of the Gauß divergence theorem where the vector field is replace by a general solution of the linearized field equations. The formulation with surface layer integrals has the further advantage that the result can be generalized in a straightforward way to non-smooth (for example discrete) spacetimes.

9.4. The Commutator Inner Product for Causal Fermion Systems

As a concrete example of a conservation law, we now consider *current conservation*. To this end, we consider the setting of causal fermion systems. As in Section 8.2 we again let $\mathcal{A}$ be a symmetric operator of finite rank on $\mathcal{H}$ and $\mathcal{U}_\tau$ be the corresponding one-parameter family of unitary transformations (8.2.2). Infinitesimally, this one-parameter family is described by the *commutator jet* $\mathfrak{v}$ (8.2.3). The unitary invariance of the causal action implies that the commutator jets satisfy the linearized field equations (see Lemma 8.2.1). Moreover, using that the scalar component of commutator jets vanishes, Proposition 9.3.1 gives for any compact $\Omega \subset M$ the conservation law

$$\gamma^\Omega_\rho(\mathfrak{v}) := \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) = 0\,. \tag{9.4.1}$$

In order to understand the significance of this conservation law, it is useful to choose $\mathcal{A}$ more specifically as an operator of rank one. More precisely, given a non-zero vector $\psi \in \mathcal{H}$, we form the symmetric linear operator $\mathcal{A} \in \mathrm{L}(\mathcal{H})$ of rank one by

$$\mathcal{A}u := \langle u|\psi\rangle_{\mathcal{H}}\, \psi \tag{9.4.2}$$

(thus in bra/ket notation, $\mathcal{A} = |\psi\rangle\langle\psi|$). We now form the corresponding commutator jet (8.2.3). Varying the vector ψ, we obtain a mapping

$$\mathfrak{j} \,:\, \mathcal{H} \to \mathfrak{J}^{\text{lin}}\,, \qquad \psi \mapsto \mathfrak{v}\,. \tag{9.4.3}$$

Moreover, we choose Ω again as the past of a Cauchy surface (as shown in Figure 9.2 (c)). We write the corresponding conserved surface layer integral in (9.4.1) as

$$\mathfrak{C}^\Omega_\rho(u) := \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \big(D_{1,\mathfrak{j}(u)} - D_{2,\mathfrak{j}(u)}\big)\mathcal{L}(x,y) \qquad \text{with } u \in \mathcal{H}\,, \tag{9.4.4}$$

where for technical simplicity we assume smoothness in order to interchange differentiation with integration. Clearly, the mapping $\mathfrak{j}$ in (9.4.3), and consequently also the mapping $\mathfrak{C}^\Omega_\rho$, are homogeneous of degree two, i.e.

$$\mathfrak{C}^\Omega_\rho\big(\lambda u\big) = |\lambda|^2\, \mathfrak{C}^\Omega_\rho(u) \qquad \text{for all } u \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}\,.$$

Therefore, we can use the polarization formula to define a sesquilinear form on the Hilbert space $\mathcal{H}$,

$$\langle u|v\rangle^\Omega_\rho := \frac{1}{4}\left(\mathfrak{C}^\Omega_\rho(u+v) - \mathfrak{C}^\Omega_\rho(u-v)\right) - \frac{\mathrm{i}}{4}\left(\mathfrak{C}^\Omega_\rho(u+\mathrm{i}v) - \mathfrak{C}^\Omega_\rho(u-\mathrm{i}v)\right). \tag{9.4.5}$$

This sesquilinear form is referred to as the *commutator inner product* (for details see [**64**, Section 3]). In [**70**, Section 5.2] it is shown that for Dirac systems describing the Minkowski vacuum, the commutator inner product coincides (up to an irrelevant prefactor) with the scalar product on Dirac solutions (1.3.12). In this way, the conservation law for the commutator inner product gives back the conservation of the Dirac current (1.3.11). We thus recover current conservation as a special case of a more general conservation law for causal fermion systems. Since in examples of physical interest, the conserved surface layer integral $\mathfrak{C}^\Omega_\rho(u,v)$ gives back the Hilbert space scalar product, we give this property a name:

DEFINITION 9.4.1. *Given a critical measure ρ and a subset $\Omega \subset M$, the surface layer integral $\mathfrak{C}^\Omega_\rho$ is said to* **represent the scalar product** *on the subspace $\mathcal{H}^\mathrm{f} \subset \mathcal{H}$ if there is a non-zero real constant c such that the sesquilinear form $\langle .|. \rangle^\Omega_\rho$ defined by* (9.4.5) *has the property*

$$\langle u|u\rangle^\Omega_\rho = c\,\|u\|^2_\mathcal{H} \qquad \textit{for all } u \in \mathcal{H}^\mathrm{f}\,. \tag{9.4.6}$$

In view of the conservation law of Proposition 9.3.1, this property remains valid if Ω is changed by a compact subset of M. We point out that the representation (9.4.6) *cannot* hold on the whole Hilbert space, i.e. for all $u \in \mathcal{H}$; for details see Exercise 9.5 and [**53**, Appendix A].

At present there is no general argument why the surface layer integral $\mathfrak{C}^\Omega_\rho$ should represent the scalar product on a non-trivial subspace $\mathcal{H}^\mathrm{f} \subset \mathcal{H}$. Therefore, in this book we shall not assume that this property holds. Instead, we make the following weaker assumption. We assume that the sesquilinear form $\mathfrak{C}^\Omega_\rho$ is equivalent to the scalar product in the sense that

$$\langle u|v\rangle^\Omega_\rho = \langle u \,|\, \mathcal{C}_\rho\, v\rangle_\mathcal{H} \qquad \text{for all } u, v \in \mathcal{H}^\mathrm{f}\,,$$

where $\mathcal{C}_\rho$ is a bounded linear operator on $\mathcal{H}$ with bounded inverse. Under this assumption, the Hilbert space scalar product can be expressed by

$$\langle u \,|\, v\rangle_\mathcal{H} = \langle u \,|\, \mathcal{C}_\rho^{-1}\, v\rangle^\Omega_\rho \qquad \text{for all } u, v \in \mathcal{H}^\mathrm{f}\,.$$

In this way, the Hilbert space scalar product can be represented by a surface layer integral involving the physical wave functions in spacetime.

We conclude this section with a remark on the connection between the commutator inner product and the scalar product on solutions of the Dirac equation. As already mentioned after (9.4.5), for Dirac systems describing the Minkowski vacuum, the commutator inner product (9.4.5) coincides with the scalar product (1.3.12). Since both inner products are conserved, the same is true for any Dirac system which evolved from the vacuum (for example by "turning on" an interaction). The basic shortcoming of this correspondence is that it holds only for the physical wave functions, i.e. for all occupied one-particle states of the system. Thus, in the example of the Minkowski vacuum, the

connection between (9.4.5) and (1.3.12) can be made only for the negative-energy solutions of the Dirac equation. The positive-energy solutions, however, do not correspond to physical wave functions, so that the commutator inner product is undefined. In order to improve the situation, one would like to extend the commutator inner product to more general wave functions, in such a way that it still agrees with (1.3.12). This construction is carried out in [**64, 53**]. Current conservation continues to hold for the extension, provided that the wave functions satisfy the so-called *dynamical wave equation*

$$\int_M Q^{\text{dyn}}(x,y)\,\psi(y)\;\mathrm{d}\rho(y) = 0\,. \tag{9.4.7}$$

Here the integral kernel Q^{dyn} is constructed from first variations of the causal Lagrangian. In this formulation, the commutator inner product takes the form

$$\langle\psi|\phi\rangle_\rho^\Omega := -2\mathrm{i}\left(\int_\Omega \mathrm{d}\rho(x)\int_{M\setminus\Omega}\mathrm{d}\rho(y) - \int_{M\setminus\Omega}\mathrm{d}\rho(x)\int_\Omega \mathrm{d}\rho(y)\right) \prec\!\psi(x)\,|\,Q^{\text{dyn}}(x,y)\,\phi(y)\!\succ_x . \tag{9.4.8}$$

For some more details on these connections see Exercises 9.3 and 9.4.

After these extensions have been made, the dynamical wave equation (9.4.7) can be regarded as the generalization of the Dirac equation to causal fermion systems. Moreover, the commutator inner product (9.4.8) generalizes the scalar product on Dirac solutions (1.3.12), thereby also extending current conservation to dynamical waves.

9.5. The Symplectic Form and the Surface Layer Inner Product

For the applications, the most important surface layer integrals are I_1^Ω (also denoted by γ_ρ^Ω; see Proposition 9.3.1 and Theorem 9.3.2 in the case $k=0$) and I_2^Ω (see Theorem 9.3.2 in the case $k=1$). We now have a closer look at the surface layer integral I_2^Ω. It is defined by

$$\begin{aligned} I_2^\Omega = \int_\Omega \mathrm{d}\rho(x)\int_{M\setminus\Omega}\mathrm{d}\rho(y) \\ \times\big(\partial_{1,s}-\partial_{2,s}\big)\big(\partial_{1,t}+\partial_{2,t}\big) f_{s,t}(x)\,\mathcal{L}\big(F_{s,t}(x),F_{s,t}(y)\big)\,f_{s,t}(y)\Big|_{s=t=0} \end{aligned} \tag{9.5.1}$$

and satisfies for any compact subset $\Omega\subset M$ the identity

$$I_2^\Omega = \mathfrak{s}\int_\Omega \partial_s\partial_t f_{s,t}(x)\Big|_{s=t=0}\,\mathrm{d}\rho(x)\,. \tag{9.5.2}$$

These formulas simplify considerably if we *anti-symmetrize* in the parameters s and t. Namely, the formula for I_2^Ω reduces to the surface layer integral

$$\int_\Omega \mathrm{d}\rho(x)\int_{M\setminus\Omega}\mathrm{d}\rho(y)\big(\partial_{1,s}\partial_{2,t}-\partial_{1,s}\partial_{2,t}\big) f_{s,t}(x)\,\mathcal{L}\big(F_{s,t}(x),F_{s,t}(y)\big)\,f_{s,t}(y)\Big|_{s=t=0}\,.$$

Since this expression involves only first partial derivatives, it can be rewritten with jet derivatives as

$$\sigma_\rho^\Omega(\mathfrak{u},\mathfrak{v}) := \int_\Omega \mathrm{d}\rho(x)\int_{M\setminus\Omega}\mathrm{d}\rho(y)\big(\nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{v}}-\nabla_{1,\mathfrak{v}}\nabla_{2,\mathfrak{u}}\big)\mathcal{L}(x,y)\,, \tag{9.5.3}$$

where the jets $\mathfrak{u}$ and $\mathfrak{v}$ are the linearized solutions

$$\mathfrak{u} = \partial_s\big(f_{s,t},F_{s,t}\big)\big|_{s=t=0} \qquad\text{and}\qquad \mathfrak{v} = \partial_t\big(f_{s,t},F_{s,t}\big)\big|_{s=t=0}\,. \tag{9.5.4}$$

Moreover, the right side of (9.5.2) vanishes when anti-symmetrizing in s and t. We conclude that

$$\sigma_\rho^\Omega(\mathfrak{u},\mathfrak{v}) = 0 \qquad \text{for every compact } \Omega \subset M\,.$$

Choosing Ω again as explained in Figure 9.2, we obtain a conservation law for a surface layer integral over a neighborhood of a hypersurface N which extends to spatial infinity. We refer to σ_ρ^Ω as the *symplectic form* (the connection to symplectic geometry will be explained after (9.5.7) below).

Symmetrizing I_2^Ω in the parameters s and t gives the surface layer integral

$$\int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y) \big(\partial_{1,s}\partial_{1,t} - \partial_{2,s}\partial_{2,t}\big) f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y)\Big|_{s=t=0}\,. \tag{9.5.5}$$

This expression has a more difficult structure because it involves second partial derivatives. Such second partial derivatives cannot be expressed directly in terms of second jet derivatives, because the derivatives of the jets also need to be taken into account. In a differential geometric language, defining second derivatives would make it necessary to introduce a connection on $\mathcal{F}$. As explained after (ii) on page 143, we here use the simpler method of taking second partial derivatives in distinguished charts (for example, symmetric wave charts for causal fermion systems; see the remark after Proposition 3.1.3 and [**69**, Section 6.1] or [**76**, Section 3]). Then it is useful to introduce the *surface layer inner product* $(.,.)_\rho^\Omega$ as the contribution to (9.5.5) involving second derivatives of the Lagrangian, i.e.

$$(\mathfrak{u},\mathfrak{v})_\rho^\Omega := \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y) \big(\nabla_{1,\mathfrak{u}}\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{u}}\nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y)\,, \tag{9.5.6}$$

where the jets $\mathfrak{u}$ and $\mathfrak{v}$ are again the linearized solutions (9.5.4). We point out that, in contrast to the symplectic form, the surface layer inner product does *not* correspond to a conservation law. This has two reasons: First because the right side of (9.5.2) gives rise to a volume term, and second because the derivatives of the jets $\mathfrak{u}$ and $\mathfrak{v}$ give additional correction terms. For the details and the interpretation of these correction terms we refer to [**72**]. Here we only remark that the significance of the surface layer inner product is that it is an approximate conservation law. In particular, it can be used for estimating solutions of the linearized field equations and for proving existence results. We will come back to these applications in Section 14.

We finally comment on the name *symplectic form*. Clearly, this name is taken from symplectic geometry, where it refers to a closed and non-degenerate two-form σ on a manifold which we denote by $\mathcal{B}$. The connection to the surface layer integral (9.5.3) is obtained if we assume that the set of all critical measures of the form (8.1.1) forms a smooth manifold $\mathcal{B}$ (which may be a infinite-dimensional Banach manifold). In this case, a jet $\mathfrak{v}$ describing first variations of a measure (8.1.5) is a tangent vector in $T_\rho\mathcal{B}$. Consequently, the jet space $\mathfrak{J}$ can be identified with the tangent space $T_\rho\mathcal{B}$. The surface layer integral (9.5.3) can be regarded as a mapping

$$\sigma_\rho^\Omega \;:\; T_\rho\mathcal{B} \times T_\rho\mathcal{B} \to \mathbb{R}\,. \tag{9.5.7}$$

Being antisymmetric, it can be regarded as a two-form. Similarly, the conserved surface layer integral γ_ρ^Ω in (9.3.1) can be regarded as a one-form. Moreover, the t-derivative in (9.5.1) can be regarded as a directional derivative acting on $I_1^\Omega = \gamma_\rho^\Omega$. Anti-symmetrizing in s and t corresponds to taking the outer derivative. We thus obtain

$$\sigma_\rho^\Omega = \mathrm{d}\gamma_\rho^\Omega\,, \tag{9.5.8}$$

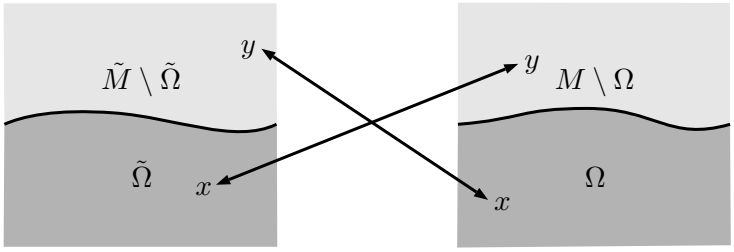


FIGURE 9.3. The nonlinear surface layer integral.

which also shows again that σ_ρ^Ω is closed. Thus, exactly as in symplectic geometry, the symplectic form defined as the surface layer integral (9.5.7) is a closed two-form. In contrast to symplectic geometry, it does not need to be non-degenerate. But this can be arranged by restricting attention to a more specific class of measures of the form (9.3.2). We refer to [**71**] for a more general discussion of this point.

We finally note that the relation (9.5.8) resembles the representation of the symplectic potential as the derivative of the *symplectic potential* (sometimes also referred to as the tautological one-form or canonical one-form). It is a major difference to classical mechanics and classical field theory that, in the setting of causal variational principles, the one-form γ_ρ^Ω is canonically defined and represented by a conserved surface layer integral in spacetime.

9.6. The Nonlinear Surface Layer Integral

We now introduce a different type of surface layer integral, which can be regarded as a generalization of the surface layer integrals considered so far. In order to explain the basic concept, we return to the general structure of a surface layer integral (9.1.2). The differential operator $(\cdots)$ in the integrand can be regarded as describing first or second variations of the measure ρ. As we saw above, the resulting surface layer integrals give rise to conserved currents, the symplectic form and inner products. Instead of considering first or second variations of a measure ρ, we now consider an additional measure $\tilde{\rho}$ which can be thought of as a finite perturbation of the measure ρ. Consequently, we also have two spacetimes

$$M := \operatorname{supp} \rho \qquad \text{and} \qquad \tilde{M} := \operatorname{supp} \tilde{\rho}\,.$$

Choosing two compact subsets $\Omega \subset M$ and $\tilde{\Omega} \subset \tilde{M}$ of the corresponding spacetimes, we form the *nonlinear surface layer integral* by

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\rho) := \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) - \int_{\Omega} \mathrm{d}\rho(x) \int_{\tilde{M}\setminus\tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(x,y)\,. \tag{9.6.1}$$

Note that one argument of the Lagrangian is in M, whereas the other is in $\tilde{M}$. Moreover, one argument lies inside the set Ω respectively $\tilde{\Omega}$, whereas the other argument lies outside. In this way, the nonlinear surface layer integral "compares" the two spacetimes near the boundaries of Ω and $\tilde{\Omega}$, as is illustrated in Figure 9.3. If $\tilde{\rho}$ is a first or second variation of ρ, one recovers surface layer integrals of the form (9.1.2). In this way, the nonlinear surface layer integral can be regarded as a generating functional for the previous surface layer integrals. Moreover, it has the advantage that it does not rely on continuous variations or a perturbative treatment. Instead, it can be used for comparing two arbitrary measures ρ

and $\tilde{\rho}$. This nonlinear surface layer integral was introduced in [**61**]. It plays a central role for getting the connection to quantum field theory (as will be outlined in Chapter 22).

The nonlinear surface layer integral comes with a corresponding conservation law, as we now explain. For technical simplicity, we assume that the measure $\tilde{\rho}$ can be obtained from ρ by multiplication with a weight function and a push-forward, i.e.

$$\tilde{\rho} = F_*(f\rho)$$

with smooth functions $f \in C^\infty(M, \mathbb{R}^+)$ and $F \in C^\infty(M, \mathcal{F})$. We use the mapping F in order to identify M with $\tilde{M}$. In particular, we choose

$$\tilde{\Omega} = F(\Omega)\,.$$

Then, using the definition of the push-forward measure, the nonlinear surface layer integral can be written alternatively as

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \rho) = \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y) \Big(f(x)\, \mathcal{L}\big(F(x), y\big) - \mathcal{L}\big(x, F(y)\big)\, f(y) \Big)\,. \tag{9.6.2}$$

Similar as explained in Section 9.2 in the connection of Noether-like theorems, by a "conservation law" we mean that the nonlinear surface layer integral should vanish for all compact Ω. In preparation for analyzing how to satisfy this condition, we rewrite the nonlinear surface layer integral as a volume integral by using the antisymmetry of the integrand in (9.6.2),

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \rho) = \int_\Omega \mathrm{d}\rho(x) \int_{M} \mathrm{d}\rho(y) \Big(f(x)\, \mathcal{L}\big(F(x), y\big) - \mathcal{L}\big(x, F(y)\big)\, f(y) \Big)\,. \tag{9.6.3}$$

In order to write this equation in a simpler form, we introduce a measure ν on M and a measure $\tilde{\nu}$ on $\tilde{M}$ by

$$\mathrm{d}\nu(x) := \bigg(\int_{\tilde{M}} \mathcal{L}(x,y)\; \mathrm{d}\tilde{\rho}(y) \bigg)\; \mathrm{d}\rho(x) \qquad \text{and} \qquad \mathrm{d}\tilde{\nu}(x) := \bigg(\int_{M} \mathcal{L}(x,y)\; \mathrm{d}\rho(y) \bigg)\; \mathrm{d}\tilde{\rho}(x)\,.$$

Intuitively speaking, these measures describe how the measures ρ and $\tilde{\rho}$ are connected to each other by the Lagrangian. We refer to them as the *correlation measures*. Then we can rewrite (9.6.3) as

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \rho) = \tilde{\nu}\big(F(\Omega)\big) - \nu(\Omega)\,.$$

In order to obtain a conservation law, this expression should vanish for all compact Ω. In other words, the measure ν should be the push-forward of the measure $\tilde{\nu}$ under the mapping F,

$$\nu = F_* \tilde{\nu}\,.$$

In this way, the task of finding a conservation law is reduced to the following abstract problem: Given two measures ν on M and $\tilde{\nu}$ on $\tilde{M}$, under which assumptions can one measure be realized as the push-forward of the other? If ν and $\tilde{\nu}$ are volume forms on compact manifolds, such a push-forward mapping is obtained from a classical theorem of Jürgen Moser (see for example [**113**, Section XVIII, §2]). In the non-compact case, the existence of F has been proven under general assumptions in [**97**]. In this way, the conservation law for the nonlinear surface layer integral can be arranged by adjusting the identification of the spacetimes M and $\tilde{M}$.

We finally remark how the nonlinear surface layer integral can be used to "compare" two causal fermion systems $(\mathcal{H}, \mathcal{F}, \rho)$ and $(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\rho})$. In this setting, one must keep in mind that the causal fermion systems are defined on different Hilbert spaces. Therefore, before forming the nonlinear surface layer integral, we must identify the Hilbert space $\mathcal{H}$

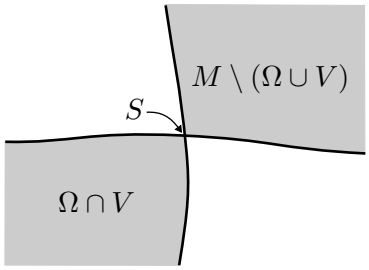


FIGURE 9.4. A two-dimensional surface layer integral.

and $\tilde{\mathcal{H}}$ by a unitary transformation $V : \mathcal{H} \to \tilde{H}$. Since this identification is not unique, we are left with the freedom to transform V according to

$$V \to V\mathcal{U} \qquad \text{with} \qquad \mathcal{U} \in \mathrm{L}(\mathcal{H}) \text{ unitary} .$$

A possible strategy for getting information independent of this freedom is to integrate over the unitary group. For example, this leads to the so-called *partition function*

$$Z^{\tilde{\Omega},\Omega}(\tilde{\rho}, \rho) := \int_{\mathcal{G}} \mathrm{e}^{\beta \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \mathcal{U}\rho)}\, \mathrm{d}\mu_{\mathcal{G}}(\mathcal{U}) ,$$

where β is a real parameter, and $\mathcal{G}$ is a compact subgroup of the unitary group on $\mathcal{H}$ with Haar measure $d\mu_{\mathcal{G}}$. Here the name "partition function" stems from an analogy to the path integral formulation of quantum field theory. For more details we refer to Chapter 22 or the research papers [**62, 65**].

9.7. Two-Dimensional Surface Layer Integrals

The surface layer integrals considered so far were intended to generalize integrals over hypersurfaces. We now explain how lower-dimensional integrals can be described by surface layer integrals. We restrict attention to two-dimensional integrals, noting that the methods can be applied similarly to one-dimensional integrals (i.e. integrals along a curve). It is most convenient to describe a two-dimensional surface $S \subset M$ as

$$S = \partial\Omega \cap \partial V ,$$

where Ω can be thought of as being the past of a Cauchy surface, and V describing a spacetime cylinder. This description has the advantage that the resulting surface layer integrals will be well-defined even in cases when spacetime is singular or discrete, in which case the boundaries $\partial\Omega$ and ∂V are no longer a sensible concept. The most obvious way of introducing a surface layer integral localized in a neighborhood of S is a double integral of the form

$$\int_{\Omega \cap V} \left(\int_{M \setminus (\Omega \cup V)} (\cdots)\, \mathcal{L}(x,y)\; \mathrm{d}\rho(y) \right) \mathrm{d}\rho(x) \tag{9.7.1}$$

(where $(\cdots)$ again stands for a differential operator acting on the Lagrangian). If the Lagrangian has short range, we only get contributions to this surface layer integral if both x and y are close to the two-dimensional surface S (see Figure 9.4).

The disadvantage of this method is that the surface layer integral (9.7.1) does not seem to fit together with the EL equations and the linearized field equations. Therefore, at present there is no corresponding conservation law. If one considers flows of two-surfaces, it seems preferable to use the following method introduced in [**21**]. We need to assume that M has a smooth manifold structure and is four-dimensional (see Definition 8.3.1)

and that v is a vector field which is transverse to the hypersurface $\partial\Omega$ and tangential to ∂V. Following Definition 9.3.3, the inner solution corresponding to v gives rise to a volume measure μ on $\partial\Omega$. Thus we can introduce a two-dimensional surface layer integral by

$$A := \int_{\partial\Omega\cap V} \mathrm{d}\mu(\mathfrak{v}, x) \int_{M\setminus V} \mathrm{d}\rho(y)\, (\cdots)\mathcal{L}(x,y)\,.$$

Applying the Gauß divergence theorem, this surface layer integral can also be written in the usual way as a double spacetime integral involving jet derivatives of the inner solution,

$$A = \int_{\Omega\cap V} \mathrm{d}\rho(x)\, \nabla_{\mathfrak{v}} \int_{M\setminus V} \mathrm{d}\rho(y)\, (\cdots)\mathcal{L}(x,y) \tag{9.7.2}$$

$$= \int_{\Omega\cap V} \mathrm{d}\rho(x) \int_{M\setminus V} \mathrm{d}\rho(y)\, \big(\nabla_{1,\mathfrak{v}} \pm \nabla_{2,\mathfrak{v}}\big)(\cdots)\mathcal{L}(x,y)\,, \tag{9.7.3}$$

where the notation $\pm$ means that the formula holds for either choice of the sign (this is because the corresponding term vanishes, as one sees after integrating by parts as in the proof of Lemma 8.3.3 and using that v is tangential to ∂V). The obtained surface layer integral (9.7.3) harmonizes with the structures of the EL equations and the linearized field equations, as is exemplified in [**21**] by a simple connection between area change and matter flux.

9.8. Exercises

EXERCISE 9.1. *(Noether-like theorems)* The goal of this exercise is to illustrate the Noether-like theorems. In order to simplify the problem as far as possible, we consider the compact setting and assume furthermore that the Lagrangian is smooth, i.e. $\mathcal{L} \in C^\infty(\mathcal{F}\times\mathcal{F}, \mathbb{R}_0^+)$. Let ρ be a minimizer of the action under variations of ρ in the class of (positive) normalized regular Borel measures. Let $u \in T\mathcal{F}$ be a vector field on $\mathcal{F}$. Assume that u is a symmetry of the Lagrangian in the sense that

$$\left(u(x)^j \frac{\partial}{\partial x^j} + u(y)^j \frac{\partial}{\partial y^j}\right) \mathcal{L}(x,y) = 0 \quad \text{for all } x, y \in \mathcal{F}. \tag{9.8.1}$$

Prove that for any measurable set $\Omega \subset \mathcal{F}$,

$$\int_\Omega \mathrm{d}\rho(x) \int_{\mathcal{F}\setminus\Omega} \mathrm{d}\rho(y)\, u(x)^j \frac{\partial}{\partial x^j}\mathcal{L}(x,y) = 0.$$

Hint: Integrate (9.8.1) over $\Omega\times\Omega$. Transform the integral using the symmetry $\mathcal{L}(x,y) = \mathcal{L}(y,x)$. Finally make use of the Euler-Lagrange equations.

EXERCISE 9.2. *(Commutator jets and conserved surface layer integrals)* Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system on a finite-dimensional Hilbert space. For any symmetric operator $S \in L(\mathcal{H})$, we define the corresponding *commutator jet* by

$$\mathfrak{C}_S := (0, \mathcal{C}_S), \quad \text{with} \quad \mathcal{C}_S(x) := \mathrm{i}[S, x] \quad \text{for all } x \in \mathcal{F}.$$

Prove the following identity between the conserved one-form and the conserved symplectic form:

$$\gamma_\rho^\Omega\big((0, [\mathcal{C}_A, \mathcal{C}_B])\big) = -\frac{1}{2}\, \sigma_\rho^\Omega(\mathfrak{C}_A, \mathfrak{C}_B),$$

where $[\mathcal{C}_A, \mathcal{C}_B]$ denotes the commutator of vector fields on $\mathcal{F}$.

EXERCISE 9.3. *(Representation of the commutator inner product)* The goal of this exercise is to represent the commutator inner product in a form similar to (9.4.8).

(a) Show that first variations of the Lagrangian can be written as

$$\delta\mathcal{L}(x,y) = 2\,\mathrm{Re}\,\mathrm{Tr}_{S_xM}\big(Q(x,y)\,\delta P(y,x)\big)$$

with a suitable kernel $Q(x,y) : S_y \to S_x$. Show that this kernel can be chosen to be symmetric, i.e. that $Q(x,y)^* = Q(y,x)$.

(b) Show that the variation described by the commutator jet in (9.4.3) and (9.4.2) corresponds to the variation of the integrand in (9.4.4)

$$\big(D_{1,\mathrm{j}(u)} - D_{2,\mathrm{j}(u)}\big)\mathcal{L}(x,y) = -2\mathrm{i}\big(\mathrm{i}\prec\psi(x)\,|\,Q(x,y)\,\psi(y)\succ_x - \mathrm{i}\prec\psi(y)\,|\,Q(y,x)\,\psi(x)\succ_y\big)\,.$$

(c) Use the polarization formula (9.4.5) to conclude that $\langle u|v\rangle_\rho^\Omega$ has the representation (9.4.8) with $\psi = \psi^u$ and $\phi = \psi^v$.

Hint: Details on this construction can be found in [**64**, Section 3].

EXERCISE 9.4. *(Extending the commutator inner product)* The goal of this exercise is to illustrate how the commutator inner product can be extended to more general wave functions. To this end, assume that we are given a space of wave function $\mathcal{W}$ which all satisfy the dynamical wave equation (9.4.7) with a suitable kernel $Q^{\mathrm{dyn}}(x,y)$. Prove that, under these assumptions, the inner product (9.4.8) is conserved for any $\psi, \phi \in \mathcal{W}$.
Hint: In a first step it seem a good idea to choose $\Omega = \Omega_t$ as the past of an equal time hypersurface and to differentiate with respect to t. More generally, one can consider the difference of (9.4.8) for two sets Ω and Ω' which differ by a compact set.

EXERCISE 9.5. *(Representing the Hilbert space scalar product in a surface layer)* The goal of this exercise is to explain why the sesquilinear form $\langle .|. \rangle_\rho^\Omega$ *cannot* represent the scalar product on the whole Hilbert space. To this end, let us assume conversely that

$$\langle u|u\rangle_\rho^\Omega = c\,\langle u|u\rangle_{\mathcal{H}} \qquad \text{for all } u \in \mathcal{H} \text{ and } c \neq 0 \tag{9.8.2}$$

and derive a contradiction. For technical simplicity, we assume that $\mathcal{H}$ is finite-dimensional and disregard all issues of convergence of integrals.

(a) Show that the surface layer integral can be written as

$$\langle u|u\rangle_\rho^\Omega = \mathrm{i}\int_M \big\langle u \,\big|\, [x, B(x)]\, u\big\rangle_{\mathcal{H}}\; \mathrm{d}\rho(x) \tag{9.8.3}$$

with $B(x)$ a suitable family of operators on the Hilbert space.

(b) Carry out the x-integral formally to obtain the representation

$$\langle u|u\rangle_\rho^\Omega = \langle u\,|\,Cu\rangle_{\mathcal{H}}\; \mathrm{d}\rho(x)$$

with a trace-free operator C. *Hint:* Make use of the commutator structure of the integrand in (9.8.3).

(c) Conclude from (9.8.2) that C is a multiple of the identity operator. Why is this a contradiction?

Hint: More details on this argument can be found in [**53**, Appendix A].

EXERCISE 9.6. *(On the surface layer inner product)* The goal of this exercise to show that, under a suitable restriction of the jet space, the surface-layer inner product is indeed positive. On $\mathcal{F} = \mathbb{R}^2$ we define the Lagrangian

$$\mathcal{L}(x,y) = \frac{1}{2}\,\eta(x_1 - y_1)\,(x_2 - y_2)^2, \quad \text{where } \eta \in C_0^\infty(\mathbb{R}, \mathbb{R}^+).$$

Let $M = \mathbb{R} \subset \mathcal{F}$ equipped with the canonical one-dimensional Lebesgue measure and consider the set of jets

$$\mathfrak{J} := \left\{ (0,u) \,\middle|\, u = \sum_{i=1}^{2} u_i \partial_i \in T\mathcal{F} \text{ with } u_1(t,0) = 0 \text{ and } \partial_1 u_2(t,0) \leq 0 \text{ for all } t \in \mathbb{R} \right\}.$$

Let $\Omega_t := (-\infty, t) \subset M$. Show that the surface-layer inner product $(\,\cdot\,,\,\cdot\,)^{\Omega_t}|_{\mathfrak{J}\times\mathfrak{J}}$ is positive semi-definite. *Hint:* Remember that jets are never differentiated in expressions like $\nabla_{i,\mathfrak{v}} \nabla_{j,\mathfrak{u}}$.

CHAPTER 10

Positive Functionals

10.1. Motivation and Setup

Many physical quantities have a definite sign (for example positive mass, positive energy, negative charge, etc.). With this in mind, it is an important question whether the structure of a causal variational principle gives rise to positive functionals. We now briefly explain the structural results known at present. These were obtained with two different methods. One method is to make us of the fact that, given a minimizer of a variational principle, second variations are always non-negative. This method was worked out in [**49**], and we will give an outline in Sections 10.2 and 10.3. The second method is to use that the action of a given minimizing measure ρ is smaller than the action of any other test measure $\tilde{\rho}$. By a suitable choice of $\tilde{\rho}$, one gets surface layer integral with a definite sign. This second method is explored in detail in [**63**] and applications are worked out. Here we only explain the basic idea in Section 10.4.

For technical simplicity, we restrict attention to causal variational principles in the *noncompact smooth setting* (see (6.2.4) and Section 6.3).

10.2. Positivity of the Hessian of ℓ

Let ρ be a minimizer of the causal action. According to the EL equations (7.1.2), the function ℓ is minimal on M. This clearly implies that its Hessian (as computed in any chart) is positive semi-definite, i.e.

$$D^2\ell(x) \geq 0 \qquad \text{for all } x \in M := \operatorname{supp} \rho\,. \tag{10.2.1}$$

This is the first non-negative quantity obtained from the fact that ρ is a minimizer. In view of the restricted EL equations (7.2.4), the zero and first order derivatives of ℓ vanish for all $x \in M$. Adding such lower derivative terms, we can write (10.2.1) with jet derivatives as

$$\nabla^2\ell|_x(\mathfrak{u},\mathfrak{u}) \geq 0 \qquad \text{for all } x \in M\,,$$

where, following our conventions (i) and (ii) on page 143,

$$\nabla^2\ell|_x(\mathfrak{u},\mathfrak{u}) := a(x)^2\,\ell(x) + 2\,a(x)\,D_u\ell(x) + D^2\ell|x(u,u)\,.$$

Integrating over M gives the following result:

PROPOSITION 10.2.1. *Let ρ be a minimizer of the causal action. Then*

$$\int_M \nabla^2\ell|_x(\mathfrak{u},\mathfrak{u})\;\mathrm{d}\rho(x) \geq 0 \qquad \textit{for all } \mathfrak{u} \in \mathfrak{J}_0\,.$$

10.3. Positivity of Second Variations Generated by Jets

We now analyze second variations for a special class of variations of the measure ρ to obtain another positive functional on jets. Similar as in [**71**, Section 3] we consider measures of the form

$$\tilde\rho_\tau = (F_\tau)_*\big(f_\tau\,\rho\big) \qquad \text{for } \tau \in (-\tau_{\max}, \tau_{\max}) \tag{10.3.1}$$

with smooth mappings

$$f \in C^\infty((-\tau_{\max}, \tau_{\max}) \times M, \mathbb{R}^+) \qquad \text{and} \qquad F \in C^\infty((-\tau_{\max}, \tau_{\max}) \times M, \mathcal{F})\,,$$

where the star denotes the push-forward measure defined by $((F_\tau)_*\mu)(\Omega) = \mu(F_\tau^{-1}(\Omega))$ (for details see the preliminaries in Section 2.3 or for example [**15**, Section 3.6]). We assume that for $\tau = 0$ the variation is trivial, (8.1.2). Moreover, for technical simplicity we assume that F_τ and f_τ are *trivial outside a compact set* $K \subset M$, meaning that

$$f_\tau|_{M\setminus K} \equiv 1 \qquad \text{and} \qquad F_\tau|_{M\setminus K} \equiv \mathbb{1}\,.$$

Finally, in order to satisfy the volume constraint on the right side of (6.3.1), we assume that

$$\int_K f_\tau(x)\; \mathrm{d}\rho(x) = \rho(K) \qquad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max})\,. \tag{10.3.2}$$

Then the transformation (10.3.1) is described infinitesimally by the smooth and compactly supported jet

$$\mathfrak{u} = (a, u) := \big(\dot f_0, \dot F_0\big) \in \mathfrak{J}_0\,,$$

where the dot denotes the τ-derivative. Moreover, we differentiate the volume constraint (10.3.2) to obtain

$$\int_K a(x)\; \mathrm{d}\rho(x) = 0\,. \tag{10.3.3}$$

We now compute the first and second variations of the action. Combining (6.3.2) with the definition of the push-forward measure, we obtain

$$\begin{aligned}\mathcal{S}\big(\tilde\rho_\tau\big) - \mathcal{S}(\rho) &= 2\int_K \mathrm{d}\rho(x) \int_{M\setminus K} \mathrm{d}\rho(y)\, \Big(f_\tau(x)\, \mathcal{L}\big(F_\tau(x), y\big) - \mathcal{L}(x,y)\Big) \\ &\quad + \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \Big(f_\tau(x)\, f_\tau(y)\, \mathcal{L}\big(F_\tau(x), F_\tau(y)\big) - \mathcal{L}(x,y)\Big)\,.\end{aligned} \tag{10.3.4}$$

The first variation vanishes because

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{S}\big(\tilde\rho_\tau\big)\Big|_{\tau=0} = 2\int_K \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}}\mathcal{L}(x,y) = 2\int_K \nabla_{\mathfrak{u}}\Big(\ell(x) + \mathfrak{s}\Big)\; \mathrm{d}\rho(x) = 0\,,$$

where in the last step we used (7.2.4) and (10.3.3) (and ∇_1 denotes the partial derivative acting on the first argument of the Lagrangian). Differentiating (10.3.4) twice, the second variation is computed to be

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathcal{S}\big(\tilde\rho_\tau\big)\Big|_{\tau=0} = 2\int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{u}}\mathcal{L}(x,y) \\ &+ 2\int_K \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \Big(a(x)\, D_{1,u}\mathcal{L}(x,y) + D_{1,u}D_{1,u}\mathcal{L}(x,y) + \big(\ddot f_0(x) + D_{1,\ddot F_0}\big)\mathcal{L}(x,y)\Big)\,.\end{aligned}$$

In the last line we can carry out the y-integration using (7.1.1). Applying the EL equations (7.2.4), we obtain

$$\begin{aligned}
\int_K \mathrm{d}\rho(x) \int_M a(x)\, D_{1,u}\mathcal{L}(x,y)\; \mathrm{d}\rho(y) &= 0 \\
\int_K \mathrm{d}\rho(x) \int_M D_{1,u} D_{1,u}\mathcal{L}(x,y)\; \mathrm{d}\rho(y) &= D^2\ell|_x(u,u) = \nabla^2\ell|_x(\mathfrak{u},\mathfrak{u}) \\
\int_K \mathrm{d}\rho(x) \int_M \big(\ddot{f}_0(x) + D_{1,\ddot{F}_0}\big)\mathcal{L}(x,y)\; \mathrm{d}\rho(y) &= \int_K \ddot{f}_0(x)\, \mathfrak{s}\; \mathrm{d}\rho(x) \overset{(10.3.2)}{=} 0\,.
\end{aligned}$$

We thus obtain the simple formula

$$\frac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathcal{S}\big(\tilde{\rho}_\tau\big)\Big|_{\tau=0} = \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{u}}\mathcal{L}(x,y) + \int_K \nabla^2\ell|_x(\mathfrak{u},\mathfrak{u})\; \mathrm{d}\rho(x)\,.$$

Since ρ is a minimizer and the first variation vanishes, the second variation is necessarily non-negative, giving rise to the inequality

$$\int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{u}}\mathcal{L}(x,y) + \int_M \nabla^2\ell|_x(\mathfrak{u},\mathfrak{u})\; \mathrm{d}\rho(x) \geq 0\,, \tag{10.3.5}$$

subject to the condition that the scalar component of the jet $\mathfrak{u}$ must satisfy the volume constraint (10.3.3). In the next proposition we remove this condition with a limiting procedure:

PROPOSITION 10.3.1. *Let ρ be a minimizer of the causal action. Then the inequality* (10.3.5) *holds for all $\mathfrak{u} \in \mathfrak{J}_0$.*

PROOF. Let $\mathfrak{u} = (a,u) \in \mathfrak{J}_0$ be a jet which violates the volume constraint (10.3.3). Then, choosing a compact set $\Omega \subset M$ with $\rho(\Omega) > 0$, the jet $\hat{\mathfrak{u}} := (\hat{a}, u)$ with

$$\hat{a}(x) = a(x) - c(\Omega)\,\chi_\Omega(x) \qquad \text{and} \qquad c(\Omega) := \frac{1}{\rho(\Omega)} \int_\Omega a(x)\; \mathrm{d}\rho(x) \tag{10.3.6}$$

(where χ_Ω is the characteristic function) does satisfy (10.3.3). Choosing the scalar variation $f_\tau = (1-\tau) + \tau\hat{a}$ and a family of diffeomorphisms F_τ with $\dot{F}_0 = u$, we obtain a variation which satisfies the volume constraint (10.3.2) (note that $\ddot{f} = 0$). Clearly, due to the characteristic function, the jet $\hat{\mathfrak{u}}$ is no longer smooth, but it has again compact support, and an approximation argument using Lebesgue's dominated convergence theorem shows that the inequality (10.3.5) also holds for $\hat{\mathfrak{u}}$. Expanding in powers of c, we thus obtain the inequality

$$\begin{aligned}
0 \leq{}& \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{u}}\mathcal{L}(x,y) + \int_K \nabla^2\ell|_x(\mathfrak{u},\mathfrak{u})\; \mathrm{d}\rho(x) \\
&- 2c \int_M \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \chi_\Omega(x) \nabla_{2,\mathfrak{u}}\mathcal{L}(x,y) \\
&+ c^2 \int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \chi_\Omega(x)\, \chi_\Omega(y)\, \mathcal{L}(x,y) \\
&+ \int_M \Big(- 2c\, \chi_\Omega(x)\, \nabla_{\mathfrak{u}}\ell(x) + c^2\, \chi_\Omega(x)^2\, \ell(x) \Big)\; \mathrm{d}\rho(x)
\end{aligned}$$

(the integrand in the last line arises from the contributions to $\nabla^2 \ell|x(\mathfrak{u},\mathfrak{u})$ involving the scalar components of the jets). The last line vanishes due to the restricted EL equations (7.2.4). Hence

$$\int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{u}} \mathcal{L}(x,y) + \int_K \nabla^2 \ell|_x(\mathfrak{u},\mathfrak{u})\; \mathrm{d}\rho(x)$$
$$\geq 2c \int_K \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) - c^2 \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \mathcal{L}(x,y) =: A(\Omega)\,.$$

We now let $(\Omega_n)_{n\in\mathbb{N}}$ be an exhaustion of M by compact sets. We distinguish the two cases when $\rho(M)$ is finite and infinite and treat these cases separately. If the total volume $\rho(M)$ is finite, one can take the limit $n \to \infty$ with Lebesgue's dominated convergence theorem to obtain

$$\lim_{n\to\infty} \int_K \mathrm{d}\rho(x) \int_{\Omega_n} \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) = \int_K \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y)$$
$$= \int_K \nabla_{\mathfrak{u}} \Big(\ell(x) + \mathfrak{s} \Big)\; \mathrm{d}\rho(x) = \mathfrak{s} \int_K a(x)\; \mathrm{d}\rho(x)$$
$$\lim_{n\to\infty} A(\Omega_n) = 2\, c(M)\, \mathfrak{s} \int_K a(x)\; \mathrm{d}\rho(x) - c(M)^2\, \rho(M)\, \mathfrak{s}$$
$$= \frac{\nu}{2\rho(M)} \left(\int_K a(x)\; \mathrm{d}\rho(x) \right)^2 \geq 0\,,$$

where in the last line we substituted the value of $c(M)$ in (10.3.6).

In the remaining case that the volume $\rho(M)$ is infinite, we estimate the terms as follows,

$$c(\Omega_n)^2 \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \mathcal{L}(x,y)$$
$$\leq c(\Omega_n)^2 \int_K \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) = c(\Omega_n)^2\, \mathfrak{s}\, \rho(K) \to 0$$
$$\int_K \mathrm{d}\rho(x) \int_{\Omega_n} \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) \to \int_K \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y)$$
$$= \int_K \nabla_{\mathfrak{u}} \Big(\ell(x) + \mathfrak{s} \Big)\; \mathrm{d}\rho(x) = \mathfrak{s} \int_K a(x)\; \mathrm{d}\rho(x)\,.$$

As a consequence, $A(\Omega_n)$ converges to zero as $n \to \infty$. This concludes the proof. □

We note that, restricting attention to scalar jets, i.e. $\mathfrak{u} = (a, 0)$ with a a real-valued function on M, the inequality in Proposition 10.3.1 reduces to

$$\int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, a(x)\, \mathcal{L}(x,y)\, a(y) \geq 0 \qquad \text{for all } a \in C_0^\infty(M)\,. \tag{10.3.7}$$

This inequality was first derived in [**84**, Lemma 3.5] and used for the analysis of minimizing measures. For more details see also Exercise 10.3.

10.3.1. Application: Hilbert Spaces of Jets. As an application, we now explain how our positive functionals can be used to endow spaces of jets in spacetime with Hilbert space structures. These Hilbert space structures should be very useful because they make functional analytic tools applicable to the analysis of the jet spaces and the causal action principle.

We introduce the following bilinear forms on $\mathfrak{J}_0$,

$$\langle \mathfrak{u}, \mathfrak{v} \rangle := \int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} \mathcal{L}(x,y) + \int_M \nabla^2 \ell|_x(\mathfrak{u}, \mathfrak{v})\ \mathrm{d}\rho(x) \tag{10.3.8}$$

$$\big\langle\!\big\langle \mathfrak{u}, \mathfrak{v} \big\rangle\!\big\rangle := \langle \mathfrak{u}, \mathfrak{v} \rangle + \int_M \nabla^2 \ell|_x(\mathfrak{u}, \mathfrak{v})\ \mathrm{d}\rho(x)\,. \tag{10.3.9}$$

By Propositions 10.2.1 and 10.3.1, both bilinear forms are positive semi-definite. The second bilinear form has the advantage that it is bounded from below by the bilinear form introduced in Proposition 10.2.1. Dividing out the null space and forming the completion gives real Hilbert spaces of jets denoted by $\mathcal{H}^{\langle .,. \rangle}$ and $\mathcal{H}^{\langle\!\langle .,. \rangle\!\rangle}$, respectively. Obviously,

$$\langle \mathfrak{u}, \mathfrak{u} \rangle \leq \big\langle\!\big\langle \mathfrak{u}, \mathfrak{u} \big\rangle\!\big\rangle\,,$$

giving rise to a norm-decreasing mapping $\mathcal{H}^{\langle\!\langle .,. \rangle\!\rangle} \to \mathcal{H}^{\langle .,. \rangle}$.

For the scalar components of the jets, the two scalar products (10.3.8) and (10.3.9) obviously agree. But they are quite different for the vector components. In order to understand this difference, it is instructive to consider a jet $\mathfrak{u} = (0, u)$ which describes a *symmetry* of the Lagrangian, i.e. (for details see [**70**, Section 3.1])

$$\big(D_{1,u} + D_{2,u}\big) \mathcal{L}(x,y) = 0 \qquad \text{for all } x, y \in M\,.$$

For this jet, a direct computation shows that

$$\langle \mathfrak{u}, \mathfrak{u} \rangle = 0\,.$$

Hence symmetry transformations lie in the kernel of the bilinear form $\langle .,. \rangle$ and thus correspond to the zero vector in $\mathcal{H}^{\langle .,. \rangle}$. Generally speaking, the scalar product $\langle .,. \rangle$ makes it possible to disregard symmetry transformations of the causal Lagrangian. However, jets describing symmetry transformations do in general correspond to non-zero vectors of the Hilbert space $\mathcal{H}^{\langle\!\langle .,. \rangle\!\rangle}$.

10.3.2. Application: A Positive Surface Layer Integral. We now derive a surface layer integral which is not necessarily conserved, but which has a definite sign. Similar as explained at the beginning of Section 10.3.1, this can be used to endow the jet space with a Hilbert structure. But in contrast to the scalar products in Section 10.3.1, where the jets were integrated over spacetime, here the scalar product is given as a surface layer integral. This should be useful for analyzing the dynamics of jets in spacetime.

PROPOSITION 10.3.2. *Let $\mathfrak{v}$ be a solution of the linearized field equations* (8.1.7)*. Then for any compact $\Omega \subset M$, the following surface layer integral is positive,*

$$-\int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}} \nabla_{2,\mathfrak{v}} \mathcal{L}(x,y) \geq 0\,.$$

PROOF. Denoting the components of $\mathfrak{v}$ by $\mathfrak{v} = (b, v)$, we evaluate (8.1.7) for $\mathfrak{u} = \mathfrak{v}$ and integrate over Ω. The resulting integrals can be rewritten as follows,

$$\begin{aligned} 0 &= \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}}\big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) - \mathfrak{s} \int_\Omega b(x)^2\, \mathrm{d}\rho(x) \\ &= \int_\Omega \nabla^2\ell|_x(\mathfrak{v},\mathfrak{v})\; \mathrm{d}\rho(x) + \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}}\nabla_{2,\mathfrak{v}}\mathcal{L}(x,y) \\ &= \int_\Omega \nabla^2\ell|_x(\mathfrak{v},\mathfrak{v})\; \mathrm{d}\rho(x) + \int_\Omega \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}}\nabla_{2,\mathfrak{v}}\mathcal{L}(x,y) \qquad (10.3.10) \\ &\quad + \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}}\nabla_{2,\mathfrak{v}}\mathcal{L}(x,y)\,. \qquad (10.3.11) \end{aligned}$$

Using characteristic functions, the expression (10.3.10) can be written as

$$\int_M \nabla^2\ell|_x(\chi_\Omega\mathfrak{v},\chi_\Omega\mathfrak{v})\; \mathrm{d}\rho(x) + \int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\chi_\Omega\mathfrak{v}}\nabla_{2,\chi_\Omega\mathfrak{v}}\mathcal{L}(x,y)\,.$$

Approximating the jet $\chi_\Omega\mathfrak{v}$ by smooth jets with compact support, one finds that the integrals in (10.3.10) are non-negative by Proposition 10.3.1. Therefore, the last summand (10.3.11) must be non-positive. This gives the result. □

We finally remark that in [**50**, Section 6] the surface layer integral in the last proposition is computed in Minkowski space.

10.4. A Positive Nonlinear Surface Layer Integral

In this section we briefly mention another method for obtaining a positive surface layer integral. This method and the corresponding positivity results will not be used later in this book. We refer the reader interested in more explanations and applications of this method to [**63**].

As in Section 9.6 we again consider two measures: A measure ρ which describes the vacuum spacetime, and another measure $\tilde\rho$ which typically describes an interacting spacetime. We assume that the vacuum measure is a *minimizer* of the causal action principle as defined in Section 6.3 (see (6.3.3) and (6.3.2)). We choose subsets $\Omega \subset M$ and $\tilde\Omega \subset \tilde M$ having the same finite volume,

$$\rho(\Omega) = \tilde\rho(\tilde\Omega) < \infty\,.$$

In order to construct an admissible test measure $\hat\rho$, we "cut out" Ω from ρ and "glue in" the set $\tilde\Omega$, i.e.

$$\hat\rho := \chi_{\tilde\Omega}\, \tilde\rho + \chi_{M\setminus\Omega}\, \rho\,.$$

The measure $\hat\rho$ differs from ρ only on a set of finite volume and preserves the volume constraint (see (6.3.1)). Therefore, we obtain from (6.3.3) and (6.3.2) (with $\tilde\rho$ replaced

by $\hat{\rho}$) that

$$\begin{aligned}
0 &\leq \big(\mathcal{S}(\hat{\rho}) - \mathcal{S}(\rho)\big) \\
&= 2 \int_{\mathcal{F}} \mathrm{d}(\hat{\rho}-\rho)(x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) + \int_{\mathcal{F}} \mathrm{d}(\hat{\rho}-\rho)(x) \int_M \mathrm{d}(\hat{\rho}-\rho)(y)\, \mathcal{L}(x,y) \\
&= 2 \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) - 2 \int_{\Omega} \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&\quad + \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(x,y) - 2 \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&\quad + \int_{\Omega} \mathrm{d}\rho(x) \int_{\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&= 2 \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) - 2 \int_{\Omega} \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&\quad + \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(x,y) - \int_{\Omega} \mathrm{d}\rho(x) \int_{\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y)\,.
\end{aligned}$$

We thus obtain the inequality

$$\begin{aligned}
2 \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) &\leq 2 \int_{\Omega} \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&\quad - \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(x,y) - \int_{\Omega} \mathrm{d}\rho(x) \int_{\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y)\,.
\end{aligned} \tag{10.4.1}$$

The left side of this inequality coincides with the first summand in the nonlinear surface layer integral as introduced in (9.6.1). However, the second summand in (9.6.1) is now missing. We can regard the left side of (10.4.1) again as a nonlinear surface layer integral, but with a somewhat different mathematical structure. It is not conserved, but it satisfies instead an *in*equality. The first summand on the right side of (10.4.1) can be interpreted as the surface area of $\partial\Omega$. The two other summands in (10.4.1), on the other hand, can be regarded as volume integrals over $\tilde{\Omega}$ and Ω, respectively.

This method can be generalized and adapted in various ways, also to cases when $\tilde{\Omega}$ and Ω do not have the same volume. Moreover, the resulting inequality can be written in a particularly useful form if also the measure $\tilde{\rho}$ satisfies the EL equations. We finally remark that, assuming that $\tilde{\rho}$ is again of the form (10.3.1) and expanding in powers of τ, one gets inequalities for surface layer integrals involving jet derivatives.

10.5. Exercises

EXERCISE 10.1. *(Positive functionals for the causal variational principle on $\mathbb{R}$)* We return to the causal variational principles on $\mathbb{R}$ corresponding to the Lagrangian $\mathcal{L}_2$ introduced in Exercise 6.4. Let $\rho = \delta$ be the unique minimizer.

(a) Compute the function $\ell(x)$ and verify that its Hessian is positive (see (10.2.1)).
(b) Compute the functional in Proposition 10.3.1 for the jets $\mathfrak{u} = (0, \partial_x)$ and $\mathfrak{u} = (1, 0)$.
(c) What are the resulting scalar products (10.3.8) and (10.3.9)?

EXERCISE 10.2. *(Positive functionals for the causal variational principle on S^1)* We return to the causal variational principle on $\mathbb{R}$ introduced in Exercise 6.5. Let ρ be a minimizing measure (6.6.4) for $0 < \tau < 1$. We choose $\mathfrak{J}_0^{\text{test}}$ as the four-dimensional vector space generated by the scalar jet $(1, 0$ and the vector jet $(0, \partial_\varphi)$ at the two points.

(a) Compute the function $\ell(x)$ and verify that its Hessian.

(b) Compute the bilinear form in Proposition 10.3.1.
(c) What are the resulting scalar products (10.3.8) and (10.3.9)? What are the resulting Hilbert spaces of jets $\mathcal{H}^{\langle \cdot,\cdot \rangle}$ and $\mathcal{H}^{\langle\!\langle \cdot,\cdot \rangle\!\rangle}$? Which dimensions do they have? How can this result be understood in view of the space of linearized solutions as computed in Exercise 8.6?

EXERCISE 10.3. *(A positive operator on scalar jets)* In this exercise we specialize the statement of Proposition 10.3.1 to scalar jets and work out a few consequences.

(a) Show that for jets of the form $\mathfrak{u} = (a, 0)$, the statement of Proposition 10.3.1 reduces to the inequality (10.3.7).
(b) Let ρ be a minimizing measure and $x_0, \ldots, x_N \in M$ be a finite number of spacetime points. Show that the *Gram matrix* L defined by
$$L = \Big(\mathcal{L}(x_i, x_j) \Big)_{i,j=0,\ldots,N}$$
is symmetric and positive semi-definite.
(c) Show that the operator $\mathcal{L}_\rho$ defined by
$$\mathcal{L}_\rho \,:\, C_0^\infty(M) \subset L^2(M, \mathrm{d}\rho) \to L^2(M, \mathrm{d}\rho)\,, \quad (\mathcal{L}_\rho \psi)(x) := \int_{\mathcal{F}} \mathcal{L}(x,y)\, \psi(y)\; \mathrm{d}\rho(y)$$
is a symmetric, densely defined operator on the Hilbert space $L^2(M, \mathrm{d}\rho)$. Prove that this operator is positive semi-definite.

EXERCISE 10.4. The goal of this exercise is to explore the positive nonlinear surface layer integral of Section 10.4 in the limiting case when the measures of the sets Ω and $\tilde{\Omega}$ tend to zero. For technical simplicity, let us assume that for given $x \in M$ and $y \in \tilde{M}$, there are sequences of open neighborhoods Ω_k of x and $\tilde{\Omega}_k$ of y with $\rho(\Omega_k) = \tilde{\rho}(\tilde{\Omega}_k)$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \rho(\Omega_k) = 0$. Show that, in the limit $k \to \infty$, the inequality (10.4.1) reduces to the inequality
$$\ell(y) \geq \ell(x)\,.$$
Thus we get back the EL equation (7.1.6).

In view of this limiting case, the positive nonlinear surface layer integral in Section 10.4 can be regarded as a refined nonlinear version of the EL equations.

CHAPTER 11

Topological and Geometric Structures

This chapter is devoted to the topological and geometric structures of a causal fermion system. We closely follow the presentation in [**60**] and [**56**].

11.1. A Topological Vector Bundle

For the topological structures, it is not essential that the operators in $\mathcal{F}$ have at most n positive and at most n negative eigenvalues (see Definition 5.4.1). Instead, it is preferable for the sake of greater generality and broader applicability to relax this condition in the following way.

DEFINITION 11.1.1. *Given a complex Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ and parameters $\mathfrak{p}, \mathfrak{q} \in \mathbb{N}_0$ with $\mathfrak{p} \leq \mathfrak{q}$, we let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all symmetric operators on $\mathcal{H}$ of finite rank, which (counting multiplicities) have at most $\mathfrak{p}$ positive and at most $\mathfrak{q}$ negative eigenvalues. On $\mathcal{F}$ we are given a positive measure ρ (defined on a σ-algebra of subsets of $\mathcal{F}$). We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a* **topological fermion system** *of spin signature $(\mathfrak{p}, \mathfrak{q})$.*

If $\mathfrak{p} = 0$, we call $(\mathcal{H}, \mathcal{F}, \rho)$ a **Riemannian fermion system** *of spin dimension $n := \mathfrak{q}$.*

Clearly, the case $\mathfrak{p} = \mathfrak{q}$ gives back a causal fermion system (see Definition 5.4.1). It should be noted that the assumption $\mathfrak{p} \leq \mathfrak{q}$ merely is a convention, because otherwise one may replace $\mathcal{F}$ by $-\mathcal{F}$. The name *Riemannian* fermion system comes from the fact that in examples on Riemannian manifolds, the inner product on the fibers is positive definite, implying that the resulting local correlation operators are negative semi-definite. For details see [**60**] or the examples in Exercises 11.1 and 11.2. We also note that for Riemannian fermion systems, the causal structure (according to Definition 5.6.1) is trivial; see Exercise 11.3.

In Section 2.5 the notion of a *topological vector bundle* was introduced (see Definition 2.5.2). Again setting $M = \operatorname{supp} \rho$, we want to construct a topological vector bundle having the spin space $S_x := x(\mathcal{H})$ as the fiber at the point $x \in M$. To this end, all the spin spaces must have the same dimension and signature, making it necessary to impose the following condition:

DEFINITION 11.1.2. *The topological fermion system is called* **regular** *if for all $x \in M$, the operator x has the maximal possible rank $\mathfrak{p} + \mathfrak{q}$.*

We note that most of our constructions can be extended to non-regular topological fermion systems by decomposing M into subsets on which x has fixed rank and a fixed number of positive and negative eigenvalues (for details see [**60**, Section 7]).

We define $\mathcal{B}$ as the set of pairs

$$\mathcal{B} = \{(x, \psi) \,|\, x \in M,\ \psi \in S_x\}$$

and let π be the projection onto the first component. Moreover, we let $(Y, \prec .|. \succ)$ be an indefinite inner product space of signature $(\mathfrak{q}, \mathfrak{p})$, and choose $G = \mathrm{U}(\mathfrak{q}, \mathfrak{p})$ as the

group of unitary transformations on Y. In order to construct the bundle charts, for any given $x \in M$ we choose a unitary mapping $\sigma : S_x \to Y$. By restricting the projection π_x in (5.7.2) to S_y, we obtain the mapping

$$\pi_x|_{S_y} \;:\; S_y \to S_x \,.$$

In order to compute its adjoint with respect to the spin inner product (5.7.8), for $\psi \in S_x$ and $\phi \in S_y$ we make the computation

$$\begin{aligned} \prec\!\psi \,|\, \pi_x|_{S_y}\, \phi\!\succ_x &= -\langle \psi | x \phi \rangle_{\mathcal{H}} = -\langle x\psi | \phi \rangle_{\mathcal{H}} = -\langle \pi_y\, x\, \psi | \phi \rangle_{\mathcal{H}} = -\langle y\, (y|_{S_y})^{-1}\, \pi_y\, x\, \psi | \phi \rangle_{\mathcal{H}} \\ &= -\big\langle (y|_{S_y})^{-1}\, \pi_y\, x\, \psi \,\big|\, y \phi \big\rangle_{\mathcal{H}} = \prec\!(y|_{S_y})^{-1}\, \pi_y\, x\, \psi \,|\, \phi\!\succ_y . \end{aligned}$$

Hence

$$\big(\pi_x|_{S_y}\big)^* = (y|_{S_y})^{-1}\, \pi_y\, x|_{S_x} \,.$$

We now introduce the operator

$$T_{xy} = \big(\pi_x|_{S_y}\big)\big(\pi_x|_{S_y}\big)^* = \pi_x\, (y|_{S_y})^{-1}\, \pi_y\, x|_{S_x} \;:\; S_x \to S_x \,.$$

By construction, this operator is symmetric and $T_{xx} = \mathbb{1}$. We now form the polar decomposition of T_{xy} to obtain a unitary operator U_{xy}: By continuity, there is a neighborhood U of x such that for all $y \in U$, the operator T_{xy} is invertible and has a unique square root ρ_{xy} (defined for example by the power series $\sqrt{T_{xy}} = \sqrt{\mathbb{1} + (T_{xy} - \mathbb{1})} = \mathbb{1} + \frac{1}{2}(T_{xy} - \mathbb{1}) + \cdots$). Introducing the mapping

$$U_{x,y} = \rho_{xy}^{-1}\, \pi_x|_{S_y} : S_y \to S_x \,,$$

the calculation

$$U_{x,y}\, U_{x,y}^* = \rho_{xy}^{-1}\, \pi_x|_{S_y} \big(\pi_x|_{S_y}\big)^* \rho_{xy}^{-1} = \rho_{xy}^{-1}\, T_{xy}\, \rho_{xy}^{-1} = \mathbb{1}_{S_x}$$

shows that the mapping U_{xy} is unitary. Moreover, it clearly depends continuously on $y \in U$.

We define the bundle chart ϕ_U by

$$\phi_U(y, v) = \big(y, (\sigma \circ U_{x,y})(v)\big) \,.$$

The commutativity of the diagram (2.5.8) is obvious because ϕ is the identity map in the first component. Moreover, the transition functions g_{UV} in (2.5.9) are in G because we are working with unitary mappings of the fibers throughout. We choose the topology on $\mathcal{B}$ such that all the bundle charts are homeomorphisms.

DEFINITION 11.1.3. *The topological vector bundle $\mathcal{B} \to M$ is referred to as the vector bundle associated to the regular topological fermion system $(\mathcal{H}, \mathcal{F}, \rho)$, or simply the* **associated vector bundle**.

The next result shows that every vector bundle over a manifold can be realized as the associated vector bundle of a corresponding topological fermion system. In other words, working with topological fermion systems poses no topological constraints for the associated vector bundles.

THEOREM 11.1.4. *Let $X \to \mathscr{M}$ be a vector bundle over a k-dimensional topological manifold $\mathscr{M}$, whose fibers are isomorphic to an indefinite inner product space of signature $(\mathfrak{q}, \mathfrak{p})$. Then there is a regular topological fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of signature $(\mathfrak{p}, \mathfrak{q})$ such that the associated vector bundle (see Definition 11.1.3) is isomorphic to X. If $\mathscr{M}$ is compact, the Hilbert space $\mathcal{H}$ can be chosen to be finite-dimensional.*

The proof can be found in [**60**, Section 3.3].

11.2. Geometric Structures of a Causal Fermion System

We now outline constructions from [**56**] which give general notions of a connection and curvature (see Theorem 11.2.9, Definition 11.2.10 and Definition 11.2.11). So far, these constructions have been carried out only in the case of spin dimension $n = 2$. This is the most important case because it allows for the description of Dirac spinors in a four-dimensional spacetime.

11.2.1. Construction of the Spin Connection. Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system of spin dimension $n = 2$. Moreover, we assume that it is *regular* (see Definition 11.1.2).

An important structure from spin geometry missing so far is Clifford multiplication. To this end, we need a Clifford algebra represented by symmetric operators on S_x. For convenience, we first consider Clifford algebras with the maximal number of five generators; later we reduce to four spacetime dimensions (see Definition 11.2.13 below). We denote the set of symmetric linear endomorphisms of $(S_x, \prec.|.\succ_x)$ by $\mathrm{Symm}(S_x)$; it is a 16-dimensional real vector space.

DEFINITION 11.2.1. *A five-dimensional subspace $K \subset \mathrm{Symm}(S_x)$ is called a* **Clifford subspace** *if the following conditions hold:*

(i) *For any $u, v \in K$, the anti-commutator $\{u, v\} \equiv uv + vu$ is a multiple of the identity on S_x.*

(ii) *The bilinear form $\langle ., . \rangle$ on K defined by*

$$\frac{1}{2}\{u, v\} = \langle u, v \rangle \mathbb{1} \qquad \text{for all } u, v \in K$$

is non-degenerate and has signature $(1, 4)$.

In view of the situation in spin geometry, we would like to distinguish a specific Clifford subspace. In order to partially fix the freedom in choosing Clifford subspaces, it is useful to impose that K should contain a given operator, as is made precise in the next definitions.

DEFINITION 11.2.2. *An operator $v \in \mathrm{Symm}(S_x)$ is called a* **sign operator** *if $v^2 = \mathbb{1}$ and if the inner product $\prec.|v\,.\succ : S_x \times S_x \to \mathbb{C}$ is positive definite.*

DEFINITION 11.2.3. *Given a sign operator v, the set of* **Clifford extensions** *$\mathcal{T}^v$ is defined as the set of all Clifford subspaces containing v,*

$$\mathcal{T}^v = \{K \text{ Clifford subspace with } v \in K\}\,.$$

Considering x as an operator on S_x, this operator has by definition of the spin dimension two positive and two negative eigenvalues. Moreover, the calculation

$$\prec u|(-x)\,u\succ_x \overset{(5.7.8)}{=} \langle u|x^2 u\rangle_{\mathcal{H}} > 0 \quad \text{for all } u \in S_x \setminus \{0\}$$

shows that the operator $(-x)$ is positive definite on S_x. Thus we can introduce a unique sign operator s_x by demanding that the eigenspaces of s_x corresponding to the eigenvalues ± 1 are precisely the positive and negative spectral subspaces of the operator $(-x)$. More constructively, this operator is obtained by diagonalizing $(-x)$ and replacing all positive matrix entries by plus one and all negative matrix entries by minus one (for details see Exercise 11.4). This sign operator is referred to as the *Euclidean sign operator* s_x. It is worth noting that for Clifford extensions of the Euclidean sign operator, the bilinear form $\langle ., . \rangle$ always has Lorentzian signature (see Exercise 11.5).

A straightforward calculation shows that for two Clifford extensions $K, \tilde{K} \in \mathcal{T}^v$, there is a unitary transformation $U \in \mathrm{e}^{\mathrm{i}\mathbb{R}v}$ such that $\tilde{K} = UKU^{-1}$ (for details see [**56**, Section 3]). By dividing out this group action, we obtain a five-dimensional vector space, endowed with the inner product $\langle ., . \rangle$. Taking for v the Euclidean signature operator, we regard this vector space as a generalization of the usual tangent space.

DEFINITION 11.2.4. *The* **tangent space** T_x *is defined by*

$$T_x = \mathcal{T}_x^{s_x} / \exp(i\mathbb{R} s_x) \,.$$

It is endowed with an inner product $\langle ., . \rangle$ *of signature* $(1, 4)$.

Here the name "tangent space" requires an explanation. Since M does not need to have a manifold structure, the geometric tangent space (as introduced in Section 2.5 of the preliminaries) cannot be introduced at this stage. Definition 11.2.4 gives another notion, which does make sense without assuming that M is a manifold (for example, it applies also to discrete spacetimes). The basic reason is that our definition of a tangent space merely describes Clifford structures at each spacetime point, but without using the relations to neighboring spacetime points. In other words, our definition does not incorporate the usual notion that spacetime is "approximated infinitesimally" by the tangent space. The reason why our notion nevertheless makes sense is that in the example of a causal fermion system constructed on a smooth spin manifold, the geometric tangent space $T_x\mathscr{M}$ of a spacetime point $x \in \mathscr{M}$ (defined for example as in Section 2.5 as equivalence classes of curves through x) can be identified with a distinguished subspace of $\mathrm{Symm}(S_x\mathscr{M})$ via Clifford multiplication $v \in T_x\mathscr{M} \mapsto \gamma(v) \in \mathrm{Symm}(S_x\mathscr{M})$. After identifying $S_x\mathscr{M}$ with S_xM, we thus obtain a distinguished representative of the tangent space T_x. With this in mind, Definition 11.2.4 can be regarded as a generalization of the tangent space of a spin manifold, keeping only those structures which can be defined on the present level of generality. We remark that a connection between the tangent space T_x and the local geometry of M can be made even in the non-smooth setting by working with so-called tangent cone measures (see Section 12.9 for the basic concept and [**60**, Section 6.2] for the detailed construction).

We next consider two spacetime points. We need the following assumption.

DEFINITION 11.2.5. *Two points* $x, y \in M$ *are said to be* **properly timelike** *separated if the closed chain* A_{xy} *has a strictly positive spectrum and if the corresponding eigenspaces are definite subspaces of* S_x.

This definition clearly implies that x and y are timelike separated (see Definition 5.6.1 and Exercise 11.7). Moreover, the eigenspaces of A_{xy} are definite if and only if those of A_{yx} are, showing that Definition 11.2.5 is again symmetric in x and y. As a consequence, the spin space can be decomposed uniquely into an orthogonal direct sum $S_x = I^+ \oplus I^-$ of a positive definite subspace I^+ and a negative definite subspace I^- of A_{xy}. This allows us to introduce a unique sign operator v_{xy} by demanding that its eigenspaces corresponding to the eigenvalues ± 1 are the subspaces $I^\pm$. This sign operator is referred to as the *directional sign operator* of A_{xy}. Having two sign operators s_x and v_{xy} at our disposal, we can distinguish unique corresponding Clifford extensions, provided that the two sign operators satisfy the following generic condition.

DEFINITION 11.2.6. *Two sign operators* $v, \tilde{v}$ *are said to be* **generically separated** *if their commutator* $[v, \tilde{v}]$ *has rank four.*

LEMMA 11.2.7. *Assume that the sign operators s_x and v_{xy} are generically separated. Then there are unique Clifford extensions $K_x^{(y)} \in \mathcal{T}^{s_x}$ and $K_{xy} \in \mathcal{T}^{v_{xy}}$ and a unique operator $\rho \in K_x^{(y)} \cap K_{xy}$ with the following properties:*

(i) *The relations $\{s_x, \rho\} = 0 = \{v_{xy}, \rho\}$ hold.*

(ii) *The operator $U_{xy} := \mathrm{e}^{\mathrm{i}\rho}$ transforms one Clifford extension to the other,*

$$K_{xy} = U_{xy}\, K_x^{(y)}\, U_{xy}^{-1} .$$

(iii) *If $\{s_x, v_{xy}\}$ is a multiple of the identity, then $\rho = 0$.*

The operator ρ depends continuously on s_x and v_{xy}.

We refer to U_{xy} as the *synchronization map*. Exchanging the roles of x and y, we also have two sign operators s_y and v_{yx} at the point y. Assuming that these sign operators are again generically separated, we also obtain a unique Clifford extension $K_{yx} \in \mathcal{T}^{v_{yx}}$.

After these preparations, we can now explain the construction of the spin connection D (for details see [**56**, Section 3]). For two spacetime points $x, y \in M$ with the above properties, we want to introduce an operator

$$D_{x,y} \,:\, S_y \to S_x$$

(generally speaking, by the subscript $_{xy}$ we always denote an object at the point x, whereas the additional comma $_{x,y}$ denotes an operator which maps an object at y to an object at x). It is natural to demand that $D_{x,y}$ is unitary, that $D_{y,x}$ is its inverse, and that these operators map the directional sign operators at x and y to each other,

$$D_{x,y} = (D_{y,x})^* = (D_{y,x})^{-1} \tag{11.2.1}$$

$$v_{xy} = D_{x,y}\, v_{yx}\, D_{y,x} . \tag{11.2.2}$$

The obvious idea for constructing an operator with these properties is to take a polar decomposition of $P(x,y)$; this amounts to setting

$$D_{x,y} = A_{xy}^{-\frac{1}{2}}\, P(x,y) . \tag{11.2.3}$$

This definition has the shortcoming that it is not compatible with the chosen Clifford extensions. In particular, it does not give rise to a connection on the corresponding tangent spaces. In order to resolve this problem, we modify (11.2.3) by the ansatz

$$D_{x,y} = \mathrm{e}^{\mathrm{i}\varphi_{xy}\, v_{xy}}\, A_{xy}^{-\frac{1}{2}}\, P(x,y) \tag{11.2.4}$$

with a free real parameter φ_{xy}. In order to comply with (11.2.1), we need to demand that

$$\varphi_{xy} = -\varphi_{yx} \mod 2\pi \,; \tag{11.2.5}$$

then (11.2.2) is again satisfied. We can now use the freedom in choosing φ_{xy} to arrange that the distinguished Clifford subspaces K_{xy} and K_{yx} are mapped onto each other,

$$K_{xy} = D_{x,y}\, K_{yx}\, D_{y,x} . \tag{11.2.6}$$

It turns out that this condition determines φ_{xy} up to multiples of $\frac{\pi}{2}$. In order to fix φ_{xy} uniquely in agreement with (11.2.5), we need to assume that φ_{xy} is not a multiple of $\frac{\pi}{4}$. This leads us to the following definition.

DEFINITION 11.2.8. *Two points $x, y \in M$ are called* **spin connectable** *if the following conditions hold:*

(a) *The points x and y are properly timelike separated (note that this already implies that x and y are regular as defined in Section 11.2.1).*
(b) *The Euclidean sign operators s_x and s_y are generically separated from the directional sign operators v_{xy} and v_{yx}, respectively.*
(c) *Employing the ansatz* (11.2.4)*, the phases φ_{xy} which satisfy condition* (11.2.6) *are not multiples of $\frac{\pi}{4}$.*

We denote the set of points which are spin connectable to x by $\mathcal{I}(x)$. It is straightforward to verify that $\mathcal{I}(x)$ is an open subset of M.

Under these assumptions, we can fix φ_{xy} uniquely by imposing that

$$\varphi_{xy} \in \Big(-\frac{\pi}{2}, -\frac{\pi}{4} \Big) \cup \Big(\frac{\pi}{4}, \frac{\pi}{2} \Big) , \tag{11.2.7}$$

giving the following result (for the proofs see [**56**, Section 3.3]).

THEOREM 11.2.9. *Assume that two points $x, y \in M$ are spin connectable. Then there is a unique* **spin connection** $D_{x,y} : S_y \to S_x$ *of the form* (11.2.4) *having the properties* (11.2.1)*,* (11.2.2)*,* (11.2.6) *and* (11.2.7)*.*

11.2.2. The Metric Connection and Curvature. We now outline a few further constructions from [**56**, Section 3]. The spin connection induces a connection on the corresponding tangent spaces, as we now explain. Suppose that $u_y \in T_y$. Then, according to Definition 11.2.4 and Lemma 11.2.7, we can consider u_y as a vector of the representative $K_y^{(x)} \in \mathcal{T}^{s_y}$. By applying the synchronization map, we obtain a vector in K_{yx},

$$u_{yx} := U_{yx}\, u_y\, U_{yx}^{-1} \in K_{yx} .$$

According to (11.2.6), we can now "parallel transport" the vector to the Clifford subspace K_{xy},

$$u_{xy} := D_{x,y}\, u_{yx}\, D_{y,x} \in K_{xy} .$$

Finally, we apply the inverse of the synchronization map to obtain the vector

$$u_x := U_{xy}^{-1}\, u_{xy}\, U_{xy} \in K_x^{(y)} .$$

As $K_x^{(y)}$ is a representative of the tangent space T_x and all transformations were unitary, we obtain an isometry from T_y to T_x.

DEFINITION 11.2.10. *The isometry between the tangent spaces defined by*

$$\nabla_{x,y} \,:\, T_y \to T_x \,:\, u_y \mapsto u_x$$

is referred to as the **metric connection** *corresponding to the spin connection D.*

We next introduce a notion of curvature.

DEFINITION 11.2.11. *Suppose that three points $x, y, z \in M$ are pairwise spin connectable. Then the associated* **metric curvature** *R is defined by*

$$R(x,y,z) = \nabla_{x,y}\, \nabla_{y,z}\, \nabla_{z,x} \,:\, T_x \to T_x . \tag{11.2.8}$$

The metric curvature $R(x,y,z)$ can be thought of as a discrete analog of the holonomy of the Levi-Civita connection on a manifold, where a tangent vector is parallel transported along a loop starting and ending at x. On a manifold, the curvature at x is immediately obtained from the holonomy by considering the loops in a small neighborhood of x. With this in mind, Definition 11.2.11 indeed generalizes the usual notion of curvature to causal fermion systems.

The following construction relates directional sign operators to vectors of the tangent space. Suppose that y is spin connectable to x. By synchronizing the directional sign operator v_{xy}, we obtain the vector

$$\hat{y}_x := U_{xy}^{-1}\, v_{xy}\, U_{xy} \in K_x^{(y)} \,. \tag{11.2.9}$$

As $K_x^{(y)} \in \mathcal{T}^{s_x}$ is a representative of the tangent space, we can regard $\hat{y}_x$ as a tangent vector. We thus obtain a mapping

$$\mathcal{I}(x) \to T_x \;:\; y \mapsto \hat{y}_x \,.$$

We refer to $\hat{y}_x$ as the *directional tangent vector* of y in T_x. As v_{xy} is a sign operator and the transformations in (11.2.9) are unitary, the directional tangent vector is a timelike unit vector with the additional property that the inner product $\prec .|\hat{y}_x .\succ_x$ is positive definite.

We finally explain how to reduce the dimension of the tangent space to four, with the desired Lorentzian signature $(1,3)$.

DEFINITION 11.2.12. *The fermion system is called* **chirally symmetric** *if to every $x \in M$ we can associate a spacelike vector $u(x) \in T_x$ which is orthogonal to all directional tangent vectors,*

$$\langle u(x), \hat{y}_x \rangle = 0 \qquad \text{for all } y \in \mathcal{I}(x) \,,$$

and is parallel with respect to the metric connection, i.e.

$$u(x) = \nabla_{x,y}\, u(y)\, \nabla_{y,x} \qquad \text{for all } y \in \mathcal{I}(x) \,.$$

DEFINITION 11.2.13. *For a chirally symmetric fermion system, we introduce the* **reduced tangent space** T_x^{red} *by*

$$T_x^{\text{red}} = \langle u_x \rangle^\perp \subset T_x \,.$$

Clearly, the reduced tangent space has dimension four and signature $(1,3)$. Moreover, the operator $\nabla_{x,y}$ maps the reduced tangent spaces isometrically to each other. The local operator $\Gamma := -\mathrm{i} u / \sqrt{-u^2}$ takes the role of the *pseudo-scalar matrix*.

11.3. Correspondence to Lorentzian Spin Geometry

We now explain how the above geometric notions correspond to the usual objects of differential geometry in Minkowski space (Theorem 11.3.1) and on a globally hyperbolic Lorentzian manifold (Theorem 11.3.2). We closely follow the presentation in [**57**, Section 3.3]; see also the review [**47**].

We let $(\mathscr{M}, g)$ be a time-oriented Lorentzian spin manifold with spinor bundle $S\mathscr{M}$ (for basic definitions see Section 4.5). In order to obtain a corresponding causal fermion system, we adapt the construction in Minkowski space given in Section 5.5: First, we choose a closed subspace $\mathcal{H}$ of the Hilbert space of Dirac solutions $(\mathcal{H}_m, (.|.))$ (as introduced in Section 4.5). Endowed with the induced scalar product $\langle .|. \rangle_{\mathcal{H}} := (.|.)|_{\mathcal{H}\times\mathcal{H}}$, we obtain a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. Next, one introduces a regularization operator (5.5.1), for example by mollifying the initial data on a Cauchy surface (as is explained in [**80**, Section 4]). Introducing the local correlation operator $F^\varepsilon(x)$ for every $x \in \mathscr{M}$ again by (5.5.3), we define the measure ρ on $\mathcal{F}$ as the push-forward of the volume measure μ on $\mathscr{M}$, i.e.

$$\rho = (F^\varepsilon)_* \mu \,,$$

where, in local coordinates, the measure μ has the form

$$\mathrm{d}\mu = \sqrt{|\det g|}\; \mathrm{d}^4x\,.$$

We thus obtain a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ describing the curved spacetime $(\mathscr{M}, g)$. The basic inherent structures of this causal fermion system (like the spin space S_x and the spin inner product as defined in Section 5.7) can be identified canonically with the corresponding objects of spin geometry (like the spinor space $S_x\mathscr{M}$ with spin inner product; for details see [**45**, §1.2.4] or [**60**]). With this identification, Clifford multiplication gives rise to a canonical identification of the tangent space $T_x\mathscr{M}$ with a distinguished Clifford subspace.

Let $\gamma(t)$ be a smooth, future-directed and timelike curve, for simplicity parametrized by the arc length, defined on the interval $[0, T]$ with $\gamma(0) = y$ and $\gamma(T) = x$. Then the parallel transport of tangent vectors along γ with respect to the Levi-Civita connection ∇^{LC} gives rise to the isometry

$$\nabla^{\text{LC}}_{x,y} \;:\; T_y \to T_x\,.$$

In order to compare with the metric connection ∇ of Definition 11.2.10, we subdivide γ (for simplicity with equal spacing, although a non-uniform spacing would work just as well). Thus for any given N, we define the points $x_0, \ldots, x_N$ by

$$x_n = \gamma(t_n) \qquad \text{with} \qquad t_n = \frac{nT}{N}\,.$$

We define the parallel transport $\nabla^N_{x,y}$ by successively composing the parallel transport between neighboring points,

$$\nabla^N_{x,y} := \nabla_{x_N, x_{N-1}} \nabla_{x_{N-1}, x_{N-2}} \cdots \nabla_{x_1, x_0} \;:\; T_y \to T_x\,.$$

We first state a result in the *Minkowski vacuum*. We choose $\mathcal{H}$ as the subspace of all negative-energy solutions of the Dirac equation (describing the Dirac sea; see the preliminaries in Section 1.5). For technical simplicity, we choose the *iε-regularization*, where the regularization operator (5.5.1) is the multiplication operator by $\mathrm{e}^{\varepsilon\omega}$ in momentum space in (5.5.2).

THEOREM 11.3.1. *For every $\varepsilon > 0$ we consider the causal fermion systems $(\mathcal{F}, \mathcal{H}, \rho)$ describing the vacuum with iε-regularization. Then for a generic curve γ and for every $N \in \mathbb{N}$, there is ε_0 such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $n = 1, \ldots, N$, the points x_n and x_{n-1} are spin connectable. Moreover,*

$$\nabla^{\text{LC}}_{x,y} = \lim_{N\to\infty} \lim_{\varepsilon \searrow 0} \nabla^N_{x,y}\,.$$

By a *generic curve* we mean that the admissible curves are dense in the C^∞-topology (i.e., for any smooth γ and every $K \in \mathbb{N}$, there is a sequence γ_ℓ of admissible curves such that $D^k\gamma_\ell \to D^k\gamma$ uniformly for all $k = 0, \ldots, K$). The restriction to generic curves is needed in order to ensure that the Euclidean and directional sign operators are generically separated (see Definition 11.2.8 (b)). The proof of the above theorem is given in [**56**, Section 4].

Clearly, in this theorem the connection $\nabla^{\text{LC}}_{x,y}$ is trivial. In order to show that our connection also coincides with the Levi-Civita connection in the case with curvature, in [**56**, Section 5] a globally hyperbolic Lorentzian manifold is considered. For technical simplicity, we assume that the manifold is flat Minkowski space in the past of a given Cauchy hypersurface.

THEOREM 11.3.2. *Let $(\mathscr{M}, g)$ be a globally hyperbolic manifold which is isometric to Minkowski space in the past of a given Cauchy-hypersurface $\mathcal{N}$. For given γ, for any $\varepsilon > 0$ we consider the causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$ which in the past of $\mathcal{N}$ coincides with the causal fermion system in Minkowski space considered in Theorem 11.3.1. Then for a generic curve γ and for every sufficiently large N, there is ε_0 such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $n = 1, \ldots, N$, the points x_n and x_{n-1} are spin connectable. Moreover,*

$$\lim_{N\to\infty} \lim_{\varepsilon \searrow 0} \nabla^N_{x,y} - \nabla^{\text{LC}}_{x,y} = \mathcal{O}\left(L(\gamma)\, \frac{\nabla R}{m^2} \right) \left(1 + \mathcal{O}\Big(\frac{\text{scal}}{m^2} \Big) \right),$$

where R denotes the Riemann curvature tensor, scal *is scalar curvature, and $L(\gamma)$ is the length of the curve γ.*

Thus the metric connection of Definition 11.2.10 indeed coincides with the Levi-Civita connection, up to higher order curvature corrections. For detailed explanations and the proof we refer to [**56**, Section 5].

We conclude this section with a few remarks on further constructions [**56**]. First, there is the subtle point that the unitary transformation $U \in \exp(\mathrm{i}\mathbb{R} s_x)$ which is used to identify two representatives $K, \tilde{K} \in T_x$ via the relation $\tilde{K} = UKU^{-1}$ (see Definition 11.2.4) is not unique. More precisely, the operator U can be transformed according to

$$U \to -U \qquad \text{and} \qquad U \to s_x\, U\,.$$

As a consequence, the metric connection (see Definition 11.2.10) is defined only up to the transformation

$$\nabla_{x,y} u \to s_x\, (\nabla_{x,y} u)\, s_x\,.$$

Note that this transformation maps representatives of the same tangent vector into each other, so that $\nabla_{x,y} u \in T_x$ is still a well-defined tangent vector. But we get an ambiguity when composing the metric connection several times (as for example in the expression for the metric curvature in Definition 11.2.11). This ambiguity can be removed by considering *parity-preserving systems* as introduced in [**56**, Section 3.4].

At first sight, one might conjecture that Theorem 11.3.2 should also apply to the spin connection in the sense that

$$D^{\text{LC}}_{x,y} = \lim_{N\to\infty} \lim_{\varepsilon \searrow 0} D^N_{x,y}\,,$$

where D^{LC} is the spin connection on $S\mathscr{M}$ induced by the Levi-Civita connection and

$$D^N_{x,y} := D_{x_N, x_{N-1}} D_{x_{N-1}, x_{N-2}} \cdots D_{x_1, x_0} \;:\; S_y \to S_x \tag{11.3.1}$$

(and D is the spin connection of Theorem 11.2.9). It turns out that this conjecture is false. But the conjecture becomes true if we replace (11.3.1) by the operator product

$$D^N_{(x,y)} := D_{x_N, x_{N-1}} U^{(x_N | x_{N-2})}_{x_{N-1}} D_{x_{N-1}, x_{N-2}} U^{(x_{N-1} | x_{N-3})}_{x_{N-2}} \cdots U^{(x_2 | x_0)}_{x_1} D_{x_1, x_0}\,.$$

Here the intermediate factors $U^{(\cdot|\cdot)}_{\cdot}$ are the so-called *splice maps* given by

$$U^{(z|y)}_x = U_{xz}\, V\, U^{-1}_{xy}\,,$$

where U_{xz} and U_{xy} are synchronization maps, and $V \in \exp(i\mathbb{R} s_x)$ is an operator which identifies the representatives $K_{xy}, K_{xz} \in T_x$ (for details see [**56**, Section 3.7 and Section 5]). The splice maps also enter the *spin curvature* $\mathfrak{R}$, which is defined in analogy to the metric curvature (11.2.8) by

$$\mathfrak{R}(x,y,z) = U^{(z|y)}_x\, D_{x,y}\, U^{(x|z)}_y\, D_{y,z}\, U^{(y|x)}_z\, D_{z,x} \;:\; S_x \to S_x\,.$$

11.4. Exercises

EXERCISE 11.1. *(Vector fields on a closed Riemannian manifold)* Let $(\mathscr{M}, g)$ be a smooth compact Riemannian manifold of dimension k and Δ the covariant Laplacian on smooth vector fields. We complexify the vector fields and endow them with the L^2-scalar product

$$\langle u | v \rangle_{L^2} := \int_{\mathscr{M}} g_{jk}\, \overline{u^j}\, v^k \, \mathrm{d}\mu_{\mathscr{M}} \,,$$

where $\mathrm{d}\mu_{\mathscr{M}} = \sqrt{\det g}\, \mathrm{d}^k x$ is the volume measure on $\mathscr{M}$. Show the following:

(a) The operator $-\Delta$ is essentially selfadjoint and has smooth eigenfunctions.

(b) We choose a parameter $L > 0$ and choose $\mathcal{H}$ as the spectral subspace of the Laplacian

$$\mathcal{H} = \mathrm{rg}\, \chi_{[0,L]}(-\Delta) \,.$$

Show that $\mathcal{H}$ is finite-dimensional.

(c) For any $p \in \mathscr{M}$ we define the local correlation operator $F(p) \in \mathrm{L}(\mathcal{H})$ by

$$-g_{ij}\, \overline{u^i(p)}\, v^j(p) = \langle u | F(p) v \rangle_{L^2} \qquad \text{for all } u, v \in \mathcal{H} \,.$$

Show that this operator is well-defined, negative semi-definite and has rank at most k.

(d) We again introduce the measure by $\rho = F_* \mu$. Show that $(\mathcal{H}, \mathcal{F}, \rho)$ is a Riemannian fermion system of spin dimension k.

Hint: For (a) and (b) one can use properties of elliptic operators on compact domains, as can be found for example in [**32, 143**].

EXERCISE 11.2. *(Spinors on a closed Riemannian manifold)* Let $(\mathscr{M}, g)$ be a compact Riemannian spin manifold of dimension $k \geq 1$. Then the spinor bundle $S\mathscr{M}$ is a vector bundle with fiber $S_p\mathscr{M} \simeq \mathbb{C}^n$ with $n = 2^{[k/2]}$ (see for example [**115, 90**]). Moreover, the spin inner product $\prec . | . \succ_p : S_p\mathscr{M} \times S_p\mathscr{M} \to \mathbb{C}$ is positive definite. On the smooth sections $\Gamma(S\mathscr{M})$ of the spinor bundle we can thus introduce the scalar product

$$\langle \psi | \phi \rangle = \int_{\mathscr{M}} \prec \psi | \phi \succ_p \, \mathrm{d}\mu_{\mathscr{M}}(p) \,.$$

Forming the completion gives the Hilbert space $L^2(\mathscr{M}, S\mathscr{M})$.

(a) The Dirac operator $\mathcal{D}$ with domain of definition $\Gamma(S\mathscr{M})$ is an essentially selfadjoint operator on $L^2(\mathscr{M}, S\mathscr{M})$. It has a purely discrete spectrum and finite-dimensional eigenspaces.

(b) Given a parameter $L > 0$, we let $\mathcal{H}$ be the space spanned by all eigenvectors whose eigenvalues lie in the interval $[-L, 0]$,

$$\mathcal{H} = \mathrm{rg}\, \chi_{[-L,0]}(\mathcal{D}) \subset L^2(\mathscr{M}, S\mathscr{M}) \,.$$

Denoting the restriction of the L^2-scalar product to $\mathcal{H}$ by $\langle . | . \rangle_{\mathcal{H}}$, we obtain a finite-dimensional Hilbert space $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$. Show that this Hilbert space is finite-dimensional and consists of smooth wave functions.

(c) For every $p \in \mathscr{M}$ we introduce the local correlation operator $F(p)$ by

$$-\prec \psi | \phi \succ_p = \langle \psi | F(p) \phi \rangle_{\mathcal{H}} \qquad \text{for all } \psi, \phi \in \mathcal{H} \,.$$

Show that this operator is negative semi-definite and has rank at most n.

(d) We again introduce the measure by $\rho = F_* \mu$. Show that $(\mathcal{H}, \mathcal{F}, \rho)$ is a Riemannian fermion system of spin dimension n.

Hint: The Dirac operator on Riemannian manifolds of general dimension is introduced in [**115, 90**]. For (a) and (b) one can again use properties of elliptic operators on compact domains, as can be found for example in [**32, 143**] or, more specifically for Dirac operators, in [**145**, Chapter 20].

EXERCISE 11.3. *(Causal structure of a Riemannian fermion system)* Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a Riemannian fermion system of spin dimension n (see Definition 11.1.1).

(a) Show that for every $y \in \mathcal{F}$, the operator $-y$ is positive semi-definite. How can its square root $\sqrt{-y}$ be defined?
(b) Show that the operator product $\sqrt{-y}\,(-x)\sqrt{-y}$ (with $x \in \mathcal{F}$) is positive semi-definite.
(c) Show that the eigenvalues of the operator product xy are all real and non-negative. *Hint:* Use the relation $xy = -x\,\sqrt{-y}\,\sqrt{-y}$ together with the fact that the spectrum is invariant under cyclic permutations.
(d) What does this mean for the causal structure of Definition 5.6.1?

EXERCISE 11.4. *(The Euclidean sign operator)* Let $x \in \mathcal{F}$ be a regular spacetime point. For convenience, we choose a basis of the Hilbert space where x is diagonal, i.e.

$$(-x) = \text{diag}\big(\nu_1, \nu_2, \nu_3, \nu_3, 0, \ldots\big) \qquad \text{with } \nu_1, \nu_2 > 0 \text{ and } \nu_3, \nu_4 < 0\,.$$

(a) Show that the eigenspaces corresponding to the positive (negative) eigenvalues of $(-x)$ are positive (respectively negative) definite w.r.t. the spin inner product.
(b) We define the Euclidean sign operator in the above basis by

$$s_x = \text{diag}\big(1, 1, -1, -1, 0, \ldots\big)\,.$$

Show that this operator is uniquely defined without referring to bases by demanding that s_x commutes with x, that its eigenvalues are $\{1, -1, 0\}$ and that the eigenspaces corresponding to 1 (and -1) are negative (respectively positive) definite with respect to the spin inner product.

EXERCISE 11.5. *(Signature of Clifford extensions)*

(a) Let $\mathcal{T}^{s_x}$ be a Clifford extension of the Euclidean sign operator s_x. Show that resulting bilinear form $\langle .,. \rangle$ on $\mathcal{T}^{s_x}$ is Lorentzian, i.e. that it has signature $(1, k)$ with $k \in \mathbb{N}$. *Hint:* it is most convenient to work in an orthonormal eigenvector basis of the Euclidean sign operator. You also find the proof in [**60**, Lemma 4.4].
(b) Now let $\mathcal{T}^v$ be a the Clifford extension of a general sign operator v. Is the signature of $\langle .,. \rangle$ necessarily Lorentzian? *Hint:* It may be helpful to have a look at [**56**, Lemma 3.2].

EXERCISE 11.6. *(Clifford extensions on the Dirac sphere)* We return to the Dirac sphere considered in Exercise 5.14. Thus we let $F : S^2 \to \mathcal{F}$ and $M := \text{supp}\,\rho = F(S^2)$.

(a) Given $p \in S^2$, we consider the spacetime point $x = F(p) \in M$. Construct the Euclidean sign operator s_x at x.
(b) What is the maximal dimension of Clifford extensions of the Euclidean sign operator? Show that the Clifford extension of maximal dimension is unique.
(c) Give an explicit parametrization of this Clifford extension. How does the inner product $\langle .,. \rangle$ look like in your parametrization?

EXERCISE 11.7. *(Stability of the causal structure)* A binary relation P on $\mathcal{F}$ is said to be *stable under perturbations* if

$$(x_0, y_0) \in P \;\Longrightarrow\; \exists\, r > 0 \;:\; B_r(x_0) \times B_r(y_0) \subset P.$$

Following Definition 11.2.5, two points $x, y \in \mathcal{F}$ are said to be *properly timelike* separated if the closed chain A_{xy} has a strictly positive spectrum and if all eigenspaces are definite subspaces of $(S_x, \prec \cdot, \cdot \succ)$.

(a) Show that proper timelike separation implies timelike separation.
(b) Show by a counterexample with 3×3 matrices that the notion of timelike separation is *not* stable under perturbations.
(c) Show that the notion of properly timelike separation is stable under perturbations.

Part 3

Mathematical Methods and Analytic Constructions

CHAPTER 12

Measure-Theoretic Methods

The main goal of this chapter is to prove the existence of minimizers for the causal action principle in the case that $\mathcal{H}$ is *finite dimensional* and ρ is *normalized*, i.e.

$$\dim \mathcal{H} =: f < \infty \quad \text{and} \qquad \rho(\mathcal{F}) = 1 \,. \tag{12.0.1}$$

After introducing the necessary methods (Sections 12.1 and 12.2) we first apply them to prove existence of minimizers for causal variational principles in the compact setting (Section 12.3). In preparation for the proof for the causal action principle, we illustrate the constraints by a few examples (Section 12.4). The difficulties revealed by these examples can be resolved by working with the so-called moment measures. After introducing the needed mathematical methods (Section 12.5), the moment measures are introduced (Section 12.6). Then the existence proof is completed (Section 12.7). In order to give a first idea for how to deal with an infinite total volume, we finally prove existence of minimizers for causal variational principles in the non-compact setting (Section 12.8).

Our general strategy is to apply the *direct method in the calculus of variations*, which can be summarized as follows:

(a) Choose a *minimizing sequence*, i.e. a sequence of measures (ρ_k) which satisfy the constraints such that

$$\mathcal{S}(\rho_k) \to \inf_{\rho} \mathcal{S}(\rho) \,.$$

Such a minimizing sequence always exists by definition of the infimum (note that the action and therefore also its infimum are non-negative).

(b) Show that a subsequence of the measures converges in a suitable sense,

$$\rho_{k_l} \text{ “}\longrightarrow\text{” } \rho \,.$$

Here the quotation marks indicate that we still need to specify in which sense the sequence should converge (convergence in which space, strong or weak convergence, etc.).

(c) Finally, one must show that the action is lower semi-continuous, i.e.

$$\mathcal{S}(\rho) \leq \liminf_{l \to \infty} \mathcal{S}(\rho_{k_l}) \,.$$

Also, one must prove that the limit measure ρ satisfies the constraints.

Once these three steps have been carried out, the measure ρ is a desired minimizer. We point out that this procedure does *not* give a *unique* minimizer, simply because there may be different minimizing sequences, and because the choice of the subsequences may involve an arbitrariness. Indeed, for the causal action principle we do not expect uniqueness. There should be many different minimizers, which describe different physical systems (like the vacuum, a system involving particles and fields, etc.). This intuitive picture is confirmed by the numerical studies in [**84, 59**] which show that, even if the dimension of $\mathcal{H}$ is small, there are many different minimizers.

12.1. The Banach-Alaoglu Theorem

For our purposes, it suffices to consider the case that the Banach space is separable, in which case the theorem was first proved by Banach (Alaoglu proved the generalization to non-separable Banach spaces; this makes use of Tychonoff's theorem and goes beyond what we need here). Indeed, the idea of proof of the theorem can be traced back to Eduard Helly's doctoral thesis in 1912, where the closely related "Helly's selection theorem" is proved (of course without reference to Banach spaces, which were introduced later). We closely follow the presentation in [**116**, Section 10.3].

Let $(E, \|.\|_E)$ be a separable (real or complex) Banach space and $(E^*, \|.\|_{E^*})$ its dual space with the usual sup-norm, i.e.

$$\|\phi\|_{E^*} = \sup_{u\in E, \|u\|=1} \big|\phi(u)\big| \,. \tag{12.1.1}$$

A sequence $(\phi_n)_{n\in\mathbb{N}}$ in E^* is said to be *weak*-convergent* to $\phi \in E^*$ if

$$\lim_{n\to\infty} \phi_n(u) = \phi(u) \qquad \text{for all } u \in E \,.$$

THEOREM 12.1.1. **(Banach-Alaoglu theorem in the separable case)** *Let E be a separable Banach space. Then every bounded sequence in E^* has a weak*-convergent subsequence.*

PROOF. Let ϕ_n be a bounded sequence in E^*, meaning that there is a constant $c > 0$ with

$$\|\phi_n\|_{E^*} \leq c \qquad \text{for all } n \in \mathbb{N} \,. \tag{12.1.2}$$

We let $(u_\ell)_{\ell\in\mathbb{N}}$ be a sequence in E which is dense in E. Combining (12.1.2) with (12.1.1), the estimate

$$|\phi_n(u_1)| \leq \|\phi_n\|_{E^*} \, \|u_1\|_E \leq c \, \|u_1\|_E \tag{12.1.3}$$

shows that $(\phi_n(u_1))_{n\in\mathbb{N}}$ is a bounded sequence. Thus we can choose a convergent subsequence. By inductively choosing subsequences and taking the diagonal sequence, we obtain a subsequence (ϕ_{n_j}) such that the limit $\lim_{j\to\infty} \phi_{n_j}(u_\ell)$ exists for all $\ell \in \mathbb{N}$. Hence setting

$$\phi(u_\ell) := \lim_{j\to\infty} \phi_{n_j}(u_\ell) \,,$$

we obtain a densely defined functional. Taking the limit in (12.1.3) (and the similar inequalities for $u_2, u_3, \ldots$), one sees that this functional is again continuous. Therefore, it has a unique continuous extension to E. By continuity, the resulting functional $\phi \in E^*$ satisfies the relations

$$\phi(u) = \lim_{j\to\infty} \phi_{n_j}(u) \qquad \text{for all } u \in \mathcal{H} \,.$$

In particular, it is again a linear. This concludes the proof. □

12.2. The Riesz Representation Theorem

In this section and Section 12.5, we shall introduce the methods from measure theory needed for the existence proofs. Apart from the books already mentioned in the preliminaries (Section 2.3), we also recommend the book [**33**] (this book is only concerned with measures in $\mathbb{R}^n$, but otherwise goes far beyond what we need here).

For our purposes, it suffices to restrict attention to the case that the base space $\mathcal{K}$ is a *compact* topological space. We always consider *bounded regular Borel measures* on $\mathcal{K}$ (for the preliminaries see Section 2.3). In order to avoid confusion, we note that by a

measure we always mean a *positive* measure (signed measures will not be considered in this book). A *bounded* measure is also referred to as a measure of *finite total volume*. Often, we *normalize* the measure such that $\mu(\mathcal{K}) = 1$.

In words, the Riesz representation theorem makes it possible to represent a linear functional on the Banach space of continuous functions of a topological space by a regular Borel measure on this topological space. We remark that we already came across the Riesz representation theorem in Section 3.2, where it was needed for the construction of spectral measures. However, in this context we only needed the special case that the topological space was an interval of the real line. We now state the general theorem and outline its proof, mainly following the presentation in [**101**, §56]. More details can be found in [**8**, §IV.29].

As a simple example, one can choose $\mathcal{K}$ as the closed unit ball in $\mathbb{R}^n$. Restricting the Lebesgue measure to the Borel subsets of $\mathcal{K}$ gives a Radon measure. The Lebesgue measure itself is a completion of this Radon measure obtained by extending the σ-algebra of measurable sets by all subsets of Borel sets of measure zero. Since this completion is a rather trivial extension, in what follows we prefer to work with Radon measures or, equivalently, with normalized regular Borel measures.

THEOREM 12.2.1. **(Riesz representation theorem)** *Let $\mathcal{K}$ be a compact topological space, and $E = C^0(\mathcal{K}, \mathbb{R})$ the Banach space of continuous functions on $\mathcal{K}$ with the usual sup-norm,*

$$\|f\| = \sup_{x \in \mathcal{K}} |f(x)| \,.$$

Let $\Lambda \in E^*$ be a continuous linear functional which is positive in the sense that

$$\Lambda(f) \geq 0 \qquad \text{for all nonnegative functions } f \in C^0(\mathcal{K}, \mathbb{R}) \,.$$

Then there is a unique regular Borel measure μ such that

$$\Lambda(f) = \int_{\mathcal{K}} f \, \mathrm{d}\mu \qquad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R}) \,.$$

OUTLINE OF THE PROOF. We follow the strategy in [**101**, §56]. Given a Borel set $A \subset \mathcal{K}$, we set

$$\lambda(A) = \inf \big\{ \Lambda(f) \,\big|\, f \in C^0(\mathcal{K}, \mathbb{R}) \text{ and } f \geq \chi_A \big\} \in \mathbb{R}_0^+ \,.$$

Intuitively speaking, λ gives us the desired "volume" of the set A. But there is the technical problem that λ is in general not a regular Borel measure. Instead, it merely is a *content*, meaning that it has the following properties:

(i) *non-negative* and *finite*: $0 \leq \lambda(A) < \infty$
(ii) *monotone*: C, D compact and $C \subset D \implies \lambda(C) \leq \lambda(D)$
(iii) *additive*: C, D compact and disjoint $\implies \lambda(C \cup D) = \lambda(C) + \lambda(D)$
(iv) *subadditive*: C, D compact $\implies \lambda(C \cup D) \leq \lambda(C) + \lambda(D)$

At this stage, we are in a similar situation as in the elementary measure theory course after saying that a cube of length ℓ in $\mathbb{R}^3$ should have volume ℓ^3. In order to get from this "volume measure" to a measure in the mathematical sense, one has to proceed in several steps invoking the subtle and clever constructions of measure theory (due to Lebesgue, Hahn, Carathéodory and others) in order to get a mapping from a σ-algebra to the non-negative real numbers which is σ-additive. In simple terms, repeating these constructions starting from the above content gives the desired Borel measure μ. For brevity, we here merely outline the constructions and refer for details to text books on measure theory (like for example [**101**, Chapter X]).

The first step is to approximate (or exhaust) from inside by compact sets. Thus one introduces the *inner content* λ_* by

$$\lambda_*(U) = \sup \big\{ \lambda(C) \,\big|\, C \subset U \text{ compact} \big\} .$$

This inner content is monotone and countably additive. The second step is to exhaust from outside by open sets. This gives the *outer measure* μ^*,

$$\mu^*(U) = \inf \big\{ \lambda_*(\Omega) \,\big|\, \Omega \supset U \text{ open} \big\} .$$

The outer measure is defined for any subset of $\mathcal{K}$. Therefore, it remains to distinguish the measurable sets. This is accomplished by Carathéodory's criterion, which defines a set $A \subset \mathcal{K}$ to be *measurable* if

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B)$$

for every subset $B \subset \mathcal{K}$. Then Carathéodory's lemma (for a concise proof see for example [**18**, Lemma 2.8]) implies that the measurable sets form a σ-algebra, and that the restriction of μ^* to the measurable sets is indeed a measure, denoted by μ.

In order to complete the proof, one still needs to verify that every Borel set is μ-measurable. Moreover, it remains to show that the resulting Borel measure is regular. To this end, one first needs to show that the content λ is regular in the following sense:

(v) *regular*: For every compact C,

$$\lambda(C) = \inf \Big\{ \lambda(D) \,\Big|\, D \text{ compact and } C \subset \overset{\circ}{D} \Big\} .$$

As the proofs of these remaining points are rather straightforward and not very instructive, we refer for the details to [**101**, §54–§56]. □

12.3. Existence of Minimizers for Causal Variational Principles in the Compact Setting

We now apply the above methods to prove existence of minimizers for causal variational principles in the compact setting. Our strategy is to apply the Banach-Alaoglu theorem to a specific Banach space, namely the continuous functions on a compact metric space. We first verify that this Banach space is separable.

PROPOSITION 12.3.1. *Let $\mathcal{K}$ be a compact metric space. Then $C^0(\mathcal{K}, \mathbb{R})$ is a separable Banach space.*

PROOF. The proposition is a consequence of the Stone-Weierstrass theorem, whose proof can be found for example in [**26**, 7.3.1] We closely follow the proof given in [**26**, 7.4.4].

Covering $\mathcal{K}$ by a finite number of open balls of radii 1, 1/2, 1/3, ..., one gets an enumerable basis of the open sets $(U_n)_{n \in \mathbb{N}}$. For any $n \in N$, we let g_n be the continuous function

$$g_n(x) = d\big(x, \mathcal{K} \setminus U_n\big) .$$

Clearly, the algebra generated by these functions (by taking finite products and finite linear combinations) is again separable. Therefore, it suffices to show that this algebra is dense in $C^0(\mathcal{K}, \mathbb{R})$. To this end, we need to verify the assumptions of the Stone-Weierstrass theorem. The only assumption which is not obvious is that the algebra separates the points. This can be seen as follows: Let x and y be two distinct points in $\mathcal{K}$. Since the (U_n) are a basis of the topology, there is U_n with $x \in U_n$ and $y \notin U_n$. As a consequence, $g_n(x) > 0$ but $g_n(y) = 0$. □

We proceed by proving a compactness result for Radon measures.

THEOREM 12.3.2. *Let ρ_n be a series of regular Borel measures on $C^0(\mathcal{K}, \mathbb{R})$ which are bounded in the sense that there is a constant $c > 0$ with*

$$\rho_n(\mathcal{K}) \leq c \qquad \text{for all } n\,.$$

Then there is a subsequence (ρ_{n_k}) which converges as a measure, i.e.

$$\lim_{k\to\infty} \int_{\mathcal{K}} f \,\mathrm{d}\rho_{n_k} = \int_{\mathcal{K}} f \,\mathrm{d}\rho \qquad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R})\,. \tag{12.3.1}$$

Moreover, the total volume converges, i.e.

$$\rho(\mathcal{K}) = \lim_{k\to\infty} \rho_{n_k}(\mathcal{K})\,. \tag{12.3.2}$$

PROOF. Via

$$\phi_n(f) := \int_{\mathcal{K}} f \,\mathrm{d}\rho_n\,,$$

every measure can be identified with a positive linear functional on $E := C^0(\mathcal{K}, \mathbb{R})$. Since E is separable (Proposition 12.3.1), we can apply the Banach-Alaoglu theorem in the separable case (Theorem 12.1.1) to conclude that there is a weak*-convergent subsequence, i.e.

$$\lim_{k\to\infty} \phi_{n_k}(f) = \phi(f) \qquad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R})\,.$$

Clearly, since all ϕ_{n_k} are positive, the same is true for the limit ϕ. Therefore, the Riesz representation theorem (Theorem 12.2.1) makes it possible to represent ϕ by a regular Borel measure ρ, i.e.

$$\phi(f) = \int_{\mathcal{K}} f \,\mathrm{d}\rho \qquad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R})\,.$$

Choosing f as the constant function, one obtains (12.3.2). This concludes the proof. □

THEOREM 12.3.3. *Assume that $\mathcal{F}$ is a compact topological space and the Lagrangian is continuous,*

$$\mathcal{L} \in C^0(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0)\,.$$

Then the causal variational principle where the causal action (6.2.2) *is minimized in the class of regular Borel measures under the volume constraint* (6.2.3) *is well-posed in the sense that every minimizing sequence $(\rho_n)_{n\in\mathbb{N}}$ has a subsequence which converges as a measure to a minimizer ρ.*

PROOF. The existence of a convergent subsequence $(\rho_{n_k})_{k\in\mathbb{N}}$ is proven in Theorem 12.3.2. It remains to show that the action is continuous, i.e.

$$\lim_{k\to\infty} \mathcal{S}\big(\rho_{n_k}\big) = \mathcal{S}(\rho)\,.$$

This is verified in detail as follows. Using that the Lagrangian is continuous in its second argument, we know that

$$\lim_{k\to\infty} \int_{\mathcal{F}} \mathcal{L}(x,y) \,\mathrm{d}\rho_{n_k}(y) = \int_{\mathcal{F}} \mathcal{L}(x,y) \,\mathrm{d}\rho(y) \qquad \text{for all } x \in \mathcal{F}\,. \tag{12.3.3}$$

Next, since $\mathcal{F}$ is compact, the Lagrangian is even uniformly continuous on $\mathcal{F} \times \mathcal{F}$. Therefore, given $\varepsilon > 0$, every point $x \in \mathcal{F}$ has an open neighborhood $U(x) \subset \mathcal{F}$ such that

$$\big|\mathcal{L}(\hat{x}, y) - \mathcal{L}(x,y)\big| < \varepsilon \qquad \text{for all } \hat{x} \in U(x) \text{ and } y \in \mathcal{F}\,.$$

Integrating over y with respect to any normed regular Borel measure $\tilde{\rho}$, it follows that

$$\left| \int_{\mathcal{F}} \mathcal{L}(\hat{x}, y)\ \mathrm{d}\tilde{\rho}(y)\ -\ \int_{\mathcal{F}} \mathcal{L}(x, y)\ \mathrm{d}\tilde{\rho}(y) \right| \leq \varepsilon \qquad \text{for all } \hat{x} \in U(x)\,. \tag{12.3.4}$$

Covering $\mathcal{F}$ by a finite number of such neighborhoods $U(x_1), \ldots, U(x_N)$, one can combine the pointwise convergence (12.3.3) for $x = x_1, \ldots, x_N$ with the estimate (12.3.4) to conclude that for any $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that

$$\left| \int_{\mathcal{F}} \mathcal{L}(x, y)\, \mathrm{d}\rho_{n_k}(y)\ -\ \int_{\mathcal{F}} \mathcal{L}(x, y)\, \mathrm{d}\rho(y) \right| \leq 3\varepsilon \qquad \text{for all } x \in \mathcal{F} \text{ and } k \geq k_0\,.$$

Integrating over y with respect to ρ_{n_k} and ρ gives for all $k \geq k_0$ the respective inequalities

$$\left| \mathcal{S}\big(\rho_{n_k}\big)\ -\ \int_{\mathcal{F}} \mathrm{d}\rho_{n_k}(x) \int_{\mathcal{F}} \mathrm{d}\rho(y)\, \mathcal{L}(x, y) \right| \leq 3\varepsilon\,,$$

$$\left| \int_{\mathcal{F}} \mathrm{d}\rho(x) \int_{\mathcal{F}} \mathrm{d}\rho_{n_k}(y)\, \mathcal{L}(x, y)\ -\ \mathcal{S}(\rho) \right| \leq 3\varepsilon\,.$$

Combining these inequalities and using that the Lagrangian is symmetric in its two arguments, we conclude that

$$\big| \mathcal{S}\big(\rho_{n_k}\big) - \mathcal{S}(\rho) \big| \leq 6\varepsilon\,.$$

This gives the result. □

We finally remark that the statement of this theorem also holds if the Lagrangian merely is lower semi-continuous, as is worked out in [**75**, Section 3.2].

12.4. Examples Illustrating the Constraints

Compared to causal variational principles in the compact setting, the existence proof for the causal action principle is considerably harder because we need to handle the constraints (5.6.3)–(5.6.5) and face the difficulty that the set $\mathcal{F}$ is unbounded and therefore non-compact. We now explain the role of the constraints in a few examples. The necessity of the volume constraint is quite obvious: If we dropped the constraint of fixed total volume (5.6.3), the measure $\rho = 0$ would be a trivial minimizer. The role of the trace constraint is already less obvious. It is explained in the next two examples.

EXAMPLE 12.4.1. **(necessity of the trace constraint)** Let x be the operator with the matrix representation

$$x = \mathrm{diag}\big(\underbrace{1, \ldots, 1}_{n \text{ times}}, \underbrace{-1, \ldots, -1}_{n \text{ times}}, 0, 0, \ldots\big)\,.$$

Moreover, we choose ρ as a multiple of the Dirac measure supported at x. Then the action $\mathcal{S}$ vanishes (see (5.6.2)), whereas the constraint $\mathcal{T}$ is strictly positive (see (5.6.5)). ♢

EXAMPLE 12.4.2. **(non-triviality of the action with trace constraint)** Let ρ be a normalized measure which satisfies the trace constraint in a non-trivial way, i.e.

$$\int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho(x) = \mathrm{const} \neq 0\,.$$

Let us prove that the action is non-zero. This will show that the trace constraint really avoids trivial minimizers of the causal action principle.

(a) Since the integral over the trace is non-zero, there is a point x in the support of ρ with $\mathrm{tr}(x) \neq 0$. We denote the non-trivial eigenvalues of x by $\nu_1, \ldots, \nu_{2n}$ and order them according to

$$\nu_1 \leq \cdots \leq \nu_n \;\leq\; 0 \;\leq\; \nu_{n+1} \leq \cdots \leq \nu_{2n} \,.$$

The fact that the trace of x is non-zero clearly implies that the ν_i do not all have the same absolute value. As a consequence, the nontrivial eigenvalues of the operator product x^2 given by $\lambda_j^{xx} = \nu_j^2$ are all non-negative and not all equal. Using the form of the Lagrangian in (5.6.1), we conclude that $\mathcal{L}(x,x) > 0$.

(b) Since the Lagrangian is continuous in both arguments, there is an open neighborhood $U \subset \mathcal{F}$ of x such that $\mathcal{L}(y,z) > 0$ for all $y, z \in U$. Since x lies in the support of ρ, we know that $\rho(U) > 0$. As a consequence,

$$\mathcal{S} \geq \int_U \mathrm{d}\rho(x) \int_U \mathrm{d}\rho(y)\, \mathcal{L}(x,y) > 0$$

(because if the integrals vanished, then the integrand would have to be zero almost everywhere, a contradiction).

We remark that this argument is quantified in [**42**, Proposition 4.3]. ◊

We now come to the boundedness constraint. In order to explain how it comes about, we give an explicit example with (4×4)-matrices (for a similar example with (2×2)-matrices see Exercise 6.2).

EXAMPLE 12.4.3. **(necessity of the boundedness constraint)** The following example explains why the *boundedness constraint* (5.6.5) is needed in order to ensure the existence of minimizers. It was first given in [**43**, Example 2.9]. Let $\mathcal{H} = \mathbb{C}^4$. We introduce the four 4×4-matrices acting on $\mathcal{H}$ by

$$\gamma^\alpha = \begin{pmatrix} \sigma^\alpha & 0 \\ 0 & -\sigma^\alpha \end{pmatrix}, \quad \alpha = 1,2,3 \qquad \text{and} \qquad \gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

(where the σ^α are again the Pauli matrices (1.3.4)). Given a parameter $\tau > 1$, we consider the following mapping from the sphere $S^3 \subset \mathbb{R}^4$ to the linear operators on $\mathcal{H}$,

$$F \;:\; S^3 \to \mathrm{L}(\mathcal{H})\,, \qquad F(x) = \sum_{i=1}^4 \tau\, x^i \gamma^i + \mathbb{1}\,.$$

(a) *The matrices $F(x)$ have two positive and two negative eigenvalues:*
Since the computation of the eigenvalues of 4×4-matrices is tedious, it is preferable to proceed as follows. The matrices γ^j are the Dirac matrices of Euclidean $\mathbb{R}^4$, satisfying the anti-commutation relations

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\, \mathbb{1} \qquad (i,j = 1, \ldots, 4)\,.$$

As a consequence,

$$\begin{aligned} F(x) - \mathbb{1} &= \sum_{i=1}^4 \tau\, x^i \gamma^i \\ \big(F(x) - \mathbb{1}\big)^2 &= \sum_{i,j=1}^4 \tau^2\, x^i\, x^j \gamma^i \gamma^j = \frac{\tau^2}{2} \sum_{i,j=1}^4 x^i\, x^j \big\{\gamma^i, \gamma^j\big\} \\ &= \frac{\tau^2}{2} \sum_{i,j=1}^4 x^i\, x^j\, 2\, \delta_{ij}\, \mathbb{1} = \tau^2\, \mathbb{1}\,. \end{aligned}$$

Hence the matrix $F(x)$ satisfies the polynomial equation

$$\big(F(x) - \mathbb{1}\big)^2 = \tau^2\, \mathbb{1}\,.$$

We conclude that $F(x)$ has the eigenvalues

$$\nu_\pm = 1 \pm \tau\,.$$

Since $F(x) - \mathbb{1}$ is trace-free, each eigenvalue must appear with multiplicity two. Using that $\tau > 1$, we conclude that $F(x)$ really has two positive and two negative eigenvalues.

(b) *Construction of a causal fermion system:*
Let μ be the normalized Lebesgue measure on $S^3 \subset \mathbb{R}^4$. Setting $\rho = F_* \mu$ defines a causal fermion system of spin dimension two and total volume one. Since the matrices $F(x)$ all have trace four, we also know that

$$\int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho(x) = \int_{S^3} \mathrm{tr}(F(x))\, \mathrm{d}\mu(x) = 4\,.$$

Therefore, the volume constraint (5.6.3) and the trace constraint (5.6.4) are satisfied, both with constants independent of τ.

(c) *Computation of the eigenvalues of $F(x)\, F(y)$:*
Again, this can be calculated most conveniently using the Clifford relations.

$$\begin{aligned} F(x)\, F(y) &= \Big(\sum_{i=1}^4 \tau\, x^i \gamma^i + \mathbb{1}\Big)\Big(\sum_{j=1}^4 \tau\, y^j \gamma^j + \mathbb{1}\Big) \\ &= \big(1 + \tau^2\, \langle x, y\rangle\big)\mathbb{1} + \tau \sum_{i=1}^4 (x^i + y^i)\gamma^i + \frac{\tau^2}{2} \sum_{i,j=1}^4 x^i y^j \big[\gamma^i, \gamma^j\big]\,. \end{aligned} \tag{12.4.1}$$

Using that

$$\gamma^i \big[\gamma^i, \gamma^j\big] = -\big[\gamma^i, \gamma^j\big]\, \gamma^i\,,$$

we conclude that

$$\Big(F(x)\, F(y) - \big(1 + \tau^2\, \langle x, y\rangle\big)\mathbb{1}\Big)^2 = \tau^2 \sum_{i=1}^4 (x^i + y^i)^2 + \left(\frac{\tau^2}{2} \sum_{i,j=1}^4 x^i y^j \big[\gamma^i, \gamma^j\big]\right)^2.$$

This can be simplified with the help of the relations

$$\sum_{i=1}^{5}(x^i + y^i)^2 = 2 + 2\,\langle x, y\rangle \tag{12.4.2}$$

$$\Big(\sum_{i,j=1}^{4} x^i y^j \,\big[\gamma^i, \gamma^j\big]\Big)^2 = -4\sin^2\vartheta = -4\,\big(1 - \langle x, y\rangle^2\big)\,, \tag{12.4.3}$$

where ϑ is the angle between the vectors $x, y \in \mathbb{R}^4$. The relation (12.4.3) can be verified in detail as follows. The rotational symmetry of the Euclidean Dirac operator on $\mathbb{R}^5$ means that for every rotation $O \in \mathrm{SO}(4)$ there is a unitary operator $U \in \mathrm{SU}(4)$ such that

$$O^i_j\, \gamma^j = U \gamma^i U^{-1}\,.$$

Making use of this rotational symmetry, we can arrange that the vector x is the basis vector e_1 and that $y = \cos\vartheta\, e_1 + \sin\vartheta\, e_2$. As a consequence,

$$\sum_{i,j=1}^{5} x^i y^j \,\big[\gamma^i, \gamma^j\big] = \sin\vartheta\,[\gamma^1, \gamma^2] = 2\,\sin\vartheta\,\gamma^1\gamma^2$$

$$\Big(\sum_{i,j=1}^{5} x^i y^j \,\big[\gamma^i, \gamma^j\big]\Big)^2 = 4\,\sin^2\vartheta\,\gamma^1\gamma^2\gamma^1\gamma^2\,,$$

and applying the anti-commutation relations gives (12.4.3).

Combining the above equations, we conclude that the product $F(x)\,F(y)$ satisfies the polynomial equation

$$\begin{aligned}\Big(F(x)\,F(y) - \big(1 + \tau^2\,\langle x, y\rangle\big)\mathbb{1}\Big)^2 &= 2\,\tau^2\big(1 + \langle x, y\rangle\big) - \tau^4\,\big(1 - \langle x, y\rangle^2\big) \\ &= \tau^2\,\Big(1 + \langle x, y\rangle\Big)\Big(2 - \tau^2\,\big(1 - \langle x, y\rangle\big)\Big)\,.\end{aligned}$$

Taking the square root, the zeros of this polynomial are computed by

$$\lambda_{1/2} = 1 + \tau^2\,\langle x, y\rangle \pm \tau\,\sqrt{1 + \langle x, y\rangle}\,\sqrt{2 - \tau^2\,(1 - \langle x, y\rangle)}\,. \tag{12.4.4}$$

Moreover, taking the trace of (12.4.1), one finds

$$\mathrm{tr}\,\big(F(x)\,F(y)\big) = 4\,\big(1 + \tau^2\,\langle x, y\rangle\big)\,.$$

This implies that each eigenvalue in (12.4.4) has algebraic multiplicity two.

(d) *Computation of the Lagrangian:*
We again denote the angle between the vectors $x, y \in \mathbb{R}^4$ by ϑ. If ϑ is sufficiently small, the term $(1 - \langle x, y\rangle)$ is close to zero, and thus the arguments of the square roots in (12.4.4) are all positive. However, if ϑ becomes so large that

$$\vartheta \;\geq\; \vartheta_{\max} := \arccos\!\left(1 - \frac{2}{\tau^2}\right),$$

then the argument of the last square root in (12.4.4) becomes negative, so that the $\lambda_{1/2}$ form a complex conjugate pair. Moreover, a short calculation shows that

$$\lambda_1\lambda_2 = (1+\tau)^2(1-\tau)^2 > 0\,,$$

implying that if the $\lambda_{1/2}$ are both real, then they have the same sign. Using this information, the Lagrangian simplifies to

$$\begin{aligned}\mathcal{L}(F(x),F(y)) &= \frac{1}{8}\sum_{i,j=1}^{4}\Big(|\lambda_i^{xy}| - |\lambda_j^{xy}|\Big)^2 = \frac{1}{2}\sum_{i,j=1}^{2}\Big(|\lambda_i| - |\lambda_j|\Big)^2 \\ &= \frac{1}{2}\,\Theta(\vartheta_{\max}-\vartheta)\sum_{i,j=1}^{2}\Big(\lambda_i-\lambda_j\Big)^2 = \Theta(\vartheta_{\max}-\vartheta)\big(\lambda_1-\lambda_2\big)^2 \\ &= 4\tau^2\,(1+\cos\vartheta)\,\big(2-\tau^2\,(1-\cos\vartheta)\big)\;\Theta(\vartheta_{\max}-\vartheta)\,.\end{aligned}$$

(e) *Computation the action:*
Inserting this Lagrangian in (5.6.2) and using the definition of the push-forward measure, we obtain

$$\begin{aligned}\mathcal{S} &= \int_{S^3} \mathrm{d}\mu(x) \int_{S^3} \mathrm{d}\mu(y)\; \mathcal{L}(F(x),F(y)) \\ &= \int_{S^3} \mathrm{d}\mu(y)\; \mathcal{L}(F(x),F(y)) = \frac{2}{\pi}\int_0^{\vartheta_{\max}} \mathcal{L}(\cos\vartheta)\,\sin^2\vartheta\;\mathrm{d}\vartheta \\ &= \frac{512}{15\pi}\,\frac{1}{\tau} + \mathcal{O}(\tau^{-2})\,.\end{aligned}$$

Thus setting $F_k = F|_{\tau=k}$, we have constructed a divergent minimizing sequence. However, the integral in the boundedness constraint (5.6.5) also diverges as $k\to\infty$. This example shows that, leaving out the boundedness constraint, there is no minimizer. ◊

We finally remark that this example is not as artificial or academic as it might appear at first sight. Indeed, as observed in the master thesis [**109**], when discretizing a Dirac system in $\mathbb{R}\times S^3$ (where the sphere can be thought of as a spatial compactification of Minkowski space), then in the simplest case of four occupied Dirac states (referred to as "one shell," i.e. $\dim\mathcal{H}=4$), this system reduces precisely to the last example. In simple terms, this observation can be summarized by saying that Clifford structures tend to make the causal action small.

12.5. The Radon-Nikodym Theorem

As already mentioned at the beginning of the previous section, one difficulty in the existence proof for the causal action principle is the fact that the set $\mathcal{F}$ is unbounded and thus non-compact. In order to deal with this difficulty, we need one more mathematical tool: the Radon-Nikodym theorem. We now give the proof of the Radon-Nikodym theorem by von Neumann following the presentation in [**136**, Chapter 6]. An alternative method of proof is given in [**101, 33**]. As in Section 12.2, it again suffices to consider the case that the base space $\mathcal{K}$ is a *compact* topological space.

DEFINITION 12.5.1. *A Radon measure λ is* **absolutely continuous** *with respect to another Radon measure ν, denoted by*

$$\lambda \ll \nu\,,$$

if the implication

$$\nu(E)=0 \quad\Longrightarrow\quad \lambda(E)=0$$

holds for any Borel set E. The measure λ is **concentrated** *on the Borel set A if $\lambda(E) = \lambda(E \cap A)$ for all Borel sets E. The measures λ and μ are* **mutually singular**, *denoted by*

$$\lambda \perp \nu \,,$$

if there are disjoint Borel sets A and B such that λ is concentrated in A and ν is concentrated in B.

In order to avoid confusion, we point out that the supports of two mutually singular measures are not necessarily disjoint, as one sees in the simple example of the Lebesgue measure on $(0,1)$ and the Dirac measure supported at the origin,

$$\lambda := \mathrm{d}x|_{(0,1)} \qquad \text{and} \qquad \mu = \delta_0 \,.$$

Since the support is by definition a closed set (see (2.3.4)), the support of $dx|_{(0,1)}$ contains the origin, which is precisely the support of the Dirac measure. But clearly, the two measures are concentrated on the sets $(0,1)$ and $\{0\}$, respectively, and are thus mutually singular.

THEOREM 12.5.2. **(Radon-Nikodym)** *Let μ and λ be Radon measures on the compact topological space $\mathcal{K}$.*

(a) *There is a unique pair of Borel measures λ_a and λ_s such that*

$$\lambda = \lambda_a + \lambda_s \qquad \text{and} \qquad \lambda_a \ll \mu \,, \quad \lambda_s \perp \mu \,.$$

(b) *There is a unique function $h \in L^1(\mathcal{K}, \mathrm{d}\mu)$ such that*

$$\lambda_a(E) = \int_E h \, \mathrm{d}\mu \qquad \text{for every Borel set } E \,. \tag{12.5.1}$$

The pair (λ_a, λ_s) is also referred to as the **Lebesgue decomposition** of λ with respect to μ.

PROOF OF THEOREM 12.5.2. The uniqueness of the decomposition is easily seen as follows: Suppose that (λ'_a, λ'_s) is another Lebesgue decomposition. Then

$$\lambda'_a - \lambda_a = \lambda_s - \lambda'_s \,. \tag{12.5.2}$$

Since $\lambda_s \perp \mu$ and $\lambda'_s \perp \mu$, the measures λ_s and $\lambda_{s'}$ are concentrated in a Borel set A with $\mu(A) = 0$. Evaluating (12.5.2) on Borel subsets of A, the left side vanishes, because λ_a and λ'_a are both absolutely continuous with respect to μ. Hence $\lambda'_s - \lambda_s = 0$. Using this relation in (12.5.2), we also conclude that $\lambda'_a - \lambda_a = 0$. This proves uniqueness.

For the existence proof, we let ρ be the measure $\rho = \lambda + \mu$. Then

$$\int_{\mathcal{K}} f \, \mathrm{d}\rho = \int_{\mathcal{K}} f \, \mathrm{d}\lambda + \int_{\mathcal{K}} f \, \mathrm{d}\mu \tag{12.5.3}$$

for any non-negative Borel function f. If $f \in L^2(\mathcal{K}, \mathrm{d}\rho)$, the Schwarz inequality gives

$$\left| \int_{\mathcal{K}} f \, \mathrm{d}\lambda \right| \leq \int_{\mathcal{K}} |f| \, \mathrm{d}\rho \leq \sqrt{\rho(\mathcal{K})} \, \|f\|_{L^2(\mathcal{K}, \mathrm{d}\rho)} \,.$$

Therefore, the mapping $f \mapsto \int_{\mathcal{K}} f \, d\lambda$ is a bounded linear functional on $L^2(\mathcal{K}, \mathrm{d}\rho)$. By the Fréchet-Riesz theorem, we can represent this linear functional by a function $g \in L^2(\mathcal{K}, \mathrm{d}\rho)$, i.e.

$$\int_{\mathcal{K}} f \, \mathrm{d}\lambda = \int_{\mathcal{K}} g \, f \, \mathrm{d}\rho \qquad \text{for all } f \in L^2(\mathcal{K}, \mathrm{d}\rho) \,. \tag{12.5.4}$$

We next want to show that, by modifying g on a set of ρ-measure zero, we can arrange that g takes values in the interval $[0,1]$. To this end, we let E be any Borel set with $\rho(E) > 0$. Evaluating (12.5.4) for $f = \chi_E$, we obtain

$$0 \leq \frac{1}{\rho(E)} \int_E g \, \mathrm{d}\rho = \frac{\lambda(E)}{\rho(E)} \leq 1 \,.$$

Now the claim follows from elementary measure theory (see for example [**136**, Theorem 1.40]).

Using (12.5.3), we can rewrite (12.5.4) as

$$\int_{\mathcal{K}} (1-g)\, f \;\mathrm{d}\lambda = \int_{\mathcal{K}} g\, f \,\mathrm{d}\mu \qquad \text{for all non-negative } f \in L^2(\mathcal{K}, \mathrm{d}\rho) \,. \tag{12.5.5}$$

We introduce the Borel sets

$$A = \{x \in \mathcal{K} \,|\, 0 \leq g(x) < 1\} \qquad \text{and} \qquad B = \{x \in \mathcal{K} \,|\, g(x) = 1\} \,.$$

and define the measures λ_a and λ_s by

$$\mathrm{d}\lambda_a = \chi_A \;\mathrm{d}\lambda \qquad \text{and} \qquad \mathrm{d}\lambda_s = \chi_B \;\mathrm{d}\lambda \,.$$

Choosing $f = \chi_B$ in (12.5.5), one sees that $\mu(B) = 0$, implying that $\lambda_s \perp \mu$.

Moreover, since g is bounded, we can evaluate (12.5.5) for

$$f = \big(1 + g + \cdots + g^n\big)\, \chi_E$$

for any $n \in \mathbb{N}$ and any Borel set E. Using the same transformation with "telescopic sums" as in the evaluation of the geometric or Neumann series, we obtain

$$\int_E \big(1 - g^{n+1}\big) \;\mathrm{d}\lambda = \int_E g \,\big(1 + g + \cdots + g^n\big) \,\mathrm{d}\mu \,. \tag{12.5.6}$$

At every point of B, the factor $(1-g^{n+1})$ in the integrand on the left vanishes. At every point of A, on the other hand, the factor $(1-g^{n+1})$ is monotone increasing in n and converges to one. Hence Lebesgue's monotone convergence theorem implies that the left side of (12.5.6) converges to

$$\lim_{n\to\infty} \int_E \big(1 - g^{n+1}\big) \;\mathrm{d}\lambda = \lambda\big(E \cap A\big) \,.$$

The integrand on the right side of (12.5.6), on the other hand, is monotone increasing in n, so that the limit

$$h(x) := \lim_{n\to\infty} g(x) \,\big(1 + g(x) + \cdots + g^n(x)\big) \quad \text{exists in } \mathbb{R}_0^+ \cup \{\infty\} \,.$$

Moreover, the monotone convergence theorem implies that

$$\lim_{n\to\infty} \int_E g \,\big(1 + g + \cdots + g^n\big) \,\mathrm{d}\mu = \int_E h \,\mathrm{d}\mu \;\in\; \mathbb{R}_0^+ \cup \{\infty\} \,.$$

We conclude that, in the limit $n \to \infty$, the relation (12.5.6) yields

$$\lambda_a(E) = \lambda\big(E \cap A\big) = \int_E h \,\mathrm{d}\mu \qquad \text{for any Borel set } E \,.$$

Choosing $E = \mathcal{K}$, one sees that $h \in L^1(\mathcal{K}, \mathrm{d}\mu)$. This concludes the proof of (12.5.1). Finally, the representation (12.5.1) implies that $\lambda_a \ll \mu$. □

12.6. Moment Measures

We now introduce an important concept needed for the existence proof: the moment measures. We again assume that the Hilbert space is finite-dimensional and that the measure ρ is normalized (12.0.1). We consider $\mathcal{F}$ with the metric induced by the sup-norm on $\mathrm{L}(\mathcal{H})$, i.e.

$$d(p,q) = \|p-q\|$$

(and $\|.\|$ as in (5.6.6)). The basic difficulty in applying the abstract theorems is that $\mathcal{F}$ is *not compact* (indeed, it is a star-shaped in the sense that $p \in \mathcal{F}$ implies $\lambda p \in \mathcal{F}$ for all $\lambda \in \mathbb{R}$). If the metric space is non-compact, our abstract results no longer apply, as becomes clear in the following simple example.

EXAMPLE 12.6.1. Consider the Banach space $C_0^0(\mathbb{R},\mathbb{R})$ of compactly supported continuous functions. Let $\rho_n = \delta_n$ be the sequence of Dirac measures supported at $n \in \mathbb{N}$. Then for any $f \in C_0^0(\mathbb{R},\mathbb{R})$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f \,\mathrm{d}\rho_n = \lim_{n\to\infty} f(n) = 0\,.$$

Hence the sequence $(\rho_n)_{n\in\mathbb{R}}$ converges as a measure to zero. Thus the limiting measure is no longer normalized. This shows that Theorem 12.3.2 fails to hold if the base space is non-compact. ◊

The way out is to make use of the fact that the causal action as well as the constraints are formed of functionals which are homogeneous under the scaling $p \to \lambda p$ of degree zero, one or two. This makes it possible to restrict attention to a compact subset of $\mathcal{F}$, and to consider three measures on this compact set. We now give the needed definitions.

DEFINITION 12.6.2. *Let $\mathcal{K}$ be the compact metric space*

$$\mathcal{K} = \{p \in \mathcal{F} \text{ with } \|p\| = 1\} \cup \{0\}\,.$$

For a given measure ρ on $\mathcal{F}$, we define the measurable sets of $\mathcal{K}$ by the requirement that the sets $\mathbb{R}^+\Omega = \{\lambda p \,|\, \lambda \in \mathbb{R}^+, p \in \Omega\}$ and $\mathbb{R}^-\Omega$ should be ρ-measurable in $\mathcal{F}$. We introduce the measures $\mathfrak{m}^{(0)}$, $\mathfrak{m}_\pm^{(1)}$ and $\mathfrak{m}^{(2)}$ by

$$\mathfrak{m}^{(0)}(\Omega) = \frac{1}{2}\,\rho\big(\mathbb{R}^+\Omega \setminus \{0\}\big) + \frac{1}{2}\,\rho\big(\mathbb{R}^-\Omega \setminus \{0\}\big) + \rho\big(\Omega \cap \{0\}\big) \tag{12.6.1}$$

$$\mathfrak{m}_+^{(1)}(\Omega) = \frac{1}{2} \int_{\mathbb{R}^+\Omega} \|p\| \,\mathrm{d}\rho(p) \tag{12.6.2}$$

$$\mathfrak{m}_-^{(1)}(\Omega) = \frac{1}{2} \int_{\mathbb{R}^-\Omega} \|p\| \,\mathrm{d}\rho(p) \tag{12.6.3}$$

$$\mathfrak{m}^{(2)}(\Omega) = \frac{1}{2} \int_{\mathbb{R}^+\Omega} \|p\|^2 \,\mathrm{d}\rho(p) \;+\; \frac{1}{2} \int_{\mathbb{R}^-\Omega} \|p\|^2 \,\mathrm{d}\rho(p)\,. \tag{12.6.4}$$

The measures $\mathfrak{m}^{(l)}$ and $\mathfrak{m}_\pm^{(l)}$ are referred to as the l^{th} **moment measure**.

As a short notation, it is convenient to abbreviate the difference of the first moment measures by

$$\mathfrak{m}^{(1)}(\Omega) := \mathfrak{m}_+^{(1)}(\Omega) - \mathfrak{m}_-^{(1)}(\Omega)\,. \tag{12.6.5}$$

We remark that $\mathfrak{m}^{(1)}$ can be regarded as a *signed measure* (see for example [**101**, §28] or [**136**, Chapter 6]). For simplicity, we here avoid the concept of signed measures by working instead with the two (positive) measures $\mathfrak{m}_\pm^{(1)}$. Nevertheless, we introduce an

$\mathfrak{m}^{(1)}$-integral as a short notation for the difference of the integrals with respect to $\mathfrak{m}_+^{(1)}$ and $\mathfrak{m}_-^{(1)}$, i.e.

$$\int_{\mathcal{K}} h \, \mathrm{d}\mathfrak{m}^{(1)} := \int_{\mathcal{K}} h \, \mathrm{d}\mathfrak{m}_+^{(1)} - \int_{\mathcal{K}} h \, \mathrm{d}\mathfrak{m}_-^{(1)} ,$$

where for simplicity we always assume that h is continuous.

The ρ-integrals of homogeneous functions can be rewritten as integrals over $\mathcal{K}$ using the moment measures, as we now make precise.

DEFINITION 12.6.3. *A function $h \in C^0(\mathcal{F})$ is called* **homogeneous of degree ℓ** *with $\ell \in \{0,1,2\}$ if*

$$h(\nu x) = \nu^\ell \, h(x) \qquad \text{for all } \nu \in \mathbb{R} \text{ and } x \in \mathcal{F} . \tag{12.6.6}$$

LEMMA 12.6.4. *Let $h \in C^0(\mathcal{F})$ be a function which is homogeneous of degree $\ell \in \{0,1,2\}$. Then*

$$\int_{\mathcal{F}} h \, \mathrm{d}\rho = \int_{\mathcal{K}} h \, \mathrm{d}\mathfrak{m}^{(l)} .$$

PROOF. We first note that, using the homogeneity (12.6.6), the function h is uniquely determined by its restriction to $\mathcal{K}$. Moreover, using an approximation argument with Lebesgue's dominated convergence theorem, it suffices to consider a function h which is homogeneous of degree ℓ and *simple* in the sense that its restriction to $\mathcal{K}$ takes a finite number of values, i.e.

$$h\big|_{\mathcal{K}} = \sum_{i=1}^{N} c_i \, \chi_{\Omega_i}$$

with suitable Borel sets $\Omega_1, \dots, \Omega_N \subset \mathcal{K}$ and real coefficients $c_1, \dots, c_N$. For such simple functions, the integrals go over to finite sums, and we obtain

$$\int_{\mathcal{F}} h \, \mathrm{d}\rho = \sum_{i=1}^{N} c_i \int_{\mathbb{R}^+ \Omega_i} \|p\|^\ell \, \mathrm{d}\rho(p) = \sum_{i=1}^{N} c_i \, \mathfrak{m}^{(\ell)}\big(\Omega_i\big) = \int_{\mathcal{K}} h \, \mathrm{d}\mathfrak{m}^{(l)} ,$$

as desired. This concludes the proof. □

Applying this lemma, the normalization $\rho(\mathcal{F}) = 1$ can be expressed in terms of the moment measures as

$$\mathfrak{m}^{(0)}(\mathcal{K}) = 1 , \tag{12.6.7}$$

whereas the action (5.6.2) as well as the functionals in the constraints (5.6.5) and (5.6.4) can be written as

$$\mathcal{S}(\rho) = \iint_{\mathcal{K}\times\mathcal{K}} \mathcal{L}(p,q) \, \mathrm{d}\mathfrak{m}^{(2)}(p) \, \mathrm{d}\mathfrak{m}^{(2)}(q) \tag{12.6.8}$$

$$\mathcal{T}(\rho) = \iint_{\mathcal{K}\times\mathcal{K}} |p\,q|^2 \, \mathrm{d}\mathfrak{m}^{(2)}(p) \, \mathrm{d}\mathfrak{m}^{(2)}(q) \tag{12.6.9}$$

$$\int_{\mathcal{F}} \mathrm{tr}(x) \, \mathrm{d}\rho(x) = \int_{\mathcal{K}} \mathrm{tr}(p) \, \mathrm{d}\mathfrak{m}_+^{(1)}(p) - \int_{\mathcal{K}} \mathrm{tr}(p) \, \mathrm{d}\mathfrak{m}_-^{(1)}(p) . \tag{12.6.10}$$

Working with the moment measures has the advantage that they are measures on the *compact* space $\mathcal{K}$. We also learn that two measures ρ and $\tilde\rho$ whose moment measures coincide yield the same values for the functionals $\mathcal{S}$ and $\mathcal{T}$ as well as for the integral (12.6.10) entering the trace constraint. It is most convenient to work exclusively with the moment measures. At the end, we shall construct a suitable representative ρ of the limiting

moment measures. A key step for making this method work is the following a-priori estimate.

LEMMA 12.6.5. *There is a constant $\varepsilon = \varepsilon(f, n) > 0$ such that for every measure ρ on $\mathcal{F}$ the corresponding moment measures (see Definition 12.6.2) satisfy for all measurable $\Omega \subset \mathcal{K}$ the following inequalities:*

$$\Big(\mathfrak{m}_+^{(1)}(\Omega) + \mathfrak{m}_-^{(1)}(\Omega)\Big)^2 \leq \mathfrak{m}^{(0)}(\Omega)\, \mathfrak{m}^{(2)}(\Omega) \tag{12.6.11}$$

$$\mathfrak{m}^{(2)}(\mathcal{K}) \leq \frac{\sqrt{\mathcal{T}(\rho)}}{\varepsilon}\,. \tag{12.6.12}$$

PROOF. The inequality (12.6.11) follows immediately from Hölder's inequality,

$$\begin{aligned}\big|2\big(\mathfrak{m}_+^{(1)}(\Omega) + \mathfrak{m}_-^{(1)}(\Omega)\big)\big|^2 &\leq \left(\int_{\mathbb{R}\Omega} \|p\|\, \mathrm{d}\rho(p)\right)^2 \\ &\leq \rho(\mathbb{R}\Omega) \int_{\mathbb{R}\Omega} \|p\|^2\, \mathrm{d}\rho(p) \leq 4\mathfrak{m}^{(0)}(\Omega)\, \mathfrak{m}^{(2)}(\Omega)\,.\end{aligned}$$

In order to prove (12.6.12), we introduce the mapping

$$\phi \,:\, \mathcal{K} \times \mathcal{K} \to \mathbb{R} \,:\, (p, q) \mapsto |p\, q|\,.$$

Clearly, ϕ is continuous and

$$\phi(p, p) = |p^2| = \mathrm{Tr}(p^2) \geq \|p\|^2 = 1$$

(here we used that the Hilbert-Schmidt norm is larger than the absolute square of each eigenvalue). Thus every point $r \in \mathcal{K}$ has a neighborhood $U(r) \subset \mathcal{K}$ with

$$\phi(p, q) \geq \frac{1}{2} \qquad \text{for all } p, q \in U(r)\,. \tag{12.6.13}$$

Since $\mathcal{K}$ is compact, there is a finite number of points $r_1, \ldots, r_N$ such that the corresponding sets $U_i := U(r_i)$ cover $\mathcal{K}$. Due to the additivity property of measures, there is an index $i \in \{1, \ldots, N\}$ such that

$$\mathfrak{m}^{(2)}(U_i) \geq \frac{\mathfrak{m}^{(2)}(\mathcal{K})}{N}\,. \tag{12.6.14}$$

We write $\mathcal{T}$ in the form (12.6.9) and apply (12.6.13) as well as (12.6.14) to obtain

$$\mathcal{T}(\rho) \geq \iint_{U_i \times U_i} |p\, q|^2 \;\mathrm{d}\mathfrak{m}^{(2)}(p)\; \mathrm{d}\mathfrak{m}^{(2)}(q) \geq \frac{1}{2}\, \mathfrak{m}^{(2)}(U_i)^2 \geq \frac{\mathfrak{m}^{(2)}(\mathcal{K})^2}{2N^2}\,.$$

Setting $\varepsilon = 1/(\sqrt{2}N)$, the result follows. □

12.7. Existence of Minimizers for the Causal Action Principle

After the above preparations, we can follow the strategy of the direct method in the calculus of variations described at the beginning of Chapter 12 to obtain the following result.

THEOREM 12.7.1. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space and $n \in \mathbb{N}$. Let ρ_k be a minimizing sequence of regular Borel measures on $\mathcal{F}$ satisfying our constraints* (5.6.3), (5.6.4) *and* (5.6.5) *(for fixed and finite constants). Then there is a regular Borel measure ρ which also satisfies the constraints (again with the same constants) and*

$$\mathcal{S}(\rho) = \liminf_{n \to \infty} \mathcal{S}(\rho_n)\,.$$

In short, the method for constructing ρ is to take a limit of the moment measures of the ρ_k and to realize this limit by the measure ρ. In more detail, we proceed as follows. In view of Lemma 12.6.5, we know that the moment measures are uniformly bounded measures on the compact metric space $\mathcal{K}$. Applying the compactness result of Theorem 12.3.2 (based on the Banach-Alaoglu theorem and the Riesz representation theorem), we conclude that that for a suitable subsequence (which we again denote by (ρ_k)), the moment measures converge in the $C^0(\mathcal{K})^*$-topology to regular Borel measures,

$$\mathfrak{m}_k^{(0)} \to \mathfrak{m}^{(0)}\,, \qquad \mathfrak{m}_{k,\pm}^{(1)} \to \mathfrak{m}_\pm^{(1)} \qquad \text{and} \qquad \mathfrak{m}_k^{(2)} \to \mathfrak{m}^{(2)}\,,$$

which again have the properties (12.6.7), (12.6.11) and (12.6.12).

We next form the Radon-Nikodym decompositions of $\mathfrak{m}_\pm^{(1)}$ and $\mathfrak{m}^{(2)}$ with respect to $\mathfrak{m}^{(0)}$. The inequality (12.6.11) shows that every set of $\mathfrak{m}^{(0)}$-measure zero is also a set of measure zero with respect to $\mathfrak{m}_+^{(1)}$ and $\mathfrak{m}_-^{(1)}$. In other words, the measures $\mathfrak{m}_\pm^{(1)}$ are absolutely continuous with respect to $\mathfrak{m}^{(0)}$. Hence, applying Theorem 12.5.2, we obtain the Radon-Nikodym decompositions

$$\mathrm{d}\mathfrak{m}_\pm^{(1)} = f_\pm \, \mathrm{d}\mathfrak{m}^{(0)} \qquad \text{with} \qquad f_\pm \in L^1(\mathcal{K}, \mathrm{d}\mathfrak{m}^{(0)})\,.$$

As a consequence, the difference of measures $\mathfrak{m}^{(1)}$ in (12.6.5) has the decomposition

$$\mathrm{d}\mathfrak{m}^{(1)} = f^{(1)} \, \mathrm{d}\mathfrak{m}^{(0)} \qquad \text{with} \qquad f^{(1)} := f_+ - f_- \in L^1(\mathcal{K}, \mathrm{d}\mathfrak{m}^{(0)})\,. \tag{12.7.1}$$

As we do not know whether also $\mathfrak{m}^{(2)}$ is absolutely continuous with respect to $m^{(0)}$, Theorem 12.5.2 gives the decomposition

$$\mathrm{d}\mathfrak{m}^{(2)} = f^{(2)} \, \mathrm{d}\mathfrak{m}^{(0)} + \mathrm{d}\mathfrak{m}_{\mathrm{sing}}^{(2)} \qquad \text{with} \qquad f^{(2)} \in L^1(\mathcal{K}, \mathrm{d}\mathfrak{m}^{(0)})\,, \tag{12.7.2}$$

where the measure $\mathfrak{m}_{\mathrm{sing}}^{(2)}$ is singular with respect to $\mathfrak{m}^{(0)}$.

LEMMA 12.7.2. *The two functions $f^{(1)}$ and $f^{(2)}$ in the Radon-Nikodym decompositions* (12.7.1) *and* (12.7.2) *can be chosen such that*

$$\bigl|f^{(1)}\bigr|^2 \le f^{(2)}\,.$$

PROOF. Since $\mathfrak{m}_{\mathrm{sing}}^{(2)} \perp \mathfrak{m}^{(0)}$, there is a Borel set V such that

$$\chi_V \, \mathrm{d}\mathfrak{m}^{(0)} = \mathrm{d}\mathfrak{m}^{(0)} \qquad \text{and} \qquad \chi_V \, \mathrm{d}\mathfrak{m}_{\mathrm{sing}}^{(2)} = 0\,.$$

Then, using the Radon-Nikodym decompositions (12.7.1) and (12.7.2) in (12.6.11), we obtain for any Borel set $U \subset V$ the inequality

$$\left| \int_U f^{(1)} \, \mathrm{d}\mathfrak{m}^{(0)} \right|^2 \le \mathfrak{m}^{(0)}(U) \int_U f^{(2)} \, \mathrm{d}\mathfrak{m}^{(0)}\,.$$

If the function $f^{(1)}$ does not change signs on U, we conclude that

$$\inf_U \bigl|f^{(1)}\bigr|^2 \le \sup_U f^{(2)}\,.$$

By decomposing U into the two sets where $f^{(1)}$ is positive and negative, respectively, one readily sees that this inequality even holds for any Borel set $U \subset V$. As a consequence, the inequality $|f^{(1)}|^2 \le f^{(2)}$ holds almost everywhere (with respect to the measure $\mathfrak{m}^{(0)}$), concluding the proof. □

In particular, we conclude that $f^{(1)}$ even lies in $L^2(\mathcal{K}, \mathrm{d}\mathfrak{m}^{(0)})$. Setting $f = f^{(1)}$ and $\mathrm{d}\mathfrak{n} = (f^{(2)} - |f|^2)\, \mathrm{d}\mathfrak{m}^{(0)} + \mathrm{d}\mathfrak{m}^{(2)}_{\text{sing}}$, we obtain the decomposition

$$\mathrm{d}\mathfrak{m}^{(1)} = f\, \mathrm{d}\mathfrak{m}^{(0)}\,, \qquad \mathrm{d}\mathfrak{m}^{(2)} = |f|^2\, \mathrm{d}\mathfrak{m}^{(0)} + \mathrm{d}\mathfrak{n}\,, \tag{12.7.3}$$

where $f \in L^2(\mathcal{K}, \mathrm{d}\mathfrak{m}^{(0)})$, and $\mathfrak{n}$ is a positive measure which need not be absolutely continuous with respect to $\mathfrak{m}^{(0)}$. From the definition (12.6.5) it is clear that f is odd, i.e.

$$f(-p) = -f(p) \quad \text{for all } p \in \mathcal{K}\,.$$

The remaining task is to represent the limiting moment measures $\mathfrak{m}^{(l)}$ in (12.7.3) by a measure ρ. Unfortunately, there is the basic problem that such a measure can exist only if $\mathfrak{m}^{(2)}$ is absolutely continuous with respect to $\mathfrak{m}^{(0)}$, as the following consideration shows: Assume conversely that $\mathfrak{m}^{(2)}$ is not absolutely continuous with respect to $\mathfrak{m}^{(0)}$. Then there is a measurable set $\Omega \subset \mathcal{K}$ with $\mathfrak{m}^{(0)}(\Omega) = 0$ and $\mathfrak{m}^{(2)}(\Omega) \neq 0$. Assume furthermore that there is a measure ρ on $\mathcal{F}$ which represents the limiting moment measures in the sense that (12.6.1)–(12.6.4) hold. From (12.6.1) we conclude that the set $\mathbb{R}\Omega \subset \mathcal{F}$ has ρ-measure zero. But then the integral (12.6.4) also vanishes, a contradiction.

This problem can also be understood in terms of the limiting sequence ρ_k. We cannot exclude that there is a star-shaped region $\mathbb{R}\Omega \subset \mathcal{F}$ such that the measures $\rho_k(\mathbb{R}\Omega)$ tend to zero, but the corresponding moment integrals (12.6.4) have a non-zero limit. Using a notion from the calculus of variations for curvature functionals, we refer to this phenomenon as the possibility of *bubbling*. This bubbling effect is illustrated by the following example.

EXAMPLE 12.7.3. *(Bubbling)* We choose $f = 2$ and $n = 1$. Furthermore, we let $\mathscr{M} = [0, 1)$ with μ the Lebesgue measure. For any parameters $\kappa \geq 0$ and $\varepsilon \in (0, \frac{1}{2})$, we introduce the mapping $F_\varepsilon : \mathscr{M} \to \mathcal{F}$ by

$$F_\varepsilon(x) = \frac{1}{1-2\varepsilon} \times \begin{cases} -\kappa\, \varepsilon^{-\frac{1}{2}}\, \sigma^3 & \text{if } x \leq \varepsilon \\ \mathbb{1} + \sigma^1 \cos(\nu x) + \sigma^2 \sin(\nu x) & \text{if } \varepsilon < x \leq 1-\varepsilon \\ \kappa\, \varepsilon^{-\frac{1}{2}}\, \sigma^3 & \text{if } x > 1-\varepsilon\,, \end{cases}$$

where we set $\nu = 2\pi/(1-2\varepsilon)$ (and σ^j are the Pauli matrices). The corresponding measure ρ_ε on $\mathcal{F}$ has the following properties. On the set

$$S := \{\mathbb{1} + v^1\sigma^1 + v^2\sigma^2 \text{ with } (v^1)^2 + (v^2)^2 = 1\}\,,$$

which can be identified with a circle S^1, ρ_ε is a multiple of the Lebesgue measure. Moreover, ρ_ε is supported at the two points

$$p_\pm := \pm \frac{\kappa\, \varepsilon^{-\frac{1}{2}}}{1-2\varepsilon}\, \sigma^3 \qquad \text{with} \qquad \rho_\varepsilon(\{p_+\}) = \rho_\varepsilon(\{p_-\}) = \varepsilon\,. \tag{12.7.4}$$

A short calculation shows that the trace constraint is satisfied. Furthermore, the separations of the points p_+ and p_- from each other and from S are either spacelike or just in the boundary case between spacelike and timelike. Thus for computing the action, we only need to take into account the pairs (p_+, p_+), (p_-, p_-) as well as pairs (x, y) with $x, y \in S$. A straightforward computation yields

$$\mathcal{S}(\rho_\varepsilon) = \frac{3}{(1-2\varepsilon)^2}\,, \qquad \mathcal{T}(\rho_\varepsilon) = \frac{6}{(1-2\varepsilon)^2} + \frac{16\kappa^2}{(1-2\varepsilon)^3} + \frac{16\kappa^4}{(1-2\varepsilon)^4}\,. \tag{12.7.5}$$

Let us consider the limit $\varepsilon \searrow 0$. From (12.7.5) we see that the functionals $\mathcal{S}$ and $\mathcal{T}$ converge,

$$\lim_{\varepsilon \searrow 0} \mathcal{S} = 3\,, \qquad \lim_{\varepsilon \searrow 0} \mathcal{T} = 6 + 16\,(\kappa^2 + \kappa^4)\,. \tag{12.7.6}$$

Moreover, there are clearly no convergence problems on the set S. Thus it remains to consider the situation at the two points $p_\pm$, (12.7.4), which move to infinity as ε tends to zero. These two points enter the moment measures only at the corresponding normalized points $\hat{p}_\pm = p_\pm/\|p_\pm\| \in \mathcal{K}$. A short calculation shows that the limiting moment measures $\mathfrak{m}^{(l)} = \lim_{\varepsilon \searrow 0} \mathfrak{m}_\varepsilon^{(l)}$ satisfy the relations

$$\mathfrak{m}^{(0)}(\{\hat{p}_\pm\}) = \mathfrak{m}^{(1)}(\{\hat{p}_\pm\}) = 0 \qquad \text{but} \qquad \mathfrak{m}^{(2)}(\{\hat{p}_\pm\}) = \kappa^2 > 0\,.$$

Hence $\mathfrak{m}^{(2)}$ is indeed *not* absolutely continuous with respect to $\mathfrak{m}^{(0)}$.

In order to avoid misunderstandings, we point out that this example does *not* show that bubbling really occurs for minimizing sequences, because we do not know whether the family $(\rho_\varepsilon)_{0<\varepsilon<1/2}$ is minimizing. But at least, our example shows that bubbling makes it possible to arrange arbitrary large values of $\mathcal{T}$ without increasing the action $\mathcal{S}$ (see (12.7.6) for large κ). ◊

In order to handle possible bubbling phenomena, it is important to observe that the second moment measure does not enter the trace constraint. Therefore, by taking out the term $d\mathfrak{n}$ in (12.7.3) we *decrease* the functionals $\mathcal{S}$ and $\mathcal{T}$ (see (12.6.8) and (12.6.9)), without affecting the trace constraint. It remains to show that the resulting moment measure can indeed be realized by a measure ρ. This is proven in the next lemma.

LEMMA 12.7.4. *For any normalized regular Borel measure $\mathfrak{m}^{(0)}$ on $\mathcal{K}$ and any function $f \in L^2(\mathcal{K}, \mathbb{R})$, there is a normalized regular Borel measure $\tilde{\rho}$ whose moment measures $\tilde{\mathfrak{m}}^{(l)}$ are given by*

$$\tilde{\mathfrak{m}}^{(0)} = \mathfrak{m}^{(0)}\,, \qquad \mathrm{d}\tilde{\mathfrak{m}}^{(1)} = f\; \mathrm{d}\mathfrak{m}^{(0)}\,, \qquad \mathrm{d}\tilde{\mathfrak{m}}^{(2)} = |f|^2\; \mathrm{d}\mathfrak{m}^{(0)}\,. \tag{12.7.7}$$

PROOF. We introduce the mapping

$$F \;:\; \mathcal{K} \to \mathcal{F}\,, \qquad F(x) = f(x)\, x\,.$$

Choosing $\tilde{\rho} := F_* \mathfrak{m}^{(0)}$, a direct computation shows that the corresponding moment measures indeed satisfy (12.7.7). □

This concludes the proof of Theorem 12.7.1. We finally remark that the fact that we dropped the measure $d\mathfrak{n}$ in (12.7.3) implies that the value of $\mathcal{T}$ might decrease in the limit. This is the only reason why the boundedness constraint (5.6.5) is formulated as an inequality, rather than an equality. It is not clear if the causal action principle also admits minimizers if the inequality in (5.6.5) is replaced by an equality. We conjecture that the answer is yes. But at present, there is no proof. We note that, for the physical applications, it makes no difference if (5.6.5) is an equality or an inequality, because in this case one works with the corresponding Euler-Lagrange equations, where the constraints enter only via Lagrange multiplier terms (for details see [**13**]).

12.8. Existence of Minimizers for Causal Variational Principles in the Non-Compact Setting

In Theorem 12.7.1 the existence of minimizers was established in the case that the Hilbert space $\mathcal{H}$ is finite-dimensional and the total volume $\rho(\mathcal{F})$ of spacetime is finite. From the physical point of view, this existence result is quite satisfying, because one

can take the point of view that it should be possible to describe our universe at least approximately by a causal fermion system with $\dim \mathcal{H} < \infty$ and $\rho(\mathcal{F}) < \infty$. From the mathematical point of view, however, it is interesting and important to also study the cases of an infinite-dimensional Hilbert space and/or infinite total volume. The case $\dim \mathcal{H} < \infty$ and $\rho(\mathcal{F}) = \infty$ is not sensible (see Exercise 12.5). In the case $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) < \infty$, on the other hand, there are minimizing sequences which converge to zero. Therefore, it remains to study the *infinite-dimensional setting* $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) = \infty$ already mentioned in Section 5.6. In this setting, the existence theory is difficult and has not yet been developed. Therefore, our strategy is to approach the problem in two steps. The first step is to deal with infinite total volume; this has been carried out in [**75**]. The second step, which involves the difficulty of dealing with non-locally compact spaces, is currently under investigation (for first results see [**114**]).

We now outline the basic strategy in the simplest possible case (more details and a more general treatment can be found in [**75**]). We consider *causal variational principles in the non-compact setting* as introduced in Section 6.3. Moreover, we consider the *smooth setting* by assuming that the Lagrangian is smooth,

$$\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)\,,$$

and has the properties (i) and (ii) on page 124. Moreover, we again assume that the Lagrangian has *compact range* (see Definition 8.1.1). The goal of this section is to prove the following theorem.

THEOREM 12.8.1. *Under the above assumptions, there is a regular Borel measure ρ on $\mathcal{F}$ (not necessarily bounded) which satisfies the EL equations*

$$\ell\big|_{\text{supp}\,\rho} \equiv \inf_M \ell = 0 \qquad \textit{with} \qquad \ell(x) := \int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - 1\,. \tag{12.8.1}$$

For the proof, we first exhaust $\mathcal{F}$ by compact sets $(K_j)_{j\in\mathbb{N}}$, i..e

$$K_1 \subset K_2 \subset \cdots \subset \mathcal{F} \qquad \text{and} \qquad \bigcup_{j=1}^\infty K_j = \mathcal{F}\,.$$

On each K_j we consider the restricted variational principle where we minimize the action

$$\mathcal{S}_{K_j}(\rho) = \int_{K_j} \mathrm{d}\rho(x) \int_{K_j} \mathrm{d}\rho(y)\, \mathcal{L}(x,y)$$

under variations of ρ within the class of normalized regular Borel measure on K_j. Using the existence theory in the compact setting (see Theorem 12.3.3), each of these restricted variational principles has a minimizer, which we denote by ρ_j. Each of these measures satisfies the EL equations stated in Theorem 7.1.1. Thus, introducing the functions

$$\ell_j \in C^0(K_j, \mathbb{R})\,, \qquad \ell(x) := \int_{K_j} \mathcal{L}(x,y)\, \mathrm{d}\rho_j(y) - \mathfrak{s}_j\,,$$

one can choose the parameters $s_j > 0$ such that

$$\ell_j\big|_{\text{supp}\,\rho_j} \equiv \inf_{K_j} \ell_j = 0\,.$$

Typically, the support of the measures ρ_j will be "spread out" over larger and larger subsets of $\mathcal{F}$. This also means that, working with normalized measures, the measures ρ_j

typically converge to the trivial measure $\rho = 0$. In order to ensure a non-trivial measure, we must perform a suitable *rescaling*. To this end, we introduce the measures

$$\tilde{\rho}_j = \frac{\rho_j}{\mathfrak{s}_j} .$$

These new measures are no longer normalized, but they satisfy the EL equations with $\tilde{\mathfrak{s}}_j = 1$, i.e.

$$\tilde{\ell}_j\big|_{\operatorname{supp}\tilde{\rho}_j} \equiv \inf_{K_j} \tilde{\ell}_j = 0 \qquad \text{with} \qquad \tilde{\ell}_j(x) := \int_{K_j} \mathcal{L}(x,y)\, d\tilde{\rho}_j(y) - 1 . \tag{12.8.2}$$

Our next task is to construct a limit measure ρ of the measures $\tilde{\rho}_j$. We first extend the measures $\tilde{\rho}_j$ by zero to all of $\mathcal{F}$ and denote them by $\rho^{[j]}$,

$$\rho^{[j]}(U) := \rho_j(U \cap K_j) \qquad \text{for any Borel subset } U \subset \mathcal{F} .$$

In the next lemma we show that these measures are bounded on every compact set.

LEMMA 12.8.2. *For every compact subset $K \subset \mathcal{F}$ there is a constant $C_K > 0$ such that*

$$\rho^{[j]}(K) \leq C_K \qquad \textit{for all } j \in \mathbb{N} . \tag{12.8.3}$$

PROOF. Since $\mathcal{L}(x,.)$ is continuous and strictly positive at x, there is an open neighborhood $U(x)$ of x with

$$\mathcal{L}(y,z) \geq \frac{\mathcal{L}(x,x)}{2} > 0 \qquad \text{for all } y,z \in U(x) .$$

Covering K by a finite number of such neighborhoods $U(x_1), \ldots, U(x_L)$, it suffices to show the inequality (12.8.3) for the sets $K \cap U(x_\ell)$ for any $\ell \in \{1, \ldots, L\}$. Moreover, we choose N so large that $K_N \supset K$ and fix $k \geq N$. If $K \cap \operatorname{supp}\rho^{[k]} = \varnothing$, there is nothing to prove. Otherwise, there is a point $z \in K \cap \operatorname{supp}\rho^{[k]}$. Using the EL equations (12.8.2) at z, it follows that

$$1 = \int_{\mathcal{F}} \mathcal{L}(z,y)\, \mathrm{d}\rho^{[k]}(y) \geq \int_{U(x_\ell)} \mathcal{L}(z,y)\, \mathrm{d}\rho^{[k]}(y) \geq \frac{\mathcal{L}(x_\ell,x_\ell)}{2}\, \rho^{[k]}(U(x_\ell)) .$$

Hence

$$\rho^{[k]}(U(x_\ell)) \leq \frac{2}{\mathcal{L}(x_\ell,x_\ell)} .$$

This inequality holds for any $k \geq N$. We introduce the constants $c(x_\ell)$ as the maximum of $2/\mathcal{L}(x_\ell,x_\ell)$ and $\rho^{[1]}(U(x_\ell)), \ldots, \rho^{[N-1]}(U(x_\ell))$. Since the open sets $U(x_1), \ldots, U(x_L)$ cover K, we finally introduce the constant C_K as the sum of the constants $c(x_1), \ldots, c(x_L)$. □

Given a compact set K, combining the result of the previous lemma with the compactness of measures on compact topological spaces (see Theorem 12.3.2), we conclude that there is a subsequence $(\rho^{[j_n]})$ whose restrictions to K converge as a measure (i.e. in the sense (12.3.1)). Proceeding inductively for the compact sets $K_1, K_2, \ldots$ and choosing a diagonal sequence, one gets a subsequence of measures on $\mathcal{F}$, denoted by $\rho^{(k)}$, whose restriction to any compact set K_j converges, i.e.

$$\rho^{(k)}\big|_{K_j} \text{ converges as } k \to \infty \text{ to } \rho|_{K_j} \text{ for all } j \in \mathbb{N} , \tag{12.8.4}$$

where ρ is a regular Borel measure on $\mathcal{F}$ (typically of infinite total volume). The convergence of measures in (12.8.4) is referred to as *vague convergence* (for more details see [**8**, Definition 30.1] or [**75**, Section 4.1]).

It remains to show that the obtained measure ρ is a non-trivial minimizer. In order to show that it is non-trivial, we make use of the EL equations (12.8.2). Let $x \in \mathcal{F}$. Then (12.8.2) implies that

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho^{(k)}(y) \geq 1\,.$$

Since $\mathcal{L}$ has compact range, we may pass to the limit to conclude that

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) \geq 1\,. \tag{12.8.5}$$

This shows (in a quantitative way) that the measure ρ is non-zero.

Our final step for proving the EL equations (12.8.1) is to show that the EL equations (12.8.2) are preserved in the limit. In view of the lower bound (12.8.5), it remains to show that ℓ vanishes on the support of ρ. Thus let $x \in \operatorname{supp}\rho$. We choose a compact subset $K \subset \mathcal{F}$ such that x lies in its interior. Again using that the Lagrangian has compact range, there is another compact subset $K' \subset \mathcal{F}$ such that (8.1.1) holds. The fact that x lies in the support and that the measures $\rho^{(k)}$ converge vaguely to ρ implies that there is a sequence $x_k \in \operatorname{supp}\rho^{(k)}$ which converges to x. The EL equations for each $\rho^{(k)}$ imply that, for sufficiently large k,

$$\int_{K'} \mathcal{L}(x_k,y)\, \mathrm{d}\rho^{(k)}(y) = 1\,.$$

Taking the limit is a bit subtle because both the argument x_k of the Lagrangian and the integration measure depend on k. Therefore, we begin with the estimate

$$\begin{aligned}\left| \int_{K'} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \int_{K'} \mathcal{L}(x_k,y)\, \mathrm{d}\rho^{(k)}(y) \right| \\ \leq \left|\ell(x) - \ell^{(k)}(x)\right| - \sup_j \left|\ell^{(j)}(x) - \ell^{(j)}(x_k)\right|,\end{aligned} \tag{12.8.6}$$

where we set

$$\ell^{(k)}(z) := \int_{K'} \mathcal{L}(z,y)\, \mathrm{d}\rho^{(k)}(y) - 1\,.$$

The first summand on the right side of (12.8.6) tends to zero because the measures $\rho^{(k)}$ converge vaguely to ρ. The second summand, on the other hand, tends to zero because the functions $\ell^{(j)}$ are equicontinuous (for more details on this argument see [**75**, Section 4.2]). This concludes the proof of Theorem 12.8.1.

We finally note that, starting from the EL equations (12.8.1), one can also show that ρ is a *minimizer under variations of finite volume*, meaning that for every regular Borel measure $\tilde{\rho}$ satisfying (6.3.1), the difference of actions (6.3.2) is nonnegative (6.3.3). The proof can be found in [**75**, Section 4.3].

12.9. Tangent Cones and Tangent Cone Measures

In the previous sections, measure-theoretic methods have been used in order to prove existence of minimizers. But methods of measure theory are also useful for analyzing the structure of the minimizing measure. Since these methods might be important for the future development of the theory, we now briefly explain the concept of a tangent cone measure (more details and applications can be found in [**60**, Section 6]). We have

the situation in mind that spacetime M does not have a smooth manifold structure, so that the powerful methods of differential topology and geometry (like the tangent space, the exponential map, etc.) cannot be used in spacetime. Nevertheless, the structure of spacetime can be analyzed locally as follows. Let $x \in M$ be a spacetime point. We want to analyze a neighborhood of x in M. To this end, it is useful to consider a continuous mapping $\mathcal{A}$ from M to the symmetric operators on the spin space at x. We always assume that this mapping vanishes at x, i.e.

$$\mathcal{A} : M \to \text{Symm}(S_x) \qquad \text{with} \qquad \mathcal{A}(x) = 0 \,.$$

There are different possible choices for $\mathcal{A}$. The simplest choice is

$$\mathcal{A} \;:\; y \mapsto \pi_x \,(y-x)\, x|_{S_x} \,.$$

Here the factor x on the right is needed for the operator to be symmetric, because

$$\begin{aligned} \prec\!\psi|\mathcal{A}\phi\!\succ_x &\overset{(5.7.8)}{=} -\langle \psi \,|\, x\,(\pi_x\,(y-x)\,x)\,\phi\rangle_{\mathcal{H}} = -\langle \psi \,|\, x\;(y-x)\,x\,\phi\rangle_{\mathcal{H}} \\ &= -\langle \pi_x\,(y-x)\,x\,\psi \,|\, x\,\phi\rangle_{\mathcal{H}} = \prec\!\mathcal{A}\psi|\phi\!\succ_x \,. \end{aligned}$$

Alternatively, one can consider mappings involving the operators s_y or π_y, like for example

$$\mathcal{A} \;:\; y \mapsto \pi_x\,(s_y - s_x)\, x|_{S_x}$$
$$\mathcal{A} \;:\; y \mapsto \pi_x\,(\pi_y - \pi_x)\, x|_{S_x}$$

(where π_x again denotes the orthogonal projection in $\mathcal{H}$ on S_x). But, of course, many other choices of $\mathcal{A}$ are possible. The detailed choice of $\mathcal{A}$ depends on the application in mind.

A *conical set* is a set of the form $\mathbb{R}^+ A$ with $A \subset \text{Symm}(S_x)$. We denote the conical sets whose pre-images under $\mathcal{A}$ are both ρ–measurable by $\mathfrak{M}$. For a conical set $A \subset \text{Symm}(S_x)$ we consider countable coverings by measurable conical sets,

$$A \subset \bigcup_{k=1}^{\infty} A_k \qquad \text{with} \qquad A_k \in \mathfrak{M} \,.$$

We denote the set of such coverings by $\mathcal{P}$. We define

$$\mu^*_{\text{con}}(A) = \inf_{\mathcal{P}} \sum_{k=1}^{\infty} \liminf_{\delta \searrow 0} \frac{1}{\rho\big(B_\delta(x)\big)}\, \rho\Big(\mathcal{A}^{-1}\big(A_k\big) \cap B_\delta(x)\Big)$$

(where $B_\delta(x) \subset \text{L}(\mathcal{H})$ is the Banach space ball). We remark for clarity that, since $x \in M := \text{supp}\,\rho$, it follows that the measure $\rho(B_\delta(x))$ is non-zero for all $\delta > 0$.

The mapping μ^*_{con} defines an outer measure on the conical sets in $\text{Symm}(S_x)$. By applying the Carathéodory extension lemma (see for example [**8, 15**]), one can construct a corresponding measure denoted by μ_{con}. By restriction one obtains a Borel measure (for details see [**60**, Section 6.1]).

DEFINITION 12.9.1. *We denote the conical Borel sets of* $\text{Symm}(S_x)$ *by* $\mathfrak{B}_{\text{con}}(x)$. *We denote the measure obtained by applying the above construction by*

$$\mu_x \;:\; \mathfrak{B}_{\text{con}}(x) \to [0, \infty] \,.$$

It is referred to as the **tangent cone measure** *corresponding to* $\mathcal{A}$. *The* **tangent cone** $\mathcal{C}_x$ *is defined as the support of the tangent cone measure,*

$$\mathcal{C}_x := \text{supp}\,\mu_x \subset \text{Symm}(S_x) \,.$$

In simple terms, the tangent cone $\mathcal{C}_x$ distinguishes directions in which the measure ρ is non-zero. The tangent cone measure, on the other hand, is a measure supported on the tangent cone. By integrating functionals on conical subsets of $\mathrm{Symm}(S_x)$ with respect to this measure, one can can get fine information on the structure of the measure ρ in different directions. For example, one can set up a variational principle by maximizing a suitable integral of this type under variations of the Clifford section at x. As a concrete application, this method is used in [**60**, Section 6.2] in order to choose a distinguished Clifford section at x.

12.10. Exercises

EXERCISE 12.1. Let Λ be the functional

$$\Lambda \,:\, C^0\big([0,1],\mathbb{R}\big) \to \mathbb{R}\,, \quad \Lambda(f) = \sup_{x\in[0,1]} f(x)\,.$$

Can this functional be represented by a measure? Analyze how your findings are compatible with the Riesz representation theorem.

EXERCISE 12.2. Let ρ be the Borel measure on $[0,\pi]$ given by

$$\rho(\Omega) = \int_\Omega \sin x \;\mathrm{d}x \;+\; \sum_{n=1}^\infty \frac{1}{n^2}\, \chi_\Omega\Big(\frac{1}{n}\Big)\,.$$

Compute the Lebesgue decomposition of ρ with respect to the Lebesgue measure.

EXERCISE 12.3. *(Normalized regular Borel measures: compactness results)*

(a) Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of normalized regular Borel measures on $\mathbb{R}$ with the property that there is a constant $c>0$ such that

$$\int_{-\infty}^\infty x^2 \,\mathrm{d}\rho_n(x) \leq c \qquad \text{for all } n\,.$$

Show that a subsequence converges again to a normalized Borel measure on $\mathbb{R}$.
Hint: Apply the compactness result in Theorem 12.3.2 to the measures restricted to the interval $[-L,L]$ and analyze the behavior as $L\to\infty$.

(b) More generally, assume that for a given non-negative function $f(x)$,

$$\int_{-\infty}^\infty f(x) \,\mathrm{d}\rho_n(x) \leq c \qquad \text{for all } n\,.$$

Which condition on f ensures that the a subsequence of the measures converges to a normalized Borel measure? Justify your answer by a counter example.

EXERCISE 12.4. Let $\mathscr{M}\subset\mathbb{R}$ be a closed embedded submanifold of $\mathbb{R}^3$. We choose a compact set $K\subset\mathbb{R}^3$ which contains $\mathscr{M}$. On $C^0(K,\mathbb{R})$ we introduce the functional

$$\Lambda \,:\, C^0(K,\mathbb{R}) \to \mathbb{R}\,, \quad \Lambda(f) = \int_{\mathscr{M}} f(x)\,\mathrm{d}\mu_{\mathscr{M}}(x)$$

(where $\mathrm{d}\mu_{\mathscr{M}}$ is the volume measure corresponding to the induced Riemannian metric on $\mathscr{M}$). Show that this functional is linear, bounded and positive. Apply the Riesz representation theorem to represent this functional by a Borel measure on K. What is the support of this measure?

EXERCISE 12.5. This exercise explains why the causal action principle is ill-posed in the case $\dim\mathcal{H}=\infty$ and $\rho(\mathcal{F})<\infty$. The underlying estimates were first given in the setting of discrete spacetimes in [**42**, Lemma 5.1].

(a) Let $\mathcal{H}_0$ be a finite-dimensional Hilbert space of dimension n and $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$ be a causal fermion system of finite total volume $\rho_0(\mathcal{F}_0)$. Let $\iota : \mathcal{H}_0 \to \mathcal{H}$ be an isometric embedding. Construct a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ which has the same action, the same total volume and the same values for the trace and boundedness constraints as the causal fermion system $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$.

(b) Let $\mathcal{H}_1 = \mathcal{H}_0 \oplus \mathcal{H}_0$. Construct a causal fermion system $(\mathcal{H}_1, \mathcal{F}_1, \rho_1)$ which has the same total volume and the same value of the trace constraint as $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$ but a smaller action and a smaller value of the boundedness constraint. *Hint:* Let $F_{1/2} : \mathrm{L}(\mathcal{H}_0) \to \mathrm{L}(\mathcal{H}_1)$ be the linear mappings

$$\big(F_1(A)\big)(u \oplus v) = (Au) \oplus 0\,, \qquad \big(F_2(A)\big)(u \oplus v) = 0 \oplus (Av)\,.$$

Show that $F_{1/2}$ map $\mathcal{F}_0$ to $\mathcal{F}_1$. Define ρ_1 by

$$\rho_1 = \frac{1}{2}\Big((F_1)_* \rho + (F_2)_* \rho\Big)\,.$$

(c) Iterate the construction in (b) and apply (a) to obtain a series of measures on $\mathcal{F}$ of fixed total volume and with fixed value of the trace constraint, for which the action and the values of the boundedness constraint tend to zero. Do these measures converge? If yes, what is the limit?

EXERCISE 12.6. *(Riesz representation theorem - part 1)* Let Λ be the functional

$$\Lambda : C^0([0,1], \mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) := \sup_{x \in [0,1]} f(x).$$

Can this functional be represented by a measure? Analyze how your findings are compatible with the Riesz representation theorem.

EXERCISE 12.7. *(Riesz representation theorem - part 2)* Let $\mathbb{M}$ be a closed embedded submanifold of $\mathbb{R}^3$. We choose a compact set $K \subset \mathbb{R}^3$ which contain $\mathbb{M}$. On $C^0(K, \mathbb{R})$ we introduce the functional

$$\Lambda : C^0(K, \mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) = \int_{\mathbb{M}} f(x)\, \mathrm{d}\mu_{\mathbb{M}}(x),$$

where $\mathrm{d}\mu_{\mathbb{M}}$ is the volume measure corresponding to the induced Riemannian metric on $\mathbb{M}$. Show that this functional is linear, bounded and positive. Apply the Riesz representation theorem to represent this functional by a Borel measure on K. What is the support of this measure?

EXERCISE 12.8. *(Radon-Nikodym decomposition)* Let ρ be the Borel measure on $[0, \pi]$ given by

$$\rho(\Omega) := \int_\Omega \sin x \;\mathrm{d}x + \sum_{n=1}^{\infty} \frac{1}{n^2}\, \chi_\Omega\Big(\frac{1}{n}\Big)\,.$$

Compute the Radon-Nikodym decomposition of ρ with respect to the Lebesgue measure.

EXERCISE 12.9. *(Derivative of measures)* Let μ be the counting measure on the σ-algebra $\mathcal{P}(\mathbb{N})$. Consider the measure

$$\lambda(\varnothing) := 0, \quad \lambda(E) := \sum_{n \in E} (1+n)^2, \quad E \in \mathcal{P}(\mathbb{N}).$$

Show that μ and λ are equivalent (one is absolutely continuous with respect to the other) and determine the Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{d\lambda}$.

EXERCISE 12.10. *(Minimizers)* Let M denote the 2-sphere $S^2 \subset \mathbb{R}^3$ and let $\mathrm{d}\mu_M$ be the normalized canonical surface measure. Consider a Lagrangian on $M \times M$ defined by

$$\mathcal{L}(x, y) := \frac{1}{1 + \|x - y\|_{\mathbb{R}^3}} \quad \text{for all } x, y \in M.$$

Show that the action $\mathcal{S}(\mu_M)$ is minimal under variations of the form

$$\mathrm{d}\rho_{x_0,t} := (1 - t)\mathrm{d}\mu_M + t\ \mathrm{d}\delta_{x_0}, \quad \text{with } t \in [0, 1),$$

where δ_{x_0} is the Dirac measure centered at $x_0 \in M$.

EXERCISE 12.11. *(Moment measures)* Let $\mathcal{F} = \mathbb{R}^2$ and $K = S^1 \cup \{0\}$ be a compact subset of $\mathcal{F}$. Moreover, let ρ be a Borel measure ρ on $\mathcal{F}$. Compute the moment measures $\mathfrak{m}^{(0)}$, $\mathfrak{m}^{(1)}$ and $\mathfrak{m}^{(2)}$ for the following choices of ρ:

(a) $\rho = F_*(\mu_{S^1})$, where $F : S^1 \hookrightarrow \mathbb{R}^2$ is the natural injection and μ_{S^1} is the normalized Lebesgue measure on S^1.

(b) $\rho = \delta_{(0,0)} + \delta_{(1,1)} + \delta_{(5,0)}$ (where $\delta_{(x,y)}$ denote the Dirac measure supported at $(x, y) \in \mathbb{R}^2$).

(c) $\rho = F_*(\mu_{\mathbb{R}})$, where $\mu_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$ and

$$F : \mathbb{R} \to \mathbb{R}^2, \quad F(x) = (x, 2).$$

CHAPTER 13

Methods of Hyperbolic Partial Differential Equations

The structures of a causal fermion system are all encoded in the family of physical wave functions (see Section 5.7). Consequently, the dynamics of the causal fermion is understood once we know how each physical wave function propagates in time. In many examples, the physical wave functions satisfy the *Dirac equation* (for the simplest example of this type see Section 5.5). More abstractly, the dynamics of the physical wave functions is described by the *dynamical wave equation* (9.4.7). Moreover, we also encountered the *linearized field equations* (see Definition 8.1.2). We now turn attention to methods for solving these equations. In this chapter, we begin with linear partial differential equations like the Dirac equation. Causality is reflected in these equations in the fact that they are *hyperbolic*. As we shall see, the methods developed here will also be fruitful for the study of the linearized field equations, as will be explained in the next chapter (Chapter 14). With this in mind, the constructions here can be regarded as a technical preparation for Chapter 14. We remark that the adaptation of the methods to the dynamical wave equation will not be covered in this book; we refer the interested reader instead to [**64**].

13.1. The Cauchy Problem, Linear Symmetric Hyperbolic Systems

In this section, we shall prove that the Cauchy problem for the Dirac equation in the presence of an external potential has a unique global solution. Moreover, we will show that the finite speed of propagation as postulated by special relativity is indeed respected by the solutions of the Dirac equation. For later purposes, it is preferable to include an inhomogeneity. Thus we consider the Cauchy problem in Minkowski space

$$(\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m)\,\psi = \phi \in C^\infty(\mathscr{M}, S\mathscr{M})\,, \qquad \psi|_{t_0} = \psi_0 \in C^\infty(\mathbb{R}^3, S\mathscr{M}) \tag{13.1.1}$$

for a given inhomogeneity ϕ and initial data ψ_0. In order to make the standard methods available, we multiply the equation by $-i\gamma^0$,

$$\mathbb{1}_{\mathbb{C}^4}\,\partial_t\psi + \gamma^0\vec{\gamma}\vec{\nabla}\psi - \mathrm{i}\gamma^0(\mathcal{B} - m)\psi = -\mathrm{i}\gamma^0\phi\,. \tag{13.1.2}$$

Now the matrices in front of the derivatives are all Hermitian (with respect to the standard scalar product on $\mathbb{C}^4$). Moreover, the matrix in front of the time derivative is positive definite. Kurt Otto Friedrichs [**91**] observed that these properties are precisely what is needed in order to get a well-posed Cauchy problem. He combined these properties in the notion of a *symmetric hyperbolic system*. We now give its general definition. More specifically, we consider a system of N complex-valued equations with spatial coordinates $\vec{x} \in \mathbb{R}^m$ and time t in an interval $[0, T]$ with $T > 0$. The initial data will always be prescribed at time $t = 0$. For notational clarity, we denote partial derivatives in spatial directions by ∇.

DEFINITION 13.1.1. *A linear system of differential equations of the form*

$$A^0(t,\vec{x})\,\partial_t u(t,\vec{x}) + \sum_{\alpha=1}^{m} A^\alpha(t,\vec{x})\,\nabla_\alpha u(t,\vec{x}) + B(t,\vec{x})\,u(t,\vec{x}) = w(t,\vec{x}) \tag{13.1.3}$$

with

$$A^0, A^\alpha, B \in C^\infty\big([0,T]\times\mathbb{R}^m, \mathrm{L}(\mathbb{C}^N)\big)\,, \qquad w \in C^\infty\big([0,T]\times\mathbb{R}^m, \mathbb{C}^N\big)\,.$$

is called **symmetric hyperbolic** *if*

(i) *The matrices A^0 and A^α are Hermitian,*

$$(A^0)^\dagger = A^0 \qquad \text{and} \qquad (A^\alpha)^\dagger = A^\alpha$$

(where $\dagger$ denotes the adjoint with respect to the canonical scalar product on $\mathbb{C}^N$).

(ii) *The matrix A^0 is uniformly positive definite, i.e. there is a positive constant C such that*

$$A^0(t,\vec{x}) > C\,\mathbb{1}_{\mathbb{C}^N} \quad \text{for all } (t,\vec{x}) \in ([0,T]\times\mathbb{R}^m)\,.$$

In the case $w \equiv 0$, the linear system is called **homogeneous**.

A good reference for linear symmetric hyperbolic systems is the book by Fritz John [**107**, Section 5.3] (who was Friedrichs' colleague at the Courant Institute). Our presentation was also influenced by [**133**, Chapter 8]. We remark that the concept of a symmetric hyperbolic system can be extended to nonlinear equations of the form

$$A^0(t,\vec{x},u)\,\partial_t u(t,\vec{x}) + \sum_{\alpha=1}^{m} A^\alpha(t,\vec{x},u)\,\nabla_\alpha u(t,\vec{x}) + B(t,\vec{x},u) = 0\,,$$

where the matrices A^0 and A^α should again satisfy the above conditions (i) and (ii). For details we refer to [**144**, Section 16] or [**135**, Section 7]. Having the Dirac equation in mind, we always restrict attention to linear systems. We also note that an alternative method for proving existence of fundamental solutions is to work with the so-called Riesz distributions (for a good textbook see [**6**]). Yet another method is to work with estimates in the interaction picture [**25**]. For completeness, we finally note that the concept of symmetric hyperbolic systems was extended by Friedrichs to so-called symmetric *positive* systems [**92**].

It is a remarkable fact that all partial differential equations in relativistic physics as well as most wave-type equations can be rewritten as a symmetric hyperbolic system. As an illustration, we now explain this reformulation in the example of a scalar hyperbolic equation.

EXAMPLE 13.1.2. Consider a scalar hyperbolic equation of the form

$$\partial_{tt}\phi(t,\vec{x}) = \sum_{\alpha,\beta=1}^{m} a_{\alpha\beta}(t,\vec{x})\,\nabla_{\alpha\beta}\phi + \sum_{\alpha=1}^{m} b_\alpha(t,\vec{x})\,\nabla_\alpha\phi + c(t,\vec{x})\,\partial_t\phi + d(t,\vec{x})\,\phi \tag{13.1.4}$$

with $(a_{\alpha\beta})$ a symmetric, uniformly positive matrix (in the case $a_{\alpha\beta} = \delta_{\alpha\beta}$ and $b, c, d = 0$, one gets the scalar wave equation). Now the initial data prescribes ϕ and its first time derivatives,

$$\phi|_{t=0} = \phi_0 \in C^\infty(M)\,, \qquad \partial_t\phi|_{t=0} = \phi_1 \in C^\infty(M)\,. \tag{13.1.5}$$

In order to rewrite the equation as a symmetric hyperbolic system, we introduce the vector u with $k := m+2$ components by

$$u_1 = \nabla_1\phi,\ \ldots,\ u_m = \nabla_m\phi, \quad u_{m+1} = \partial_t\phi, \quad u_{m+2} = \phi\,. \tag{13.1.6}$$

Then the system

$$\left\{\begin{array}{cccc} \displaystyle\sum_{\beta=1}^{m} a_{\alpha\beta}\,\partial_t u_\beta & \displaystyle -\sum_{\beta=1}^{m} a_{\alpha\beta}\,\nabla_\beta u_{m+1} & & = 0 \\ \displaystyle -\sum_{\alpha,\beta=1}^{m} a_{\alpha\beta}\nabla_\beta u_\alpha - \sum_{\alpha=1}^{m} b_\alpha\, u_\alpha & +\partial_t u_{m+1} - c\, u_{m+1} & -d\, u_{m+2} & = 0 \\ 0 & -u_{m+1} & +\partial_t u_{m+2} & = 0 \end{array}\right. \tag{13.1.7}$$

is symmetric hyperbolic (as one verifies by direct inspection). Also, a short calculation shows that if ϕ is a smooth solution of the scalar equation (13.1.4), then the corresponding vector u is a solution of the system (13.1.7). Conversely, assume that u is a smooth solution of (13.1.7) which satisfies the initial condition $u|_{t=0} = u_0$, where u_0 is determined by ϕ_0 and ϕ_1 via (13.1.6). Setting $\phi = u_{m+2}$, the last line in (13.1.7) shows that $u_{m+1} = \partial_t\phi$. Moreover, the first line in (13.1.7) implies that $\partial_t u_\beta = \nabla_\beta u_{m+1} = \partial_t \nabla_\beta \phi$. Integrating over t and using that the relation $u_\beta = \nabla_\beta\phi$ holds initially, we conclude that this relation holds for all times. Finally, the second line in (13.1.7) yields that ϕ satisfies the scalar hyperbolic equation (13.1.4). In this sense, the Cauchy problem for the system (13.1.7) is equivalent to that for the scalar equation (13.1.5). ◊

This procedure works similarly for other physical equations like the Klein-Gordon or Maxwell equations. Exercise 13.1 is concerned with the example of the homogeneous Maxwell equations.

13.2. Finite Propagation Speed and Uniqueness of Solutions

For what follows, it is convenient to combine the time and spatial coordinates to a spacetime vector $x = (t, \vec{x}) \in [0, T] \times \mathbb{R}^m$. We denote the spacetime dimension by $n = m + 1$. Moreover, setting $\partial_0 \equiv \partial_t$, we use latin spacetime indices $i \in \{0, \ldots, m\}$ and employ the Einstein summation convention. Then our linear system (13.1.3) can be written in the compact form

$$A^j(x)\,\partial_j u(x) + B(x)\,u(x) = w(x)\,. \tag{13.2.1}$$

Next, a direction in spacetime can be described by a vector $\xi = (\xi_i)_{i=0,\ldots,m} \in \mathbb{R}^{m+1}$. Contracting with the matrices $A^j(x)$, we obtain the Hermitian $N \times N$-matrix

$$A(x, \xi) := A^j(x)\,\xi_j\,,$$

referred to as the *characteristic matrix*. Note that in the example of the Dirac equation (13.1.2), the index i is a vector index in Minkowski space, and ξ should be regarded as a co-vector (i.e. a vector in the cotangent bundle). One should keep in mind that, despite the suggestive notation, the equation (13.2.1) should not be considered as being manifestly covariant, because it corresponds to the Hamiltonian formulation (13.1.2), where a time direction is distinguished.

The determinant of the characteristic matrix is referred to as the *characteristic polynomial*, being a polynomial in the components ξ_i. For our purposes, it is most helpful to consider whether the characteristic matrix is positive or negative definite. If the vector $\xi = (\tau, \vec{0})$ points in the time direction, then $A(x, \xi) = \tau A^0$, which in view of Definition 13.1.1 is a definite matrix. By continuity, $A(x, \xi)$ is definite if the spatial components of ξ are sufficiently small. In the example of the Dirac equation (13.1.2), the fact that

$$A(x, \xi) = \mathbb{1}\xi_0 + \gamma^0\vec{\gamma}\vec{\xi} \qquad \text{has eigenvalues} \qquad \xi^0 \pm |\vec{\xi}| \tag{13.2.2}$$

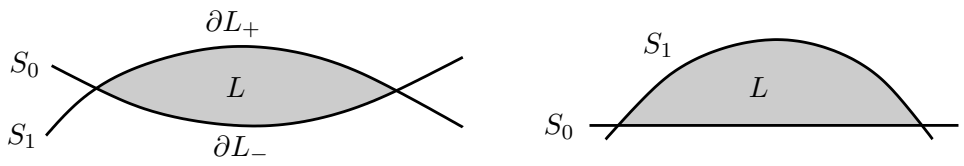


FIGURE 13.1. Lens-shaped regions.

shows that $A(x,\xi)$ is definite if and only if ξ is a timelike vector. Moreover, it is positive definite if and only if ξ is future-directed and timelike. This suggests that the causal properties of the equation are encoded in the positivity of the characteristic matrix. We simply use this connection to *define* the causal structure for a general symmetric hyperbolic system.

DEFINITION 13.2.1. *The vector $\xi \in \mathbb{R}^{m+1}$ is called* **timelike** *at the spacetime point x if the characteristic matrix $A(x,\xi)$ is definite. A timelike vector is called* **future-directed** *if $A(x,\xi)$ is positive definite. If the characteristic polynomial vanishes, then the vector ξ is called* **lightlike**. *A hypersurface $\mathcal{H} \subset [0,T] \times \mathbb{R}^m$ with normal ν is called* **spacelike** *if the matrix $A(x,\nu)$ is positive definite for all $x \in \mathcal{H}$.*

The notion of a *normal* used here requires an explanation. The simplest method is to represent the hypersurface locally as the zero set of a function $\phi(x)$. Then the normal can be defined as the gradient of ϕ. In this way, the gradient is a co-vector, so that the contraction $A^j \nu_j = A^j \partial_j \phi$ is well-defined without referring to a scalar product. In particular, the last definition is independent of the choice of a scalar product on spacetime vectors in $\mathbb{R}^n$. We always choose the normal to be future-directed, and we normalize it with respect to the Euclidean scalar product on $\mathbb{R}^{m+1}$, but these are merely conventions.

We shall now explain why and in which sense the solutions of symmetric hyperbolic systems comply with this notion of causality.

DEFINITION 13.2.2. *Let u be a smooth solution of the linear symmetric hyperbolic system* (13.2.1). *A subset K of the initial value surface $\{t = 0\}$ determines the solution at a spacetime point $x \in [0,T] \times \mathbb{R}^m$ if every smooth solution of the system which coincides on K with u, also coincides with u at x. The* **domain of determination** *of K is the set of all spacetime points at which the solution is determined by the initial data on K.*

DEFINITION 13.2.3. *An open subset $L \subset (0,T) \times \mathbb{R}^m$ is called a* **lens-shaped region** *if L is relatively compact in $\mathbb{R}^n$ and if its boundary ∂L is contained in the union of two smooth hypersurfaces S_0 and S_1 whose intersection with $\overline{L}$ is spacelike. We set $(\partial L)_+ = \partial L \cap S_1$ and $(\partial L)_- = \partial L \cap S_0$, where we adopt the convention that $(\partial L)_+$ lies to the future of $(\partial L)_-$.*

Figure 13.1 shows typical examples of lens-shaped regions. Often, one chooses the initial data surface as $S_0 = \{t = 0\}$. Moreover, it is often convenient to write the hypersurface S_1 as a graph $S_1 = \{(t,\vec{x}) \,|\, t = f(\vec{x})\}$. In this case, S_1 is the zero set of the function $\phi(t,\vec{x}) = t - f(\vec{x})$, and the normal ν is the gradient of this function, i.e.

$$(\nu_j)_{j=0,\ldots,m} = (1, \nabla_1 f, \ldots, \nabla_m f)\,.$$

We first consider the homogeneous equation

$$(A^j \partial_j + B)\, u = 0\,. \tag{13.2.3}$$

The idea for analyzing the domain of determination is to multiply this equation by a suitable test function and to integrate over a lens-shaped region. More precisely, we consider the equation

$$0 = \int_L \mathrm{e}^{-Kt}\, 2\mathrm{Re}\langle u, (A^j\partial_j + B)u\rangle \, \mathrm{d}^n x \,, \tag{13.2.4}$$

where $\langle .,. \rangle$ denotes the canonical scalar product on $\mathbb{C}^N$, and $K > 0$ a positive parameter to be determined later. Since the A^j are Hermitian, we have

$$\partial_j \langle u, A^j u\rangle = 2\,\mathrm{Re}\,\langle u, A^j\partial_j u\rangle + \langle u, (\partial_j A^j)u\rangle \,, \tag{13.2.5}$$

and using this identity in (13.2.4) gives

$$0 = \int_L \mathrm{e}^{-Kt} \Big(\partial_j \langle u, A^j u\rangle + \big\langle u, \big(B + B^* - (\partial_j A^j)\big)u\big\rangle\Big) \, \mathrm{d}^n x \,. \tag{13.2.6}$$

In the first term we integrate by parts with the Gauß divergence theorem,

$$\begin{aligned} \int_L \mathrm{e}^{-Kt}\partial_j \langle u, A^j u\rangle \, \mathrm{d}^n x &= K \int_L \mathrm{e}^{-Kt} \, \langle u, A^0 u\rangle \, \mathrm{d}^n x \\ &+ \int_{(\partial L)_+} \mathrm{e}^{-Kt} \, \langle u, \nu_j A^j u\rangle \, \mathrm{d}\mu_{\partial L_+} - \int_{(\partial L)_-} \mathrm{e}^{-Kt} \, \langle u, \nu_j A^j u\rangle \, \mathrm{d}\mu_{\partial L_-} \,. \end{aligned} \tag{13.2.7}$$

We now use (13.2.7) in (13.2.6) and solve for the surface integral over $(\partial L)_+$,

$$\begin{aligned} \int_{(\partial L)_+} \mathrm{e}^{-Kt} \, \langle u, \nu_j A^j u\rangle \, \mathrm{d}\mu_{\partial L_+} &= \int_{(\partial L)_-} \mathrm{e}^{-Kt} \, \langle u, \nu_j A^j u\rangle \, \mathrm{d}\mu_{\partial L_-} \\ &+ \int_L \mathrm{e}^{-Kt} \, \big\langle u, \big(-K - B - B^* + (\partial_j A^j)\big)\, u\big\rangle \, \mathrm{d}^n x \,. \end{aligned} \tag{13.2.8}$$

This identity is the basis for the following uniqueness results.

THEOREM 13.2.4. *Let u_1 and u_2 be two smooth solutions of the linear symmetric hyperbolic system* (13.1.3) *which coincide on the past boundary of a lens-shaped region L,*

$$u_1|_{(\partial L)_-} = u_2|_{(\partial L)_-} \,.$$

Then u_1 and u_2 coincide in the whole set L.

PROOF. The function $u := u_1 - u_2$ is a solution of the homogeneous system (13.2.3) with $u|_{(\partial L)_-} = 0$. Hence (13.2.8) simplifies to

$$\int_{(\partial L)_+} \mathrm{e}^{-Kt} \, \langle u, \nu_j A^j u\rangle \, \mathrm{d}\mu_{\partial L_+} = \int_L \mathrm{e}^{-Kt} \, \big\langle u, \big(-K - B - B^* + \partial_j A^j\big)\, u\big\rangle \, \mathrm{d}^n x \,.$$

Assume that u does not vanish identically in L. By choosing K sufficiently large, we can then arrange that the right side becomes negative. However, since ∂L_+ is a spacelike hypersurface, the left side is non-negative. This is a contradiction. □

As an immediate consequence, we obtain the following uniqueness result for solutions of the Cauchy problem.

COROLLARY 13.2.5. *Let u_1 and u_2 be two smooth solutions of the linear symmetric hyperbolic system* (13.1.3) *with the same initial at time $t = 0$. Then $u_1 \equiv u_2$ in a neighborhood of the initial data surface.*

If the matrices A^j are uniformly bounded and A^0 is uniformly positive, then $u_1 \equiv u_2$ in $[0, T] \times \mathbb{R}^m$.

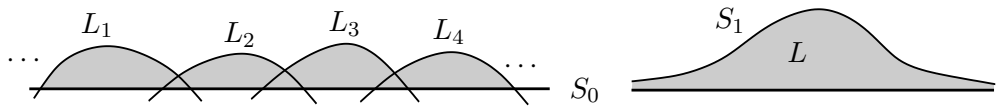


FIGURE 13.2. Coverings by lens-shaped regions.

PROOF. The local uniqueness result follows immediately by covering the initial data surface by lens-shaped regions (see the left of Figure 13.2). For the global uniqueness, for any $x_0 = (t_0, \vec{x}_0) \in [0,T] \times \mathbb{R}^m$ our task is to choose a lens-shaped region which contains x_0 and whose past boundary S_0 is contained in the surface $\{t = 0\}$. We need to rule out the situation shown on the right of Figure 13.2 that the hypersurface S_1 does not intersect S_0, in which case we would not get a relatively compact lens-shaped region. To this end, we must use that the matrices A^j are uniformly bounded by assumption. As a consequence, there is $\varepsilon > 0$ such that the inequality $\|\nabla f\| \leq \varepsilon$ implies that the hypersurface $S_1 = \{(t = f(\vec{x}), \vec{x})\}$ is spacelike. Possibly after decreasing ε, we may choose

$$f(\vec{x}) = t_0 + \varepsilon \left(1 - \sqrt{1 + \|\vec{x} - \vec{x}_0\|^2}\right).$$

This concludes the proof. □

By a suitable choice of lens-shaped region one can get an upper bound for the maximal propagation speed. For the Dirac equation, where the causal structure of Definition 13.2.1 coincides in view of (13.2.2) with that of Minkowski space, one can choose for S_1 a family of spacelike hypersurfaces which converge to the boundary of a light cone (see Figure 13.3). This shows that the maximal propagation speed for Dirac waves is indeed the speed of light (which, according to our conventions, is equal to one).

13.3. Global Existence of Smooth Solutions

In this section we will show that, by refining the above uniqueness argument, we even obtain an existence proof. The close connection between existence and uniqueness for linear equations is a familiar theme in mathematics. The simplest setting where it occurs is in the study of the linear equation $Au = v$ with a given vector $v \in \mathbb{R}^n$ and a quadratic matrix A. In this case, the uniqueness of the solution implies that the matrix A is invertible, which in turn ensures existence. A more interesting example is Fredholm's alternative for compact operators (see for example [**131**, Section VI.5]). The procedure for globally hyperbolic systems follows somewhat similar ideas. Here the general strategy is to construct a bounded linear functional on a Hilbert space in such a way that the Fréchet-Riesz theorem (see Theorem 2.2.4) gives the desired solution.

Before beginning, we point out that, in view of uniqueness and finite propagation speed, it suffices to consider the problem in a bounded spatial region. Indeed, once we have constructed "local solutions" in small lens-shaped regions as shown on the left of Figure 13.2, uniqueness implies that these solutions agree in the overlap of the lens-shaped regions, making it possible to "glue them together" to obtain the desired solution which is global in space. We will come back to this construction in more detail in the context of the Dirac equation in Sections 13.4 and 13.6 (see also Figure 13.4). Having this construction in mind, we may start from a local problem and to extend the coefficients of the symmetric hyperbolic system in an arbitrary way outside. Therefore, it is no loss of generality to consider a problem in the whole space $\mathbb{R}^m$. Choosing a bounded time

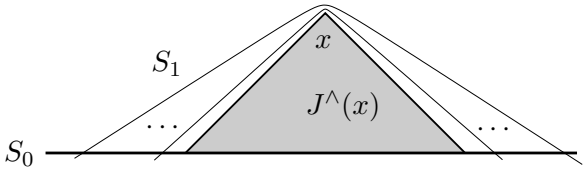


FIGURE 13.3. Approximating the light cone by lens-shaped regions.

interval $t \in [0,T]$ (where $t=0$ is the initial time), we are led to considering the time strip

$$R_T := [0,T] \times \mathbb{R}^m \,.$$

We now write the linear system (13.2.1) as

$$Lu = w \qquad \text{with} \qquad L := A^j \partial_j + B \,, \tag{13.3.1}$$

where we again sum over $j = 0, \ldots, m$. Again using that the system can be extended arbitrarily outside a bounded spatial region, we may assume that that the functions A^j, B and w are uniformly bounded in R_T and that w has spatially compact support (meaning that $w(t,.) \in C_0^\infty(\mathbb{R}^m)$ for all $t \in [0,T]$). Moreover, for convenience we again assume smoothness of A^j, B and w. In the *Cauchy problem* one seeks for a solution of the equation (13.3.1) with prescribed initial data $u_0 \in C^\infty(\mathbb{R}^m)$ at time $t=0$,

$$Lu = w \,, \qquad u|_{t=0} = u_0 \in C_0^\infty(\mathbb{R}^m) \tag{13.3.2}$$

in $C^s(R_T)$. First of all, we may restrict attention to the case $u_0 \equiv 0$,

$$Lu = w \,, \qquad u|_{t=0} \equiv 0 \,. \tag{13.3.3}$$

In order to see this, let u be a solution of the above Cauchy problem. Choosing a function $v \in C^\infty(R_T)$ which at $t=0$ coincides with u_0. Then the function $\tilde{u} := (u-v)$ satisfies the equation $L\tilde{u} = \tilde{w}$ with $\tilde{w} = w + A^j \partial_j v + Bv$ and vanishes at $t=0$. If conversely $\tilde{u}$ is a solution of the corresponding Cauchy problem with zero initial data, then $u := \tilde{u} + v$ is a solution of the original problem (13.3.2).

In preparation of the existence proof, we need to introduce the notion of a *weak solution*. In order to get into the weak formulation, we multiply the equation (13.3.1) by a test function $v(t,\vec{x})$ and integrate over R_T, giving rise to the equation

$$\langle v, Lu \rangle_{L^2(R_T)} = \langle v, w \rangle_{L^2(R_T)}$$

with the L^2-scalar product defined by

$$\langle v, w \rangle_{L^2(R_T)} := \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} \langle v(t,\vec{x}), w(t,\vec{x}) \rangle \; \mathrm{d}^m x \,. \tag{13.3.4}$$

The next step is to integrate by parts, so that the derivatives act on the test function v. Before doing so, we need to specify the regularity of the test functions. To this end, for $\lambda \in [0,T]$ we consider the time strip

$$R_\lambda := [0,\lambda] \times \mathbb{R}^m \,.$$

We denote the s-times continuously differential functions on R_λ with spatially compact support by $C^s(R_\lambda)$. The function spaces

$$\underline{C^s(R_\lambda)} \qquad \text{and} \qquad \overline{C^s(R_\lambda)}$$

are defined as the functions which in addition vanish at $t=0$ and $t=\lambda$, respectively. As the space of test functions we choose $\overline{C^1(R_T)}$; this guarantees that integrating by parts

does not yield boundary terms at $t = T$. For a classical solution $u \in \underline{C^1(R_T)}$ (i.e. a solution with zero Cauchy data (13.3.3)), also the boundary term at $t = 0$ vanishes. We thus obtain

$$\langle v, w\rangle_{L^2(R_T)} = \langle \tilde{L}v, u\rangle_{L^2(R_T)} \qquad \text{for all } v \in \overline{C^1(R_T)}\,, \tag{13.3.5}$$

where $\tilde{L}$ is the formal adjoint of L with respect to the scalar product (13.3.4), i.e.

$$\tilde{L} := \tilde{A}^j \partial_j + \tilde{B} \qquad \text{with} \qquad \tilde{A}^j = -A^j \quad \text{and} \quad \tilde{B} = B^\dagger - \big(\partial_j A^j\big)\,. \tag{13.3.6}$$

Now suppose that a function $u \in C^1(R_T)$ satisfies (13.3.5). Testing with functions $v \in \overline{C^1(R_T)} \cap \underline{C^1(R_T)}$ which vanish both at times $t = 0$ and $t = T$, we can integrate by parts without boundary terms. Using a standard denseness argument, one finds that u solves the symmetric hyperbolic system (13.3.1). Next, testing with a function $v \in \overline{C^1(R_T)}$ which does *not* vanish at $t = 0$, only the boundary term remains, giving the equation

$$\int_{\mathbb{R}^m} \langle v(0,\vec{x}), u(0,\vec{x})\rangle \, \mathrm{d}^m x = 0 \qquad \text{for all } v \in \overline{C^1(R_T)}\,,$$

which in turn implies that u vanishes initially. Thus u is a solution of the Cauchy problem (13.3.3). To summarize, for functions $u \in \overline{C^1(R_T)}$, the weak formulation (13.3.5) is equivalent to our Cauchy problem (13.3.1) and (13.3.3). Therefore, it is sensible to take (13.3.5) as the definition of a weak solution of the Cauchy problem. The main advantage of the weak formulation (13.3.5) is that it is well-defined even for functions which are not differentiable.

Our next step is to derive so-called *energy estimates* for a given solution $u \in \underline{C^1(R_T)}$. To this end, we return to the formula for the divergence (13.2.5) and using the equation (13.3.1), we obtain

$$\partial_j \langle u, A^j u\rangle + \langle u, Cu\rangle = 2\,\mathrm{Re}\,\langle u, w\rangle, \tag{13.3.7}$$

$$C := B + B^* - (\partial_j A^j)\,. \tag{13.3.8}$$

Next, we integrate (13.3.7) over R_λ, integrate by parts and use that the initial values at $t = 0$ vanish. We thus obtain

$$E(\lambda) := \int_{t=\lambda} \langle u, A^0 u\rangle \, \mathrm{d}^m x = \int_0^\lambda \mathrm{d}t \int_{\mathbb{R}^m} \Big(2\,\mathrm{Re}\,\langle u, w\rangle - \langle u, Cu\rangle\Big) \, \mathrm{d}^m x\,. \tag{13.3.9}$$

Since the matrix C is uniformly bounded and A_0 is uniformly positive, there is a constant $K > 1$ such that

$$|\langle u, Cu\rangle| \le K \langle u, A^0 u\rangle\,.$$

Moreover, the linear term in u can be estimated with the Schwarz inequality by

$$2\,\mathrm{Re}\,\langle u, w\rangle \le \mu\,\langle u, u\rangle + \frac{1}{\mu}\,\langle w, w\rangle \le \langle u, A^0 u\rangle + \frac{1}{\mu^2}\,\langle w, A^0 w\rangle$$

with a suitable constant $\mu > 0$. Applying these estimates in (13.3.9) gives

$$E(\lambda) \le (K+1) \int_0^\lambda E(t) \, \mathrm{d}t + \frac{1}{\mu^2} \int_{R_\lambda} \langle w, A^0 w\rangle \, \mathrm{d}^n x\,.$$

Writing this inequality as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{e}^{-(K+1)\lambda} \int_0^\lambda E(t) \, \mathrm{d}t \le \mathrm{e}^{-(K+1)\lambda} \frac{1}{\mu^2} \int_{R_T} \langle w, A^0 w\rangle \, \mathrm{d}^n x\,,$$

we can integrate over λ to obtain

$$\int_0^T E(\lambda)\ \mathrm{d}\lambda \leq \frac{\mathrm{e}^{(K+1)T}-1}{K+1}\,\frac{1}{\mu^2}\int_{R_T} \langle w, A^0 w\rangle\ \mathrm{d}^n x\,.$$

Finally, we apply the mean value theorem and use that the exponential function is monotone to conclude that

$$\int_0^T E(\lambda)\,d\lambda \leq \frac{T}{\mu^2}\,\mathrm{e}^{(K+1)T}\int_{R_T} \langle w, A^0 w\rangle\ \mathrm{d}^n x\,. \tag{13.3.10}$$

This is the desired energy estimate.

Before going on, we point out that the notion of "energy" used for the quantity $E(\lambda)$ does in general *not* coincide with the physical energy. In fact, for the Dirac equation (13.1.2), $E(\lambda)$ has the interpretation as the electric charge. Following Example 13.1.2, for the scalar wave equation $\Box\phi = 0$, we find

$$E(\lambda) = \int_{\mathbb{R}^m} \Big(|\partial_t \phi|^2 + |\nabla\phi|^2 + |\phi|^2\Big)\,\mathrm{d}^m x\,. \tag{13.3.11}$$

This differs from the physical energy by the last summand $|\phi|^2$ (and an overall factor of two). The name "energy" for $E(\lambda)$ was motivated by the fact, considering only the highest derivative terms, the expression (13.3.11) is indeed the physical energy. We point out that, in contrast to the physical energy, the quantity $E(\lambda)$ does in general depend on time. The point is that (13.3.10) gives an a-priori control of the energy in terms of the inhomogeneity. The exponential factor in (13.3.10) can be understood in analogy to a Grönwall estimate (for the classical Grönwall estimate see for example [**1**, Lemma 1.15 in Section VII.1]).

For the following construction, it is convenient to introduce on $C^1(R_T)$ the scalar product

$$(u, v) = \int_{R_T} \langle u, A^0 v\rangle\ \mathrm{d}^n x\,.$$

We denote the corresponding norm by $\|\cdot\|$. Setting furthermore

$$\Gamma^2 = \frac{T}{\mu^2}\mathrm{e}^{(K+1)T}\,,$$

the energy estimate can be written in the compact form

$$(u, u) \leq \Gamma^2\,(w, w)\,.$$

This inequality holds for every solution u of the differential equation $Lu = w$ which vanishes at $t = 0$. Noting that every function $u \in \underline{C^1(R_T)}$ is a solution of this differential equation with inhomogeneity $w := Lu$, we obtain

$$\|u\| \leq \Gamma\,\|Lu\| \qquad \text{for all } u \in \underline{C^1(R_T)}\,. \tag{13.3.12}$$

This is the form of the energy estimates suitable for an abstract existence proof.

Note that the operator $\tilde{L}$ in (13.3.6) is also symmetric hyperbolic and has the same boundedness and positivity properties as L. Hence, repeating the above arguments, we obtain similar to (13.3.12) the "dual estimate"

$$\|v\| \leq \tilde{\Gamma}\,\|\tilde{L}v\| \qquad \text{for all } v \in \overline{C^1(R_T)}\,. \tag{13.3.13}$$

We now want to show the existence of weak solutions with the help of the Fréchet-Riesz theorem (see Theorem 2.2.4 in the preliminaries or for example [**131, 116**]). To this end, we first introduce on $\overline{C^1(R_T)}$ yet another scalar product denoted by

$$\langle v, v' \rangle = (\tilde{L}v, \tilde{L}v)\,. \tag{13.3.14}$$

This scalar product is indeed positive definite, because for any $v \neq 0$,

$$\langle v, v \rangle = (\tilde{L}v, \tilde{L}v) \geq \tilde{\Gamma}^{-2}\,(v, v) \neq 0\,,$$

where in the last step we applied (13.3.13). Forming the completion, we obtain the Hilbert space $(\mathcal{H}, \langle .,. \rangle)$. We denote the corresponding norm by $|\!|\!|\,.\,|\!|\!|$. In view of (13.3.13) and (13.3.14), we know that every vector $v \in \mathcal{H}$ is a function in $L^2(R_T,\, \mathrm{d}^n x)$. Moreover, we know from (13.3.14) that $\tilde{L}v$ is also in $L^2(R_T,\, \mathrm{d}^n x)$. We remark that, in the language of functional analysis, the space $\mathcal{H}$ can be identified with the Sobolev space $W^{1,2}(R_T)$, but we do not need this here.

We now consider for $w \in C^0(R_T)$ and $v \in \overline{C^1(R_T)}$ the linear functional $\langle v, w \rangle_{L^2(R_T)}$. In view of the estimate

$$\big|\langle v, w \rangle_{L^2(R_T)}\big| \leq \|v\|_{L^2(R_T)}\, \|w\|_{L^2(R_T)} \leq \frac{\tilde{\Gamma}}{C}\, \|w\|_{L^2(R_T)}\; |\!|\!|\, v \,|\!|\!|\,,$$

this functional is continuous in $v \in \mathcal{H}$. The Fréchet-Riesz theorem shows that there is $U \in \mathcal{H}$ with

$$\langle v, w \rangle_{L^2(R_T)} = \langle v, U \rangle = (\tilde{L}v, \tilde{L}U) \qquad \text{for all } v \in \mathcal{H}\,.$$

Rewriting the last scalar product as

$$(\tilde{L}v, \tilde{L}U) = \langle Lv, A^0 \tilde{L}U \rangle_{L^2(R_T)}\,,$$

one sees that the function $u := A^0 \tilde{L}U \in L^2(R_T,\, \mathrm{d}^n x)$ satisfies the equation (13.3.5) and is thus the desired weak solution. Note that all our methods apply for arbitrarily large T. We have thus proved the global existence of weak solutions.

We next want to show that the solutions are smooth. Thus our task is to show that our constructed weak solution u is of the class $C^s(R_\lambda)$, where $s \geq 1$ can be chosen arbitrarily large. We first show that a linear symmetric hyperbolic system can be "enlarged" to include the partial derivatives of ϕ.

LEMMA 13.3.1. *Suppose that the system $A^j \partial_j u + Bu = w$ is symmetric hyperbolic. Then there is a symmetric hyperbolic system of the form*

$$\tilde{A}^j \partial_j \Psi + \tilde{B}\Psi = \tilde{w} \tag{13.3.15}$$

for the vector $\Psi := (\partial_t u, \nabla_1 u, \ldots, \nabla_m u, u) \in \mathbb{C}^{(n+1)N}$.

PROOF. Let i be a fixed spacetime index. We differentiate the equation $Lu = w$,

$$\partial_i w = \partial_i L u = L \partial_i u + (\partial_i A^j)\, \partial_j u + (\partial_i B)\, u\,.$$

This equation can be written as

$$A^j \partial_j \Psi_i + \sum_{j=1}^{n} \tilde{B}_i^j\, \Psi_j + (\partial_i B)\, u = \tilde{w}_i\,,$$

where we set

$$\tilde{B}_i^j = B\, \delta_i^j + (\partial_i A^j) \qquad \text{and} \qquad \tilde{w}_i = \partial_i w\,.$$

Combining these equations with the equation $Lu = w$, we obtain a system of the form (13.3.15), where the matrices A^j are block diagonal in the sense that

$$\tilde{A}^j = \big((\tilde{A}^j)^\alpha_\beta\big)_{\alpha,\beta=0,\ldots,m+1} \qquad \text{with} \qquad (\tilde{A}^j)^\alpha_\beta = A^j\, \delta^\alpha_\beta\,.$$

Obviously, this system is again symmetric hyperbolic. □

Iterating this lemma, we obtain (at least in principle) a symmetric hyperbolic system for u and all its partial derivatives up to any given order s. Since the corresponding weak solution is in $L^2(R_T)$, we conclude that u and all its weak partial derivatives are square integrable. The next lemma, which is a special case of the general Sobolev embedding theorems (see for example [**32**, Section II.5.] or [**143**, Section 4]), gives smoothness of the solution.

LEMMA 13.3.2. *Let $s > \frac{m}{2}$ be an integer. If a function g on $\mathbb{R}^m$ is s times weakly differentiable and*

$$\int_{\mathbb{R}^m} |\nabla^\alpha g|^2 \, \mathrm{d}^m x < C \tag{13.3.16}$$

for all multi-indices α with $|\alpha| \leq s$, then g is bounded, $g \in L^\infty(\mathbb{R}^m)$. Likewise, if g is $s+l+1$ times weakly differentiable with $l \geq 1$ and (13.3.16) *holds for all α with $|\alpha| \leq s+l+1$, then $g \in C^l(\mathbb{R}^m)$.*

PROOF. We apply the Schwarz inequality to the Fourier transform,

$$\begin{aligned}
|g(x)|^2 &= \left| \int_{\mathbb{R}^m} \frac{\mathrm{d}^m k}{(2\pi)^m}\, \hat{g}(k)\, \mathrm{e}^{-\mathrm{i}kx} \right|^2 \\
&= \left| \int_{\mathbb{R}^m} \frac{\mathrm{d}^m k}{(2\pi)^m} (1+|k|^2)^{-\frac{s}{2}}\, (1+|k|^2)^{\frac{s}{2}}\, \hat{g}(k) \mathrm{e}^{-\mathrm{i}kx} \right|^2 \\
&\leq c_m \int_{\mathbb{R}^m} \frac{\mathrm{d}^m k}{(2\pi)^m} (1+|k|^2)^s\, |\hat{g}(k)|^2\,,
\end{aligned}$$

where the constant c_m is finite due to our choice of s,

$$c_m = \int_{\mathbb{R}^m} \frac{\mathrm{d}^m k}{(2\pi)^m}\, (1+|k|^2)^{-s} < \infty\,.$$

Using the Plancherel formula together with the fact that a factor k^2 corresponds to a Laplacian in position space, we obtain

$$\int_{\mathbb{R}^m} \frac{\mathrm{d}^m k}{(2\pi)^m} (1+|k|^2)^s\, |\hat{g}(k)|^2 = \sum_{\ell=0}^{s} \binom{n}{\ell} \|\nabla^\ell g\|^2_{L^2(\mathbb{R}^m)} < c\,.$$

Hence $\sqrt{c_m\, c}$ is an L^∞-bound for g.

Next, if g is $s+1$ times weakly differentiable, then $\|Dg\|_{L^\infty}(\mathbb{R}^m) < c$. As a consequence, the mean value theorem yields $|g(x) - g(y)| \leq c|x-y|$, so that g is Lipschitz continuous. Finally, if g is $s+l+1$ times weakly differentiable, then all partial derivatives $\nabla^\alpha g$ of order $|\alpha| \leq l$ are Lipschitz continuous, so that $g \in C^l(\mathbb{R}^m)$. □

More precisely, in order to apply this lemma, we fix a time t and consider the solution $u(\lambda, .)$. The identity (13.3.9) implies that $E(\lambda)$ is controlled in terms of $\|w\|$ and $\|u\|$. After iteratively applying Lemma 13.3.1, we conclude that the weak derivatives of $u(\lambda, .)$ exist to any order and are in $L^2(\mathbb{R}^m)$. It follows that $u(\lambda, .)$ is smooth. Finally, one uses the equation to conclude that u is also smooth in the time variable.

The results of this section can be summarized as follows.

THEOREM 13.3.3. *Consider the Cauchy problem*

$$\big(A^0 \partial_t + \sum_{\alpha=1}^{m} A^\alpha \nabla_\alpha + B\big) u = w \in C_0^\infty([0,T] \times \mathbb{R}^m)\,, \qquad u|_{t=0} = u_0 \in C_0^\infty(\mathbb{R}^m)\,.$$

Assume that the matrices A^0, A^j and B as well as the functions w and u_0 are smooth. Moreover, assume that all these functions as well as as all their partial derivatives are uniformly bounded (where the bound may depend on the order of the derivatives). Then the Cauchy problem has a smooth solution on $[0,T] \times \mathbb{R}^m$.

This theorem also applies in the case $T = \infty$, giving global existence of a smooth solution.

We finally show that the solutions depend smoothly an parameters.

COROLLARY 13.3.4. *Suppose that the matrices A^j, B and the functions w, u_0 depend smoothly on a parameter λ. Then the family of solutions $u(\lambda)$ is also smooth in λ.*

PROOF. First, similar as explained after (13.3.3), we may restrict attention to the case $u_0 = 0$. Differentiating the equation $Lu = w$ with respect to λ, we obtain

$$Lu_\lambda = (\partial_\lambda L) u + \partial_\lambda w =: \tilde{w}\,,$$

where u_λ stands for the formal derivative $\partial_\lambda u$. This is a symmetric hyperbolic system for u_λ. According to Theorem 13.3.3, we know that u and therefore $\tilde{w}$ are smooth. Hence, applying this theorem again, we conclude that there exists a smooth solution u_λ. Considering the limit of the difference quotients, one verifies that u_λ really coincides with $\partial_\lambda u(\lambda)$ for our given family of solutions $u(\lambda)$. The higher λ-derivatives can be treated inductively. □

13.4. The Causal Dirac Green's Operators in Minkowski Space

We now want to apply the previous general existence and uniqueness results to the Cauchy problem (13.1.1) for the Dirac equation in Minkowski space in the presence of an external potential $\mathcal{B}$.

THEOREM 13.4.1. *Consider the Cauchy problem for the Dirac equation* (13.1.1) *for smooth initial data ψ_0, a smooth inhomogeneity ϕ and a smooth matrix-valued potential $\mathcal{B} \in C^\infty(\mathscr{M}, \mathbb{C}^{4\times 4})$. Then there is a unique global smooth solution $\psi \in C^\infty(\mathscr{M}, S\mathscr{M})$.*

PROOF. Writing the Dirac equation in the Hamiltonian form (13.1.2), we obtain a symmetric hyperbolic system. In view of the uniqueness result for smooth solutions of Corollary 13.2.5, it suffices to construct a smooth solution at any given time $T \in \mathbb{R}$. It suffices to consider the case $T > t_0$, because otherwise we reverse the time direction. Moreover, we can arrange by a time shift that $t_0 = 0$.

We cannot apply Theorem 13.3.3 directly because the coefficient functions in (13.1.2) do not need to be bounded, nor are our initial values compactly supported. For this reason, we need to construct local solutions and "glue them together" using linearity: We first extend the initial data ψ_0 smoothly to the time strip R_T and consider the Cauchy problem for $\tilde{\psi} := \psi - \psi_0$,

$$(\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{\psi} = \tilde{\phi} \in C^\infty(\mathscr{M}, S\mathscr{M})\,, \qquad \tilde{\psi}|_{t_0} = 0\,.$$

We let $(\eta_k)_{k\in\mathbb{N}}$ be a smooth partition of unity of $\mathbb{R}^m$ with $\eta_k \in C_0^\infty(\mathbb{R}^m)$ (for details see for example [**136**, Theorem 2.13]). We extend these functions to static functions on R_T (i.e. $\eta_k(t, \vec{x}) := \eta_k(\vec{x})$. Given $k \in \mathbb{N}$, we first solve the Cauchy problem for the

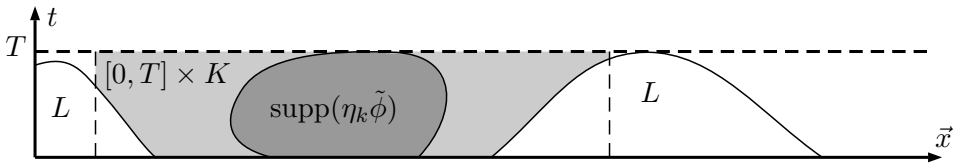


FIGURE 13.4. Construction of local solutions.

inhomogeneity $\eta_k \tilde{\phi}$. We choose a compact set $K \subset \mathbb{R}^m$ such that $[0,T] \times K$ contains the causal future of the support of $(\eta_k \tilde{\phi})$ (see Figure 13.4; more specifically, we could choose $K = B_{2T}(\operatorname{supp} \eta_k)$). Next, we choose a smooth, compactly supported function $\theta \in C_0^\infty(\mathbb{R}^m)$ with $\theta|_K \equiv 1$. We again extend θ to a static function on R_T.

We now consider the modified Cauchy problem

$$\Big(\mathbb{1}_{\mathbb{C}^4}\, \partial_t \psi + \gamma^0 \vec{\gamma} \vec{\nabla} + \theta\big(-\mathrm{i}\gamma^0 (\mathcal{B} - m)\big)\Big) \tilde{\psi}_k = -\mathrm{i}\gamma^0\, \eta_k\, \tilde{\phi}\,, \qquad \tilde{\psi}_k|_{t_0} = 0\,.$$

Now the coefficients in the PDE are uniformly bounded, and the inhomogeneity has compact support. Therefore, we can apply Theorem 13.3.3 to obtain a global smooth solution. Due to finite propagation speed (see Theorem 13.2.4, where we choose lens-shaped regions L as shown in Figure 13.4), this solution vanishes outside K. Therefore, it is also a solution of the unmodified Dirac equation, with initial data $\eta_k \tilde{\phi}$.

Finally, summing over k gives the desired solution of the original Cauchy problem,

$$\psi := \sum_{k=1}^{\infty} \tilde{\psi}_k\,.$$

Here the series converges because, again due to finite propagation speed, it is locally finite. □

We next explain how the previous existence and uniqueness results give rise to the existence of causal Green's operators, being defined as integral operators with distributional kernels. These kernels are often referred to as *Green's functions*. Our main tool is the *Schwartz kernel theorem*. We do not give a proof of this more advanced result of distribution theory but refer instead to [**105**, Section 5.2] or [**143**, Section 4.6]. For better consistency with the notation in the perturbative treatment in Section 18, from now on we denote the objects in the presence of an external potential with an additional tilde. We begin with a representation formula for the solution of the Cauchy problem in terms of a distribution.

THEOREM 13.4.2. *Assume that the external potential $\mathcal{B}$ is smooth and that $\mathcal{B}$ and all its partial derivatives are uniformly bounded in Minkowski space. Then for any t, t_0 there is a unique distribution $\tilde{k}_m(t, .; t_0, .) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)$ such that the solution of the Cauchy problem* (17.0.1) *has the representation*

$$\psi(t, \vec{x}) = 2\pi \int_N \tilde{k}_m(t, \vec{x}; t_0, \vec{y})\, \gamma^0\, \psi_0(\vec{y})\; \mathrm{d}^3 y\,. \tag{13.4.1}$$

The integral kernel k_m is also a distribution in spacetime, $k_m \in \mathcal{D}'(M \times M)$ It is a distributional solution of the Dirac equation,

$$(\mathrm{i}\partial\!\!\!/_x + \mathcal{B} - m)\, \tilde{k}_m(x, y) = 0\,. \tag{13.4.2}$$

PROOF. The energy estimates combined with the Sobolev embedding of Lemma 13.3.2 showed that there is $k \in \mathbb{N}$ and a constant $C = C(t, t_0, \vec{x}, \mathcal{B})$ such that the solution $\psi(t,.)$ of the Cauchy problem is bounded in terms of the initial data by

$$|\psi(t, \vec{x})| \leq C\, |\psi_0|_{C^k}\,, \tag{13.4.3}$$

where $|\psi|^2 := \prec \psi | \gamma^0 \psi \succ$, and the C^k-norm is defined by

$$|\psi_0|_{C^k} = \max_{|\beta| \leq k} \sup_{\vec{x} \in \mathbb{R}^3} |\nabla^\beta \psi_0(\vec{x})|\,.$$

Moreover, this estimate is locally uniform in $\vec{x}$, meaning that for any compact set $K \subset \mathbb{R}^3$, there is a constant C such that (13.4.3) holds for all $\vec{x} \in K$. This makes it possible to apply the Schwartz kernel theorem [**105**, Theorem 5.2.1], showing that $\tilde{k}_m(t, .; t_0, .) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)$.

Next, we note that the constant C in (13.4.3) can also be chosen locally uniformly in t and t_0. Thus, after evaluating weakly in t and t_0, we may again apply the Schwartz kernel theorem to obtain that $\tilde{k}_m \in \mathcal{D}'(M \times M)$. Finally, the distributional equation (13.4.2) follows immediately from the fact that (13.4.1) is satisfies the Dirac equation for any choice of ψ_0. □

The distribution $\tilde{k}_m$ is referred to as the *causal fundamental solution*. Encoding the whole Dirac dynamics, it plays a fundamental role in the analysis of the Dirac equation. In the next step, we introduce the *causal Green's operators* by decomposing $\tilde{k}_m$ in time. Namely, for any t, t_0 we introduce the distribution $\tilde{s}^\vee_m(t, .; t_0, .), \tilde{s}^\wedge_m(t, .; t_0, .) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$\begin{cases} \tilde{s}^\vee_m(t, .; t_0, .) = \quad 2\pi \mathrm{i}\, \tilde{k}_m(t, .; t_0, .)\, \Theta(t_0 - t) \\ \tilde{s}^\wedge_m(t, .; t_0, .) = -2\pi \mathrm{i}\, \tilde{k}_m(t, .; t_0, .)\, \Theta(t - t_0) \end{cases} \tag{13.4.4}$$

(where Θ denotes the Heaviside function). In this way, we introduce the causal fundamental solutions for any given t_0 and t as distributions on $\mathbb{R}^3 \times \mathbb{R}^3$. Alternatively, they can also be introduced as bi-distributions in spacetime, as is shown in the next lemma.

THEOREM 13.4.3. *Assume that the external potential $\mathcal{B}$ is smooth and that $\mathcal{B}$ and all its partial derivatives are uniformly bounded in Minkowski space. Then there are unique distributions*

$$\tilde{s}^\vee_m, \tilde{s}^\wedge_m \in \mathcal{D}'(\mathscr{M} \times \mathscr{M})$$

which satisfy the distributional equations

$$(\mathrm{i}\partial\!\!\!/_x + \mathcal{B} - m)\, \tilde{s}_m(x, y) = \delta^4(x - y) \tag{13.4.5}$$

and are supported in the upper respectively lower light cone,

$$\operatorname{supp} \tilde{s}^\vee_m(x, .) \subset J^\vee_x\,, \qquad \operatorname{supp} \tilde{s}^\wedge_m(x, .) \subset J^\wedge_x\,. \tag{13.4.6}$$

PROOF. It is clear by construction and the fact that the constant C in (13.4.3) can be chosen locally uniformly in x and y that the causal Green's operators are well-defined distributions in $\mathcal{D}'(M \times M)$. The support property (13.4.6) follows immediately from finite propagation speed as explained at the end of Section 13.2. The uniqueness of the causal Green's operators is clear from the uniqueness of solutions of the Cauchy problem. In order to derive the distributional equations (13.4.5), we only consider the retarded

Green's operator (the argument for the advanced Green's operator is analogous). Then, according to (13.4.1) and (13.4.4),

$$\Theta(t-t_0)\,\psi(t,\vec{x}) = \mathrm{i}\int_N \tilde{s}^\wedge_m(t,\vec{x};t_0,\vec{y})\,\gamma^0\,\psi_0(\vec{y})\;\mathrm{d}^3y\,,$$

where ψ is the solution of the corresponding Cauchy problem. Applying the Dirac operator in the distributional sense yields

$$\mathrm{i}\gamma^0\delta(t-t_0)\,\psi_0(t,\vec{x}) = \mathrm{i}(\mathcal{D}_x - m)\int_N \tilde{s}^\wedge_m(t,\vec{x};t_0,\vec{y})\,\gamma^0\,\psi_0(\vec{y})\;\mathrm{d}^3y\,.$$

We now choose the initial values as the restriction of a test function in spacetime, $\psi_0 = \phi|_{t=t_0}$ with $\phi \in C_0^\infty(M, SM)$. Then we can integrate over t_0 to obtain

$$\mathrm{i}\gamma^0\phi(x) = (\mathcal{D}_x - m)\int_M \tilde{s}^\wedge_m(x,y)\,\mathrm{i}\gamma^0\phi(y)\;\mathrm{d}^4y\,.$$

This gives the result. □

We remark that, turning the above argument around, we can also use the causal Green's operators in order to define the causal fundamental solution as a bi-distribution in spacetime,

$$\tilde{k}_m := \frac{1}{2\pi\mathrm{i}}\left(\tilde{s}^\vee_m - \tilde{s}^\wedge_m\right) \in \mathcal{D}'(\mathscr{M}\times\mathscr{M})\,. \tag{13.4.7}$$

The causal fundamental solution has the remarkable property that it relates the scalar product with the inner product obtained by integrating the spin inner product over spacetime. We now explain this relation step by step. Given two wave functions ψ and ϕ (not necessarily solutions of the Dirac equation), we want to integrate their pointwise inner product $\prec\psi|\phi\succ_x$ over spacetime (similar as already done in the preliminaries in (1.3.16) and (4.2.33)). In order to ensure that this integral is well-defined, it suffices to assume that one of the functions is compactly supported. We thus obtain the sesquilinear pairing

$$\begin{aligned} <.|.> \;&:\; C^\infty(\mathscr{M}, S\mathscr{M}) \times C_0^\infty(\mathscr{M}, S\mathscr{M}) \to \mathbb{C}\,, \\ <\psi|\phi> &= \int_{\mathscr{M}} \prec\psi|\phi\succ_x \mathrm{d}\mu_{\mathscr{M}} \end{aligned} \tag{13.4.8}$$

(here $C^\infty(\mathscr{M}, S\mathscr{M})$ are again the smooth sections of the spinor bundle, and $C_0^\infty(\mathscr{M}, S\mathscr{M})$ denotes the smooth sections with compact support). Restricting the first argument to compactly supported wave functions, we obtain an inner product,

$$<.|.> \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \times C_0^\infty(\mathscr{M}, S\mathscr{M}) \to \mathbb{C}\,,$$

referred to as the *spacetime inner product* (we remark that this inner product space can be extended to a Krein space; we refer the interested reader to [**45**, §1.1.5]). Alternatively, one can also restrict the first argument of $<.|.>$ to smooth Dirac solutions and extend by approximation to the whole Hilbert space $\mathcal{H}_m$, giving the sesquilinear pairing

$$<.|.> \;:\; \mathcal{H}_m \times C_0^\infty(\mathscr{M}, S\mathscr{M}) \to \mathbb{C}\,.$$

The following proposition goes back to John Dimock (see [**27**, Proposition 2.2]).

PROPOSITION 13.4.4. *For any $\psi_m \in \mathcal{H}_m$ and $\phi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$,*

$$(\psi_m \,|\, \tilde{k}_m\phi)_m = <\psi_m|\phi>\,. \tag{13.4.9}$$

PROOF. We first give the proof under the additional assumption $\psi_m \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})$ that the Dirac solution has spatially compact support. We choose Cauchy surfaces $\mathscr{N}_+$ and $\mathscr{N}_-$ lying in the future and past of $\operatorname{supp}\phi$, respectively. Let Ω be the spacetime region between these two Cauchy surfaces, i.e. $\partial\Omega = \mathscr{N}_+ \cup \mathscr{N}_-$. Then, according to (13.6.2) and using again the notation (4.2.32),

$$\begin{aligned}(\psi_m \,|\, \tilde{k}_m\, \phi)_m &= (\psi_m \,|\, \tilde{k}_m\, \phi)_{\mathscr{N}_+} = \frac{\mathrm{i}}{2\pi}\, (\psi_m \,|\, \tilde{s}^\wedge_m\, \phi)_{\mathscr{N}_+} \\ &= \frac{\mathrm{i}}{2\pi}\Big[(\psi_m \,|\, \tilde{s}^\wedge_m\, \phi)_{\mathscr{N}_+} - (\psi_m \,|\, \tilde{s}^\wedge_m\, \phi)_{\mathscr{N}_-}\Big] \\ &= \mathrm{i}\int_\Omega \nabla_j \prec \psi_m \,|\, \gamma^j\, \tilde{s}^\wedge_m \phi \succ_x \mathrm{d}\mu(x)\,,\end{aligned}$$

where in the last line we applied the Gauß divergence theorem and used (15.1.2). Using that ψ_m satisfies the Dirac equation, a calculation similar to (1.3.10) yields

$$(\psi_m \,|\, \tilde{k}_m\, \phi)_m = \int_\Omega \prec \psi_m \,|\, (\mathcal{D} - m)\, \tilde{s}^\wedge_m \phi \succ_x \mathrm{d}\mu(x) \overset{(16.2.1)}{=} \int_\Omega \prec \psi_m | \phi \succ_x \mathrm{d}\mu(x)\,.$$

As ϕ is supported in Ω, we can extend the last integration to all of $\mathscr{M}$, giving the result.

In order to extend the result to general $\psi_m \in \mathcal{H}_m$, we use the following approximation argument. Let $\psi_m^{(n)} \in \mathcal{H}_m \cap C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})$ be a sequence which converges in $\mathcal{H}_m$ to ψ_m. Then obviously $(\psi_m^{(n)} \,|\, \tilde{k}_m\, \phi)_m \to (\psi_m \,|\, \tilde{k}_m\, \phi)_m$. In order to show that the right side of (13.4.9) also converges, it suffices to prove that $\psi_m^{(n)}$ converges in $L^2_{\text{loc}}(\mathscr{M}, S\mathscr{M})$ to ψ_m. Thus let $K \subset \mathscr{M}$ be a compact set contained in the domain of a chart (x, U). Using Fubini's theorem, we obtain for any $\psi \in \mathcal{H}_m \cap C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})$ the estimate

$$\int_K \prec \psi | \psi \psi \succ \mathrm{d}\mu_{\mathscr{M}} = \int \mathrm{d}x^0 \int \prec \psi | \psi \psi \succ \sqrt{|g|}\; \mathrm{d}^3x \le C(K)\, (\psi|\psi)_m\,.$$

Applying this estimate to the functions $\psi = \psi_m^{(n)} - \psi_m^{(n')}$, we see that $\psi_m^{(n)}$ converges in $L^2(K, S\mathscr{M})$ to a function $\tilde{\psi}$. This implies that $\psi_m^{(n)}$ converges to $\tilde{\psi}$ pointwise almost everywhere (with respect to the measure $\mathrm{d}\mu_{\mathscr{M}}$). Moreover, the convergence of $\psi_m^{(n)}$ in $\mathcal{H}_m$ to ψ_m implies that the restriction of $\psi_m^{(n)}$ to any Cauchy surface $\mathscr{N}$ converges to $\psi_m|_{\mathscr{N}}$ pointwise almost everywhere (with respect to the measure $\mathrm{d}\mu_{\mathscr{N}}$). It follows that $\tilde{\psi} = \psi_m|_K$, concluding the proof. □

COROLLARY 13.4.5. *The operator $\tilde{k}_m$,* (13.6.2)*, is symmetric with respect to the inner product* (13.4.8)*.*

PROOF. Using Proposition 13.4.4, we obtain for all $\phi, \psi \in C^\infty_0(\mathscr{M}, S\mathscr{M})$,

$$<\tilde{k}_m \phi \,|\, \psi> = (\tilde{k}_m \phi \,|\, \tilde{k}_m \psi)_m = <\phi \,|\, \tilde{k}_m \psi>\,,$$

concluding the proof. □

13.5. A Polynomial Estimate in Time

We now derive an estimate which shows that the solutions of the Dirac equation increase at most polynomially in time. This result will be needed in Section 17.2.1. For

the proof we adapt standard methods of the theory of partial differential equations to the Dirac equation. In generalization of (16.3.4), we denote the spatial Sobolev norms by

$$\|\phi\|^2_{W^{a,2}} = \sum_{\alpha \text{ with } |\alpha| \leq a} \int_{\mathbb{R}^3} |\nabla^\alpha \phi(\vec{x})|^2 \, \mathrm{d}^3x \,.$$

LEMMA 13.5.1. *We are given two non-negative integers a and b as well as a smooth time-dependent potential $\mathcal{B}$. In the case $a > 0$ and $b \geq 0$, we assume furthermore that the spatial derivatives of $\mathcal{B}$ decay faster than linearly for large times in the sense that*

$$|\nabla \mathcal{B}(t)|_{C^{a-1}} \leq \frac{c}{1+|t|^{1+\varepsilon}} \tag{13.5.1}$$

for suitable constants $c, \varepsilon > 0$. Then there is a constant $C = C(c, \varepsilon, a, b)$ such that every family of solutions $\psi \in \mathcal{H}^\infty$ of the Dirac equation (1.3.14) for varying mass parameter can be estimated for all times in terms of the boundary values at $t = 0$ by

$$\big\|\partial_m^b \psi_m|_t\big\|_{W^{a,2}} \leq C\left(1+|t|^b\right) \sum_{p=0}^{b} \big\|\partial_m^p \psi_m|_{t=0}\big\|_{W^{a,2}} \,.$$

PROOF. We choose a multi-index α of length $a := |\alpha|$ and a non-negative integer b. Differentiating the Dirac equation (1.3.14) with respect to the mass parameter and to the spatial variables gives

$$(\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m)\, \nabla^\alpha \partial_m^b \psi_m = b\, \nabla^\alpha \partial_m^{b-1} \psi_m - \nabla^\alpha \big(\mathcal{B}\, \partial_m^b \psi_m\big) + \mathcal{B}\, \nabla^\alpha \partial_m^b \psi_m \,.$$

Introducing the abbreviations

$$\Xi := \nabla^\alpha \partial_m^b \psi_m \qquad \text{and} \qquad \phi := b\, \nabla^\alpha \partial_m^{b-1} \psi_m - \nabla^\alpha \big(\mathcal{B}\, \partial_m^b \psi_m\big) + \mathcal{B}\, \nabla^\alpha \partial_m^b \psi_m \,,$$

we rewrite this equation as the inhomogeneous Dirac equation

$$(\mathcal{D} - m)\, \Xi = \phi \,.$$

A calculation similar to current conservation yields

$$-\mathrm{i}\partial_j \prec \Xi | \gamma^j \Xi \succ = \prec (\mathcal{D}-m)\Xi \,|\, \Xi \succ - \prec \Xi \,|\, (\mathcal{D}-m)\Xi \succ = \prec \phi | \Xi \succ - \prec \Xi | \phi \succ \,.$$

Integrating over the equal time hypersurfaces and using the Schwarz inequality, we obtain

$$\big|\partial_t \big(\Xi|_t | \Xi|_t\big)_t\big| \leq 2 \big\|\Xi|_t\big\|_t \, \big\|\phi|_t\big\|_t$$

and thus

$$\Big|\partial_t \big\|\Xi|_t\big\|\Big| \leq \big\|\phi|_t\big\|_t \,.$$

Substituting the specific forms of Ξ and ϕ and using the Schwarz and triangle inequalities, we obtain the estimate

$$\Big|\partial_t \big\|\nabla^\alpha \partial_m^b \psi_m|_t\big\|_t\Big| \leq b\, \big\|\nabla^\alpha \partial_m^{b-1} \psi_m|_t\big\|_t + c\, a\, |\nabla \mathcal{B}(t)|_{C^{a-1}} \, \big\|\partial_m^b \psi_m|_t\big\|_{W^{a-1,2}} \,, \tag{13.5.2}$$

where we used the notation (17.2.2).

We now proceed inductively in the maximal total order $a+b$ of the derivatives. In the case $a = b = 0$, the claim follows immediately from the unitarity of the time evolution. In order to prove the induction step, we note that in (13.5.2), the order of differentiation of the wave function on the right hand side is smaller than that on the left hand side at least by one. In the case $a = 0$ and $b \geq 0$, the induction hypothesis yields the inequality

$$\big|\partial_t \|\partial_m^b \psi_m|_t\|\big| \leq b\, \big\|\partial_m^{b-1} \psi_m|_t\big\| \leq b\, C\left(1+|t|^{b-1}\right) \sum_{p=0}^{b-1} \big\|\partial_m^p \psi_m|_{t=0}\big\| \,,$$

and integrating this inequality from 0 to t gives the result. In the case $a > 0$ and $b \geq 0$, we apply (13.5.1) together with the induction hypothesis to obtain

$$\begin{aligned}\Big|\partial_t \big\|\partial_m^b \psi_m|_t\big\|_{W^{a,2}}\Big| &\leq b\, C\, \big(1+|t|^{b-1}\big) \sum_{p=0}^{b-1} \big\|\partial_m^p \psi_m|_{t=0}\big\|_{W^{a,2}} \\ &\quad + c\, C\, \frac{1+|t|^b}{1+|t|^{1+\varepsilon}} \sum_{p=0}^{b} \big\|\partial_m^p \psi_m|_{t=0}\big\|_{W^{a-1,2}} .\end{aligned}$$

Again integrating over t gives the result. □

13.6. The Cauchy Problem in Globally Hyperbolic Spacetimes

We conclude this chapter by extending the global existence and uniqueness result for the Dirac equation to curved spacetime. These results were already stated in Section 4.5. We are now in the position for giving the proof. The reader not interested in or not familiar with curved spacetime may skip this section. We note that more details on the geometric properties of globally hyperbolic spacetimes can be found in [**6**, Section 3.2].

Proof of Theorem 4.5.1. Exactly as explained in the proof of Theorem 13.4.1, by considering the Cauchy problem for $\psi - \psi_0$ one may reduce the problem to that of zero initial data zero. Moreover, choosing a partition of unity (η_k) subordinate to the charts of a given atlas, it suffices to consider the compactly supported inhomogeneity $\eta_k \phi$ (the sum over k is again locally finite, similar as explained in the proof of Theorem 13.4.1). In view of these constructions, it remains to consider the Cauchy problem

$$(\mathcal{D} - m)\psi = \phi \in C_0^\infty(\mathscr{M}, S\mathscr{M}) , \qquad \psi|_{\mathcal{N}_{t_0}} = 0 . \tag{13.6.1}$$

We denote the support of ϕ by K.

Clearly, in local charts the Dirac equation can be written as a symmetric hyperbolic system. Therefore, the results in Sections 13.2 and 13.3 yield existence and uniqueness of solutions of the Cauchy problem in local charts. This also yields global uniqueness: Let ψ and $\tilde{\psi}$ be two smooth solutions of the Cauchy problem (13.6.1). Then their difference $\Xi := \tilde{\psi} - \psi$ is a homogeneous solution which vanishes at time t_0. In view of a possible time reversal, it suffices to consider the solution in the future of t_0. Thus let $x \in \mathscr{M}$ be in the future of t_0. Then the past light cone $J^\wedge(x)$ intersects the future of t_0 in a compact set,

$$J^\wedge(x) \cap \Big(\bigcup\nolimits_{t \geq t_0} \mathcal{N}_t \Big) \qquad \text{is compact} .$$

Therefore, we can choose $\delta > 0$ such that for every $\hat{t}$, there is a finite number of lens-shaped regions which cover the time strip

$$J^\wedge(x) \cap \Big(\bigcup\nolimits_{t=\hat{t}}^{\hat{t}+\delta} \mathcal{N}_t \Big) .$$

On each lens-shaped regions, the solution for the Cauchy problem with zero initial data vanishes identically. Therefore, we can proceed inductively to conclude that $\Xi(x) = 0$. Since x is arbitrary, the solution Ξ vanishes identically on $\mathscr{M}$.

In order to prove global existence, we proceed indirectly. In view of a possible time reversal, it suffices to consider the Cauchy problem to the future. Thus suppose that the solution exists only up to finite time $t_{\max}$ (see Figure 13.5). Due to finite propagation

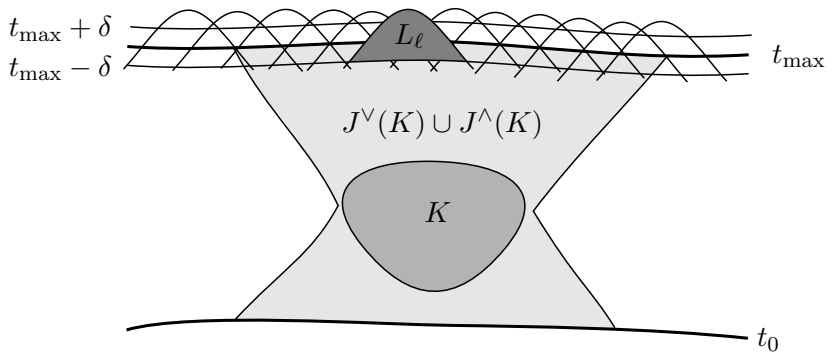


FIGURE 13.5. Global solutions in globally hyperbolic spacetimes.

speed, the solution is supported in the domain of causal dependence of K,

$$\operatorname{supp} \psi \subset J^\vee(K) \cup J^\wedge(K) .$$

By properties of globally hyperbolic spacetimes, the intersection D of this set with the Cauchy surface $\mathcal{N}_{t_{\max}}$ is compact. Covering D by a finite number of charts, we choose δ such that the sets $J^\vee(D) \cup J^\wedge(D) \cap \mathcal{N}_t$ lie in the domain of these charts for all $t \in [t_{\max} - \delta, t_{\max} + \delta]$. Next we choose a finite number of lens-shaped regions L_ℓ which also cover all these sets (see again Figure 13.5). In each of these lens-shaped regions we can solve the Cauchy problem with initial data at time $t_{\max} - \delta$. In this way, we get a solution up to time $t_{\max} + \delta$. This is a contradiction, thereby proving that the solution must exist for all times. □

PROOF OF THEOREM 4.5.2. By extending the initial data ψ_0 to a smooth and compactly supported function in spacetime and considering the Cauchy problem for $\psi - \psi_0$, it again suffices to consider the case of zero initial data (13.6.1). The solution constructed subsequently the proof of Theorem 4.5.1 was supported in $J^\vee(K) \subset J^\wedge(K)$. By general properties of globally hyperbolic manifolds, the intersection of this set with every Cauchy surface is compact. This concludes the proof. □

Similar as explained in Section 13.4 in Minkowski space, also in curved spacetime the solution of the Cauchy problem can be expressed in terms of the causal fundamental solution k_m, as we now explain. Similar as explained in Minkowski space in Section 13.4, the *retarded* and *advanced Green's operators* $s_m^\wedge$ and $s_m^\vee$ are linear mappings (for details see for example [**27, 6**])

$$s_m^\wedge, s_m^\vee \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \to C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) .$$

They satisfy the defining equation of the Green's operator

$$(\mathcal{D} - m)\left(s_m^{\wedge,\vee} \phi\right) = \phi .$$

Moreover, they are uniquely determined by the condition that the support of $s_m^\wedge \phi$ (or $s_m^\vee \phi$) lies in the future (respectively the past) of $\operatorname{supp} \phi$. The *causal fundamental solution* k_m is introduced by

$$k_m := \frac{1}{2\pi \mathrm{i}} \left(s_m^\vee - s_m^\wedge\right) \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \to C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m . \tag{13.6.2}$$

Note that it maps to solutions of the Dirac equation.

PROPOSITION 13.6.1. *The solution of the Cauchy problem* (4.5.1) *has the representation*

$$\psi(x) = 2\pi \int_{\mathcal{N}} k_m(x,y)\, \not{\psi}\, \psi_{\mathcal{N}}(y)\, \mathrm{d}\mu_{\mathcal{N}}(y)\,,$$

where $k_m(x,y)$ *is the causal fundamental solution* (13.6.2).

PROOF. Let us consider a point x in the future of $\mathcal{N}$ (the case for the past is analogous). In this case, due to (13.6.2), the lemma simplifies to

$$\psi(x) = \mathrm{i} \int_{\mathcal{N}} s_m^\wedge(x,y)\, \not{\psi}(y)\, \psi_{\mathcal{N}}(y)\, \mathrm{d}\mu_{\mathcal{N}}(y)\,. \tag{13.6.3}$$

In preparation, we want to prove that for any $\phi \in C^\infty(\mathscr{M}, S\mathscr{M})$ which has compact support to the past of $\mathcal{N}$ and with the property that $(\mathcal{D}-m)\phi$ has compact support the equation

$$\phi = s_m^\wedge\big((\mathcal{D}-m)\phi\big) \tag{13.6.4}$$

holds. To this end, we consider the function

$$\Xi(x) := \phi - s_m^\wedge\big((\mathcal{D}-m)\phi\big)\,.$$

Applying the operator $(\mathcal{D}-m)$ and using the defining equation of the Green's operators, one sees that Ξ is a solution of the Dirac equation. Moreover, Ξ obviously vanishes in the past of the support of ϕ. The uniqueness of the solution of the Cauchy problem implies that Ξ vanishes identically, proving (13.6.4).

In order to derive equation (13.6.3), we let $\eta \in C^\infty(\mathscr{M})$ be a function which is identically equal to one at x and on $\mathcal{N}$, but such that the function $\eta\psi$ has compact support to the past. (For example, in a foliation $(\mathcal{N}_t)_{t\in\mathbb{R}}$ with $\mathcal{N} = \mathcal{N}_{t_0}$ one can take $\eta = \eta(t)$ as a smooth function with $\eta|_{[t_0,\infty)} \equiv 1$ which vanishes if $t < t_0 - 1$). Then we can apply (13.6.4) to the wave function $\phi = \eta\psi$. We thus obtain for any x in the future of $\mathcal{N}$ the relations

$$\psi(x) = (\eta\psi)(x) = \Big(s_m^\wedge\big((\mathcal{D}-m)(\eta\psi)\big)\Big)(x) = \Big(s_m^\wedge\big(\mathrm{i}\gamma^j(\partial_j\eta)\,\psi\big)\Big)(x)\,, \tag{13.6.5}$$

where we have used that ψ is a solution of the Dirac equation.

To conclude the proof, for η in (13.6.5) we choose a sequence η_ℓ which converges in the distributional sense to the function which in the future $\mathcal{N}$ is equal to one and in the past of $\mathcal{N}$ is equal to zero. This yields $\partial_j\eta_\ell \to \nu$, and thus the right-hand-side of (13.6.5) goes over to the right-hand-side of (13.6.3). □

13.7. Exercises

EXERCISE 13.1. The *homogeneous Maxwell equations* for the electric field $E : \mathbb{R}^3 \to \mathbb{R}^3$ and the magnetic field $B : \mathbb{R}^3 \to \mathbb{R}^3$ read

$$\nabla \times B = \partial_t E\,, \quad \nabla \times E = -\partial_t B\,,$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. Rewrite these equations as a symmetric hyperbolic system. *Remark:* We here ignore the equations $\operatorname{div} E = \operatorname{div} B = 0$. The reason is that these equations hold automatically if they are satisfied initially.

EXERCISE 13.2. Consider the scalar wave equation $(\partial_{tt} - \Delta_{\mathbb{R}^m})\phi(t,x) = 0$.

(a) Rewrite the equation as a symmetric hyperbolic system

$$A^0\partial_t u + \sum_{\alpha=1}^{m} A^\alpha \nabla_\alpha u + Bu = 0.$$

(b) Determine the timelike and future-directed directions. Which directions ξ are characteristic (meaning that the characteristic polynomial $\det A(x,\xi)$ vanishes)?

(c) Express the "energy"

$$E(t) = \int_{\mathbb{R}^m} \langle u, A^0 u\rangle \, \mathrm{d}^m x$$

in terms of $\phi(t,x)$. Compare the resulting expression with the conserved physical energy

$$\int_{\mathbb{R}^m} (|\partial_t \phi|^2 + |\nabla \phi|^2) \, \mathrm{d}^m x \,.$$

(d) Compute $\frac{\mathrm{d}E(t)}{\mathrm{d}t}$. Prove the inequality

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq E(t)$$

and integrate it (Grönwall's lemma).

EXERCISE 13.3. Consider the solution of the homogeneous wave equation

$$(\partial_{tt} - \Delta_{\mathbb{R}^n})\phi(t,x) = 0$$

for smooth initial data $\phi(0,x) = f(x)$ and $\partial_t \phi(0,x) = g(x)$.
Show by a suitable choice of lens-shaped regions that $\phi(t_0,x_0)$ depends only on the initial data in the closed ball $\{x \in \mathbb{R}^n : |x - x_0| \leq t_0\}$.

EXERCISE 13.4. We consider the system

$$\begin{aligned} \partial_t u_1(t,x) + \partial_x u_1(t,x) + 4\partial_x u_2(t,x) &= 0 \\ \partial_t u_2(t,x) + 4\partial_x u_1(t,x) + \partial_x u_2(t,x) &= 0 \end{aligned}$$

(a) Write the system in symmetric hyperbolic form.

(b) Compute the solution of the Cauchy problem for the initial data $u_1(0,x) = \sin x$ and $u_2(0,x) = \cos x$.

EXERCISE 13.5. *(The Euler equations)* The evolution equation for an isentropic compressible fluid reads

$$\begin{cases} \partial_t v + \nabla_v v + \frac{1}{\rho}\mathrm{grad}(p) &= 0 \\ \partial_t \rho + \nabla_v \rho + \rho \,\mathrm{div}(v) &= 0. \end{cases} \tag{13.7.1}$$

Here $v : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity vector field, $\rho : \mathbb{R}^+ \times \mathbb{R}^3 \to (0,\infty)$ the density and $p = A\rho^\gamma$ the pressure (where $A > 0$ and $\gamma > 1$).

(a) Show that (13.7.1) is equivalent to a quasilinear symmetric hyperbolic system, provided that ρ is bounded away from zero.

(b) Show that for smooth solutions, the system (13.7.1) is equivalent to

$$\begin{cases} \partial_t v + \nabla_v v + \mathrm{grad}(h(\rho)) &= 0 \\ \partial_t \rho + \mathrm{div}(\rho v) &= 0, \end{cases} \tag{13.7.2}$$

where $h \in C^\infty(\mathbb{R})$ satisfies the equation $h'(\rho) = \rho^{-1} p'(\rho)$.

(c) Let (v,ρ) be a solution of (13.7.2) with $v(t,x) = \nabla_x \varphi(t,x)$ for a real-valued potential φ. Prove *Bernoulli's law*: If φ and ρ decay at infinity sufficiently fast and if $h(0) = 0$, then

$$\partial_t \varphi + \frac{1}{2}|\nabla_x \varphi|^2 + h(\rho) = 0 \,.$$

(d) Show that (13.7.1) can also be rewritten as a system for (p, v),

$$\begin{cases} \partial_t v + \nabla_v v + \rho(p)^{-1}\mathrm{grad}(p) & = 0 \\ \partial_t p + \nabla_v p + (\gamma p)\,\mathrm{div}(v) & = 0\,. \end{cases}$$

Rewrite this system in symmetric hyperbolic form.

EXERCISE 13.6. Let $\lambda > 0$. A symmetric hyperbolic system of the form

$$\partial_t u + A^\alpha(u)\partial_\alpha u + \lambda u = 0\,,$$

where the matrices A^α are smooth, uniformly bounded and uniformly positive, is an example of a so-called *dissipative system*.

(a) Prove that for spatially compact solutions, the following energy estimate holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^p} \leq \Big(-2\lambda + c\|u(t)\|_{C^1}\Big)\|u(t)\|^2_{H^p}\,.$$

(b) Prove: If the initial data u_0 is sufficiently small in the C^1-norm, then there exists a global solution.
Hint: Choose p sufficiently large and use the Sobolev embedding theorem.

EXERCISE 13.7. *(Causality in the setting of symmetric hyperbolic systems)* The Dirac equation $(\mathrm{i}\partial\!\!\!/ - m)\psi = 0$ can be rewritten as a *symmetric hyperbolic system*, i.e. in the form $(c > 0)$

$$(A^0(x)\,\partial_0 + A^\alpha(x)\,\partial_\alpha + B(x))\psi = 0, \quad \text{with } \ (A^i)^\dagger = A^i \ \text{ and } \ A^0(x) \geq c\mathbb{I}.$$

For such systems a notion of *causality* can be introduced: a vector $\xi \in \mathbb{R}^4$ is said to be *timelike* or *lightlike* at $x \in \mathbb{R}^4$, if the matrix $A(x,\xi) := A^i(x)\,\xi_i$ is definite (either positive or negative) or singular, respectively.

Find the matrices A^i and B for the Dirac equation and show that the above notions of timelike and lightlike vectors coincide with the corresponding notions in Minkowski space.

CHAPTER 14

Energy Methods for the Linearized Field Equations

In the previous chapter, we used energy methods in order to study the Cauchy problem for linear symmetric hyperbolic systems. We now briefly explain how these methods can be adapted to the linearized field equations for causal variational principles as introduced in Chapter 7. These constructions are carried out in detail in [**22**]; for later developments see [**64, 74**]. Here we do not aim for the largest generality, but instead explain the basic ideas in the simplest possible setting. We also note that some of the constructions in this section will be illustrated in Chapter 20 by simple concrete examples.

14.1. Local Foliations by Surface Layers

We consider *causal variational principles in the compact setting* (see Section 6.3). Moreover, for technical simplicity we again restrict attention to the *smooth setting* by assuming that the Lagrangian is smooth (6.2.4). Following our procedure for symmetric hyperbolic systems, we want to analyze the initial problem "locally" in an open subset U of spacetime M. In analogy to the time function in a lens-shaped region L (see Section 13.2) we here choose a foliation of a compact subset $L \subset U$ by surface layers. This motivates the following definition.

DEFINITION 14.1.1. *Let $U \subset M$ be an open subset of spacetime and $I := [t_{\min}, t_{\max}]$ a compact interval. Moreover, we let $\eta \in C^\infty(I \times U, \mathbb{R})$ be a function with $0 \leq \eta \leq 1$ which for all $t \in I$ has the following properties:*

(i) *The function $\theta(t,.) := \partial_t \eta(t,.)$ is non-negative and compactly supported in U.*
(ii) *For all $x \in \operatorname{supp} \theta(t,.)$ and all $y \in M \setminus U$, the function $\mathcal{L}(x,y)$ as well as its first and second derivatives vanish.*

We also write $\eta(t,x)$ as $\eta_t(x)$ and $\theta(t,x)$ as $\theta_t(x)$. We refer to $(\eta_t)_{t \in I}$ as a **local foliation** *inside U.*

The situation in mind is shown in Figure 14.1. The parameter t can be thought of as the time of a local observer and will be referred to simply as *time*. The support of the

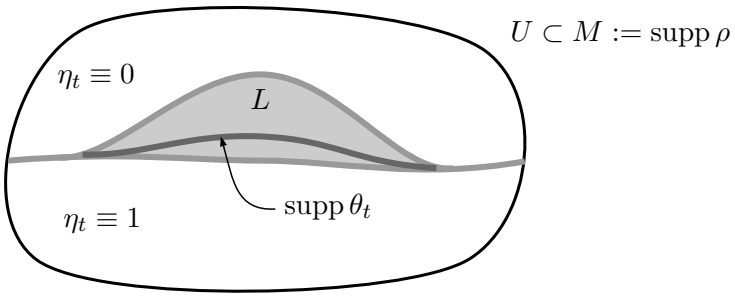


FIGURE 14.1. A local foliation.

function θ_t is a *surface layer*. The function η_t should be thought of as being equal to one in the past and equal to zero in the future of this surface layer. The condition (i) implies that the set L defined by

$$L := \bigcup_{t \in I} \operatorname{supp} \theta_t \tag{14.1.1}$$

is compact. It is the region of spacetime described by the local foliation. The condition (ii) has the purpose to ensure that the dynamics in the region L does not depend on the jets outside U, making it possible to restrict attention to the spacetime region U. Sometimes, we refer to this property that L is *$\mathcal{L}$-localized* in U. One way of satisfying (ii) is to simply choose $U = M$. However, in the applications it may be desirable to "localize" the problem for example by choosing U as the domain of a coordinate chart.

Following the procedure for hyperbolic partial differential equations, our first goal is to analyze the initial value problem For the following constructions, it will be useful to combine the functions η_t and θ_t with the measure ρ such as to form new measures: The measure

$$\mathrm{d}\rho_t(x) := \theta_t(x)\, \mathrm{d}\rho(x) \tag{14.1.2}$$

with $t \in I$ is supported in the surface layer at time t. Likewise, the measures

$$\eta_t\, \mathrm{d}\rho \qquad \text{and} \qquad \big(1 - \eta_t\big)\, \mathrm{d}\rho$$

are supported in the past respectively future of the surface layer at time t. For the measures supported in a spacetime strip, we use the notation

$$\eta_{[t_0,t_1]}\, \mathrm{d}\rho \qquad \text{with} \qquad \eta_{[t_0,t_1]} := \eta_{t_1} - \eta_{t_0} \in C_0^\infty(U)\,, \tag{14.1.3}$$

where we always choose $t_0, t_1 \in I$ with $t_0 \leq t_1$. Note that the function $\eta_{[t_0,t_1]}$ is supported in L.

14.2. Energy Estimates and Hyperbolicity Conditions

For the analysis of the linearized field equations it is helpful to study the surface layer integrals as introduced in Section 9.5 for our local foliation $(\eta_t)_{t \in I}$. It is useful to "soften" these surface layer integrals by rewriting the integration domains with characteristic functions and replacing the characteristic functions by smooth cutoff functions formed of η_t, i.e. symbolically

$$\begin{aligned}\int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y) \cdots &= \int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \chi_\Omega(x) \big(1 - \chi_\Omega(y)\big) \cdots \\ &\longrightarrow \int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \eta_t(x) \big(1 - \eta_t(y)\big) \cdots\,.\end{aligned}$$

We thus define the *softened symplectic form* and the *softened surface layer inner product* by

$$(\mathfrak{u}, \mathfrak{v})^t = \int_U \mathrm{d}\rho(x)\, \eta_t(x) \int_U \mathrm{d}\rho(y)\, \big(1 - \eta_t(y)\big) \Big(\nabla_{1,\mathfrak{u}} \nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{u}} \nabla_{2,\mathfrak{v}}\Big) \mathcal{L}(x,y) \tag{14.2.1}$$

$$\sigma^t(\mathfrak{u}, \mathfrak{v}) = \int_U \mathrm{d}\rho(x)\, \eta_t(x) \int_U \mathrm{d}\rho(y)\, \big(1 - \eta_t(y)\big) \Big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{1,\mathfrak{v}} \nabla_{2,\mathfrak{u}}\Big) \mathcal{L}(x,y)\,. \tag{14.2.2}$$

The quantity $(\mathfrak{u}, \mathfrak{u})^t$ is of central importance for the following constructions, because it will play the role of the energy used in our energy estimates. In preparation of these estimates, we derive an energy identity:

LEMMA 14.2.1. **(energy identity)** *For any jet* $\mathfrak{u} = (a, u) \in \mathfrak{J}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathfrak{u},\mathfrak{u})^t &= 2\int_U \langle \mathfrak{u}, \Delta\mathfrak{u}\rangle(x)\,\mathrm{d}\rho_t(x) \\ &\quad - 2\int_U \Delta_2[\mathfrak{u},\mathfrak{u}]\,\mathrm{d}\rho_t(x) + \mathfrak{s}\int_U a(x)^2\,\mathrm{d}\rho_t(x)\,, \end{aligned} \tag{14.2.3}$$

where the operator $\Delta_2 : \mathfrak{J}\times\mathfrak{J} \to \mathfrak{J}^*$ *defined by*

$$\begin{aligned} &\big\langle \mathfrak{u}, \Delta_2[\mathfrak{u}_1,\mathfrak{u}_2]\big\rangle(x) \\ &= \frac{1}{2}\,\nabla_{\mathfrak{u}}\bigg(\int_M \big(\nabla_{1,\mathfrak{u}_1}+\nabla_{2,\mathfrak{u}_1}\big)\big(\nabla_{1,\mathfrak{u}_2}+\nabla_{2,\mathfrak{u}_2}\big)\mathcal{L}(x,y)\,\mathrm{d}\rho(y) \;-\nabla_{\mathfrak{u}_1}\nabla_{\mathfrak{u}_2}\,\mathfrak{s}\bigg)\,. \end{aligned} \tag{14.2.4}$$

PROOF. Differentiating (14.2.1) with respect to t gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathfrak{u},\mathfrak{u})^t &= \int_U \mathrm{d}\rho(x)\,\theta_t(x)\int_U \mathrm{d}\rho(y)\,\big(1-\eta_t(y)\big)\,\big(\nabla_{1,\mathfrak{u}}^2-\nabla_{2,\mathfrak{u}}^2\big)\mathcal{L}(x,y) \\ &\quad - \int_U \mathrm{d}\rho(x)\,\eta_t(x)\int_U \mathrm{d}\rho(y)\,\theta_t(y)\,\big(\nabla_{1,\mathfrak{u}}^2-\nabla_{2,\mathfrak{u}}^2\big)\mathcal{L}(x,y) \\ &= \int_U \mathrm{d}\rho(x)\,\theta_t(x)\int_U \mathrm{d}\rho(y)\,\big(\nabla_{1,\mathfrak{u}}^2-\nabla_{2,\mathfrak{u}}^2\big)\mathcal{L}(x,y)\,. \end{aligned} \tag{14.2.5}$$

Next, for all $x \in L$ we may use Definition 14.1.1 (ii) to change the integration range in (8.1.7) from M to U,

$$\langle \mathfrak{u}, \Delta\mathfrak{u}\rangle(x) = \int_U \nabla_{1,\mathfrak{u}}\big(\nabla_{1,\mathfrak{u}}+\nabla_{2,\mathfrak{u}}\big)\,\mathcal{L}(x,y)\,\mathrm{d}\rho(y) - \mathfrak{s}\,a(x)^2\,.$$

Multiplying by θ_t and integrating, we obtain

$$\begin{aligned} 0 &= \int_U \theta_t(x)\,\langle \mathfrak{u}, \Delta\mathfrak{u}\rangle(x)\,\mathrm{d}\rho(x) + \mathfrak{s}\int_U \theta_t(x)\,a(x)^2\,\mathrm{d}\rho(x) \\ &\quad - \int_U \mathrm{d}\rho(x)\,\theta_t(x)\int_U \mathrm{d}\rho(y)\,\big(\nabla_{1,\mathfrak{u}}^2+\nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{u}}\big)\,\mathcal{L}(x,y)\,. \end{aligned}$$

We multiply this equation by two and add (14.2.5). This gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathfrak{u},\mathfrak{u})^t &= -\int_U \mathrm{d}\rho(x)\,\theta_t(x)\int_U \mathrm{d}\rho(y)\,\big(\nabla_{1,\mathfrak{u}}+\nabla_{2,\mathfrak{u}}\big)^2\mathcal{L}(x,y) \\ &\quad + 2\int_U \theta_t(x)\,\langle \mathfrak{u}, \Delta\mathfrak{u}\rangle(x)\,\mathrm{d}\rho(x) + 2\mathfrak{s}\int_U \theta_t(x)\,a(x)^2\,\mathrm{d}\rho(x)\,. \end{aligned}$$

Using the property in Definition 14.1.1 (ii), in the y-integral we may replace the integration range U by M, making it possible to apply (14.2.4). Rewriting the obtained integrals using the notation (14.1.2) gives (14.2.3). □

In order to make use of this energy identity, we need to impose a condition which we call hyperbolicity condition. This notion can be understood as follows. As explained in Chapter 13, in the theory of hyperbolic partial differential equations the hyperbolicity of the equations gives rise to a positive energy (see $E(\lambda)$ in (13.3.9)). The positivity of this energy was a consequence of the structure of the equations (more precisely, for symmetric linear hyperbolic systems, it is a consequence of the positivity statement in Definition 13.1.1 (ii)). The positivity of the energy is crucial for the analysis of hyperbolic equations, because it gives both uniqueness (see Section 13.2) and existence of weak solutions (see Section 13.3). With this in mind, our strategy is to express the hyperbolic

nature of the linearized field equations by imposing a positivity condition for our "energy" $(\mathfrak{u},\mathfrak{u})^t$. As we shall see, this so-called hyperbolicity condition is precisely what is needed in order to obtain existence and uniqueness of solutions. For Dirac systems in Minkowski space, the hyperbolicity conditions can be verified by direct computation (for details see [**50**]). With this in mind, our hyperbolicity conditions are physically sensible. But in most situation, imposing the hyperbolicity conditions for all jets in $\mathfrak{J}$ is a too strong assumption. Instead, these conditions will or can be satisfied only on a suitably chosen subspace of jets, which we denote by

$$\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}\,. \tag{14.2.6}$$

Clearly, the smaller the jet space $\mathfrak{J}^{\text{vary}}$ is chosen, the easier it is to satisfy (14.2.8). The drawback is that the Cauchy problem will be solved in a weaker sense.

In order to define the hyperbolicity conditions, for all $x \in M$ we choose the subspace of the tangent space spanned by the test jets,

$$\Gamma_x := \big\{ u(x) \,|\, u \in \Gamma^{\text{test}} \big\} \;\subset\; T_x\mathcal{F}\,.$$

We introduce a Riemannian metric g_x on Γ_x. The choice of the Riemannian metric is arbitrary; the resulting freedom can be used in order to satisfy the hyperbolicity conditions below (note, however, that for causal fermion systems a canonical Riemannian metric is obtained form the Hilbert-Schmidt scalar product; see [**69, 76**]). This Riemannian metric also induces a pointwise scalar product on the jets. Namely, setting

$$\mathfrak{J}_x := \mathbb{R} \oplus \Gamma_x\,,$$

we obtain the scalar product on $\mathfrak{J}_x$

$$\langle .,. \rangle_x \,:\, \mathfrak{J}_x \times \mathfrak{J}_x \to \mathbb{R}\,, \qquad \langle \mathfrak{u}, \tilde{\mathfrak{u}} \rangle_x := a(x)\, \tilde{a}(x) + g_x\big(u(x), \tilde{u}(x)\big) \tag{14.2.7}$$

(where we again denote the scalar and vector components of the jet by $\mathfrak{u} = (a, u)$). We denote the corresponding norm by $\|.\|_x$.

Definition 14.2.2. *The local foliation $(\eta_t)_{t\in I}$ inside U satisfies the* **hyperbolicity condition** *if there is a constant $C > 0$ such that for all $t \in I$,*

$$(\mathfrak{u},\mathfrak{u})^t \geq \frac{1}{C^2} \int_U \Big(\|\mathfrak{u}(x)\|_x^2 \,+ \big|\Delta_2[\mathfrak{u},\mathfrak{u}]\big| \Big)\, \mathrm{d}\rho_t(x) \qquad \textit{for all } \mathfrak{u} \in \mathfrak{J}^{\text{vary}}\,. \tag{14.2.8}$$

A compact set $L \subset M$ is a **lens-shaped region** *inside U if there is a local foliation $(\eta_t)_{t\in I}$ inside U satisfying* (14.1.1) *which satisfies the hyperbolicity conditions.*

We point out that these hyperbolicity conditions also pose constraints for the choice of the functions η_t; these constraints can be understood as replacing the condition in the theory of hyperbolic PDEs that the initial data surface be spacelike. In general situations, the inequality (14.2.8) is not obvious and must be arranged and verified in the applications. More specifically, one can use the freedom in choosing the jet space $\mathfrak{J}^{\text{vary}}$, the Riemannian metric in the scalar product (14.2.7) and the functions η_t in Definition 14.1.1 in order to ensure that (14.2.8) holds.

We now explain how the above hyperbolicity condition can be used to derive energy estimates. We let L be a lens-shaped region inside U with the local foliation $(\eta_t)_{t\in I}$. We denote the norm corresponding to the jet scalar product by $\|\mathfrak{u}\|^t := \sqrt{(\mathfrak{u},\mathfrak{u})^t}$. We begin with a simple estimate of the energy identity in Lemma 14.2.1.

LEMMA 14.2.3. *Assume that the hyperbolicity condition of Definition 14.2.2 holds. Then for every $t \in I$ and all $\mathfrak{u} \in \mathfrak{J}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\mathfrak{u}\|^t \leq C \, \|\Delta \mathfrak{u}\|_{L^2(U, \mathrm{d}\rho_t)} + c \, \|\mathfrak{u}\|^t \tag{14.2.9}$$

with

$$c := C^2 + \frac{C^2 \, \mathfrak{s}}{2} \,.$$

PROOF. Applying (14.2.8) in (14.2.3), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} (\mathfrak{u}, \mathfrak{u})^t &\leq 2 \int_U \langle \mathfrak{u}, \Delta \mathfrak{u} \rangle_x \, \mathrm{d}\rho_t(x) - 2 \int_U \Delta_2[\mathfrak{u}, \mathfrak{u}] \, \mathrm{d}\rho_t(x) + \mathfrak{s} \int_U b(x)^2 \, \mathrm{d}\rho_t(x) \\
&\leq 2 \int_U \langle \mathfrak{u}, \Delta \mathfrak{u} \rangle_x \, \mathrm{d}\rho_t(x) + \Big(2C^2 + C^2 \, \mathfrak{s} \Big) \, (\mathfrak{u}, \mathfrak{u})^t \\
&\leq 2 \, \|\mathfrak{u}\|_{L^2(U, \mathrm{d}\rho_t)} \, \|\Delta \mathfrak{u}\|_{L^2(U, \mathrm{d}\rho_t)} + 2c \, (\mathfrak{u}, \mathfrak{u})^t \\
&\leq 2C \, \|\mathfrak{u}\|^t \, \|\Delta \mathfrak{u}\|_{L^2(U, \mathrm{d}\rho_t)} + 2c \, (\mathfrak{u}, \mathfrak{u})^t \,,
\end{aligned}$$

where in the last line we applied (14.2.8). Using the relation $\partial_t \|\mathfrak{u}\|^t = \partial_t (\mathfrak{u}, \mathfrak{u})^t / (2 \|\mathfrak{u}\|^t)$ gives the result. □

Applying Grönwall-type estimates (see for example [**1**, Lemma 1.15 in Section VII.1] or the proof of Proposition 14.2.4 below), the inequality (14.2.9) shows that $\|\mathfrak{u}\|^t$ grows at most exponentially in time, provided that $\Delta \mathfrak{u}$ decays in time sufficiently fast. We here make this statement precise by estimates in Hilbert spaces of jets with zero initial values. In the lens-shaped region L we work with the L^2-scalar product

$$\langle \mathfrak{u}, \mathfrak{v} \rangle_{L^2(L)} := \int_L \langle \mathfrak{u}(x), \mathfrak{v}(x) \rangle_x \, \eta_I(x) \, \mathrm{d}\rho(x) \,, \tag{14.2.10}$$

which, according to (14.1.2) and (14.1.3), can also be written in terms of a time integral,

$$\langle \mathfrak{u}, \mathfrak{v} \rangle_{L^2(L)} = \int_{t_0}^{t_{\max}} \langle \mathfrak{u}, \mathfrak{v} \rangle_{L^2(U, \mathrm{d}\rho_t)} \, \mathrm{d}t \,. \tag{14.2.11}$$

The corresponding norm is denoted by $\|.\|_{L^2(L)}$.

PROPOSITION 14.2.4. **(energy estimate)** *Assume that the hyperbolicity condition of Definition 14.2.2 holds. Then, choosing*

$$\Gamma = 2 \, C \, \mathrm{e}^{2c \, (t_{\max} - t_0)} \, (t_{\max} - t_0) \,, \tag{14.2.12}$$

the following estimate holds,

$$\|\mathfrak{u}\|_{L^2(L)} \leq \Gamma \, \|\Delta \mathfrak{u}\|_{L^2(L)} \qquad \textit{for all } \mathfrak{u} \in \mathfrak{J} \textit{ with } \|\mathfrak{u}\|^{t_0} = 0 \,.$$

PROOF. We write the energy estimate of Lemma 14.2.3 as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{-2ct} \, (\mathfrak{u}, \mathfrak{u})^t \right) \leq 2 \, \mathrm{e}^{-2ct} \, C \, \|\mathfrak{u}\|^t \, \|\Delta \mathfrak{u}\|_{L^2(U, \mathrm{d}\rho_t)} \,.$$

Integrating over t from t_0 to some $t \in I$ and using the hyperbolicity condition (14.2.8), we obtain

$$\begin{aligned}
\mathrm{e}^{-2ct} \, (\mathfrak{u}, \mathfrak{u})^t &= \int_{t_0}^t \frac{\mathrm{d}}{\mathrm{d}t'} \left(\mathrm{e}^{-2ct'} (\mathfrak{u}, \mathfrak{u})^{t'} \right) \, \mathrm{d}t' \\
&\leq 2 \, C \int_{t_0}^t \mathrm{e}^{-2ct'} \, \|\mathfrak{u}\|^{t'} \, \|\Delta \mathfrak{u}\|_{L^2(U, \mathrm{d}\rho_{t'})} \, \mathrm{d}t' \,.
\end{aligned}$$

Multiplying by e^{2ct} gives the inequality

$$\begin{aligned}(\mathfrak{u},\mathfrak{u})^t &\leq 2\,C \int_{t_0}^{t} \mathrm{e}^{2c\,(t-t')}\, \|\mathfrak{u}\|^{t'}\, \|\Delta\mathfrak{u}\|_{L^2(U,\mathrm{d}\rho_{t'})}\, \mathrm{d}t' \\ &\leq 2\,C\,\mathrm{e}^{2c\,(t_{\max}-t_0)} \int_{t_0}^{t_{\max}} \|\mathfrak{u}\|^{t'}\, \|\Delta\mathfrak{u}\|_{L^2(U,\mathrm{d}\rho_{t'})}\, \mathrm{d}t' \\ &\leq 2\,C\,\mathrm{e}^{2c\,(t_{\max}-t_0)}\, \|\Delta\mathfrak{u}\|_{L^2(L)} \left(\int_{t_0}^{t_{\max}} (\mathfrak{u},\mathfrak{u})^{t'}\, \mathrm{d}t' \right)^{\frac{1}{2}},\end{aligned}$$

where in the last step we used the Schwarz inequality and (14.2.11). Integrating once again over t from t_0 to $t_{\max}$ gives

$$\left(\int_{t_0}^{t_{\max}} (\mathfrak{u},\mathfrak{u})^t\, \mathrm{d}t \right)^{\frac{1}{2}} \leq 2\,C\,\mathrm{e}^{2c\,(t_{\max}-t_0)}\, (t_{\max}-t_0)\, \|\Delta\mathfrak{u}\|_{L^2(L)}\,. \tag{14.2.13}$$

Finally, we apply the hyperbolicity condition (14.2.8) in (14.2.11),

$$\|v\|_{L^2(L)} = \left(\int_{t_0}^{t_{\max}} \|\mathfrak{u}\|^2_{L^2(U,\mathrm{d}\rho_t)}\, \mathrm{d}t \right)^{\frac{1}{2}} \leq C \left(\int_{t_0}^{t_{\max}} (\mathfrak{u},\mathfrak{u})^t\, \mathrm{d}t \right)^{\frac{1}{2}}.$$

Combining this inequality with (14.2.13) gives the result. □

14.3. Uniqueness of Strong Solutions

Based on the above energy estimates, we can now prove uniqueness of strong solutions of the Cauchy problem. The methods is quite similar to that employed in Section 13.2 for symmetric hyperbolic systems. In preparation of formulating the Cauchy problem, we need to introduce jets which vanish at initial time $t_{\min}$. To this end, we demand that the jet vanishes in the surface layer and that the corresponding softened surface layer integrals (14.2.1) and (14.2.2) vanish,

$$\underline{\mathfrak{J}}_{t_{\min}} := \left\{ \mathfrak{u} \in \mathfrak{J} \;\middle|\; \eta_{t_{\min}}\, \mathfrak{u} \equiv 0 \quad \text{and} \quad (\mathfrak{u},\mathfrak{v})^{t_{\min}} = 0 = \sigma^{t_{\min}}(\mathfrak{u},\mathfrak{v}) \text{ for all } \mathfrak{v} \in \mathfrak{J} \right\}.$$

Similarly, we define the space of jets which vanish at final time $t_{\max}$ by

$$\begin{aligned}\overline{\mathfrak{J}_U}^{t_{\max}} :=& \big\{ \mathfrak{u} \in \mathfrak{J} \;\big|\; \big(1-\eta_{t_{\max}}\big)\, \mathfrak{u} \equiv 0 \\ &\text{and} (\mathfrak{u},\mathfrak{v})^{t_{\max}} = 0 = \sigma^{t_{\max}}(\mathfrak{u},\mathfrak{v}) \text{ for all } \mathfrak{v} \in \mathfrak{J} \big\}.\end{aligned}$$

A *strong solution* of the Cauchy problem is a jet $\mathfrak{u} \in \mathfrak{J}_U$ which satisfies the equations

$$\Delta\mathfrak{u} = \mathfrak{w} \quad \text{in } L \qquad \text{and} \qquad \mathfrak{u} - \mathfrak{u}_0 \in \underline{\mathfrak{J}}_{t_{\min}}\,, \tag{14.3.1}$$

where $\mathfrak{u}_0 \in \mathfrak{J}$ is the initial data and $\mathfrak{w}$ is the *inhomogeneity*. More precisely, as explained after (8.1.8), the jet $\mathfrak{w}(x)$ can be regarded as a dual jet. Here, having already introduced a scalar product on the jets at every spacetime point (see (14.2.7)), we can identify dual jets with jets. With this in mind, the inhomogeneity simply is a jet $\mathfrak{w} \in \mathfrak{J}_U$.

PROPOSITION 14.3.1. **(uniqueness of strong solutions)** *Let $(\eta_t)_{t\in I}$ be a local foliation inside U which satisfies the hyperbolicity conditions (as stated in Definitions 14.1.1 and 14.2.2). Then the Cauchy problem* (14.3.1) *with $\mathfrak{u}_0, \mathfrak{w} \in \mathfrak{J}_U$ has at most one solution $\mathfrak{u}$ in L.*

PROOF. Let $\mathfrak{u}$ be the difference of two solutions. Then $\mathfrak{u}$ is a solution of the homogeneous equation with zero initial data. Applying Lemma 14.2.3, we obtain

$$\Big| \frac{\mathrm{d}}{\mathrm{d}t} \|\mathfrak{u}\|^t \Big| \leq c\, \|\mathfrak{u}\|^t \qquad \text{and thus} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{-ct}\, \|\mathfrak{u}\|^t \right) \leq 0\,.$$

It follows that $\|\mathfrak{u}\|^t$ vanishes for all t in the respective interval. Using (14.2.8), we conclude that $\mathfrak{u}$ vanishes identically in L. This gives the result. □

Similar as explained in Section 13.2 for symmetric hyperbolic systems, this uniqueness statement also gives information on the speed of propagation and the resulting causal structure. For details we refer to [**22, 24**].

14.4. Existence of Weak Solutions

Our existence proof is inspired by the method invented by K.O. Friedrichs for symmetric hyperbolic systems in [**91**] as outlined in Section 13.3. Our first step is to formulate the linearized field equations weakly. To this end, we need to "integrate by parts" with the help of the following lemma.

LEMMA 14.4.1. **(Green's formula)** *For all* $\mathfrak{u}, \mathfrak{v} \in \mathfrak{J}$,

$$\sigma^{t_{\max}}(\mathfrak{u}, \mathfrak{v}) - \sigma^{t_{\min}}(\mathfrak{u}, \mathfrak{v}) = \langle \mathfrak{u}, \Delta \mathfrak{v} \rangle_{L^2(L)} - \langle \Delta \mathfrak{u}, \mathfrak{v} \rangle_{L^2(L)}\,.$$

PROOF. Using the definition of the L^2-scalar product in (14.2.10) and the definition of the linearized field operator (8.1.7), we obtain

$$\begin{aligned}
&\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle_{L^2(L)} - \langle \Delta \mathfrak{u}, \mathfrak{v} \rangle_{L^2(L)} = \int_U \Big(\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle - \langle \Delta \mathfrak{u}, \mathfrak{v} \rangle \Big)\, \eta_I \, \mathrm{d}\rho \\
&= \int_U \mathrm{d}\rho(x)\, \eta_I(x)\, \nabla_{\mathfrak{u}} \bigg(\int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \big) \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}}\, \mathfrak{s} \bigg) \\
&\quad - \int_U \mathrm{d}\rho(x)\, \eta_I(x)\, \nabla_{\mathfrak{v}} \bigg(\int_M \big(\nabla_{1,\mathfrak{u}} + \nabla_{2,\mathfrak{u}} \big) \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{\mathfrak{u}}\, \mathfrak{s} \bigg)\,.
\end{aligned}$$

Here the spacetime point x is in L. Using Definition 14.1.1 (ii), we get a contribution to the integrals only if $y \in U$. Therefore, we may replace the integration range M by U. We thus obtain

$$\begin{aligned}
&\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle_{L^2(L)} - \langle \Delta \mathfrak{u}, \mathfrak{v} \rangle_{L^2(L)} \\
&\quad = \int_U \mathrm{d}\rho(x)\, \eta_I(x) \int_U \mathrm{d}\rho(y) \big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{2,\mathfrak{u}} \nabla_{1,\mathfrak{v}} \big) \mathcal{L}(x,y)\,, \qquad (14.4.1)
\end{aligned}$$

where we used that, following our convention (8.1), the second derivatives of the Lagrangian are symmetric. Using the definition (14.1.3) as well as the anti-symmetry of the

integrand, the term (14.4.1) can be rewritten as

$$\begin{aligned}
&\int_U \mathrm{d}\rho(x)\, \eta_I(x) \int_U \mathrm{d}\rho(y) \big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{2,\mathfrak{u}} \nabla_{1,\mathfrak{v}}\big) \mathcal{L}(x,y) \\
&= \int_U \mathrm{d}\rho(x) \int_U \mathrm{d}\rho(y)\, \eta_t(x) \, \big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{2,\mathfrak{u}} \nabla_{1,\mathfrak{v}}\big) \mathcal{L}(x,y) \Big|_{t_0}^{t_{\max}} \\
&= \int_U \mathrm{d}\rho(x) \int_U \mathrm{d}\rho(y) \Big(\eta_t(x) - \eta_t(x)\, \eta_t(y)\Big) \big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{2,\mathfrak{u}} \nabla_{1,\mathfrak{v}}\big) \mathcal{L}(x,y) \Big|_{t_0}^{t_{\max}} \\
&= \int_U \mathrm{d}\rho(x) \int_U \mathrm{d}\rho(y)\, \eta_t(x) \, \big(1 - \eta_t(y)\big) \big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{2,\mathfrak{u}} \nabla_{1,\mathfrak{v}}\big) \mathcal{L}(x,y) \Big|_{t_0}^{t_{\max}} \\
&= \sigma^{t_{\max}}(\mathfrak{u}, \mathfrak{v}) - \sigma^{t_{\min}}(\mathfrak{u}, \mathfrak{v}) \,.
\end{aligned}$$

This gives the result. □

Assume that $\mathfrak{u}$ is a strong solution of the Cauchy problem (14.3.1). As usual, replacing $\mathfrak{u}$ by $\mathfrak{u} - \mathfrak{u}_0$ and $\mathfrak{w}$ by $\mathfrak{w} - \Delta\mathfrak{u}_0 \in \mathfrak{J}$, it suffices to consider the Cauchy problem for zero initial data, i.e.

$$\Delta\mathfrak{u} = \mathfrak{w} \quad \text{in } U \qquad \text{and} \qquad \mathfrak{u} \in \underline{\mathfrak{J}_U}_{t_{\min}} \,.$$

Then, applying the above Green's formula, we obtain for any $\mathfrak{v} \in \mathfrak{J}$,

$$\langle \mathfrak{v}, \mathfrak{w} \rangle_{L^2(L)} = \langle \mathfrak{v}, \Delta\mathfrak{u} \rangle_{L^2(L)} = \langle \Delta\mathfrak{v}, \mathfrak{u} \rangle_{L^2(L)} - \sigma^{t_{\max}}(\mathfrak{v}, \mathfrak{u}) + \sigma^{t_{\min}}(\mathfrak{v}, \mathfrak{u}) \,.$$

Having implemented the vanishing initial data by the condition $\mathfrak{u} \in \underline{\mathfrak{J}}_{t_0}$, the symplectic form vanishes at time $t_{\min}$. In order to also get rid of the boundary values at time $t_{\max}$, we restrict attention to test jets which vanish at $t_{\max}$. This leads us to the following definition:

DEFINITION 14.4.2. *A jet $\mathfrak{u} \in L^2(L)$ is a* **weak solution** *of the Cauchy problem $\Delta\mathfrak{u} = \mathfrak{w}$ with zero initial data if*

$$\langle \Delta\mathfrak{v}, \mathfrak{u} \rangle_{L^2(L)} = \langle \mathfrak{v}, \mathfrak{w} \rangle_{L^2(L)} \qquad \textit{for all } \mathfrak{v} \in \overline{\mathfrak{J}}^{t_{\max}} \,. \tag{14.4.2}$$

Clearly, the energy estimate of Proposition 14.2.4 also holds if we exchange the roles of $t_{\max}$ and $t_{\min}$, i.e.

$$\|\mathfrak{u}\|_{L^2(L)} \leq \Gamma \, \|\Delta\mathfrak{u}\|_{L^2(L)} \qquad \text{for all } \mathfrak{u} \in \overline{\mathfrak{J}}^{t_{\max}} \tag{14.4.3}$$

(where the constant Γ is again given by (14.2.12)).

We introduce the positive semi-definite bilinear form

$$<.,.> \,:\, \overline{\mathfrak{J}_U}^{t_{\max}} \times \overline{\mathfrak{J}_U}^{t_{\max}} \to \mathbb{R} \,, \qquad <\mathfrak{u}, \mathfrak{v}> = \langle \Delta\mathfrak{u}, \Delta\mathfrak{v} \rangle_{L^2(L)} \,.$$

Dividing out the null space and forming the completion, we obtain a Hilbert space $(\mathcal{H}, <.,.>)$. The corresponding norm is denoted by $|||\,.\,|||$.

We now consider the linear functional $\langle \mathfrak{w}, . \rangle_{L^2(L)}$ on $\overline{\mathfrak{J}_U}^{t_{\max}}$. Applying the Schwarz inequality and (14.4.3), we obtain

$$\big| \langle \mathfrak{w}, \mathfrak{u} \rangle_{L^2(L)} \big| \leq \|\mathfrak{w}\|_{L^2(L)} \, \|\mathfrak{u}\|_{L^2(L)} \leq \Gamma \, \|\mathfrak{w}\|_{L^2(L)} \, |||\,\mathfrak{u}\,||| \,,$$

proving that the linear functional $\langle \mathfrak{w}, . \rangle_{L^2(L)}$ on $\overline{\mathfrak{J}}^{t_{\max}}$ is bounded on $\mathcal{H}$. Therefore, it can be extended uniquely to a bounded linear functional on all of $\mathcal{H}$. Moreover, by the Fréchet-Riesz theorem there is a unique vector $U \in \mathcal{H}$ with

$$\langle \mathfrak{w}, \mathfrak{v} \rangle_{L^2(L)} = <U, \mathfrak{v}> = \langle \Delta U, \Delta\mathfrak{v} \rangle_{L^2(L)} \qquad \text{for all } \mathfrak{v} \in \overline{\mathfrak{J}_U}^{t_{\max}} \,.$$

Hence $\mathfrak{u} := \Delta U \in L^2(L)$ is the desired weak solution. We point out that in the above estimates, the inhomogeneity $\mathfrak{w}$ enters only via its L^2-norm, making it possible to generalize our methods to $\mathfrak{w} \in L^2(L)$. We have obtain the following result:

THEOREM 14.4.3. *Assume that $(\eta_t)_{t\in I}$ is a local foliation satisfying the hyperbolicity conditions (see Definitions 14.1.1 and 14.2.2). Then for every $\mathfrak{w} \in L^2(L)$ there is a weak solution $\mathfrak{u} \in L^2(L)$ of the Cauchy problem* (14.4.2).

We remark that the construction of weak solutions is the starting point for the more detailed analysis of linearized fields as carried out in [**22, 64, 24, 74**]. One task is to deal with the uniqueness problem for weak solutions (see Exercise 14.3). Another issue is to construct global solutions (for various methods for doing so see [**22**, Section 4], [**64**, Section 6.3], [**24**, Section 3.3] and [**74**]). Once global advanced and retarded solutions have been obtained for a general class of inhomogeneities, one can also introduce corresponding Green's operators (see [**22**, Section 5], [**74**, Section 4] or Exercise 14.4).

14.5. Exercises

EXERCISE 14.1. *(Differentiated form of conservation laws)* Conservation laws for causal variational principles are formulated in terms of surface layer integrals (see for example Theorem 9.2.3, Theorem 9.3.2, Proposition 9.3.1 or Lemma 14.4.1). For the proofs, we rewrote the surface layer integrals as double volume integrals, making use of anti-symmetry properties (see for example the proof of Theorem 9.3.2). Alternatively, one can prove the conservation laws by computing the time derivatives. The goal of this exercise is to illustrate this method (for more details see for example [**61**, Section 2.6]).

(a) Given a local foliation $(\eta_t)_{t\in I}$ we consider the softened symplectic form (14.2.2). Given $\mathfrak{u}, \mathfrak{v} \in \mathfrak{J}$, compute
$$\frac{\mathrm{d}}{\mathrm{d}t}\, \sigma^t(\mathfrak{u}, \mathfrak{v})$$
in similar style as in the proof of Lemma 14.2.1. Use this formula to give an alternative proof of Lemma 14.4.1.

(b) Given a local foliation $(\eta_t)_{t\in I}$, formulate a softened version of the surface layer integral I^Ω_{k+1} in Theorem 9.3.4. Differentiate with respect to the time parameter t to obtain an alternative proof of Theorem 9.3.4.

EXERCISE 14.2. *(Counter example to hyperbolicity conditions)* The hyperbolicity conditions in Definition 14.2.2 were formulated only for jets in a subspace $\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}$. The goal of this exercise is to explain why it would not be sensible to impose the inequality (14.2.8) for all $\mathfrak{u} \in \mathfrak{J}$. To this end, consider for simplicity the unsoftened surface layer integral (9.5.6). Show that there is a jet $\mathfrak{u} \in C^0_0(M, \mathbb{R}) \oplus C^0_0(M, T\mathcal{F})$ with $(\mathfrak{u}, \mathfrak{u})^t < 0$. *Hint:* Choose points $x \in \Omega$ and $y \in M \setminus \Omega$ for which $\mathcal{L}(x, y) \neq 0$. Choose $\mathfrak{u}$ as a scalar jet which is supported in a small neighborhood of x and y.

EXERCISE 14.3. *(Non-uniqueness of weak solutions)* As illustrated in the previous exercise (Exercise 14.2), in order to satisfy the hyperbolicity conditions, the jet space $\mathfrak{J}^{\text{vary}}$ must not be chosen too large. In particular, in typical examples the jet space $\mathfrak{J}^{\text{vary}}$ is not dense in $L^2(L)$. This leads to a non-uniqueness issue for weak solutions, which will be illustrated in this exercise.

(a) Given an inhomogeneity $\mathfrak{w} \in L^2(L)$, to which extent are weak solutions unique? Specify the jet space of all homogeneous solutions.

(b) On the other hand, the construction before Theorem 14.4.3 gives a unique solution $\mathfrak{u} = \Delta U$. How does this result fit together with the non-uniqueness in (a)? In which sense is the solution $\mathfrak{u} = \Delta U$ distinguished?

Hint: Similar questions are analyzed in [**22**, Section 3].

EXERCISE 14.4. *(Causal Green's operators for the linearized field equations)* In order to avoid the issue of how to "glue together" local solutions obtained in different lens-shaped regions such as to obtain global solutions, one can simplify the setting by assuming that the spacetime M admits a global foliation $(\eta_t)_{t\in\mathbb{R}}$.

(a) How can Definition 14.1.1 be modified in order to describe a global foliation? What is the resulting global analog of Theorem 14.4.3? *Hint:* It might be instructive to compare your definition with Definition 6.4 in [**64**].

(b) Suppose that we know that for any compactly supported jet $\mathfrak{w} \in L^2_0(M)$ there is a distinguished global weak solution $\mathfrak{v} \in L^2_{\text{loc}}(M)$, i.e.

$$\langle \Delta\mathfrak{v}, \mathfrak{u}\rangle_{L^2(M)} = \langle \mathfrak{v}, \mathfrak{w}\rangle_{L^2(M)} \qquad \text{for all } \mathfrak{v} \in \mathfrak{J}_0 \,.$$

Then the operator $S : \mathfrak{w} \to -\mathfrak{v}$ is referred to as the Green's operator. How can one distinguish between the advanced Green's operator $S^\vee$ and the retarded Green's operator $S^\wedge$. Show that their difference $G := S^\wedge - S^\vee$ maps to homogeneous weak solutions. *Hint:* More details on Green's operators and their causal properties can be found in [**22**, Section 5], [**24**, Section 3] and [**74**, Section 4].

CHAPTER 15

Functional Analytic Methods in Spacetime

When constructing a causal fermion system in Minkowski space in Section 5.5, we chose $\mathcal{H}$ as a subspace of the solution space $\mathcal{H}_m$ of the Dirac equation. In principle, one can choose $\mathcal{H}$ as one likes, and different choices give rise to different causal fermion systems. However, if one wants to describe a given physical system, one must specify the subspace $\mathcal{H} \subset \mathcal{H}_m$, and it important to do it right. It is not obvious what "right" and "wrong" should be. Generally speaking, $\mathcal{H}$ can be thought of as the "occupied states" of the physical system under consideration. If we want to describe the vacuum in Minkowski space (i.e. no particles and no interaction is present), then the natural and only physically reasonable choice is to let $\mathcal{H}$ be the subspace of all *negative-frequency solutions* of the Dirac equation. As already explained in the Section 1.5 in the preliminaries, this choice corresponds to the physical concept of the Dirac sea as introduced by Dirac in 1930, which led to the prediction of anti-matter (discovered shortly afterward in 1932, earning Dirac the Nobel prize in 1933). Following these physical concepts, it is also clear that if particles and/or anti-particles (but no interaction of the matter) is present, then $\mathcal{H}$ is obtained from the subspace of all negative-frequency solutions by occupying additional particle states and by creating "holes" in the sea corresponding to the anti-particle states. Once an interaction (for example an electromagnetic field) is present, it is no longer clear how $\mathcal{H}$ is to be chosen. The reason is that, as soon as the fields are time-dependent, the notion of positive and negative frequency solutions breaks down, so that there is no obvious decomposition of the solution space into two subspaces. But for the description of the physical system, a decomposition of the solution space is needed, and taking the "wrong" decomposition leads to artificial mathematical and physical difficulties.

We now explain a functional analytic method which gives rise to a canonical decomposition of the solution space into two subspaces, even in the time-dependent situation. In the static situation, this decomposition reduces to the canonical frequency splitting. This splitting is "right" in the sense that it gives rise to a physically sensible ground state of the system (a so-called Hadamard state, as we will learn in Chapter 19). Moreover, when performing our construction perturbatively, one can compute the singularities of $P(x,y)$ explicitly working exclusively with bounded line integrals. These explicit computations are the backbone of the analysis of the continuum limit in [**45**]. Before outlining the perturbative treatment (see Chapter 18), we now explain the general functional analytic construction.

15.1. General Setting and Basic Ideas

In preparation, we summarize the structures of Section 1 using a more general notation, which has the advantage that our setting applies just as well if Minkowski space is replaced by a globally hyperbolic spacetime. Thus the reader who is familiar with general relativity and Lorentzian geometry, in what follows may consider $(\mathscr{M}, g)$ as a globally hyperbolic Lorentzian manifold with spinor bundle $(S\mathscr{M}, \prec.|.\succ)$. The Dirac equation is

written as

$$(\mathcal{D} - m)\, \psi_m = 0 \tag{15.1.1}$$

(here the subscript m indicates the mass of the solution; this is of advantage because later on, we shall consider families of solutions with a varying mass parameter). In Minkowski space, one chooses $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ such as to get back to (1.3.14), where $\mathcal{B}$ is an arbitrary multiplication operator satisfying the symmetry condition (1.3.13). More generally, in a globally hyperbolic spacetime, the Dirac operator is a first order differential operator, but the coefficients depend on the metric (for details see Chapter 4). Next, we let $\mathcal{N}$ be any Cauchy surface. Then the scalar product (1.3.12) on the solutions can be written more generally as

$$(\psi_m | \phi_m)_m = 2\pi \int_{\mathcal{N}} \prec \psi_m | \psi \phi_m \succ_x \mathrm{d}\mu_{\mathcal{N}}(x)\,, \tag{15.1.2}$$

where ν is the future-directed normal and $\mathrm{d}\mu_{\mathcal{N}}$ the volume measure given by the induced Riemannian metric on $\mathcal{N}$ (in Minkowski space and $\mathcal{N} = \{t = \text{const}\}$, the normal has the components $\nu^i = (1,0,0,0)$ and $\mathrm{d}\mu_{\mathcal{N}} = \mathrm{d}^3x$, giving back (1.3.12)). Similar to the computation (1.3.10), the vector field $\prec \psi_m | \gamma^j \phi_m \succ_x$ is again divergence-free, implying that the above scalar product is independent of the choice of the Cauchy surface (for details see [**80**, Section 2]). Forming the completion gives the Hilbert space $(\mathcal{H}_m, (.|.)_m)$.

For the following constructions, we again need the spacetime inner product (13.4.8). In order to explain the basic idea of the construction as first given in [**80**], let us assume for simplicity that the integral in (13.4.8) exists for all solutions $\psi_m, \phi_m \in \mathcal{H}_m$. This condition is not satisfied in Minkowski space because the time integral in (13.4.8) in general diverges. But it is indeed satisfied in spacetimes of finite lifetime (for details see [**80**, Section 3.2]). Under this assumption, the spacetime inner product can be extended by continuity to a sesquilinear form

$$<.|.> \;:\; \mathcal{H}_m \times \mathcal{H}_m \to \mathbb{C}\,,$$

which is bounded, i.e.

$$|<\phi_m | \psi_m>| \leq c\, \|\phi_m\|_m\, \|\psi_m\|_m \tag{15.1.3}$$

(where $\|.\|_m = (.|.)_m^{\frac{1}{2}}$ is the norm on $\mathcal{H}_m$). Then, applying the Fréchet-Riesz theorem (similar as explained in the construction of the local correlation operator (5.5.3) in Section 5.5), we can uniquely represent this inner product on the Hilbert space $\mathcal{H}_m$ with a signature operator $\mathcal{S}$,

$$\mathcal{S} \;:\; \mathcal{H}_m \to \mathcal{H}_m \qquad \text{with} \qquad <\phi_m | \psi_m> = (\phi_m \,|\, \mathcal{S}\, \psi_m)_m\,.$$

We refer to $\mathcal{S}$ as the **fermionic signature operator**. It is obviously a symmetric operator. Moreover, it is bounded according to (15.1.3). Therefore, the spectral theorem for selfadjoint operators gives the spectral decomposition

$$\mathcal{S} = \int_{\sigma(\mathcal{S})} \lambda\, dE_\lambda\,,$$

where E_λ is the spectral measure (see Section 3.2 or for example [**131**]). The spectral measure gives rise to the spectral calculus

$$f(\mathcal{S}) = \int_{\sigma(\mathcal{S})} f(\lambda)\; \mathrm{d}E_\lambda \;:\; \mathcal{H}_m \to \mathcal{H}_m\,,$$

where f is a bounded Borel function on $\sigma(\mathcal{S}) \subset \mathbb{R}$. Choosing f as a characteristic function, one obtains the operators $\chi_{(0,\infty)}(\mathcal{S})$ and $\chi_{(-\infty,0)}(\mathcal{S})$. Their images are referred to as the *positive* and *negative spectral subspace* of $\mathcal{H}_m$, respectively. In this way, one

obtains the desired decomposition of the solution space into two subspaces. We remark that the fermionic signature operator also gives a setting for doing spectral geometry and index theory with Lorentzian signature. We will not enter these topics here but refer the interested reader to the papers [**78, 46**].

The basic shortcoming of the above construction is that in many physically interesting spacetimes (like Minkowski space) the inequality (15.1.3) fails to be true. The idea to bypass this problem is to make use of the fact that a typical solution $\psi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$ of the Dirac equation oscillates for large times. If, instead of a single solution, we consider a *family* of solutions with a *varying mass parameter* m, then the wave functions for different values of m typically have different phases. Therefore, integrating over the mass parameter leads to dephasing (in the physics literature also referred to as destructive interference), giving rise to decay in time. In order to make this idea mathematically precise, one considers families of solutions $(\psi_m)_{m \in I}$ of the family of Dirac equations (15.1.1) with the mass parameter m varying in an open interval I. We need to assume that I does not contain the origin, because our methods for dealing with infinite lifetime do not apply in the massless case $m = 0$ (this seems no physical restriction because all known fermions in nature have a non-zero rest mass). By symmetry, it suffices to consider positive masses. Thus we choose I as the interval

$$I := (m_L, m_R) \subset \mathbb{R} \qquad \text{with parameters } m_L, m_R > 0\,. \tag{15.1.4}$$

The masses of the Dirac particles of our physical system should be contained in I. Apart from that, the choice of I is arbitrary and, as we shall see, all our results will be independent of the choice of m_L and m_R. We always choose the family of solutions $(\psi_m)_{m \in I}$ in the class $C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$ of smooth solutions with spatially compact support in Minkowski space $\mathscr{M}$ which depend smoothly on m and vanish identically for m outside a compact subset of I. Then the "decay due to destructive interference" can be made precise by demanding that there is a constant $c > 0$ such that

$$\left| < \int_I \phi_m \, \mathrm{d}m \,|\, \int_I \psi_{m'} \, \mathrm{d}m' > \right| \leq c \int_I \|\phi_m\|_m \, \|\psi_m\|_m \, \mathrm{d}m \tag{15.1.5}$$

for all families of solutions $(\psi_m)_{m \in I}, (\phi_m)_{m \in I} \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$. The point is that we integrate over the mass parameter *before* taking the spacetime inner product. Intuitively speaking, integrating over the mass parameter generates a decay of the wave function, making sure that the time integral converges. The inequality (15.1.5) is one variant of the so-called *mass oscillation property*. If (15.1.5) holds, we shall prove that there is a representation

$$< \int_I \phi_m \, \mathrm{d}m \,|\, \int_I \psi_{m'} \, \mathrm{d}m' > = \int_I (\phi_m \,|\, \tilde{\mathcal{S}}_m \, \psi_m)_m \, \mathrm{d}m\,, \tag{15.1.6}$$

which for every $m \in I$ uniquely defines the *fermionic signature operator* $\tilde{\mathcal{S}}_m$. This operator is bounded and symmetric with respect to the scalar product (15.1.2). Moreover, it does not depend on the choice of the interval I. Then the positive and negative spectral subspaces of the operator $\tilde{\mathcal{S}}_m$ again yield the desired splitting of the solution space into two subspaces.

Before entering the detailed constructions, we explain how the above integrals over the mass parameters are to be understood. At first sight, integrating over a varying mass parameter $m \in I$ may look like "smearing out" the physical mass in the Dirac equation. However, this picture is misleading. Instead, one should consider the mass integrals merely as a technical tool in order to generate decay for large times. The resulting

operators $\tilde{\mathcal{S}}_m$ in (15.1.6) act on ψ_m with the corresponding mass $m \in I$. Choosing m again as the physical mass, the operator $\tilde{\mathcal{S}}_m$ acts on standard Dirac wave functions describing physical particles, without any smearing in the mass parameter.

15.2. The Mass Oscillation Properties

In a spacetime of infinite lifetime, the spacetime inner product $<\psi_m|\phi_m>$ of two solutions $\psi_m, \phi_m \in \mathcal{H}_m$ is in general ill-defined, because the time integral in (13.4.8) may diverge. In order to avoid this difficulty, we shall consider families of solutions with a variable mass parameter. The so-called mass oscillation property will make sense of the spacetime integral in (13.4.8) after integrating over the mass parameter.

We consider the mass parameter in a bounded open interval I (15.1.4). For a given Cauchy surface $\mathcal{N}$, we consider a function $\psi_{\mathcal{N}}(x, m) \in S_x\mathscr{M}$ with $x \in \mathcal{N}$ and $m \in I$. We assume that this wave function is smooth and has compact support in both variables, $\psi_{\mathcal{N}} \in C_0^\infty(\mathcal{N} \times I, S\mathscr{M})$. For every $m \in I$, we let $\psi(., m)$ be the solution of the Cauchy problem for initial data $\psi_{\mathcal{N}}(., m)$,

$$(\mathcal{D} - m)\,\psi(x, m) = 0\,, \qquad \psi(x, m) = \psi_{\mathcal{N}}(x, m) \;\; \forall\, x \in \mathcal{N}\,. \tag{15.2.1}$$

Since the solution of the Cauchy problem is smooth and depends smoothly on parameters, we know that $\psi \in C^\infty(\mathscr{M} \times I, S\mathscr{M})$. Moreover, due to finite propagation speed, $\psi(., m)$ has spatially compact support. Finally, the solution is clearly compactly supported in the mass parameter m. We summarize these properties by writing

$$\psi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})\,, \tag{15.2.2}$$

where $C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$ denotes the smooth wave functions with spatially compact support which are also compactly supported in I. We often denote the dependence on m by a subscript, $\psi_m(x) := \psi(x, m)$. Then for any fixed m, we can take the scalar product (15.1.2). On families of solutions $\psi, \phi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$ of (15.2.1), we introduce a scalar product by integrating over the mass parameter,

$$(\psi|\phi)_I := \int_I (\psi_m|\phi_m)_m \;\mathrm{d}m \tag{15.2.3}$$

(where $\mathrm{d}m$ is the Lebesgue measure). Forming the completion, we obtain the Hilbert space $(\mathcal{H}, (.|.)_I)$. It consists of measurable functions $\psi(x, m)$ such that for almost all $m \in I$, the function $\psi(., m)$ is a weak solution of the Dirac equation which is square integrable over any Cauchy surface. Moreover, this spatial integral is integrable over $m \in I$, so that the scalar product (15.2.3) is well-defined. We denote the norm on $\mathcal{H}$ by $\|.\|_I$.

For the applications, it is useful to introduce a subspace of the solutions of the form (15.2.2):

DEFINITION 15.2.1. *We let $\mathcal{H}^\infty \subset C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$ be a subspace of the smooth solutions with the following properties:*

(i) *$\mathcal{H}^\infty$ is invariant under multiplication by smooth functions in the mass parameter,*

$$\eta(m)\,\psi(x, m) \in \mathcal{H}^\infty \qquad \forall\, \psi \in \mathcal{H}^\infty,\; \eta \in C^\infty(I)\,.$$

(ii) *For every $m \in I$, the set $\mathcal{H}^\infty_m := \{\psi(., m)\,|\,\psi \in \mathcal{H}^\infty\}$ is a dense subspace of $\mathcal{H}_m$,*

$$\overline{\mathcal{H}^\infty_m}^{(.|.)_m} = \mathcal{H}_m \qquad \forall\, m \in I\,.$$

We refer to $\mathcal{H}^\infty$ as the **domain** *for the mass oscillation property.*

The simplest choice is to set $\mathcal{H}^\infty = C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$, but in some applications it is preferable to choose $\mathcal{H}^\infty$ as a proper subspace of $C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$. (for example, in [**81**, Section 6], the space $\mathcal{H}^\infty$ was chosen as being spanned by a finite number of angular modes, making it unnecessary to prove estimates uniform in the angular mode).

Our motivation for considering a variable mass parameter is that integrating over the mass parameter should improve the decay properties of the wave function for large times (similar as explained in the introduction in the vacuum Minkowski space). This decay for large times should also make it possible to integrate the Dirac operator in the inner product (13.4.8) by parts without boundary terms,

$$<\mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi> ,$$

implying that the solutions for different mass parameters should be orthogonal with respect to this inner product. Instead of acting with the Dirac operator, it is technically easier to work with the operator of multiplication by m, which we denote by

$$T \,:\, \mathcal{H} \to \mathcal{H} , \qquad (T\psi)_m = m\, \psi_m . \tag{15.2.4}$$

In view of property (i) in Definition 15.2.1, this operator leaves $\mathcal{H}^\infty$ invariant,

$$T|_{\mathcal{H}^\infty} \,:\, \mathcal{H}^\infty \to \mathcal{H}^\infty .$$

Moreover, T is a symmetric operator, and it is bounded because the interval I is,

$$T^* = T \in \mathrm{L}(\mathcal{H}) .$$

Finally, integrating over m gives the operation

$$\mathfrak{p} \,:\, \mathcal{H}^\infty \to C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) , \qquad \mathfrak{p}\psi = \int_I \psi_m \, \mathrm{d}m . \tag{15.2.5}$$

We point out for clarity that $\mathfrak{p}\psi$ no longer satisfies a Dirac equation. The following notions were introduced in [**81**], and we refer the reader to this paper for more details.

Definition 15.2.2. *The Dirac operator $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ on Minkowski space $\mathscr{M}$ has the* **weak mass oscillation property** *in the interval $I = (m_L, m_R)$ with domain $\mathcal{H}^\infty$ if the following conditions hold:*

(a) *For every $\psi, \phi \in \mathcal{H}^\infty$, the function $\prec\mathfrak{p}\phi|\mathfrak{p}\psi\succ$ is integrable on $\mathscr{M}$. Moreover, there is a constant $c = c(\psi)$ such that*

$$|<\mathfrak{p}\psi|\mathfrak{p}\phi>| \leq c\, \|\phi\|_I \qquad \text{for all } \phi \in \mathcal{H}^\infty . \tag{15.2.6}$$

(b) *For all $\psi, \phi \in \mathcal{H}^\infty$,*

$$<\mathfrak{p}T\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi> . \tag{15.2.7}$$

Definition 15.2.3. *The Dirac operator $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ on Minkowski space $\mathscr{M}$ has the* **strong mass oscillation property** *in the interval $I = (m_L, m_R)$ with domain $\mathcal{H}^\infty$ if there is a constant $c > 0$ such that*

$$|<\mathfrak{p}\psi|\mathfrak{p}\phi>| \leq c \int_I \|\phi_m\|_m \, \|\psi_m\|_m \, \mathrm{d}m \qquad \textit{for all } \psi, \phi \in \mathcal{H}^\infty . \tag{15.2.8}$$

15.3. The Fermionic Signature Operator

In this section we give abstract constructions based on the mass oscillation property. We first assume that the *weak mass oscillation property* of Definition 15.2.2 holds. Then, in view of the inequality (15.2.6), every $\psi \in \mathcal{H}^\infty$ gives rise to a bounded linear functional on $\mathcal{H}^\infty$. By continuity, this linear functional can be uniquely extended to $\mathcal{H}$. The Fréchet-Riesz theorem allows us to represent this linear functional by a vector $u \in \mathcal{H}$, i.e.

$$(u|\phi)_I = <\mathfrak{p}\psi|\mathfrak{p}\phi> \qquad \forall\, \phi \in \mathcal{H}\,.$$

Varying ψ, we obtain the linear mapping

$$\mathcal{S} \;:\; \mathcal{H}^\infty \to \mathcal{H}\,, \qquad (\mathcal{S}\psi|\phi)_I = <\mathfrak{p}\psi|\mathfrak{p}\phi> \quad \forall\, \phi \in \mathcal{H}\,.$$

This operator is symmetric because

$$(\mathcal{S}\psi|\phi)_I = <\mathfrak{p}\psi|\mathfrak{p}\phi> = (\psi|\mathcal{S}\phi)_I \qquad \forall\, \phi, \psi \in \mathcal{H}^\infty\,.$$

Moreover, (15.2.7) implies that the operators $\mathcal{S}$ and T commute,

$$\mathcal{S}\,T = T\,\mathcal{S} \;:\; \mathcal{H}^\infty \to \mathcal{H}\,.$$

Thus the weak mass oscillation property makes it possible to introduce $\mathcal{S}$ as a densely defined symmetric operator on $\mathcal{H}$. It is indeed possible to construct a selfadjoint extension of the operator $\mathcal{S}^2$ (using the Friedrich's extension), giving rise to a functional calculus with corresponding spectral measure (for details see [**81**, Section 3]). In this setting the operator $\mathcal{S}$ and the spectral measure are operators on the Hilbert space $\mathcal{H}$ which involves an integration over the mass parameter. In simple terms, this implies that all objects are defined only for almost all values of m (with respect to the Lebesgue measure on $I \subset \mathbb{R}$), and they can be modified arbitrarily on subsets of I of measure zero. But it does not seem possible to "evaluate pointwise in the mass" by constructing operators $\mathcal{S}_m$ which act on the Hilbert space $\mathcal{H}_m$ for fixed mass.

In view of this shortcoming, we shall not enter the spectral calculus based on the weak mass oscillation operator. Instead, we move on to the *strong mass oscillation property*, which makes life much easier because it implies that $\mathcal{S}$ is a bounded operator.

THEOREM 15.3.1. *The following statements are equivalent:*

(i) *The strong mass oscillation property holds.*

(ii) *There is a constant $c > 0$ such that for all $\psi, \phi \in \mathcal{H}^\infty$, the following two relations hold:*

$$|<\mathfrak{p}\psi|\mathfrak{p}\phi>| \leq c\,\|\psi\|_I\,\|\phi\|_I \tag{15.3.1}$$

$$<\mathfrak{p}T\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi>\,. \tag{15.3.2}$$

(iii) *There is a family of linear operators $\mathcal{S}_m \in \mathrm{L}(\mathcal{H}_m)$ which are uniformly bounded,*

$$\sup_{m \in I} \|\mathcal{S}_m\| < \infty\,,$$

such that

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi_m \,|\, \mathcal{S}_m\,\phi_m)_m \;\mathrm{d}m \qquad \forall\, \psi, \phi \in \mathcal{H}^\infty\,. \tag{15.3.3}$$

PROOF. The implication (iii)⇒(i) follows immediately from the estimate

$$|<\mathfrak{p}\psi|\mathfrak{p}\phi>| \leq \int_I \big|(\psi_m|\mathcal{S}_m\phi_m)_m\big| \;\mathrm{d}m \leq \sup_{m\in I} \|\mathcal{S}_m\| \int_I \|\psi_m\|_m\,\|\phi\|_m \;\mathrm{d}m\,.$$

In order to prove the implication (i)⇒(ii), we first apply the Schwarz inequality to (15.2.8) to obtain

$$\begin{aligned} |<\mathfrak{p}\psi|\mathfrak{p}\phi>| &\leq c \int_I \|\phi_m\|_m \, \|\psi_m\|_m \, \mathrm{d}m \\ &\leq c \Big(\int_I \|\phi_m\|_m^2 \, \mathrm{d}m \Big)^{\frac{1}{2}} \Big(\int_I \|\psi_m\|_m^2 \, \mathrm{d}m \Big)^{\frac{1}{2}} = c \, \|\phi\|_I \, \|\psi\|_I \,, \end{aligned}$$

proving (15.3.1). Next, given $N \in \mathbb{N}$ we subdivide the interval $I = (m_L, m_R)$ by choosing the intermediate points

$$m_\ell = \frac{\ell}{N} \, (m_R - m_L) + m_L \,, \qquad \ell = 0, \ldots, N \,.$$

Moreover, we choose non-negative test functions $\eta_1, \ldots, \eta_N \in C_0^\infty(\mathbb{R})$ which form a partition of unity and are supported in small sub-intervals, meaning that

$$\sum_{\ell=1}^N \eta_\ell\big|_I = 1|_I \qquad \text{and} \qquad \operatorname{supp} \eta_\ell \subset (m_{\ell-1}, m_{\ell+1}) \,, \tag{15.3.4}$$

where we set $m_{-1} = m_L - 1$ and $m_{N+1} = m_R + 1$. For any smooth function $\eta \in C_0^\infty(\mathbb{R})$ we define the bounded linear operator $\eta(T) : \mathcal{H}^\infty \to \mathcal{H}^\infty$ by

$$\big(\eta(T)\psi\big)_m = \eta(m) \, \psi_m \,.$$

Then by linearity,

$$\begin{aligned} &<\mathfrak{p}T\psi|\mathfrak{p}\phi> - <\mathfrak{p}\psi|\mathfrak{p}T\phi> \\ &= \sum_{\ell,\ell'=1}^N \Big(<\mathfrak{p}\, T\, \eta_\ell(T)\, \psi \,|\, \mathfrak{p}\, \eta_{\ell'}(T)\, \phi> - <\mathfrak{p}\, \eta_\ell(T)\, \psi \,|\, \mathfrak{p}\, T\, \eta_{\ell'}(T)\, \phi> \Big) \\ &= \sum_{\ell,\ell'=1}^N \Big(<\mathfrak{p}\, \big(T - m_\ell\big)\, \eta_\ell(T)\, \psi \,|\, \mathfrak{p}\, \eta_{\ell'}(T)\, \phi> - <\mathfrak{p}\, \eta_\ell(T)\, \psi \,|\, \mathfrak{p}\, \big(T - m_\ell\big)\, \eta_{\ell'}(T)\, \phi> \Big) \,. \end{aligned}$$

Taking the absolute value and applying (15.2.8), we obtain

$$\big|<\mathfrak{p}T\psi|\mathfrak{p}\phi> - <\mathfrak{p}\psi|\mathfrak{p}T\phi>\big| \leq c \sum_{\ell,\ell'=1}^N \int_I |m - m_\ell| \, \eta_\ell(m) \, \eta_{\ell'}(m) \, \|\phi_m\|_m \, \|\psi_m\|_m \, \mathrm{d}m \,.$$

In view of the second property in (15.3.4), we only get a contribution if $|\ell - \ell'| \leq 1$. Moreover, we know that $|m - m_\ell| \leq 2\,|I|/N$ on the support of η_ℓ. Thus

$$\begin{aligned} \big|<\mathfrak{p}T\psi|\mathfrak{p}\phi> - <\mathfrak{p}\psi|\mathfrak{p}T\phi>\big| &\leq \frac{6c\,|I|}{N} \sum_{\ell=1}^N \int_I \eta_\ell(m) \, \|\phi_m\|_m \, \|\psi_m\|_m \, \mathrm{d}m \\ &= \frac{6c\,|I|}{N} \int_I \|\phi_m\|_m \, \|\psi_m\|_m \, \mathrm{d}m \,. \end{aligned}$$

Since N is arbitrary, we obtain (15.3.2).

It remains to prove the implication (ii)⇒(iii). Combining (15.3.1) with the Fréchet-Riesz theorem, there is a bounded operator $\mathcal{S} \in \mathrm{L}(\mathcal{H})$ with

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = (\psi|\mathcal{S}\phi)_I \qquad \forall\, \psi, \phi \in \mathcal{H}^\infty \,. \tag{15.3.5}$$

The relation (15.3.2) implies that the operators $\mathcal{S}$ and T commute. Moreover, these two operators are obviously symmetric. Hence the spectral theorem for commuting selfadjoint operators implies that there is a spectral measure F on $\sigma(\mathcal{S}) \times I$ such that

$$\mathcal{S}^p\, T^q = \int_{\sigma(\mathcal{S})\times I} \nu^p\, m^q \, \mathrm{d}F_{\nu,m} \qquad \forall\, p, q \in \mathbb{N}\,. \tag{15.3.6}$$

For given $\psi, \phi \in \mathcal{H}^\infty$, we introduce the Borel measure $\mu_{\psi,\phi}$ on I by

$$\mu_{\psi,\phi}(\Omega) = \int_{\sigma(\mathcal{S})\times\Omega} \nu \;\mathrm{d}(\psi|F_{\nu,m}\phi)_I\,. \tag{15.3.7}$$

Then $\mu_{\psi,\phi}(I) = (\psi|\mathcal{S}\phi)_I$ and

$$\mu_{\psi,\phi}(\Omega) = \int_{\sigma(\mathcal{S})\times I} \nu \;\mathrm{d}\big(\chi_\Omega(T)\,\psi \,\big|\, F_{\nu,m}\,\chi_\Omega(T)\,\phi\big)_I = (\chi_\Omega(T)\,\psi \,|\, \mathcal{S}\,\chi_\Omega(T)\,\phi)_I\,.$$

Since the operator $\mathcal{S}$ is bounded, we conclude that

$$\begin{aligned} |\mu_{\psi,\phi}(\Omega)| &\leq c\,\|\chi_\Omega(T)\,\psi\|_I\,\|\chi_\Omega(T)\,\phi\|_I \overset{(15.2.3)}{=} c\left(\int_\Omega \|\psi\|_m^2 \;\mathrm{d}m \int_\Omega \|\phi\|_{m'}^2 \;\mathrm{d}m'\right)^{\frac{1}{2}} \\ &\leq c\,|\Omega|\,\Big(\sup_{m\in\Omega}\|\psi_m\|_m\Big)\Big(\sup_{m'\in\Omega}\|\phi_{m'}\|_{m'}\Big)\,. \end{aligned} \tag{15.3.8}$$

This shows that the measure μ is absolutely continuous with respect to the Lebesgue measure. The Radon-Nikodym theorem (see Theorem 12.5.2) implies that there is a unique function $f_{\psi,\phi} \in L^1(I, \mathrm{d}m)$ such that

$$\mu_{\psi,\phi}(\Omega) = \int_\Omega f_{\psi,\phi}(m) \;\mathrm{d}m\,. \tag{15.3.9}$$

Using this representation in (15.3.8), we conclude that for any $\varphi \in \mathbb{R}$,

$$\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\varphi}\int_\Omega f_{\psi,\phi}(m) \;\mathrm{d}m\right) \leq \big|\mu_{\psi,\phi}(\Omega)\big| \leq c\,|\Omega|\,\Big(\sup_{m\in\Omega}\|\psi_m\|_m\Big)\Big(\sup_{m'\in\Omega}\|\phi_{m'}\|_{m'}\Big)\,.$$

As a consequence, for almost all $m \in I$ (with respect to the Lebesgue measure $\mathrm{d}m$),

$$\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\varphi}\, f_{\psi,\phi}(m)\big) \leq c\,\|\psi_m\|_m\,\|\phi_m\|_m\,.$$

Since the phase factor is arbitrary, we obtain the pointwise bound

$$|f_{\psi,\phi}(m)| \leq c\,\|\psi_m\|_m\,\|\phi_m\|_m \qquad \text{for almost all } m \in I\,.$$

Using this inequality, we can apply the Fréchet-Riesz theorem to obtain a unique operator $\mathcal{S}_m \in \mathrm{L}(\mathcal{H}_m)$ such that

$$f_{\psi,\phi}(m) = (\psi_m|\mathcal{S}_m\phi_m)_m \qquad \text{and} \qquad \|\mathcal{S}_m\| \leq c\,. \tag{15.3.10}$$

Combining the above results, for any $\psi, \phi \in \mathcal{H}^\infty$ we obtain

$$\begin{aligned} <\mathfrak{p}\psi|\mathfrak{p}\phi> &\overset{(15.3.5)}{=} (\psi|\mathcal{S}\phi)_I \overset{(15.3.6)}{=} \int_{\sigma(\mathcal{S})\times I} \nu\, d(\psi\,|\,F_{\nu,m}\,\phi)_I \\ &\overset{(15.3.7)}{=} \int_I \mathrm{d}\mu_{\psi,\phi} \overset{(15.3.9)}{=} \int_I f_{\psi,\phi}(m) \;\mathrm{d}m \overset{(15.3.10)}{=} \int_I (\psi_m|\mathcal{S}_m\phi_m)_m \;\mathrm{d}m\,. \end{aligned}$$

This concludes the proof. □

Comparing the statement of Theorem 15.3.1 (ii) with Definition 15.2.2, we immediately obtain the following result.

COROLLARY 15.3.2. *The strong mass oscillation property implies the weak mass oscillation property.*

We next show uniqueness as well as the independence of the choice of the interval I.

PROPOSITION 15.3.3. **(uniqueness of $\mathcal{S}_m$)** *The family $(\mathcal{S}_m)_{m\in I}$ in the statement of Theorem 15.3.1 can be chosen such that for all $\psi, \phi \in \mathcal{H}^\infty$, the expectation value $f_{\psi,\phi}(m) := (\psi_m | \mathcal{S}_m \phi_m)_m$ is continuous in m,*

$$f_{\psi,\phi} \in C^0_0(I)\,. \tag{15.3.11}$$

The family $(\mathcal{S}_m)_{m\in I}$ with the properties (15.3.3) and (15.3.11) is unique. Moreover, choosing two intervals $\check{I}$ and I with $m \in \check{I} \subset I$ and $0 \notin \overline{I}$, and denoting all the objects constructed in $\check{I}$ with an additional check, we have

$$\check{\mathcal{S}}_m = \mathcal{S}_m\,. \tag{15.3.12}$$

PROOF. Let us show that the function $f_{\psi,\phi}$ is continuous. To this end, we choose a function $\eta \in C^\infty_0(I)$. Then for any $\varepsilon > 0$ which is so small that $B_\varepsilon(\operatorname{supp}\eta) \subset I$, we obtain

$$\begin{aligned}\int_I \Big(f_{\psi,\phi}(m+\varepsilon) - f_{\psi,\phi}(m)\Big)\,\eta(m)\;\mathrm{d}m &= \int_I f_{\psi,\phi}(m)\,\Big(\eta(m-\varepsilon) - \eta(m)\Big)\;\mathrm{d}m \\ &\overset{(*)}{=} <\!\int_I \Big(\eta(m-\varepsilon) - \eta(m)\Big)\psi_m\;\mathrm{d}m \,|\, \mathfrak{p}\phi\!> = <\!\int_I \eta(m)\,\Big(\psi_{m+\varepsilon} - \psi_m\Big)\;\mathrm{d}m \,|\, \mathfrak{p}\phi\!>\,,\end{aligned}$$

where in $(*)$ we used (15.3.6) and (15.3.7). Applying (15.3.1), we obtain

$$\left|\int_I \Big(f_{\psi,\phi}(m+\varepsilon) - f_{\psi,\phi}(m)\Big)\,\eta(m)\;\mathrm{d}m\right| \leq c\,\|\psi_{+\varepsilon} - \psi\|_I\,\|\phi\|_I \sup_I |\eta|\,,$$

where the vector $\psi_{+\varepsilon} \in \mathcal{H}^\infty$ is defined by $(\psi_{+\varepsilon})_m := \psi_{m+\varepsilon}$. Since $\lim_{\varepsilon\searrow 0} \|\psi_{+\varepsilon} - \psi\|_I = 0$ and η is arbitrary, we conclude that $f_{\psi,\phi}$ is continuous (15.3.11). This continuity is important because it implies that the function $f_{\psi,\phi}$ is uniquely defined pointwise (whereas in (15.3.9) this function could be modified arbitrarily on sets of measure zero).

In order to prove (15.3.12), we note that the representation (15.3.5) implies that

$$(\psi|\check{\mathcal{S}}\phi)_I = (\psi|\mathcal{S}\phi)_I \qquad \text{for all } \psi, \phi \in \check{\mathcal{H}}^\infty\,.$$

Using (15.3.7) and (15.3.9), it follows that

$$\int_\Omega \check{f}_{\psi,\phi}(m)\;\mathrm{d}m = \int_\Omega f_{\psi,\phi}(m)\;\mathrm{d}m \qquad \text{for all } \Omega \subset \check{I}\,.$$

Choosing $\check{f}_{\psi,\phi}(m)$ and $f_{\psi,\phi}(m)$ as continuous functions, we conclude that they coincide for every $m \in \check{I}$. It follows from (15.3.3) that the operators $\check{\mathcal{S}}_m$ and $\mathcal{S}_m$ coincide. This concludes the proof. □

15.4. The Unregularized Kernel of the Fermionic Projector

We now explain how the fermionic signature operator can be used for the construction of the so-called fermionic projector. This will give a direct connection to the kernel of the fermionic projector introduced abstractly for causal fermion systems in Section 5 (see (5.7.3)). We will explain this connection, which will be elaborated on further in Section 21.

It follows directly from Definition 15.3.3 that the operator $\mathcal{S}_m$ is symmetric. Thus the spectral theorem gives rise to the spectral decomposition

$$\mathcal{S}_m = \int_{\sigma(\mathcal{S}_m)} \nu \, \mathrm{d}E_\nu \,,$$

where E_ν is the spectral measure (see for example [**131**]). The spectral measure gives rise to the spectral calculus

$$f(\mathcal{S}_m) = \int_{\sigma(\mathcal{S}_m)} f(\nu) \, \mathrm{d}E_\nu \,,$$

where f is a bounded Borel function.

DEFINITION 15.4.1. *Assume that the Dirac operator $\mathcal{D}$ on $(\mathscr{M}, g)$ satisfies the strong mass oscillation property (see Definition 15.2.3). We define the operators*

$$P_\pm \,:\, C_0^\infty(\mathscr{M}, S\mathscr{M}) \to \mathcal{H}_m$$

by

$$P_+ = \chi_{[0,\infty)}(\mathcal{S}_m)\, k_m \qquad \text{and} \qquad P_- = -\chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \tag{15.4.1}$$

(where χ denotes the characteristic function). The **fermionic projector** *P is defined by $P = P_-$.*

PROPOSITION 15.4.2. *For all $\phi, \psi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$, the operators $P_\pm$ are symmetric,*

$$<P_\pm \phi \,|\, \psi> = <\phi \,|\, P_\pm \psi> \,.$$

Moreover, the image of $P_\pm$ is the positive respectively negative spectral subspace of $\mathcal{S}_m$, i.e.

$$\overline{P_+(C_0^\infty(\mathscr{M}, S\mathscr{M}))} = E_{(0,\infty)}(\mathcal{H}_m) \,, \qquad \overline{P_-(C_0^\infty(\mathscr{M}, S\mathscr{M}))} = E_{(-\infty,0)}(\mathcal{H}_m) \,. \tag{15.4.2}$$

PROOF. According to Proposition 13.4.4,

$$\begin{aligned} <P_- \phi \,|\, \psi> = (P_- \phi \,|\, k_m \psi)_m &= -\big(\chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \phi \,\big|\, k_m \psi\big)_m \\ &= -\big(k_m\, \phi \,\big|\, \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \psi\big)_m = <\phi \,|\, P_- \psi> \,. \end{aligned}$$

The proof for P_+ is similar. The relations (15.4.2) follow immediately from the fact that $k_m(C_0^\infty(\mathscr{M}, S\mathscr{M}))$ is dense in $\mathcal{H}_m$. □

Similar as in [**80**, Theorem 3.12], the fermionic projector can be represented by a two-point distribution on $\mathscr{M}$. As usual, we denote the space of test functions (with the Fréchet topology) by $\mathcal{D}$ and define the space of distributions $\mathcal{D}'$ as its dual space.

THEOREM 15.4.3. *Assume that the strong mass oscillation property holds. Then there is a unique distribution $\mathcal{P} \in \mathcal{D}'(\mathscr{M} \times \mathscr{M})$ such that for all $\phi, \psi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$,*

$$<\phi | P \psi> = \mathcal{P}(\phi \otimes \psi) \,.$$

PROOF. According to Proposition 13.4.4 and Definition 15.4.1,

$$<\phi | P \psi> = (k_m \phi \,|\, P \psi)_m = -(k_m \phi \,|\, \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \psi)_m \,.$$

Since the norm of the operator $\chi_{(-\infty,0)}(\mathcal{S}_m)$ is bounded by one, we conclude that

$$|<\phi | P \psi>| \leq \|k_m \phi\|_m \, \|k_m \psi\|_m = (<\phi | k_m \phi> <\psi | k_m \psi>)^{\frac{1}{2}} \,,$$

where in the last step we again applied Proposition 13.4.4. As $k_m \in \mathcal{D}'(\mathscr{M} \times \mathscr{M})$, the right side is continuous on $\mathcal{D}(\mathscr{M} \times \mathscr{M})$. We conclude that also the functional $<\phi | P \psi>$ is continuous on $\mathcal{D}(\mathscr{M} \times \mathscr{M})$. The result now follows from the Schwartz kernel theorem

(see [**105**, Theorem 5.2.1], keeping in mind that this theorem applies just as well to bundle-valued distributions on a manifold simply by working with the components in local coordinates and a local trivialization). □

Exactly as explained in [**80**, Section 3.5], it is convenient to use the standard notation with an integral kernel $P(x,y)$,

$$<\phi|P\psi> = \iint_{\mathscr{M}\times\mathscr{M}} \prec\phi(x)\,|\,P(x,y)\,\psi(y)\succ_x \mathrm{d}\mu_{\mathscr{M}}(x)\,\mathrm{d}\mu_{\mathscr{M}}(y)$$
$$(P\psi)(x) = \int_{\mathscr{M}} P(x,y)\,\psi(y)\,\mathrm{d}\mu_{\mathscr{M}}(y)$$

(where $P(.,.)$ coincides with the distribution $\mathcal{P}$ above). In view of Proposition 15.4.2, we know that the last integral is not only a distribution, but a function which is square integrable over every Cauchy surface. Moreover, the symmetry of P shown in Proposition 15.4.2 implies that

$$P(x,y)^* = P(y,x)\,,$$

where the star denotes the adjoint with respect to the spin inner product.

We next specify the normalization of the fermionic projector. We introduce an operator Π by

$$\Pi\,:\,\mathcal{H}_m \to \mathcal{H}_m\,, \qquad (\Pi\,\psi_m)(x) = -2\pi \int_{\mathscr{N}} P(x,y)\,\not{y}\,(\psi_m)|_{\mathscr{N}}(y)\,\mathrm{d}\mu_{\mathscr{N}}(y)\,, \tag{15.4.3}$$

where $\mathscr{N}$ is any Cauchy surface.

Proposition 15.4.4. **(spatial normalization)** *The operator* Π *is a projection operator on* $\mathcal{H}_m$.

Proof. According to Theorem 13.4.2, the spatial integral in (15.4.3) can be combined with the factor k_m in (15.4.1) to give the solution of the corresponding Cauchy problem. Thus

$$\Pi\,:\,\mathcal{H}_m \to \mathcal{H}_m\,, \qquad (\Pi\,\psi_m)(x) = \chi_{(-\infty,0)}(\mathcal{S}_m)\,\psi_m\,,$$

showing that Π is a projection operator. □

Instead of the spatial normalization, one could also consider the *mass normalization* (for details on the different normalization methods see [**86**]). To this end, one needs to consider families of fermionic projectors P_m indexed by the mass parameter. Then for all $\phi, \psi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$, we can use (15.3.3) and Proposition 13.4.4 to obtain

$$\begin{aligned}<\mathfrak{p}(P_m\phi)\,|\,\mathfrak{p}(P_{m'}\psi)> &= \int_I (P_m\phi\,|\,\mathcal{S}_m P_m\psi)_m\;\mathrm{d}m = \int_I (k_m\phi\,|\,\mathcal{S}_m\chi_{(-\infty,0)}(\mathcal{S}_m)\,k_m\psi)_m\;\mathrm{d}m \\ &= \int_I <\phi\,|\,\mathcal{S}_m\chi_{(-\infty,0)}(\mathcal{S}_m)\,k_m\psi>\;\mathrm{d}m = -<\phi\,|\,\mathfrak{p}(\mathcal{S}_m P_m\psi)>\,,\end{aligned}$$

which can be written in a compact formal notation as

$$P_m\,P_{m'} = \delta(m-m')\,(-\mathcal{S}_m)\,P_m\,.$$

Due to the factor $(-\mathcal{S}_m)$ on the right, in general the fermionic projector does *not* satisfy the mass normalization condition. The mass normalization condition could be arranged by modifying the definition (15.4.1) to

$$\mathcal{S}_m^{-1}\,\chi_{(-\infty,0)}(\mathcal{S}_m)\,k_m\,.$$

Here we prefer to work with the spatial normalization. For a detailed discussion of the different normalization methods we refer to [**86**, Section 2].

Finally, the spatial normalization property of Proposition 15.4.4 makes it possible to obtain a representation of the fermionic projector in terms of one-particle states. To this end, one chooses an orthonormal basis $(\psi_j)_{j\in\mathbb{N}}$ of the subspace $\chi_{(-\infty,0)}(\mathcal{S}_m) \subset \mathcal{H}_m$. Then

$$P(x,y) = -\sum_{j=1}^{\infty} |\psi_j(x)\succ\!\prec \psi_j(y)| \tag{15.4.4}$$

with convergence in $\mathcal{D}'(\mathscr{M}\times\mathscr{M})$ (for details see [**80**, Proposition 3.13]).

This formulas is reminiscent of the decomposition of the kernel of the fermionic projector into physical wave functions in (5.7.10). Indeed, these formulas can be understood as being completely analogous, with the only difference that (15.4.4) is formed of wave functions in Minkowski space, whereas in (5.7.10) one works abstractly with the physical wave functions of a general causal fermion system. The connection can be made more precise if one identifies the structures of the causal fermion system with corresponding structures in Minkowski space. In order to avoid technicalities and too much overlap with [**45**], here we shall not enter the details of these identifications (which are worked out in [**45**, Section 1.2]). Instead, we identify (15.4.4) with (5.7.10) as describing the same object, on one side in Minkowski space, and on the other side as abstract object of the corresponding causal fermion system. With this identification, the Hilbert space $\mathcal{H}$ of the causal fermion system corresponds to the negative spectral subspace of the fermionic signature operator $\mathcal{S}_m$. In the vacuum, this gives us back the subspace of all negative frequency solutions as considered in the example of Exercise 5.15. However, the above identification has one shortcoming: the wave functions in (15.4.4) have not yet been regularized. This is why we refer to $P(x,y)$ as the *unregularized* kernel. In order to get complete agreement between (15.4.4) and (5.15), one needs to introduce an ultraviolet regularization. To this end, one proceeds similar as explained in the example in Section 5.5: One introduces regularization operators $(\mathfrak{R}_\varepsilon)_{\varepsilon>0}$, computes the local correlation operators $F^\varepsilon(x)$ and defines the measure ρ as the push-forwards $\mathrm{d}\rho = F^\varepsilon_* \mathrm{d}\mu_{\mathscr{M}}$. We will come back to this construction in Chapter 21.

15.5. Exercises

EXERCISE 15.1. Let $\mathscr{M}$ be the "spacetime strip"

$$\mathscr{M} = \{(t,\vec{x}) \in \mathbb{R}^{1,3} \text{ with } 0<t<T\}\,.$$

Show that for any solution $\psi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$ of the Dirac equation, the following inequality holds,

$$\big|<\psi|\phi>\big| \leq T\, \|\psi\|_m\, \|\phi\|_m\,.$$

This estimate illustrates why in spacetimes of finite lifetime, the spacetime inner product is a bounded sesquilinear form on $\mathcal{H}_m$.

EXERCISE 15.2. Let $\mathscr{M}$ again be the "spacetime strip" of the previous exercise. Let $\psi, \phi \in \mathcal{H}^\infty := \mathcal{H} \cap C^\infty_{\text{sc},0}(\mathscr{M}\times I, S\mathscr{M})$ be families of smooth Dirac solutions of spatially compact support, with compact support in the mass parameter. Moreover, we again define the operators $\mathfrak{p}$ and T as in (15.2.4) and (15.2.5). Does the equation

$$<\mathfrak{p}T\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi>$$

(which appears in the weak mass oscillation property) in general hold? Justify your answer by a proof or a counter example.

EXERCISE 15.3. Let $\mathscr{M}$ again be the "spacetime strip" of the previous exercises. Moreover, as in Exercise 5.11 we again let $\mathcal{H} \subset \mathcal{H}_m$ be a finite-dimensional subspace of the Dirac solution space $\mathcal{H}_m$, consisting of smooth wave functions of spatially compact support, i.e.

$$\mathcal{H} \subset C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap H_m \qquad \text{finite-dimensional}\,.$$

Show that the fermionic signature operator $\mathcal{S} \in \mathrm{L}(\mathcal{H})$ defined by

$$<\psi|\phi> = (\psi|\mathcal{S}\phi)_m \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

can be expressed within the causal fermion system by

$$\mathcal{S} = -\int_M x \,\mathrm{d}\rho(x)$$

(where ρ is again the push-forward of $\mathrm{d}\mu_{\mathscr{M}}$).

EXERCISE 15.4. Let E be the Banach space $E = C^0([0,1], \mathbb{C})$ and $\Lambda : E \times E \to \mathbb{C}$ be sesquilinear, bounded and positive semi-definite.

(a) Assume that Λ satisfies for a suitable constant $c > 0$ and all $f, g \in E$ the inequality

$$\big|\Lambda(f,g)\big| \leq c \sup_{x \in [0,1]} \big|f(x)\, g(x)\big|\,. \tag{15.5.1}$$

Show that there is a regular bounded Borel measure μ such that

$$\Lambda(f,g) = \int_0^1 \overline{f(x)}\, g(x)\, \mathrm{d}\mu(x)\,.$$

(b) Now make the stronger assumption that Λ satisfies for a suitable constant $\tilde{c} > 0$ and all $f, g \in E$ the inequality

$$\big|\Lambda(f,g)\big| \leq \tilde{c} \int_0^1 \big|f(x)\, g(x)\big|\; \mathrm{d}x\,. \tag{15.5.2}$$

Show that μ is absolutely continuous w.r.to the Lebesgue measure. Show that there is a non-negative function $h \in L^1([0,1],\, \mathrm{d}x)$ such that

$$\Lambda(f,g) = \int_0^1 \overline{f(x)}\, g(x)\, h(x)\; \mathrm{d}x\,.$$

Show that h is pointwise bounded by c.

(c) In order to clarify the different assumptions in this exercise, give an example for a sesquilinear, bounded and positive semi-definite functional Λ which violates (15.5.1). Give an example which satisfies (15.5.1) but violates (15.5.2).

EXERCISE 15.5. *(Toward the mass oscillation property - part 1)* This exercise illustrates the mass oscillation property. Let $0 < m_L < m_R$ and $\eta \in C^\infty_0((m_L, m_R))$. Show that the function f given by

$$f(t) = \int_{m_L}^{m_R} \eta(m)\, \mathrm{e}^{-\mathrm{i}\sqrt{1+m^2}\, t}\, \mathrm{d}m$$

has rapid decay. Does this result remain valid if m_L and m_R are chosen to have opposite signs? Justify your finding by a proof or a counter example.

EXERCISE 15.6. *(Toward the mass oscillation property - part 2)* Let R_T be the "spacetime strip"

$$R_T = \{(t, \vec{x}) \in \mathbb{R}^{1,3} \text{ with } 0 < t < T\} .$$

Show that for any solutions $\psi, \phi \in C^\infty_{\text{sc}}(\mathbb{R}^4, \mathbb{C}^4) \cap \mathcal{H}_m$ of the Dirac equation, the following inequality holds,

$$\big|<\psi|\phi>_T\big| \leq T \,\|\psi\|_m \,\|\phi\|_m , \quad \text{where} \quad <\psi|\phi>_T := \int_{R_T} \prec\psi(x)|\phi(x)\succ \mathrm{d}^4x.$$

This estimate illustrates how in spacetimes of finite lifetime, the spacetime inner product is a bounded sesquilinear form on $\mathcal{H}_m$.

EXERCISE 15.7. *(Toward the mass oscillation property - part 3)* Let R_T again be the "spacetime strip" of the previous exercises. Moreover, we again let $\mathcal{H} \subset \mathcal{H}_m$ be a finite-dimensional subspace of the Dirac solution space $\mathcal{H}_m$, consisting of smooth wave functions of spatially compact support, i.e.

$$\mathcal{H} \subset C^\infty_{\text{sc}}(\mathbb{R}^4, \mathbb{C}^4) \cap \mathcal{H}_m \qquad \text{finite-dimensional} .$$

Show that the fermionic signature operator $\mathcal{S} \in \mathrm{L}(\mathcal{H})$ defined by

$$<\psi|\phi>_T = (\psi|\mathcal{S}\phi)_m \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

can be expressed within the causal fermion system by

$$\mathcal{S} = -\int_{R_T} x \,\mathrm{d}\rho(x)$$

(where ρ is again the push-forward of d^4x).

EXERCISE 15.8. *(The external field problem)* In physics, the notion of "particle" and "anti-particle" is often introduced as follows: Solutions of the Dirac equation with positive frequency are called "particles" and solutions with negative frequency "anti-particles". In this exercise, we will check in how far this makes sense.
To this end, take a look at the Dirac equation in an external field:

$$(\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m)\psi = 0. \tag{15.5.3}$$

Assume that $\mathcal{B}$ is time-dependent and has the following form:

$$\mathcal{B}(t, x) = V\, \Theta(t - t_0)\Theta(t_1 - t),$$

where $V \in \mathbb{R}$, Θ denotes the Heaviside step function and $t_0 = 0$, $t_1 = 1$. In order to construct a solution thereof, for a given momentum $\vec{k}$, we use plane wave solutions of the Dirac equation,

$$\psi(t, \vec{x}) = \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\vec{k}\vec{x}} \chi_{\vec{k}},$$

where $\chi_{\vec{k}}$ is a spinor $\in \mathbb{C}^4$, and patch them together suitably. (The quantity ω is called the "frequency" or "energy", and $\vec{k}$ the"momentum".) To simplify the calculation, we set $\vec{k} = (k_1, 0, 0)^T$. Proceed as follows:

(a) First, take a look at the region $t < t_0$. Reformulate (15.5.3) such that there is only the time derivative on the left hand side. (Hint: Multiply by γ^0.)
(b) Insert the plane wave ansatz with $\vec{k} = (k_1, 0, 0)^T$ into the equation. Your equation now has the form $\omega\psi = H(k_1)\psi$. Show that the eigenvalues of $H(k_1)$ are $\pm\omega_0$ with $\omega_0 := \sqrt{(k_1)^2 + m^2}$.

(c) Show that one eigenvector belonging to $+\omega_0$ is $\chi_0^+ := (\frac{m+\omega_0}{k1}, 0, 0, 1)^T$ and that one eigenvector belonging to $-\omega_0$ is $\chi_0^- := (\frac{m-\omega_0}{k1}, 0, 0, 1)^T$. (Both eigenvalues have multiplicity two, but we do not need the other two eigenvectors here.)

(d) With this, you have constructed plane wave solutions $\mathrm{e}^{-\mathrm{i}(\pm\omega_0)t+\mathrm{i}\vec{k}\vec{x}}\chi_0^\pm$ for $t < t_0$ and also for $t > t_1$. By transforming $m \to (m - V)$, you immediately obtain plane wave solutions also for $t_0 < t < t_1$. Denote the respective quantities by ω_1 and $\chi_1^\pm$.

(e) Assume that for $t < t_0$ there is one "particle" present, i.e. set

$$\psi(t, \vec{x}) = \mathrm{e}^{-\mathrm{i}\omega_0 t+\mathrm{i}\vec{k}\vec{x}}\chi_0^+ \quad \text{for } t < t_0.$$

Assume that the solution for $t_0 < t < t_1$ takes the form

$$A\mathrm{e}^{-\mathrm{i}\omega_1 t+\mathrm{i}\vec{k}\vec{x}}\chi_1^+ + B\mathrm{e}^{-\mathrm{i}(-\omega_1)t+\mathrm{i}\vec{k}\vec{x}}\chi_1^- \quad \text{with } A, B \in \mathbb{R}.$$

Calculate A and B for the case $k_1 = 1$ and $V = m$ by demanding continuity of the solution at $t = t_0$.

(f) Assume that for $t > t_1$ the solution takes the form

$$C\mathrm{e}^{-\mathrm{i}\omega_0 t+\mathrm{i}\vec{k}\vec{x}}\chi_0^+ + D\mathrm{e}^{-\mathrm{i}(-\omega_0)t+\mathrm{i}\vec{k}\vec{x}}\chi_0^- \quad \text{with } C, D \in \mathbb{C}.$$

Calculate C and D for $m = 2$ by demanding continuity of the solution at $t = t_1$ (here you may want to use computer algebra).

(g) Interpret what you have found. Why could this be called the "external field *problem*"?

CHAPTER 16

Fourier Methods

In the previous chapter, the fermionic signature operator and the unregularized fermionic projector were constructed abstractly. We now turn to the question how to compute them in the Minkowski vacuum. This question can be addressed and answered with Fourier methods. Since these techniques are frequently used and of independent interest, we introduce them from a general perspective before entering the proof of the mass oscillation properties and the construction of the fermionic signature operator. More details can be found in [**81, 79**].

16.1. The Causal Green's Operators

We already encountered Green's operators in Chapter 13 when solving the Cauchy problem with methods of hyperbolic partial differential equations (see Theorem 13.4.3). In Minkowski space, these Green's operators can be computed in more detail with Fourier methods. Our starting point is the definition of the *Green's operator* $s_m(x,y)$ of the vacuum Dirac equation by the distributional equation

$$(\mathrm{i}\partial\!\!\!/_x - m)\, s_m(x,y) = \delta^4(x-y)\,, \tag{16.1.1}$$

where $\delta^4(x,y)$ denotes the four-dimensional δ distribution. Taking the Fourier transform of (16.1.1),

$$s_m(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, s_m(k)\, \mathrm{e}^{-\mathrm{i}k(x-y)} \tag{16.1.2}$$

(where $x, y \in \mathscr{M}$ are spacetime points, k is the four-momentum, and $k(x-y)$ denotes the Minkowski inner product) we obtain the algebraic equation

$$(k\!\!\!/ - m)\, s_m(k) = \mathbb{1}\,. \tag{16.1.3}$$

Multiplying by $k\!\!\!/ + m$ and using the identity $(k\!\!\!/ - m)(k\!\!\!/ + m) = k^2 - m^2$, one sees that if $k^2 \neq m^2$, the matrix $k\!\!\!/ - m$ is invertible. If conversely $k^2 = m^2$, we have $(k\!\!\!/ - m)^2 = -2m(k\!\!\!/ - m)$, which shows that the matrix $k\!\!\!/ - m$ is diagonalizable with eigenvalues $-2m$ and zero. Since the Dirac matrices (1.3.3) are trace-free, we have $\mathrm{Tr}(k\!\!\!/ - m) = -4m$. It follows that the matrix $k\!\!\!/ - m$ has a two-dimensional kernel if k is on the mass shell. This shows that the Green's operator of the Dirac equation is *not unique*. If we add to it any vector in the kernel of $k\!\!\!/ - m$, i.e. if we add to it a solution of the homogeneous Dirac equation, it still satisfies the defining equation (16.1.1) (for details see [**14**].)

A convenient method for solving the equation (16.1.3) for $s_m(k)$ is to use a $\pm\mathrm{i}\varepsilon$-regularization on the mass shell. Common choices are the *advanced* and the *retarded* Green's functions, which are defined by

$$s_m^\vee(k) = \lim_{\varepsilon \searrow 0} \frac{k\!\!\!/ + m}{k^2 - m^2 - \mathrm{i}\varepsilon k^0} \qquad \text{and} \qquad s_m^\wedge(k) = \lim_{\varepsilon \searrow 0} \frac{k\!\!\!/ + m}{k^2 - m^2 + \mathrm{i}\varepsilon k^0}\,, \tag{16.1.4}$$

respectively (with the limit $\varepsilon \searrow 0$ taken in the distributional sense). Computing their Fourier transform (16.1.2), one sees that they are *causal* in the sense that their supports lie in the upper and lower light cone, respectively,

$$\operatorname{supp} s_m^\vee(x,.) \subset J_x^\vee \,, \qquad \operatorname{supp} s_m^\wedge(x,.) \subset J_x^\wedge \,. \tag{16.1.5}$$

Mathematically, the formulas in (16.1.4) define the Green's operators in momentum space as tempered distributions. Taking their Fourier transform (16.1.2), the advanced and retarded Green's operators are tempered distributions in the variable $\xi := y - x$. We also regard these distributions as integral kernels of corresponding operators on the wave functions, i.e.

$$(s_m(\psi))(x) := \int_{\mathscr{M}} s_m(x,y)\, \psi(y)\; \mathrm{d}^4y \,.$$

We thus obtain operators

$$s_m^\wedge, s_m^\vee \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \to C_{\text{sc}}^\infty(\mathscr{M}, S\mathscr{M}) \,.$$

Here $C_0^\infty(\mathscr{M}, S\mathscr{M})$ denote the smooth functions with compact support in $\mathscr{M}$, taking values in the spinors, and C_{sc}^∞ denotes the smooth functions with spatially compact support.

16.2. The Causal Fundamental Solution and Time Evolution

We now state a few properties of the Green's operators and explain why they are useful. The considerations in this section are valid more generally in the presence of an external potential. Then the defining equation of the Green's operator (16.1.1) is modified similar to (1.3.14) to

$$(\mathrm{i}\partial\!\!\!/_x + \mathcal{B} - m)\, s_m(x,y) = \delta^4(x-y) \,, \tag{16.2.1}$$

where $\mathcal{B}$ is again a multiplication operator satisfying the symmetry condition (1.3.13). Then the existence of Green's operators can no longer be proven by Fourier transformation. Instead, one can use methods of hyperbolic PDEs as introduced in Chapter 13 (see Section 13.6). Here we do not assume that the reader is familiar with these methods. Instead, we simply assume that we are given advanced and retarded Green's operators.

The *causal fundamental solution* k_m is defined as the difference of the advanced and the retarded Green's operator,

$$k_m(x,y) := \frac{1}{2\pi\mathrm{i}} \left(s_m^\vee(x,y) - s_m^\wedge(x,y)\right). \tag{16.2.2}$$

It is a distribution which is causal in the sense that it vanishes if x and y have spacelike separation. Moreover, it is a distributional solution of the homogeneous Dirac equation,

$$(\mathrm{i}\partial\!\!\!/_x + \mathcal{B} - m)\, k_m(x,y) = 0 \,.$$

The unique solvability of the Cauchy problem allows us to introduce the time evolution operator of the Dirac equation as follows. Solving the Cauchy problem with initial data at time t and evaluating the solution at some other time t' gives rise to a mapping

$$U^{t',t} \;:\; \mathcal{H}_t \to \mathcal{H}_{t'} \,,$$

referred to as the *time evolution operator*. Since the scalar product (15.1.2) is time independent, the operator $U^{t',t}$ is unitary. Moreover, using that the Cauchy problem can be solved forwards and backwards in time, the unitary time evolution operators form a representation of the group $(\mathbb{R}, +)$. More precisely,

$$U^{t,t} = \mathbb{1} \qquad \text{and} \qquad U^{t'',t'}\, U^{t',t} = U^{t'',t} \,.$$

Proposition 13.6.1 immediately gives the following representation of $U^{t',t}$:

$$\big(U^{t',t}\,\psi|_t\big)(\vec{y}) = \int_{\mathbb{R}^3} U^{t',t}(\vec{y},\vec{x})\,\psi(t,\vec{x})\;\mathrm{d}^3x\,, \tag{16.2.3}$$

where the kernel $U^{t',t}(\vec{x},\vec{y})$ is defined as

$$U^{t',t}(\vec{y},\vec{x}) = 2\pi\, k_m\big((t',\vec{y}),(t,\vec{x})\big)\,\gamma^0\,. \tag{16.2.4}$$

16.3. Proof of the Weak Mass Oscillation Property in the Minkowski Vacuum

In the remainder of this chapter, we return to the Dirac equation in Minkowski space (16.1.1). An external potential will be considered in the next chapter (Chapter 17).

The mass oscillation property in the Minkowski vacuum can be proved using Fourier methods. Here we shall give two different approaches in detail. The method of the first proof (in this section) is instructive because it gives an intuitive understanding of "mass oscillations". However, this method only yields the weak mass oscillation property. The second proof (Section 16.4) is more abstract but also gives the strong mass oscillation property.

We again consider the foliation $\mathcal{N}_t = \{(t,\vec{x})\,|\,\vec{x}\in\mathbb{R}^3\}$ of constant time Cauchy hypersurfaces in a fixed reference frame $(t,\vec{x})$ and a variable mass parameter m in the interval $I=(m_L,m_R)$ with $m_L, m_R > 0$. The families of solutions $\psi=(\psi_m)_{m\in I}$ of the Dirac equations $(i\partial\!\!\!/ - m)\psi_m = 0$ are contained in the Hilbert space $(\mathcal{H},(.|.))$ with scalar product (15.2.3). The subspace $\mathcal{H}^\infty\subset\mathcal{H}$ in Definition 15.2.1 is chosen as

$$\mathcal{H}^\infty = C^\infty_{\mathrm{sc},0}(\mathscr{M}\times I, S\mathscr{M})\cap\mathcal{H}\,. \tag{16.3.1}$$

For what follows, it is convenient to work with the Fourier transform in space, i.e.

$$\hat{\psi}(t,\vec{k}) = \int_{\mathbb{R}^3}\psi(t,\vec{x})\,\mathrm{e}^{-\mathrm{i}\vec{k}\vec{x}}\;\mathrm{d}^3x\,,\qquad \psi(t,\vec{x}) = \int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\,\hat{\psi}(t,\vec{k})\,\mathrm{e}^{\mathrm{i}\vec{k}\vec{x}}\,.$$

Then a family of solutions $\psi\in\mathcal{H}^\infty$ has the representation

$$\hat{\psi}_m(t,\vec{k}) = c_+(\vec{k},m)\,\mathrm{e}^{-\mathrm{i}\omega(\vec{k},m)\,t} + c_-(\vec{k},m)\,\mathrm{e}^{\mathrm{i}\omega(\vec{k},m)\,t}\qquad\text{for all } m\in I \tag{16.3.2}$$

with suitable spinor-valued coefficients $c_\pm(\vec{k},m)$ and $\omega(\vec{k},m) := \sqrt{|\vec{k}|^2+m^2}$. Integrating over the mass parameter, we obtain a superposition of waves oscillating at different frequencies. Intuitively speaking, this leads to destructive interference for large t, giving rise to decay in time. This picture can be made precise using integration by parts in m, as we now explain. Integrating (16.3.2) over the mass by applying the operator $\mathfrak{p}$, (15.2.5), we obtain

$$\begin{aligned}
\mathfrak{p}\hat{\psi}(t,\vec{k}) &= \int_I \big(c_+\,\mathrm{e}^{-\mathrm{i}\omega t} + c_-\,\mathrm{e}^{\mathrm{i}\omega t}\big)\;\mathrm{d}m \\
&= \int_I \frac{\mathrm{i}}{t\,\partial_m\omega}\,\big(c_+\,\partial_m\mathrm{e}^{-\mathrm{i}\omega t} - c_-\,\partial_m\mathrm{e}^{\mathrm{i}\omega t}\big)\;\mathrm{d}m \\
&= -\frac{\mathrm{i}}{t}\int_I\left[\partial_m\Big(\frac{c_+}{\partial_m\omega}\Big)\,\mathrm{e}^{-\mathrm{i}\omega t} - \partial_m\Big(\frac{c_-}{\partial_m\omega}\Big)\,\mathrm{e}^{\mathrm{i}\omega t}\right]\mathrm{d}m
\end{aligned}$$

(we do not get boundary terms because $\psi\in\mathcal{H}^\infty$ has compact support in m). With $\partial_m\omega = m/\omega$, we conclude that

$$\mathfrak{p}\hat{\psi}(t,\vec{k}) = -\frac{\mathrm{i}}{t}\int_I\left[\partial_m\Big(\frac{\omega\,c_+}{m}\Big)\,\mathrm{e}^{-\mathrm{i}\omega t} - \partial_m\Big(\frac{\omega\,c_-}{m}\Big)\,\mathrm{e}^{\mathrm{i}\omega t}\right]\mathrm{d}m\,.$$

Since the coefficients $c_\pm$ depend smoothly on m, the resulting integrand is bounded uniformly in time, giving a decay at least like $1/t$, i.e. $|\mathfrak{p}\hat{\psi}(t,\vec{k})| \lesssim 1/t$. Iterating this procedure, one even can prove decay rates $\lesssim 1/t^2, 1/t^3, \ldots$ The price one pays is that higher and higher powers in ω come up in the integrand, which means that in order for the spatial Fourier integral to exist, one needs a faster decay of $c_\pm$ in $|\vec{k}|$. Expressed in terms of the initial data, this means that every factor $1/t$ gives rise to an additional spatial derivative acting on the initial data. This motivates the following basic estimate.

LEMMA 16.3.1. *For any $\psi \in \mathcal{H}^\infty$, there is a constant $C = C(m_L)$ such that*

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \frac{C\,|I|}{1+t^2} \sup_{m\in I} \sum_{b=0}^{2} \big\|(\partial_m^b \psi_m)|_{t=0}\big\|_{W^{2,2}}\,, \tag{16.3.3}$$

where $\|.\|_t$ is the norm corresponding to the scalar product

$$(.|.)|_t := 2\pi \int_{\mathbb{R}^3} \prec . \,|\gamma^0\,.\succ_{\vec{x}} \,\mathrm{d}^3x \;:\; L^2(\mathcal{N}_t, S\mathscr{M}) \times L^2(\mathcal{N}_t, S\mathscr{M}) \to \mathbb{C}$$

(which is similar to (15.1.2)*, but now applied to wave functions which do not need to be solutions), and $\|.\|_{W^{2,2}}$ is the spatial Sobolev norm*

$$\|\phi\|^2_{W^{2,2}} := \sum_{\alpha \text{ with } |\alpha|\leq 2} \int_{\mathbb{R}^3} |\nabla^\alpha \phi(\vec{x})|^2 \,\mathrm{d}^3x\,, \tag{16.3.4}$$

where α is a multi-index.

The absolute value in (16.3.4) is the norm $|\,.\,| := \sqrt{\prec .|\gamma^0 .\succ}$ on the spinors. If we again identify all spinor spaces in the Dirac representation with $\mathbb{C}^4$, this simply is the standard Euclidean norm on $\mathbb{C}^4$.

The proof of this lemma will be given later in this section. Before, we infer the weak mass oscillation property.

COROLLARY 16.3.2. *The vacuum Dirac operator $i\partial\!\!\!/$ in Minkowski space has the weak mass oscillation property with domain* (16.3.1)*.*

PROOF. For every $\psi, \phi \in \mathcal{H}^\infty$, the Schwarz inequality gives

$$|<\mathfrak{p}\psi|\mathfrak{p}\phi>| = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \big((\mathfrak{p}\psi)|_t \,\big|\, \gamma^0\,(\mathfrak{p}\phi)|_t\big)_t \,\mathrm{d}t\right| \leq \int_{-\infty}^{\infty} \big\|(\mathfrak{p}\psi)|_t\big\|_t \,\big\|(\mathfrak{p}\phi)|_t\big\|_t \,\mathrm{d}t\,. \tag{16.3.5}$$

Applying Lemma 16.3.1 together with the estimate

$$\begin{aligned}\big\|(\mathfrak{p}\phi)|_t\big\|_t^2 &= \iint_{I\times I} \big(\phi_m|_t \,\big|\, \phi_{m'}|_t\big)_t \,\mathrm{d}m\,\mathrm{d}m' \\ &\leq \frac{1}{2}\iint_{I\times I} \Big(\|\phi_m\|_m^2 + \|\phi_{m'}\|_{m'}^2\Big)\,\mathrm{d}m\,\mathrm{d}m' = |I|\,\|\phi\|^2\,,\end{aligned}$$

we obtain inequality (15.2.6) with

$$c = C\,|I|^{\frac{3}{2}} \sup_{m\in I} \sum_{b=0}^{2} \|\partial_m^b(\psi_m)|_{t=0}\|_{W^{2,2}} \int_{-\infty}^{\infty} \frac{1}{1+t^2}\,\mathrm{d}t < \infty\,. \tag{16.3.6}$$

The identity (15.2.7) follows by integrating the Dirac operator by parts,

$$\begin{aligned} <\mathfrak{p} T\psi|\mathfrak{p}\phi> = <\mathfrak{p}\mathcal{D}\psi|\mathfrak{p}\phi> = <\mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi> &= \int_{\mathscr{M}} \prec \mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi\succ_x \mathrm{d}^4x \\ &\overset{(\star)}{=} \int_{\mathscr{M}} \prec \mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi\succ_x \mathrm{d}^4x = <\mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi> . \end{aligned} \tag{16.3.7}$$

In $(\star)$, we used that the Dirac operator is symmetric with respect to the inner product $<.|.>$. Moreover, we do not get boundary terms because of the time decay in Lemma 16.3.1. □

The remainder of this section is devoted to the proof of Lemma 16.3.1. Using the result of Proposition 13.6.1, we can express the solution ψ_m of the Cauchy problem in terms of the causal fundamental solution k_m. In order to bring k_m into a more explicit form, we use (16.2.2) together with formulas for the advanced and retarded Green's operators. Indeed, these Green's operators are the multiplication operators in momentum space (16.1.4) (with the limit $\varepsilon \searrow 0$ taken in the distributional sense, and where the vector k is the four-momentum). We thus obtain in momentum space

$$\begin{aligned} k_m(p) &= \frac{1}{2\pi \mathrm{i}} \,(\not{p} + m)\, \lim_{\varepsilon \searrow 0} \left[\frac{1}{p^2 - m^2 - \mathrm{i}\varepsilon p^0} \;-\; \frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon p^0} \right] \\ &= \frac{1}{2\pi \mathrm{i}} \,(\not{p} + m)\, \lim_{\varepsilon \searrow 0} \left[\frac{1}{p^2 - m^2 - \mathrm{i}\varepsilon} \;-\; \frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon} \right] \epsilon(p^0) \end{aligned}$$

(where for notational clarity we denoted the momentum variables by p, and ϵ is the *sign function* $\epsilon(x) = 1$ if $x > 0$ and $\epsilon(x) = -1$ otherwise). Employing the distributional equation

$$\lim_{\varepsilon \searrow 0} \left(\frac{1}{x - \mathrm{i}\varepsilon} - \frac{1}{x + \mathrm{i}\varepsilon} \right) = 2\pi \mathrm{i}\, \delta(x) ,$$

we obtain the simple formula

$$k_m(p) = (\not{p} + m)\, \delta(p^2 - m^2)\, \epsilon(p^0) . \tag{16.3.8}$$

It is convenient to transform spatial coordinates of the time evolution operator to momentum space. First, in the Minkowski vacuum, the time evolution operator can be represented as in (16.2.3) with an integral kernel $U^{t,t'}(\vec{y}, \vec{x})$ which depends only on the difference vector $\vec{y} - \vec{x}$. We set

$$U^{t,t'}(\vec{k}) := \int_{\mathbb{R}^3} U^{t,t'}(\vec{y}, 0)\, \mathrm{e}^{-\mathrm{i}\vec{k}\vec{y}}\; \mathrm{d}^3y .$$

Combining (16.2.4) with (16.3.8) yields

$$U^{t,t'}(\vec{k}) = \int_{-\infty}^{\infty} (\not{k} + m)\, \gamma^0\, \delta(k^2 - m^2)\big|_{k=(\omega,\vec{k})}\, \epsilon(\omega)\, \mathrm{e}^{-\mathrm{i}\omega(t-t')}\; \mathrm{d}\omega .$$

Carrying out the ω-integral, we get

$$U^{t,t'}(\vec{k}) = \sum_{\pm} \Pi_{\pm}(\vec{k})\, \mathrm{e}^{\mp \mathrm{i}\omega(t-t')} , \tag{16.3.9}$$

where we set

$$\Pi_\pm(\vec{k}) := \pm\frac{1}{2\omega(\vec{k})}\,(\not{k}_\pm + m)\,\gamma^0 \tag{16.3.10}$$

$$\text{with} \qquad \omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2} \qquad \text{and} \qquad k_\pm = (\pm\omega(\vec{k}), \vec{k})\,.$$

Moreover, applying Plancherel's theorem, the scalar product (15.1.2) can be written in momentum space as

$$(\psi_m \,|\, \phi_m)_m = (2\pi)^{-2} \int_{\mathbb{R}^3} \prec\hat{\psi}_m(t,\vec{k}) \,|\, \gamma^0\, \hat{\phi}_m(t,\vec{k})\succ \,\mathrm{d}^3k\,.$$

The unitarity of the time evolution operator in position space implies that the matrix $U^{t,t'}(\vec{k})$ is unitary (with respect to the scalar product $\langle .\,,.\rangle_{\mathbb{C}^2} \equiv \prec .\,|\gamma^0\,.\succ$), meaning that its eigenvalues are on the unit circle and the corresponding eigenspaces are orthogonal. It follows that the operators $\Pi_\pm(\vec{k})$ in (16.3.9) are the orthogonal projection operators to the eigenspaces corresponding to the eigenvalues $\mathrm{e}^{\mp i\omega(t-t')}$, i.e.

$$\gamma^0\Pi_s^*\gamma^0 = \Pi_s \quad \text{and} \quad \Pi_s(\vec{k})\,\Pi_{s'}(\vec{k}) = \delta_{s,s'}\,\Pi_s(\vec{k}) \qquad \text{for } s,s' \in \{+,-\} \tag{16.3.11}$$

(these relations can also be verified by straightforward computations using (16.3.10); see Exercise 16.8).

The next two lemmas involve derivatives with respect to the mass parameter m. For clarity, we again denote the m-dependence of the operators by the subscript m.

LEMMA 16.3.3. *The time evolution operator in the vacuum satisfies the relation*

$$(t-t')\,U_m^{t,t'}(\vec{k}) = \frac{\partial}{\partial m} V_m^{t,t'}(\vec{k}) + W_m^{t,t'}(\vec{k})\,, \tag{16.3.12}$$

where

$$V_m^{t,t'}(\vec{k}) = \sum_\pm \frac{\mathrm{i}}{2m}\,(\not{k}_\pm + m)\gamma^0\,\mathrm{e}^{\mp\mathrm{i}\omega(t-t')} \tag{16.3.13}$$

$$W_m^{t,t'}(\vec{k}) = \sum_\pm \frac{\mathrm{i}}{2}\Big(\frac{\not{k}_\pm\gamma^0}{m^2} \mp \frac{1}{\omega}\Big)\,\mathrm{e}^{\mp\mathrm{i}\omega(t-t')}\,. \tag{16.3.14}$$

The operators $V_m^{t,t'}$ and $W_m^{t,t'}$ are estimated uniformly by

$$\|V_m^{t,t'}(\vec{k})\| + \|W_m^{t,t'}(\vec{k})\| \leq C\left(1 + \frac{|\vec{k}|}{m}\right), \tag{16.3.15}$$

where the constant C is independent of m, $\vec{k}$, t and t' (and $\|\,.\,\|$ is any norm on the 2×2-matrices).

PROOF. First, we generate the factor $t-t'$ by differentiating the exponential in (16.3.9) with respect to ω,

$$(t-t')\,U_m^{t,t'}(\vec{k}) = \sum_\pm \Pi_\pm(\vec{k})\Big(\pm\,\mathrm{i}\frac{\partial}{\partial\omega}\,\mathrm{e}^{\mp\mathrm{i}\omega(t-t')}\Big).$$

Next, we want to rewrite the ω-derivative as a derivative with respect to m. Taking the total differential of the dispersion relation $\omega^2 - |\vec{k}|^2 = m^2$ for fixed $\vec{k}$, one finds that

$$\frac{\partial}{\partial\omega} = \frac{\omega}{m}\,\frac{\partial}{\partial m}\,. \tag{16.3.16}$$

Hence

$$(t-t')\,U_m^{t,t'} = \sum_{\pm} \Pi_{\pm}\Big(\pm \mathrm{i}\,\frac{\omega}{m}\frac{\partial}{\partial m}\,\mathrm{e}^{\mp \mathrm{i}\omega(t-t')}\Big)$$
$$= \frac{\partial}{\partial m}\sum_{\pm}\Big(\pm \mathrm{i}\,\frac{\omega}{m}\,\Pi_{\pm}\,\mathrm{e}^{\mp \mathrm{i}\omega(t-t')}\Big) - \sum_{\pm}\left(\frac{\partial}{\partial m}\left[\pm \mathrm{i}\,\frac{\omega}{m}\,\Pi_{\pm}\right]\right)\mathrm{e}^{\mp \mathrm{i}\omega(t-t')}\,.$$

Computing the operators in the round brackets using (16.3.10) gives the identities (16.3.13) and (16.3.14). Estimating these formulas, one obtains bounds which are at most linear in $|\vec{k}|$, proving (16.3.15). □

This method can be iterated to generate more factors of $t-t'$. In the next lemma, we prove at least quadratic decay in time. For later use, it is preferable to formulate the result in position space.

LEMMA 16.3.4. *The time evolution operator in the vacuum has the representation*

$$U_m^{t,t'} = \frac{1}{(t-t')^2}\left(\frac{\partial^2}{\partial m^2}A_m^{t,t'} + \frac{\partial}{\partial m}B_m^{t,t'} + C_m^{t,t'}\right) \tag{16.3.17}$$

with operators

$$A_m^{t,t'}, B_m^{t,t'}, C_m^{t,t'} \;:\; W^{2,2}(\mathscr{N}_{t'}, S\mathscr{M}) \to L^2(\mathscr{N}_t, S\mathscr{M})\,,$$

which are bounded uniformly in time by

$$\|A_m^{t,t'}(\phi)\|_t + \|B_m^{t,t'}(\phi)\|_t + \|C_m^{t,t'}(\phi)\|_t \le c\,\|\phi\|_{W^{2,2}}\,, \tag{16.3.18}$$

where c is a constant which depends only on m.

PROOF. A straightforward computation using exactly the same methods as in Lemma 16.3.3 yields the representation

$$(t-t')^2\,U_m^{t,t'}(\vec{k}) = \frac{\partial^2}{\partial m^2}A_m^{t,t'}(\vec{k}) + \frac{\partial}{\partial m}B_m^{t,t'}(\vec{k}) + C_m^{t,t'}(\vec{k})\,, \tag{16.3.19}$$

where the operators $A_m^{t,t'}$, $B_m^{t,t'}$ and $C_m^{t,t'}$ are bounded by

$$\|A_m^{t,t'}(\vec{k})\| + \|B_m^{t,t'}(\vec{k})\| + \|C_m^{t,t'}(\vec{k})\| \le \frac{C}{m}\left(1 + \frac{|\vec{k}|}{m} + \frac{|\vec{k}|^2}{m^2}\right), \tag{16.3.20}$$

with a numerical constant $C > 0$. We remark that, compared to (16.3.12), the right of (16.3.20) involves an additional $1/m$. This prefactor is necessary for dimensional reasons, because the additional factor $t-t'$ in (16.3.19) (compared to (16.3.12)) brings in an additional dimension of length (and in natural units, the factor $1/m$ also has the dimension of length). The additional summand $|\vec{k}|^2/m^2$ in (16.3.20) can be understood from the fact that applying (16.3.16) generates a factor of ω/m which for large $|\vec{k}|$ scales like $|\vec{k}|/m$.

Translating this result to position space and keeping in mind that the vector $\vec{k}$ corresponds to the derivative $-\mathrm{i}\vec{\nabla}$, we obtain the result. □

PROOF OF LEMMA 16.3.1. First of all, the Schwarz inequality gives

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \le \int_I \|\psi_m\|_m\,\mathrm{d}m \le \sqrt{|I|}\;\|\psi\|\,.$$

Thus it remains to show the decay for large t, i.e.

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \frac{C\,|I|}{t^2}\,\sup_{m\in I} \sum_{b=0}^{2} \|\partial_m^b(\psi_m)|_{t=0}\|_{W^{2,2}}\,. \tag{16.3.21}$$

We apply Lemma 16.3.4 and integrate by parts in m to obtain

$$\begin{aligned}(\mathfrak{p}\psi)|_t &= \int_I U_m^{t,0}\,\psi_m|_{t=0}\;\mathrm{d}m = \frac{1}{t^2}\int_I \left(\partial_m^2 A_m^{t,0} + \partial_m B_m^{t,0} + C_m^{t,0}\right)\psi_m|_{t=0}\;\mathrm{d}m \\ &= \frac{1}{t^2}\int_I \Big(A_m^{t,0}\,(\partial_m^2\psi_m|_{t=0}) - B_m^{t,0}\,(\partial_m\psi_m|_{t=0}) + C_m^{t,0}\,\psi_m|_{t=0}\Big)\;\mathrm{d}m\,.\end{aligned}$$

Taking the norm and using (16.3.18) gives (16.3.21). □

We finally note that the previous estimates are not optimal for two reasons. First, the pointwise quadratic decay in (16.3.3) is more than what is needed for the convergence of the integral in (16.3.6). Second and more importantly, the Schwarz inequality (16.3.5) does not catch the optimal scaling behavior in $\vec{k}$. This is the reason why the constant in (15.2.6) involves derivatives of ψ_m (cf. (16.3.6)), making it impossible to prove the inequality (15.2.8) which arises in the strong mass oscillation property. In order to improve the estimates, one needs to use Fourier methods both in space and time, as will be explained in the next section.

16.4. Proof of the Strong Mass Oscillation Property in the Minkowski Vacuum

THEOREM 16.4.1. *The vacuum Dirac operator in Minkowski space has the strong mass oscillation property with domain* (16.3.1).

Our proof relies on a Plancherel argument in spacetime. It also provides an alternative method for establishing the weak mass oscillation property.

PROOF OF THEOREM 16.4.1. Let $\psi = (\psi_m)_{m\in I} \in \mathcal{H}^\infty$ be a family of solutions of the Dirac equation for a varying mass parameter in the Minkowski vacuum. Using Proposition 13.6.1, one can express ψ_m in terms of its values at time $t = 0$ by

$$\psi_m(x) = 2\pi \int_{\mathbb{R}^3} k_m(x,(0,\vec{y}))\,\gamma^0\,\psi_m|_{t=0}(\vec{y})\;\mathrm{d}^3y\,.$$

We now take the Fourier transform, denoting the four-momentum by k. Using (16.3.8), we obtain

$$\begin{aligned}\psi_m(k) &= 2\pi k_m(k)\,\gamma^0\hat{\psi}_m^0(\vec{k}) \\ &= 2\pi\,\delta(k^2-m^2)\,\epsilon(k^0)\,(\not{k}+m)\,\gamma^0\hat{\psi}_m^0(\vec{k})\,,\end{aligned}$$

where $\hat{\psi}_m^0(\vec{k})$ denotes the spatial Fourier transform of $\psi_m|_{t=0}$ (in order to avoid an ambiguity of notation, the hat of the Fourier transform in spacetime was omitted). Obviously, this is a distribution supported on the mass shell. In particular, it is not square integrable over $\mathbb{R}^4$.

Integrating over m, we obtain the following function

$$(\mathfrak{p}\psi)(k) = 2\pi\,\chi_I(m)\,\frac{1}{2m}\,\epsilon(k^0)\,(\not{k}+m)\,\gamma^0\hat{\psi}_m^0(\vec{k})\Big|_{m=\sqrt{k^2}}\,, \tag{16.4.1}$$

where m now is a function of the momentum variables. Since the function $\psi_m|_{t=0}$ is compactly supported and smooth in the spatial variables, its Fourier transform $\hat{\psi}_m^0(\vec{k})$

has rapid decay. This shows that the function (16.4.1) is indeed square integrable. Using Plancherel, we see that condition (a) in Definition 15.2.2 is satisfied. Moreover, the operator T is simply the operator of multiplication by $\sqrt{k^2}$, so that condition (b) obviously holds. This again shows the weak mass oscillation property.

In order to prove the strong mass oscillation property, we need to compute the inner product $<\mathfrak{p}\psi|\mathfrak{p}\phi>$. To this end, we first write this inner product in momentum space as

$$\begin{aligned} <\mathfrak{p}\psi|\mathfrak{p}\phi> &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} 4\pi^2\, \chi_I(m)\, \frac{1}{4m^2}\, \prec (\not{k}+m)\,\gamma^0 \hat{\psi}^0_m(\vec{k}) \,|\, (\not{k}+m)\,\gamma^0 \hat{\phi}^0_m(\vec{k}) \succ \Big|_{m=\sqrt{k^2}} \\ &= \int \frac{\mathrm{d}^4k}{4\pi^2}\, \chi_I(m)\, \frac{1}{2m}\, \prec \gamma^0 \hat{\psi}^0_m(\vec{k}) \,|\, (\not{k}+m)\,\gamma^0 \hat{\phi}^0_m(\vec{k}) \succ \Big|_{m=\sqrt{k^2}} . \end{aligned}$$

Reparametrizing the k^0-integral as an integral over m, we obtain

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \frac{1}{4\pi^2} \int_I \mathrm{d}m \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{2\,|k^0|}\, \prec \gamma^0 \hat{\psi}^0_m(\vec{k}) \,|\, (\not{k}+m)\,\gamma^0 \hat{\phi}^0_m(\vec{k}) \succ \Big|_{k^0=\pm\sqrt{|\vec{k}|^2+m^2}} . \tag{16.4.2}$$

Estimating the inner product with the Schwarz inequality and applying Plancherel's theorem, one finds

$$|<\mathfrak{p}\psi|\mathfrak{p}\phi>| \leq \frac{1}{4\pi^2} \int_I \mathrm{d}m \int_{\mathbb{R}^3} \|\hat{\psi}^0_m(\vec{k})\|\, \|\hat{\phi}^0_m(\vec{k})\|\; \mathrm{d}^3k \leq 2\pi \int_I \|\psi_m\|_m\, \|\phi_m\|_m\; \mathrm{d}m .$$

Thus the inequality (15.2.8) holds. □

Apart from completing the proof of the strong mass oscillation property, the computation in the above proof also tells us what the fermionic signature operator is. In order to see this, we return to the formula (16.4.2). Applying Plancherel's theorem and using (15.1.2), we conclude that

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi^0_m \,|\, \mathcal{S}_m\, \phi^0_m)_m\; \mathrm{d}m , \tag{16.4.3}$$

where $\mathcal{S}_m$ is the multiplication operator in momentum space

$$\mathcal{S}_m(\vec{k}) := \sum_{k^0=\pm\omega(\vec{k})} \frac{\not{k}+m}{2\,\omega(\vec{k})}\,\gamma^0 = \frac{\vec{k}\vec{\gamma}+m}{\omega(\vec{k})}\,\gamma^0 . \tag{16.4.4}$$

Comparing (16.4.3) with (15.3.3), one sees that the matrix $\mathcal{S}_m(\vec{k})$ is indeed the fermionic signature operator, considered as a multiplication operator in momentum space. By direct computation, one verifies that the matrix $\mathcal{S}_m(\vec{k})$ has eigenvalues ± 1 (here one can use that $\mathcal{S}_m = \Pi_+ - \Pi_-$ with $\Pi_\pm$ as introduced in (16.3.10)).

16.5. Exercises

EXERCISE 16.1. This exercise recalls basics on the principal value in one dimension

$$\frac{1}{2} \lim_{\varepsilon \searrow 0} \left(\frac{1}{x - \mathrm{i}\varepsilon} + \frac{1}{x + \mathrm{i}\varepsilon} \right) =: \frac{\mathrm{PP}}{x} . \tag{16.5.1}$$

(a) Repeat the method in Exercise 2.23 to show that the limit of the left side of (16.5.1) exist for any $\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Derive a corresponding estimate which shows that PP is a well-defined tempered distribution.

(b) Show that for any $\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$,

$$\mathrm{PP}(\eta) = \lim_{\varepsilon \searrow 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\eta(x)}{x}\, dx\,.$$

EXERCISE 16.2. The goal of this exercise is to justify that the one-dimensional relations

$$\lim_{\varepsilon \searrow 0} \left(\frac{1}{x - \mathrm{i}\varepsilon} - \frac{1}{x + \mathrm{i}\varepsilon} \right) = 2\pi i\, \delta(x) \tag{16.5.2}$$

$$\frac{1}{2} \lim_{\varepsilon \searrow 0} \left(\frac{1}{x - \mathrm{i}\varepsilon} + \frac{1}{x + \mathrm{i}\varepsilon} \right) =: \frac{\mathrm{PP}}{x}\,. \tag{16.5.3}$$

can be used in the four-dimensional setting to obtain the identity

$$\lim_{\varepsilon \searrow 0} \frac{1}{r^2 + (\varepsilon + \mathrm{i}t)^2} = \lim_{\varepsilon \searrow 0} \frac{1}{r^2 - t^2 + \mathrm{i}\varepsilon t} = -\frac{\mathrm{PP}}{\xi^2} - \mathrm{i}\pi\, \delta(\xi^2)\, \epsilon(\xi^0)\,, \tag{16.5.4}$$

(a) Let T be a distribution on $\mathbb{R}$, $\Omega \subset \mathscr{M}$ be an open subset of Minkowski space and $f : \Omega \to \mathbb{R}$ a smooth function with nowhere vanishing gradient. Show that the relation

$$\big(f^* T\big)(\eta) := T\big(\phi_f(\eta)\big)\,, \qquad \eta \in C_0^\infty(\Omega)$$

with

$$\phi_f(\eta)(t) := \frac{\partial}{\partial t} \int_\Omega \Theta\big(t - f(x)\big)\, \eta(x)\, \mathrm{d}^4 x$$

(where Θ is the Heaviside function) defines f^*T as a distribution on Ω (this is the so-called *pullback* of T under f; for details see [**89**, Section 7.2]).

(b) Choosing Ω as the half space in the future, $\Omega = \{x \in \mathscr{M}, x^0 > 0\}$, one can rewrite the expression on the left of (16.5.4) as

$$\lim_{\varepsilon \searrow 0} \frac{1}{r^2 - t^2 + \mathrm{i}\varepsilon}\,.$$

Use (a) to conclude that this expression is a well-defined distribution for any $\varepsilon > 0$. Show that the limit $\varepsilon \searrow 0$ exist in the distributional sense.

(c) Repeating the procedure of (b) for the half space in the past, one obtains a distribution on $\mathscr{M} \setminus \{t = 0\}$. Show that this distribution coincides with the limit in (16.5.4). *Hint:* Similar as in Exercise 2.23, one can estimate the behavior at the origin with Lebesgue's dominated convergence theorem.

EXERCISE 16.3. This exercise is devoted to the *advanced Green's operator* $s_m^\vee$.

(a) Assume that $m > 0$. Show that the limit $\nu \searrow 0$ in (16.1.4) exist in the distributional sense.

(b) Show that the limit $\nu \searrow 0$ in (16.1.4) also exists in the massless case $m = 0$ and that

$$\lim_{m \searrow 0} s_m^\vee(k) = s_0^\vee(k) \qquad \text{as a distribution}\,.$$

Hint: Proceed similar as in Exercise 16.2.

(c) Consider the Fourier integral in the q^0-variable

$$\int_{-\infty}^{\infty} \frac{1}{q^2 - m^2 - \mathrm{i}\nu q^0}\, \mathrm{e}^{\mathrm{i}q^0 t}\; \mathrm{d}q^0\,.$$

Show with residues that this integral vanishes for sufficiently small ν if $t < 0$.

(d) Argue with Lorentz invariance to prove the left side of (16.1.5).

EXERCISE 16.4. Modifying the location of the poles in (16.1.4) gives rise to the distribution

$$s_m^F(k) := \lim_{\nu \searrow 0} \frac{\not{k} + m}{k^2 - m^2 + \mathrm{i}\nu} \,.$$

This is the well-known *Feynman propagator*, which is often described intuitively by saying that "positive frequencies move to the future and negative frequencies move to the past." Make this sentence precise by a computation similar to that in Exercise 16.3 (c).

EXERCISE 16.5. Given $\omega \in \mathbb{R}$, we consider the ordinary differential operator $\mathcal{D} = \mathrm{i}\partial_t + \omega$.

(a) Construct the advanced and retarded Green's operators, which satisfy in analogy to (16.2.1) the equation

$$\mathcal{D}_t s(t,t') = \delta(t-t') \,.$$

(b) Compute the resulting causal fundamental solution according to (16.2.2). How is it related to the time evolution operator $U^{t,t'}$? On which Hilbert space does the time evolution operator act as a unitary operator?

EXERCISE 16.6. Consider the massless Dirac equation $\mathcal{D}\psi = 0$ in the two-dimensional spacetime cylinder $\mathbb{R} \times S^1$, i.e.

$$\mathcal{D} = \mathrm{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_t + \mathrm{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_\varphi$$

with $t \in \mathbb{R}$ and $\varphi \in (0, 2\pi)$.

(a) Choose the spin inner product such that the Dirac matrices become symmetric. What is the resulting spacetime inner product $<.|.>$? What is the scalar product $(.|.)$?

(b) Employ for $k \in \mathbb{Z}$ the separation ansatz

$$\psi(t,\varphi) = \mathrm{e}^{-\mathrm{i}k\varphi} \chi(t) \qquad \text{with} \qquad \chi(t) \in \mathbb{C}^2 \,.$$

Derive the resulting ODE for χ. Compute the time evolution operator for this ODE. *Hint:* Use the result of Exercise 16.5.

(c) Use a Fourier series decomposition in order to deduce a series representation of the time evolution operator of the Dirac operator on $\mathbb{R} \times S^1$. Try to carry out the infinite series to obtain a closed expression for $U^{t,t'}$. How can one see finite propagation speed?

EXERCISE 16.7. As in Exercise 16.6, we consider the two-dimensional massless Dirac equation.

(a) Adapt the formulas for the advanced and retarded Green's operators in momentum space to the two-dimensional massless case.

(b) Compute the Fourier transform to obtain $s^\vee(x,y)$ and $s^\wedge(x,y)$.

(c) Use the result of (b) to compute the causal fundamental solution and the time evolution operator.

(d) How can one see finite propagation speed? How is the obtained formula related to the formula in Exercise 16.6 (c)?

EXERCISE 16.8. Verify the relations (16.3.11) by direct computation starting from the definition (16.3.10).

EXERCISE 16.9. Verify by formal computation that in the Minkowski vacuum, the fundamental solution k_m and the Green's operator s_m defined by

$$s_m := \frac{1}{2}\left(s_m^\vee + s_m^\wedge\right)$$

satisfy the distributional relations in the mass parameters m and m'

$$k_m\, k_{m'} = \delta(m-m')\, p_m$$
$$k_m\, s_{m'} = s_{m'}\, k_m = \frac{\mathrm{PP}}{m-m'}\, k_m\ ,$$

where PP denotes the principal part, and p_m is the distribution

$$p_m(k) = (\not{k} + m)\, \delta(k^2 - m^2)\ .$$

Hint: By a "formal computation" we mean that you do not need to evaluate weakly in the mass with test functions.

EXERCISE 16.10. Proceed similar as in Exercise 16.9 to derive a relation for the operator product $s_m^\vee s_{m'}^\vee$. Derive the relation

$$s_m\, s_{m'} = \frac{\mathrm{PP}}{m-m'}\,(s_m - s_{m'}) + \pi^2\, \delta(m-m')\, p_m\ .$$

CHAPTER 17

Methods of Scattering Theory

In Chapter 15, the fermionic signature operator and the unregularized fermionic projector were introduced abstractly. In the previous chapter, we computed them in the Minkowski vacuum. It remains to construct them in the presence of an external potential. In order to prove the mass oscillation properties, our task is to analyze the Dirac solutions asymptotically for large times and near spatial infinity. This can be accomplished with methods of scattering theory, which we now briefly introduce. We follow the presentation in [**79**].

We return to the Cauchy problem in the presence of an external potential,

$$(\mathcal{D} - m)\,\psi_m = 0\,, \qquad \psi_m\big|_{t_0} = \psi_0 \in C^\infty(\mathcal{N}_{t_0} \simeq \mathbb{R}^3, S\mathscr{M})\,, \tag{17.0.1}$$

with $\mathcal{D}$ as in (1.3.14). For notational clarity, we shall often denote the objects in the presence of the external potential by a tilde (the "interacting objects"), whereas the objects without tilde refer to the Minkowski vacuum.

17.1. The Lippmann-Schwinger Equation

The Dirac dynamics can be rewritten in terms of a symmetric operator $\tilde{H}$. To this end, we multiply the Dirac equation (1.3.14) by γ^0 and bring the t-derivative separately on one side of the equation,

$$\mathrm{i}\partial_t \psi_m = \tilde{H}\psi_m\,, \qquad \text{where} \qquad \tilde{H} := -\gamma^0(\mathrm{i}\vec{\gamma}\vec{\nabla} + \mathcal{B} - m) \tag{17.1.1}$$

(note that $\gamma^j\partial_j = \gamma^0\partial_t + \vec{\gamma}\vec{\nabla}$). We refer to (17.1.1) as the Dirac equation in *Hamiltonian form*. The fact that the scalar product (15.1.2) is time independent implies that for any two solutions $\phi_m, \psi_m \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$,

$$0 = \partial_t(\phi_m \,|\, \psi_m)_m = \mathrm{i}\big((\tilde{H}\phi_m \,|\, \psi_m)_m - (\phi_m \,|\, \tilde{H}\psi_m)_m\big)\,,$$

showing that the Hamiltonian is a symmetric operator on $\mathcal{H}_m$.

The Lippmann-Schwinger equation can be used to compare the dynamics in the Minkowski vacuum with the dynamics in the presence of an external potential. We denote the time evolution operator in the Minkowski vacuum by U_m^{t,t_0}.

PROPOSITION 17.1.1. *The Cauchy problem* (17.0.1) *has a solution ψ_m which satisfies the equation*

$$\psi_m|_t = U_m^{t,t_0}\psi_0 + \mathrm{i}\int_{t_0}^{t} U_m^{t,\tau}\left(\gamma^0\mathcal{B}\,\psi_m\right)\big|_\tau \,\mathrm{d}\tau\,, \tag{17.1.2}$$

referred to as the **Lippmann-Schwinger equation**.

PROOF. Obviously, the wave function $\psi_m|_t$ given by (17.1.2) has the correct initial values at $t = t_0$. Thus it remains to show that this wave function satisfies the Dirac

equation. To this end, we rewrite the Dirac equation in the Hamiltonian form (17.1.1), and separate the vacuum Hamiltonian H from the term involving the external potential,

$$(\mathrm{i}\partial_t - H)\,\psi_m = -\gamma^0 \mathcal{B}\,\psi_m \qquad \text{with} \qquad H = -\mathrm{i}\gamma^0 \vec{\gamma}\vec{\nabla} + \gamma^0 m\,. \tag{17.1.3}$$

Applying the operator $\mathrm{i}\partial_t - H$ to (17.1.2) and observing that the time evolution operator maps to solutions of the vacuum Dirac equation, only the derivative of the upper limit of integration contributes,

$$(\mathrm{i}\partial_t - H)\,\psi_m|_t = -U_m^{t,\tau}\,\left(\gamma^0 \mathcal{B}\,\psi_m\right)\big|_{\tau=t} = -\gamma^0 \mathcal{B}\,\psi_m|_t\,,$$

so that (17.1.3) is indeed satisfied. □

We remark that one way of solving the Lippmann-Schwinger equation is to substitute the left side on the right side to obtain

$$\begin{aligned}\psi_m|_t &= U_m^{t,t_0}\psi_0 + \mathrm{i}\int_{t_0}^t U_m^{t,\tau}\left(\gamma^0 \mathcal{B}\,\psi_m\right)\big|_\tau\,\mathrm{d}\tau \\ &= U_m^{t,t_0}\psi_0 + \mathrm{i}\int_{t_0}^t U_m^{t,\tau}\,\gamma^0\mathcal{B}|_\tau\,U_m^{\tau,t_0}\psi_0\,\mathrm{d}\tau - \int_{t_0}^t U_m^{t,\tau}\,\gamma^0\mathcal{B}|_\tau \int_{t_0}^\tau U_m^{\tau,\tau'}\left(\gamma^0\mathcal{B}\,\psi_m\right)\big|_{\tau'}\,\mathrm{d}\tau'\,\mathrm{d}\tau\,.\end{aligned}$$

Iterating this procedure, one gets a series of nested integrals referred to as the *Dyson series*, which is commonly used in perturbative quantum field theory (see for example [**147**, Section 3.5]). The Dyson series can be regarded as an *ordered exponential* (see Exercise 17.1).

17.2. The Mass Oscillation Property in the Presence of an External Potential

The goal of this section is to prove the following result:

THEOREM 17.2.1. *Assume that the external potential $\mathcal{B}$ is smooth and for large times decays faster than quadratically in the sense that*

$$|\mathcal{B}(t)|_{C^2} \leq \frac{c}{1+|t|^{2+\varepsilon}} \tag{17.2.1}$$

for suitable constants $\varepsilon, c > 0$. Then the strong mass oscillation property holds.

In words, the condition (17.2.1) means that the potential and its up to second derivatives must decay for large times faster than quadratically. This condition does not seem to have any physical significance; it is needed in order for our methods to apply.

The C^2-norm in (17.2.1) is defined as follows. We denote spatial derivatives by ∇ and use the notation with multi-indices, i.e. for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_p)$ we set $\nabla^\alpha = \partial_{\alpha_1 \cdots \alpha_p}$ and denote the length of the multi-index by $|\alpha| = p$. Then the spatial C^k-norms of the potential are defined by

$$|\mathcal{B}(t)|_{C^k} := \max_{|\alpha|\leq k} \sup_{\vec{x}\in\mathbb{R}^3} |\nabla^\alpha \mathcal{B}(t,\vec{x})|\,, \tag{17.2.2}$$

where $|\,.\,|$ is the sup-norm corresponding to the norm $|\phi|^2 := \prec\phi|\gamma^0\phi\succ$ on the spinors.

17.2.1. Proof of the Weak Mass Oscillation Property. In this section, we prove the following theorem.

THEOREM 17.2.2. *Assume that the time-dependent external potential $\mathcal{B}$ is smooth and decays faster than quadratically for large times in the sense that* (17.2.1) *holds for suitable constants $c, \varepsilon > 0$. Then the Dirac operator $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ has the weak mass oscillation property.*

We expect that this theorem could be improved by weakening the decay assumptions on the potential. However, this would require refinements of our methods which would go beyond the scope of this paper. Also, using that Dirac solutions dissipate, the pointwise decay in time could probably be replaced or partially compensated by suitable spatial decay assumptions. Moreover, one could probably refine the result of the above theorem by working with other norms (like weighted C^k- or Sobolev norms).

The main step is the following basic estimate, which is the analog of Lemma 16.3.1 in the presence of an external potential.

PROPOSITION 17.2.3. *Under the decay assumptions* (17.2.1) *on the external potential $\mathcal{B}$, there are constants $c, \varepsilon > 0$ such that for every family $\psi \in \mathcal{H}^\infty$ of solutions of the Dirac equation* (1.3.14) *with varying mass,*

$$\big\| (\mathfrak{p}\psi)|_t \big\|_t \leq \frac{c}{1+|t|^{1+\varepsilon}} \sup_{m \in I} \sum_{b=0}^{2} \big\| (\partial_m^b \psi_m)|_{t=0} \big\|_{W^{2,2}} \,. \tag{17.2.3}$$

We first show that this proposition implies the weak mass oscillation property.

PROOF OF THEOREM 17.2.2 ASSUMING THAT PROPOSITION 17.2.3 HOLDS. In order to derive the inequality (15.2.6), we begin with the estimate

$$|<\mathfrak{p}\psi|\mathfrak{p}\phi>| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big| \big(\mathfrak{p}\psi|_t \,\big|\, \mathfrak{p}\phi|_t \big)\big|_t \Big| \, \mathrm{d}t \leq \sup_{t \in \mathbb{R}} \big\| \mathfrak{p}\phi|_t \big\|_t \int_{-\infty}^{\infty} \big\| \mathfrak{p}\psi|_t \big\|_t \, \mathrm{d}t \,.$$

The last integral is finite by Proposition 17.2.3. The supremum can be bounded by the Hilbert space norm using the Hölder inequality,

$$\|\mathfrak{p}\phi|_t\|_t = \left\| \int_I \phi_m|_t \, \mathrm{d}m \right\|_t \leq \int_I \big\| \phi_m|_t \big\|_t \, \mathrm{d}m \leq \sqrt{|I|} \left(\int_I \|\phi_m|_t\|_t^2 \, \mathrm{d}m \right)^{\frac{1}{2}} = \sqrt{|I|}\, \|\phi\| \,,$$

giving (15.2.6).

Using (1.3.13), the Dirac operator $i\partial\!\!\!/ + \mathcal{B}$ is symmetric with respect to the inner product $<.|.>$. Therefore, the identity (15.2.7) can be obtained just as in (16.3.7) by integrating the Dirac operator in spacetime by parts, noting that we do not get boundary terms in view of the time decay in Proposition 17.2.3. □

The remainder of this section is devoted to the proof of Proposition 17.2.3. We make use of the Lippmann-Schwinger equation (17.1.2),

$$\psi_m|_t = U_m^{t,0}\, \psi_m|_{t=0} + \mathrm{i} \int_0^t U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m \big)\big|_\tau \, \mathrm{d}\tau \,. \tag{17.2.4}$$

Since the first summand of this equation is controlled by Lemma 16.3.1, it remains to estimate the second summand. Again using (16.3.17) and integrating by parts with respect to the mass, we obtain

$$\int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m \big)\big|_\tau \, \mathrm{d}m = \frac{1}{(t-\tau)^2} \int_I \big(A_m^{t,\tau}\, \partial_m^2 - B_m^{t,\tau} \partial_m + C_m^{t,\tau} \big) \big(\gamma^0 \mathcal{B}\, \psi_m \big)\big|_\tau \, \mathrm{d}m$$

(where I is again the interval (15.1.4)) and thus

$$\left\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, \mathrm{d}m \right\|_t \leq \frac{c\,|I|}{(t-\tau)^2} \sup_{m \in I} \sum_{b=0}^{2} \big\| \mathcal{B}(\tau)\, (\partial_m^b \psi_m)|_\tau \big\|_{W^{2,2}}$$
$$\leq \frac{c\,|I|}{(t-\tau)^2}\, |\mathcal{B}(\tau)|_{C^2} \sup_{m \in I} \sum_{b=0}^{2} \big\| \partial_m^b \psi_m|_\tau \big\|_{W^{2,2}} .$$

We now bound $\mathcal{B}(\tau)$ with the help of (17.2.1). Moreover, we estimate the Sobolev norm $\big\|\partial_m^b \psi_m|_\tau\big\|_{W^{2,2}}$ at time τ by means of Lemma 13.5.1. This gives rise to the inequality

$$\left\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, \mathrm{d}m \right\|_t \leq \frac{c^2\, C\, |I|}{(t-\tau)^2}\, \frac{1+|\tau|^2}{1+|\tau|^{2+\varepsilon}} \sup_{m \in I} \sum_{b=0}^{2} \big\| \partial_m^b \psi_m|_{t=0} \big\|_{W^{2,2}} ,$$

which yields the desired decay provided that τ and t are not too close to each other. More precisely, we shall apply this inequality in the case $|\tau| \leq |t|/2$. Then the estimate simplifies to

$$\left\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, \mathrm{d}m \right\|_t$$
$$\leq \frac{\tilde{C}}{t^2\,(1+|\tau|^\varepsilon)} \sup_{m \in I} \sum_{b=0}^{2} \big\| \partial_m^b \psi_m|_{t=0} \big\|_{W^{2,2}} \qquad \text{if } |\tau| \leq |t|/2 \tag{17.2.5}$$

with a new constant $\tilde{C} > 0$. In the remaining case $|\tau| > |t|/2$, we use the unitarity of $U_m^{t,\tau}$ to obtain

$$\left\| \int_I U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m\right)|_\tau \, \mathrm{d}m \right\|_t \leq |I|\, |\mathcal{B}(\tau)|_{C^0} \sup_{m \in I} \|\psi_m\| .$$

Applying (17.2.1) together with the inequality $|\tau| > |t|/2$, this gives

$$\left\| \int_I U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m\right)|_\tau \, \mathrm{d}m \right\|_t \leq \frac{\tilde{C}}{t^{2+\varepsilon}} \sup_{m \in I} \|\psi_m\| \qquad \text{if } |\tau| > |t|/2 . \tag{17.2.6}$$

This again decays for large t because τ is close to t and because $|\mathcal{B}(\tau)|_{C^0}$ decays for large τ.

Comparing (17.2.5) and (17.2.6), we find that the inequality in (17.2.5) even holds for all τ. Thus integrating this inequality over $\tau \in [0,t]$, we obtain the following estimate for the second summand in (17.2.4),

$$\left\| \int_I \mathrm{d}m \int_0^t U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m\right)|_\tau \, \mathrm{d}\tau \right\|_t \leq \frac{C'}{t^{1+\varepsilon}} \sup_{m \in I} \sum_{b=0}^{2} \big\| \partial_m^b \psi|_{t=0} \big\|_{W^{2,2}}$$

(where $C' > 0$ is a new constant). Combining this inequality with the estimate (16.3.3) of the first summand in (17.2.4), we obtain the desired inequality (17.2.3). This concludes the proof of Proposition 17.2.3.

17.2.2. Proof of the Strong Mass Oscillation Property. In this section, we prove the following result.

THEOREM 17.2.4. *Assume that the weak mass oscillation property holds and that the external potential $\mathcal{B}$ satisfies the condition*

$$\int_{-\infty}^{\infty} |\mathcal{B}(\tau)|_{C^0} \, \mathrm{d}\tau < \infty \,. \tag{17.2.7}$$

Then the Dirac operator $\mathcal{D} = \mathrm{i}\partial\!\!\!/ + \mathcal{B}$ has the strong mass oscillation property.

Combining this theorem with Theorem 17.2.2, one immediately obtains Theorem 17.2.1.

For the proof we shall derive an explicit formula for the fermionic signature operator (Proposition 17.2.5). This formula is obtained by comparing the dynamics in the presence of the external potential with that in the Minkowski vacuum using the Lippmann-Schwinger equation, and by employing distributional relations for products of fundamental solutions and Green's operators (Lemma 17.2.8).

In order to compare the dynamics in the presence of the external potential with that in the Minkowski vacuum, we work with the Hamiltonian formulation. We decompose the Dirac Hamiltonian (17.1.1) into the Hamiltonian in the Minkowski vacuum (17.1.3) plus a potential,

$$\tilde{H} = H + \mathcal{V} \qquad \text{with} \qquad \mathcal{V} := -\gamma^0 \mathcal{B} \,.$$

PROPOSITION 17.2.5. *Assume that the potential $\mathcal{B}$ satisfies the condition* (17.2.7)*. Then for every $\psi, \phi \in \mathcal{H}^\infty$,*

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi_m \,|\, \tilde{\mathcal{S}}_m \, \phi_m)_m \, \mathrm{d}m \,, \tag{17.2.8}$$

where $\tilde{\mathcal{S}}_m : \mathcal{H}_m \to \mathcal{H}_m$ are bounded linear operators which act on the wave functions at time t_0 by

$$\tilde{\mathcal{S}}_m = \mathcal{S}_m - \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0) \left[\mathcal{S}_m \, U_m^{t_0,t} \, \mathcal{V}(t) \, \tilde{U}_m^{t,t_0} - \tilde{U}_m^{t_0,t} \, \mathcal{V}(t) \, \mathcal{S}_m \, U_m^{t,t_0} \right] \mathrm{d}t \tag{17.2.9}$$

$$+ \frac{1}{2} \left(\int_{t_0}^{\infty} \int_{t_0}^{\infty} + \int_{-\infty}^{t_0} \int_{-\infty}^{t_0} \right) \tilde{U}_m^{t_0,t} \, \mathcal{V}(t) \, \mathcal{S}_m \, U_m^{t,t'} \, \mathcal{V}(t') \, \tilde{U}_m^{t',t_0} \, \mathrm{d}t \, \mathrm{d}t' \tag{17.2.10}$$

(and $\mathcal{S}_m$ is again the fermionic signature operator of the vacuum (16.4.4)*).*

Before entering the proof of this proposition, it is instructive to verify that the above formula for $\tilde{\mathcal{S}}_m$ does not depend on the choice of t_0.

REMARK 17.2.6. **(Independence of $\tilde{\mathcal{S}}_m$ on t_0)** Our strategy is to differentiate the above formula for $\tilde{\mathcal{S}}_m$ with respect to t_0 and to verify that we obtain zero. We first observe that taking a solution $\phi_m \in \mathcal{H}_m$ of the Dirac equation in the presence of $\mathcal{B}$, evaluating at time t_0 and applying the time evolution operator $\tilde{U}_m^{t,t_0}$ gives ϕ_m at time t, i.e. $\tilde{U}_m^{t,t_0} \phi_m|_{t_0} = \phi_m|_t$. Differentiating with respect to t_0 yields

$$\partial_{t_0} \tilde{U}_m^{t,t_0} \phi_m|_{t_0} = 0 \,.$$

The situation is different when one considers the time evolution operator of the vacuum. Namely, in the expression $U_m^{t,t_0} \phi_m|_{t_0}$, the wave function ϕ_m satisfies the Dirac equation $(\mathrm{i}\partial_t - H)\phi_m = \mathcal{V}\phi_m$, whereas the time evolution operator solves the Dirac equation with $\mathcal{V} \equiv 0$. As a consequence,

$$\partial_{t_0} U_m^{t,t_0} \phi_m|_{t_0} = -\mathrm{i} U_m^{t,t_0} (\mathcal{V}\phi_m)|_{t_0} \,.$$

Using these formulas together with $U^{t_0,t_0} = \mathbb{1} = \tilde{U}^{t_0,t_0}$, a straightforward computation gives

$$\begin{aligned}
\partial_{t_0}\big(\psi_m \,|\, (17.2.9)\, \phi_m\big)\big|_{t_0} &= -\,\mathrm{i}(\psi_m \,|\, [\mathcal{S}_m, \mathcal{V}]\, \phi_m)|_{t_0} \\
&\quad - \frac{\mathrm{i}}{2}\,(-2)\big(\psi_m \,\big|\, \big(\mathcal{S}_m\, \mathcal{V}(t_0) - \mathcal{V}(t_0)\, \mathcal{S}_m\big)\, \phi_m\big)\big|_{t_0} \\
&\quad - \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big((-\mathrm{i}\mathcal{V}(t_0))\, \psi_m \,\big|\, \mathcal{S}_m\, U_m^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}_m^{t,t_0}\, \phi_m\big)\big|_{t_0}\, \mathrm{d}t \\
&\quad + \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(\psi_m \,\big|\, \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t_0}\, (-i\mathcal{V}(t_0))\, \phi_m\big)\big|_{t_0}\, \mathrm{d}t \\
\partial_{t_0}\big(\psi_m \,|\, (17.2.10)\, \phi_m\big)\big|_{t_0} &= -\,\frac{1}{2} \int_{-\infty}^{\infty} \epsilon(t'-t_0)\, \big(\psi_m \,\big|\, \mathcal{V}(t_0)\, \mathcal{S}_m\, U_m^{t_0,t'}\, \mathcal{V}(t')\, \tilde{U}_m^{t',t_0}\, \phi_m\big)\big|_{t_0}\, \mathrm{d}t' \\
&\quad - \frac{1}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(\psi_m \,\big|\, \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t_0}\, \mathcal{V}(t_0)\, \phi_m\big)\big|_{t_0}\, \mathrm{d}t\,,
\end{aligned}$$

where for notational simplicity we here omitted the restrictions $|_{t_0}$ for the solutions ψ_m and ϕ_m. Adding the terms gives zero. ◊

The remainder of this section is devoted to the proof of Proposition 17.2.5. Our strategy is to combine the Lippmann-Schwinger equation with estimates in momentum space. We begin with two technical lemmas.

LEMMA 17.2.7. *Assume that the external potential $\mathcal{B}$ satisfies condition* (17.2.7). *For any $t_0 \in \mathbb{R}$, we denote the characteristic functions in the future respectively past of this hypersurface $t = t_0$ by $\chi_{t_0}^{\pm}(x)$ (i.e. $\chi_{t_0}^{\pm}(x) = \Theta(\pm(x^0 - t_0))$, where Θ is the Heaviside function). Then for any $\psi_m \in C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$, the wave function $k_m(\chi_{t_0}^{\pm}\mathcal{B}\psi_m)$ is a well-defined vector in $\mathcal{H}_{t_0}$ and*

$$\|k_m(\chi_{t_0}^{\pm}\mathcal{B}\psi_m)\|_{t_0} \leq \frac{1}{2\pi}\, \|\psi_m\|_m \int_{-\infty}^{\infty} \chi_{t_0}^{\pm}(\tau)\, |\mathcal{B}(\tau)|_{C^0}\, \mathrm{d}\tau\,.$$

PROOF. Using the integral kernel representation (16.2.3) and (16.2.4) together with the fact that the time evolution in the vacuum is unitary, we obtain

$$\begin{aligned}
&2\pi \left\| \int_{\mathbb{R}^3} k_m\big((t_0,.),(\tau,\vec{y})\big)\, \big(\chi_{t_0}^{\pm}\mathcal{B}\psi_m\big)(\tau,\vec{y})\; \mathrm{d}^3y \right\|_{t_0} \\
&\quad = \big\|U_m^{t_0,\tau}\gamma^0(\chi_{t_0}^{\pm}\mathcal{B}\psi_m)|_\tau\big\|_{t_0} = \big\|\gamma^0(\chi_{t_0}^{\pm}\mathcal{B}\psi_m)|_\tau\big\|_\tau \leq |\mathcal{B}(\tau)|_{C^0}\, \|\psi_m\|_m\,.
\end{aligned}$$

Integrating over τ and using (17.2.7) gives the result. □

The following lemma is proved in [**55**, Eqs. (2.13)–(2.17)] (see Exercises 16.9 and 16.10).

LEMMA 17.2.8. *In the Minkowski vacuum, the fundamental solution k_m and the Green's operator s_m defined by*

$$s_m := \frac{1}{2}\,\big(s_m^\vee + s_m^\wedge\big) \tag{17.2.11}$$

satisfy the distributional relations in the mass parameters m and m'

$$\begin{aligned} k_m\, k_{m'} &= \delta(m-m')\, p_m \\ k_m\, s_{m'} &= s_{m'}\, k_m = \frac{\mathrm{PP}}{m-m'}\, k_m \\ s_m\, s_{m'} &= \frac{\mathrm{PP}}{m-m'}\, (s_m - s_{m'}) + \pi^2\, \delta(m-m')\, p_m \;, \end{aligned}$$

where PP *denotes the principal part, and p_m is the distribution*

$$p_m(k) = (\not{k} + m)\, \delta(k^2 - m^2)\;. \tag{17.2.12}$$

Proof of Proposition 17.2.5. Let $\psi \in \mathcal{H}^\infty$ be a family of solutions of the Dirac equation for varying mass. We denote the boundary values at time t_0 by $\psi_m^0 := \psi_m|_{t_0}$. Then we can write the Lippmann-Schwinger equation (17.1.2) as

$$\psi_m|_t = U_m^{t,t_0} \psi_m^0 + \mathrm{i} \int_{t_0}^t U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, \mathrm{d}\tau\;.$$

We now bring this equation into a more useful form. Expressing the time evolution operator with the help of (16.2.4) in terms of the fundamental solution, we obtain

$$\begin{aligned} \psi_m(x) = 2\pi \int_{\mathbb{R}^3} k_m\big(x, (t_0, \vec{y})\big)\, \gamma^0 \psi_m^0(t_0, \vec{y})\; \mathrm{d}^3 y \\ + 2\pi \mathrm{i} \int_{t_0}^{x^0} \mathrm{d}y^0 \int_{\mathbb{R}^3} \mathrm{d}^3 y\, k_m(x,y) (\mathcal{B}\, \psi_m)(y)\;. \end{aligned}$$

Applying (16.2.2) and using that the advanced and retarded Green's operators are supported in the future and past light cones, respectively, we can rewrite the last integral in terms of the advanced and retarded Green's operators,

$$\psi_m = 2\pi\, k_m\big(\gamma^0 \delta_{t_0} \psi_m^0\big) - s_m^\wedge\big(\chi_{t_0}^+ \mathcal{B} \psi_m\big) - s_m^\vee\big(\chi_{t_0}^- \mathcal{B} \psi_m\big)\;,$$

where $\delta_{t_0}(x) := \delta(t_0 - x^0)$ is the δ distribution supported on the hypersurface $x^0 = t_0$. Next, we express the advanced and retarded Green's operators in terms of the Green's operator (17.2.11): According to (16.2.2), we have the relations

$$s_m = s_m^\vee - \mathrm{i}\pi k_m = s_m^\wedge + \mathrm{i}\pi k_m$$

and thus

$$\psi_m = k_m g_m - s_m \mathcal{B} \psi_m \qquad \text{with} \qquad g_m := 2\pi\, \gamma^0 \delta_{t_0} \psi_m^0 + \mathrm{i}\pi\, \epsilon_{t_0} \mathcal{B} \psi_m\;, \tag{17.2.13}$$

where ϵ_{t_0} is the step function

$$\epsilon_{t_0}(x) := \epsilon(x^0 - t_0)$$

(and we omitted the brackets in expressions like $k_m g_m \equiv k_m(g_m)$). Note that the expression $k_m g_m$ is well-defined according to Lemma 17.2.7. We also remark that by applying the operator $(i\partial\!\!\!/ - m)$ to the distribution g_m in (17.2.13), one immediately verifies that ψ_m indeed satisfies the Dirac equation $(\mathrm{i}\partial\!\!\!/ - m)\psi_m = -\mathcal{B}\psi_m$.

Now we can compute the inner product $<\mathfrak{p}\psi|\mathfrak{p}\psi>$ with the help of Lemma 17.2.8. Namely, using (17.2.13),

$$\begin{aligned}<\mathfrak{p}\psi|\mathfrak{p}\psi> &= \iint_{I\times I} <k_m g_m - s_m \mathcal{B}\psi_m \,|\, k_{m'} g_{m'} - s_{m'} \mathcal{B}\psi_{m'}> \,\mathrm{d}m\; \mathrm{d}m' \\ &= \int_I \Big(<g_m \,|\, p_m g_m> + \pi^2 <\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> \Big)\, \mathrm{d}m \\ &\quad + \iint_{I\times I} \frac{\mathrm{PP}}{m-m'} \Big(<\mathcal{B}\psi_m \,|\, k_{m'} g_{m'}> - <k_m g_m \,|\, \mathcal{B}\psi_{m'}> \\ &\qquad\qquad + <\mathcal{B}\psi_m \,|\, (s_m - s_{m'}) \mathcal{B}\psi_{m'}> \Big)\; \mathrm{d}m\; \mathrm{d}m' \,.\end{aligned}$$

Note that this computation is mathematically well-defined in the distributional sense because ψ_m and g_m are smooth and compactly supported in the mass parameter m. Employing the explicit formula for g_m in (17.2.13), we obtain

$$<\mathfrak{p}\psi|\mathfrak{p}\psi> = \int_I \Big(<g_m \,|\, p_m g_m> + \pi^2 <\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> \Big)\, \mathrm{d}m \,.$$

Comparing (16.3.8) with (17.2.12) and taking into account that the operator $\mathcal{S}_m$ defined by (16.4.4) gives a minus sign for the states of negative frequency, we get

$$p_m = \mathcal{S}_m\, k_m \,.$$

Using this identity together with Proposition 13.4.4 in the vacuum yields the relations

$$\begin{aligned}<g_m \,|\, p_m g_m> &= (k_m g_m \,|\, \mathcal{S}_m\, k_m g_m)|_{t_0} \\ <\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> &= (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m\, k_m \mathcal{B}\psi_m)|_{t_0} \,.\end{aligned}$$

We finally apply Proposition 13.6.1 to obtain the representation

$$<\mathfrak{p}\psi|\mathfrak{p}\psi> = \int_I \Big((h_m \,|\, \mathcal{S}_m\, h_m)|_{t_0} + \pi^2\, (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m\, k_m \mathcal{B}\psi_m)|_{t_0} \Big)\; \mathrm{d}m \,, \tag{17.2.14}$$

where

$$h_m := \psi_m + \mathrm{i}\pi\, k_m (\epsilon_{t_0} \mathcal{B}\psi_m) \,.$$

Comparing (17.2.8) with (17.2.14), we get

$$(\psi_m \,|\, \tilde{\mathcal{S}}_m\, \psi_m)_m = (h_m \,|\, \mathcal{S}_m\, h_m)|_{t_0} + \pi^2\, (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m\, k_m \mathcal{B}\psi_m)|_{t_0} \,.$$

Expressing the operators k_m according to (16.2.4) by the time evolution operator and writing ψ_m in terms of the initial data as

$$\psi_m|_t = \tilde{U}^{t,t_0} \psi|_{t_0} \,,$$

we obtain

$$\begin{aligned}(\psi_m \,|\, \tilde{\mathcal{S}}_m\, \psi_m)_m &= (\psi|\mathcal{S}_m \psi)|_{t_0} - \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(\psi \,\big|\, \mathcal{S}_m\, U^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}^{t,t_0}\, \psi \big)\big|_{t_0}\; \mathrm{d}t \\ &\quad + \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(U^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}^{t,t_0}\, \psi \,\big|\, \mathcal{S}_m\, \psi \big)\big|_{t_0}\; \mathrm{d}t \\ &\quad + \frac{1}{4} \iint_{\mathbb{R}\times\mathbb{R}} \epsilon(t-t_0)\, \epsilon(t'-t_0)\, \big(U^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}^{t,t_0}\, \psi \,\big|\, \mathcal{S}_m\, U^{t_0,t'}\, \mathcal{V}(t')\, \tilde{U}^{t',t_0}\, \psi \big)\big|_{t_0}\; \mathrm{d}t\; \mathrm{d}t' \\ &\quad + \frac{1}{4} \iint_{\mathbb{R}\times\mathbb{R}} \big(U^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}^{t,t_0}\, \psi \,\big|\, \mathcal{S}_m\, U^{t_0,t'}\, \mathcal{V}(t')\, \tilde{U}^{t',t_0}\, \psi \big)\big|_{t_0}\; \mathrm{d}t\; \mathrm{d}t' \,.\end{aligned}$$

Rearranging the terms and polarizing gives the result. □

PROOF OF THEOREM 17.2.4. Since the time evolution operators are unitary and the operators $\mathcal{S}_m$ have norm one (see (16.4.4)), the representation (17.2.9) and (17.2.10) gives rise to the following estimate for the sup-norm of $\tilde{S}_m$,

$$\|\tilde{\mathcal{S}}_m\| \leq 1 + \int_{\mathbb{R}} |\mathcal{V}(t)|_{C^0} \, \mathrm{d}t + \iint_{\mathbb{R}\times\mathbb{R}} |\mathcal{V}(t)|_{C^0} \, |\mathcal{V}(t')|_{C^0} \, \mathrm{d}t \, \mathrm{d}t' \,.$$

The decay assumption (17.2.7) implies that the sup-norm of $\tilde{S}_m$ is bounded uniformly in m. Using this fact in (17.2.8) gives the inequality (15.2.8), thereby establishing the strong mass oscillation property. □

We finally remark that the uniqueness statement in Proposition 15.3.3 implies that the relations (17.2.9) and (17.2.10) yield an explicit representation of the fermionic signature operator in the presence of a time-dependent external potential.

17.3. Exercises

EXERCISE 17.1. For a smooth one-parameter family of matrices $F(\alpha)$, $\alpha \in \mathbb{R}$, the **ordered exponential** $\mathrm{Pexp}(\int F(\alpha)\, d\alpha)$

$$\mathrm{Pexp}\left(\int_a^b F(\alpha)\, d\alpha \right) = \mathbb{1} + \int_a^b F(t_0) \, \mathrm{d}t_0 \, + \int_a^b \mathrm{d}t_0 \, F(t_0) \int_{t_0}^b \mathrm{d}t_1 \, F(t_1) + \int_a^b \mathrm{d}t_0 \, F(t_0) \int_{t_0}^b \mathrm{d}t_1 \, F(t_1) \int_{t_1}^b \mathrm{d}t_2 \, F(t_2) + \cdots \,.$$

In this exercise we collect a few elementary properties of the ordered exponential.

(a) Assume that the matrix-valued function F is commutative in the sense that

$$\big[F(\alpha), F(\beta)\big] = 0 \qquad \text{for all } \alpha, \beta \in [a,b] \,.$$

Show that the ordered exponential reduces to the ordinary exponential,

$$\mathrm{Pexp}\left(\int_a^b F(\alpha) \, \mathrm{d}\alpha \right) = \exp\left(\int_a^b F(\alpha) \, \mathrm{d}\alpha \right) .$$

Hint: Show inductively that

$$\int_a^b \mathrm{d}t_0 \, F(t_0) \int_{t_0}^b \mathrm{d}t_1 \, F(t_1) \cdots \int_{t_{n-1}}^b \mathrm{d}t_n \, F(t_n) = \frac{1}{(n+1)!} \left(\int_a^b F(t) \, \mathrm{d}t \right)^{n+1} .$$

(b) Assume that F is continuous on $[a,b]$. Show that the Dyson series converges absolutely and that

$$\left\| \mathrm{Pexp}\left(\int_a^b F(\alpha) \, \mathrm{d}\alpha \right) \right\| \leq \exp\left(\int_a^b \|F(\alpha)\| \, \mathrm{d}\alpha \right) .$$

Hint: Estimate the integrals and apply (a).

(c) Show by direct computation that the ordered exponential satisfies the equations

$$\frac{\mathrm{d}}{\mathrm{d}a} \mathrm{Pexp}\left(\int_a^b F(\alpha) \, \mathrm{d}\alpha \right) = -F(a) \, \mathrm{Pexp}\left(\int_a^b F(\alpha) \, \mathrm{d}\alpha \right) \tag{17.3.1}$$

$$\mathrm{Pexp}\left(\int_a^a F(\alpha) \, \mathrm{d}\alpha \right) = \mathbb{1} \,. \tag{17.3.2}$$

Use the uniqueness theorem for solutions of ordinary differential equations to give an alternative definition in terms of the solution of an initial-value problem. Use this reformulation to show the group property

$$\operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha\right)\operatorname{Pexp}\left(\int_b^c F(\alpha)\ \mathrm{d}\alpha\right) = \operatorname{Pexp}\left(\int_a^c F(\alpha)\ \mathrm{d}\alpha\right). \tag{17.3.3}$$

(d) Show that

$$\frac{\mathrm{d}}{\mathrm{d}b}\operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha\right) = \operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha\right) F(b)\,. \tag{17.3.4}$$

Hint: Differentiate the identity (17.3.3) in the case $c = a$ and use the group properties (17.3.2) and (17.3.3).

(e) Show that

$$\operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha\right)^* = \operatorname{Pexp}\left(\int_b^a \big(-F(\alpha)^*\big)\ \mathrm{d}\alpha\right).$$

Deduce that if $F(\alpha)$ is an anti-Hermitian matrix, then the ordered exponential is a unitary matrix. *Hint:* There are two alternative methods. One method is to argue using the differential equations (17.3.1) and (17.3.4) or with the group property. A more computational approach is to take the adjoint of the Dyson series and reparametrize the integrals.

EXERCISE 17.2. Given $\omega \in \mathbb{R}$ and a smooth function $V(t)$, we consider the ordinary differential equation

$$\big(\mathrm{i}\partial_t + \omega\big)\phi(t) = V(t)\,\phi(t)\,.$$

(a) Write down the Lippmann-Schwinger equation, taking the right side of the equation as the perturbation. *Hint:* The free time evolution operator $U^{t,t'}$ was computed in Exercise 16.5.

(b) Express the Lippmann-Schwinger equation in the case $\omega = 0$ explicitly as an integral equation. How is it related to the integral equation used in the Picard iteration (in the proof of the Picard-Lindelöf theorem)?

CHAPTER 18

Methods of Perturbation Theory

In Chapter 15, the unregularized kernel of the fermionic projector was constructed abstractly with functional analytic methods. In order to fill these constructions with life, one can analyze this kernel with methods of perturbation theory. The resulting explicit formulas give a detailed understanding of the structure of this kernel. We now outline the perturbative methods; more details can be found in [**45**, Chapter 2] or in the original papers [**37, 55, 86**].

As the general setting we consider the Dirac equation in Minkowski space (1.3.14) in the presence of an external potential $\mathcal{B}$ which we assume to be symmetric (1.3.13). In preparation, we rewrite the definition of the fermionic signature operator constructed in Chapter 15 in a way suitable for the perturbative treatment. Our starting point is the representation (15.3.3) of the spacetime inner product in terms of the scalar product,

$$<\mathfrak{p}\psi|\mathfrak{p}\psi'> = \int_I (\psi_m \,|\, \mathcal{S}_m\, \psi'_m)_m \; \mathrm{d}m \,, \tag{18.0.1}$$

where $\mathcal{S}_m$ is the fermionic signature operator. Here $\psi = (\psi_m)_{m \in I}$ and similarly ψ' are families of solutions of the Dirac equation for a varying mass parameter. More specifically, we now consider families obtained by acting with the causal fundamental solution on given test wave functions, i.e.

$$\psi_m = \tilde{k}_m \phi \quad \text{and} \quad \psi'_m = \tilde{k}_m \phi' \qquad \text{with} \qquad \phi, \phi' \in C_0^\infty(\mathscr{M}, S\mathscr{M}) \,.$$

Using this ansatz in (18.0.1) and pulling the mass integrals outside, we obtain the formula

$$\int_I \mathrm{d}m \int_I \mathrm{d}m' <\tilde{k}_m \phi \,|\, \tilde{k}_{m'} \phi'> = \int_I (\tilde{k}_m \phi \,|\, \mathcal{S}_m\, \tilde{k}_m \phi')_m \; \mathrm{d}m \,. \tag{18.0.2}$$

Next, we rewrite the integrand on the left side by using that the fundamental solution is symmetric with respect to the spacetime inner product (see Corollary 13.4.5),

$$<\tilde{k}_m \phi | \tilde{k}_{m'} \phi'> = <\phi \,|\, \tilde{k}_m \tilde{k}_{m'} \phi'> \,.$$

Moreover, the integrand on the right can be rewritten with the help of Proposition 13.4.4 as

$$(\tilde{k}_m \phi \,|\, \mathcal{S}_m\, \tilde{k}_m \phi')_m = \langle \phi \,|\, \mathcal{S}_m\, \tilde{k}_m \phi' \rangle \,.$$

Thus (18.0.2) becomes

$$\int_I \mathrm{d}m \int_I \mathrm{d}m' <\phi \,|\, \tilde{k}_m\, \tilde{k}_{m'} \psi'> = \int_I \langle \phi \,|\, \mathcal{S}_m\, \tilde{k}_m \phi' \rangle \; \mathrm{d}m \,.$$

Here one should keep in mind the product $\tilde{k}_m\, \tilde{k}_{m'}$ is an operator product in spacetime,

$$\big(\tilde{k}_m\, \tilde{k}_{m'}\big)(x,y) = \int_{\mathscr{M}} \tilde{k}_m(x,z)\, \tilde{k}_{m'}(z,y) \; \mathrm{d}^4 z \,, \tag{18.0.3}$$

whereas in the product $\mathcal{S}_m \tilde{k}_m$ we multiply by an operator on the Hilbert space $\mathcal{H}_m$ defined for example at time t. In order to clarify the notation, we write this product as

$$\mathcal{S}_m |_t \tilde{k}_m .$$

Then the relation (18.0.3) can be written in the short form

$$\tilde{k}_m \tilde{k}_{m'} = \delta(m - m')\, \mathcal{S}_m |_t \tilde{k}_m . \tag{18.0.4}$$

In this way, one is led to considering products of operators in spacetime which involve the mass as a parameter. Carrying out the products gives rise to δ distributions in the respective mass parameters.

This computational procedure was introduced in [**37**]. In the Minkowski vacuum, it can be carried out most conveniently in momentum space. We begin with the formula for the causal fundamental solution in momentum space (16.3.8),

$$k_m(p) = (\not{p} + m)\, \delta(p^2 - m^2)\, \epsilon(p^0) .$$

Then, using Plancherel together as well as the anti-commutation relations of the Dirac matrices, we obtain

$$\begin{aligned}
\big(k_m\, k_{m'}\big)(p) &= k_m(p)\, k_{m'}(p) = (\not{p} + m)\, \delta(p^2 - m^2)\, \epsilon(p^0)\, (\not{p} + m')\, \delta(p^2 - (m')^2)\, \epsilon(p^0) \\
&= \big(p^2 + (m + m')\, \not{p} + mm'\big)\, \delta\big(m^2 - (m')^2\big)\, \delta(k^2 - m^2) \\
&= \big(p^2 + (m + m')\, \not{p} + mm'\big)\, \frac{1}{2m}\, \delta(m - m')\, \delta(k^2 - m^2) \\
&= \delta(m - m')\, (\not{p} + m)\, \delta(p^2 - m^2) = \delta(m - m')\, \epsilon\big(p^0\big)\, k_m(p) .
\end{aligned}$$

Comparing with (18.0.4), we can read off that that the fermionic signature operator simply is the operator of multiplication operator by the sign of the frequency,

$$\mathcal{S}_m(p) = \epsilon\big(p^0\big) .$$

This computation is an efficient way of seeing that, in the Minkowski vacuum, the fermionic signature operator gives back the frequency splitting.

We proceed by explaining how this computation can be extended to the situation of the Dirac equation (1.3.14) when an external potential $\mathcal{B}$ is present. We want to proceed order by order in a perturbation expansion in $\mathcal{B}$. Before entering the details, we point out that by a "perturbation expansion" we mean a formal expansion in powers of $\mathcal{B}$. The resulting formulas will be well-defined and finite to every order. But it is unknown whether the power series converges. This procedure is convincing because we already know from our functional analytic constructions in Chapter 15 that the fermionic signature operator and the unregularized fermionic projector are well-defined mathematical objects. With this in mind, the only purpose of the constructions in this chapter is to compute these objects more explicitly. For this purpose, a perturbative treatment order by order in perturbation theory is most suitable.

18.1. Perturbation Expansion of the Causal Green's Operators

We already encountered the causal Green's operators for the Dirac equation several times in this book. In Section 13.4, they were constructed with methods of hyperbolic partial differential equations (see Theorem 13.4.3). In Section 16.1, on the other hand, we used Fourier methods to derive explicit formulas for the causal Green's operator in the Minkowski vacuum (see (16.1.4)). Taking these explicit formulas as the starting point, one can also write down closed formulas for the causal Green's operators in the

presence of an external potential. In order to state these formulas, we consider the Dirac equation (1.3.14) in the presence of an external potential $\mathcal{B}$. We always denote the objects in the presence of the external potential with a tilde, whereas the objects without tilde refer to the vacuum. Then the advanced and retarded Dirac Green's operators have the perturbation expansions

$$\tilde{s}^\vee_m = \sum_{n=0}^\infty \big(- s^\vee_m \,\mathcal{B}\big)^n s^\vee_m \,, \qquad \tilde{s}^\wedge_m = \sum_{n=0}^\infty \big(- s^\wedge_m \,\mathcal{B}\big)^n s^\wedge_m \,, \tag{18.1.1}$$

as can be verified as follows. First, one sees by direct computation using the defining equation of the Green's operator (16.1.1) that they are formal solutions of (16.2.1). For example, for the advanced Green's operator,

$$\begin{aligned}
&(\mathrm{i}\partial\!\!\!/_x + \mathcal{B} - m)\Big(\sum_{n=0}^\infty \big(- s^\vee_m \mathcal{B}\big)^n s^\vee_m \Big) \\
&= (\mathrm{i}\partial\!\!\!/_x - m)\, s^\vee_m \Big(\sum_{n=0}^\infty \big(- \mathcal{B} s^\vee_m\big)^n \Big) + \mathcal{B}\Big(\sum_{n=0}^\infty \big(- s^\vee_m \mathcal{B}\big)^n s^\vee_m \Big) \\
&= \sum_{n=0}^\infty (-\mathcal{B} s^\vee_m)^n + \mathcal{B}\Big(\sum_{n=0}^\infty \big(- s^\vee_m \mathcal{B}\big)^n s^\vee_m \Big) = \mathbb{1} \,.
\end{aligned}$$

Second, the fact that the Green's operators in (18.1.1) are either all advanced or all retarded implements the causal properties of the respective Green's operators. Let us consider for example the integral kernel of the first order contribution to the advanced Green's operator

$$\big(- s^\wedge_m \mathcal{B} s^\wedge_m\big)(x,y) = - \int_{\mathscr{M}} s^\wedge_m(x,z)\, \mathcal{B}(z)\, s^\wedge_m(z,y)\; \mathrm{d}^4 z \,. \tag{18.1.2}$$

The integrand vanishes unless z lies in the causal future of y and x lies in the causal future of z. Using transitivity of the causal relations, one concludes that the integral is zero unless x lies in the causal future of y. In this sense, the expression (18.1.2) is again causal and retarded. The higher orders can be treated similarly by induction.

We finally explain in which sense the perturbation series (18.1.1) are mathematically well-defined. To every order in perturbation theory, the operator products are well-defined and finite, provided that the potential $\mathcal{B}$ is smooth and decays so fast at infinity that the functions $\mathcal{B}(x)$, $x^i \mathcal{B}(x)$, and $x^i x^j \mathcal{B}(x)$ are integrable (for an inductive proof see [**45**, Lemma 2.1.2]). Knowing that the Green's operators are well-defined non-perturbatively (see Chapter 13), we disregard the issue of convergence of the perturbation series.

18.2. The Causal Perturbation Expansion of the Fermionic Projector

Using (13.4.7), we also have a unique perturbation expansion for the causal fundamental solution,

$$\tilde{k}_m = \frac{1}{2\pi\mathrm{i}} (\tilde{s}^\vee_m - \tilde{s}^\wedge_m) \,. \tag{18.2.1}$$

Using the identities

$$s^\vee_m = s_m + \mathrm{i}\pi k_m \,, \qquad s^\wedge_m = s_m - \mathrm{i}\pi k_m \,, \tag{18.2.2}$$

where we introduced the *symmetric Green's operator*

$$s_m := \frac{1}{2} \left(s^\vee_m + s^\wedge_m \right) , \tag{18.2.3}$$

one can write the above perturbation series as operator product expansions. More precisely, the operator $\tilde{k}_m$ has the series expansion

$$\tilde{k}_m = \sum_{\beta=0}^{\infty} (\mathrm{i}\pi)^{2\beta}\, b_m^< \, k_m \, (b_m k_m)^{2\beta}\, b_m^> \,, \tag{18.2.4}$$

where the factors $b_m^\bullet$ are defined by

$$b_m^< = \sum_{n=0}^{\infty} (-s_m \mathcal{B})^n \,, \qquad b_m = \sum_{n=0}^{\infty} (-\mathcal{B} s_m)^n \mathcal{B} \,, \qquad b_m^> = \sum_{n=0}^{\infty} (-\mathcal{B} s_m)^n \,. \tag{18.2.5}$$

In the following constructions, we need to multiply the operator products in (18.2.4). These products have a mathematical meaning as distributions in the involved mass parameters,

$$p_m \, p_{m'} = k_m \, k_{m'} = \delta(m - m')\, p_m \tag{18.2.6}$$

$$p_m \, k_{m'} = k_m \, p_{m'} = \delta(m - m')\, k_m \tag{18.2.7}$$

$$k_m \, b_m^> \, b_{m'}^< \, k_{m'} = \delta(m - m') \Big(p_m + \pi^2 \, k_m \, b_m \, p_m \, b_m \, k_m \Big) \,, \tag{18.2.8}$$

where

$$p_m(q) = (\not{q} + m)\, \delta(q^2 - m^2) \tag{18.2.9}$$

$$k_m(q) = (\not{q} + m)\, \delta(q^2 - m^2)\, \epsilon(q^0) \,. \tag{18.2.10}$$

Since all these formulas involve a common prefactors $\delta(m - m')$, we can introduce a convenient notation by leaving out this factor and omitting the mass indices. For clarity, we denote this short notation with a dot, i.e. symbolically

$$A \cdot B = C \qquad \text{stands for} \qquad A_m \, B_{m'} = \delta(m - m')\, C_m \,. \tag{18.2.11}$$

With this short notation, the multiplication rules can be written in the compact form

$$p \cdot p = k \cdot k = p \,, \qquad p \cdot k = k \cdot p = k \,, \qquad k\, b^> \cdot b^< k = p + \pi^2 k b p b k \,. \tag{18.2.12}$$

In all the subsequent calculations, the operator products are well-defined provided that the potential $\mathcal{B}$ is sufficiently smooth and has suitable decay properties at infinity (for details see again [**45**, Lemma 2.1.2]). But again, all infinite series are to be understood merely as formal power series in the potential $\mathcal{B}$.

Using this notation, we can write (18.2.4) as

$$\tilde{k} = k + \Delta k \qquad \text{with} \qquad \Delta\tilde{k} = \sum_{\beta=0}^{\infty} (\mathrm{i}\pi)^{2\beta}\, b^< \, k \, (bk)^{2\beta}\, b^> \;-\; k \tag{18.2.13}$$

(note that $\Delta\tilde{k}$ is at least linear in $\mathcal{B}$). Powers of the operator $\tilde{k}$ with the product (18.2.11) are well-defined using the multiplication rules (18.2.12). This makes it possible to develop a spectral calculus for $\tilde{k}$, which is formulated most conveniently with contour integrals. To this end, we introduce the *resolvent* by

$$\tilde{R}_\lambda = (\tilde{k} - \lambda)^{-1} \,. \tag{18.2.14}$$

We choose a contour Γ_+ which encloses the point 1 in counter-clockwise direction and does not enclose the points -1 and 0. Likewise, Γ_- is chosen as a contour which encloses

the point -1 in counter-clockwise direction and does not enclose the points 1 and 0. Given a holomorphic function f we define $f(\tilde{k})$ by

$$f\big(\tilde{k}\big) := -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_+ \cup \Gamma_-} f(\lambda)\, \tilde{R}_\lambda \, \mathrm{d}\lambda \,. \tag{18.2.15}$$

Before going on, we need to explain how the resolvent and these contour integrals are to be understood mathematically. First, the resolvent can be expressed in terms of the vacuum resolvent with a a perturbation series being a formal Neumann series,

$$\tilde{R}_\lambda = (k - \lambda + \Delta k)^{-1} = (1 + R_\lambda \cdot \Delta k)^{-1} \cdot R_\lambda = \sum_{n=0}^{\infty} (-R_\lambda \cdot \Delta k)^n \cdot R_\lambda \,. \tag{18.2.16}$$

In order to define R_λ, we note that, according to (18.2.12), the operator k has the eigenvalues ± 1 and 0 with corresponding spectral projectors $(p \pm k)/2$ and $\mathbb{1} - p$. Hence we can write the free resolvent as

$$R_\lambda = \frac{p+k}{2} \left(\frac{1}{1-\lambda} \right) + \frac{p-k}{2} \left(\frac{1}{-1-\lambda} \right) - \frac{\mathbb{1}-p}{\lambda} \,. \tag{18.2.17}$$

Substituting this formula in (18.2.16), to every order in perturbation theory we obtain a meromorphic function in λ having poles only at $\lambda = 0$ and $\lambda = \pm 1$. Therefore, the contour integral in (18.2.15) can be computed with residues, and the result is independent of the choice of the contours Γ_- and Γ_+. In this way, the operator $f(\tilde{k})$ is uniquely defined as a formal perturbation series. As explained at the end of the previous section (in the paragraph after (18.1.2)), this series is well-defined and finite to every order in perturbation theory.

We now establish the functional calculus.

THEOREM 18.2.1. **(functional calculus)** *For any functions f, g which are holomorphic in discs around ± 1 which contain the contours $\Gamma_\pm$,*

$$(\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m)\, f\big(\tilde{k}\big) = 0 \tag{18.2.18}$$

$$f\big(\tilde{k}\big)^* = \overline{f}\big(\tilde{k}\big) \tag{18.2.19}$$

$$f\big(\tilde{k}\big) \cdot g\big(\tilde{k}\big) = (fg)\big(\tilde{k}\big) \,. \tag{18.2.20}$$

PROOF. Since the image of the operator $\tilde{k}$ lies in the kernel of the Dirac operator, we know that

$$(\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{R}_\lambda = (\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m) \big(-\lambda^{-1} \big).$$

Taking the contour integral (18.2.15) gives (18.2.18).

The operators p_m, k_m and s_m are obviously symmetric (see the relations (18.2.9), (18.2.10) and (18.2.3)). According to (18.2.4), the operator $\tilde{k}_m$ is also symmetric. Hence the resolvent $\tilde{R}_\lambda$ defined by (18.2.14) has the property

$$\tilde{R}_\lambda^* = \tilde{R}_{\overline{\lambda}} \,.$$

The relation (18.2.19) follows by taking the adjoint of (18.2.15) and reparametrizing the integral.

The starting point for proving (18.2.20) is the resolvent identity (see Exercise 3.3)

$$\tilde{R}_\lambda \cdot \tilde{R}_{\lambda'} = \frac{1}{\lambda - \lambda'} \left(\tilde{R}_\lambda - \tilde{R}_{\lambda'} \right) \,. \tag{18.2.21}$$

We set $\Gamma = \Gamma_+ \cup \Gamma_-$ and denote the corresponding contour for λ' by Γ'. Since the integral (18.2.15) is independent of the precise choice of the contour, we may choose

$$\Gamma = \partial B_\delta(1) \cup \partial B_\delta(-1) \qquad \text{and} \qquad \Gamma' = \partial B_{2\delta}(1) \cup \partial B_{2\delta}(-1)$$

for sufficiently small $\delta < 1/2$. Then Γ does not enclose any point of Γ', implying that

$$\oint_\Gamma \frac{f(\lambda)}{\lambda - \lambda'}\, \mathrm{d}\lambda = 0 \qquad \text{for all } \lambda' \in \Gamma' . \tag{18.2.22}$$

On the other hand, Γ' encloses every point of Γ, so that

$$\oint_{\Gamma'} f(\lambda)\, g(\lambda')\, \frac{\tilde{R}_\lambda}{\lambda - \lambda'}\, \mathrm{d}\lambda' = -2\pi\mathrm{i}\, f(\lambda)\, g(\lambda)\, \tilde{R}_\lambda \qquad \text{for all } \lambda \in \Gamma . \tag{18.2.23}$$

Combining (18.2.21) with (18.2.22) and (18.2.23), we obtain

$$\begin{aligned} f\big(\tilde{k}\big) \cdot g\big(\tilde{k}\big) &= -\frac{1}{4\pi^2} \oint_\Gamma f(\lambda)\, \mathrm{d}\lambda \oint_{\Gamma'} g(\lambda')\, \mathrm{d}\lambda'\, \frac{1}{\lambda - \lambda'} \Big(\tilde{R}_\lambda - \tilde{R}_{\lambda'} \Big) \\ &= -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma f(\lambda)\, g(\lambda)\, \tilde{R}_\lambda\, \mathrm{d}\lambda = (fg)\big(\tilde{k}\big) . \end{aligned}$$

This concludes the proof. □

This functional calculus makes it possible to compute the unregularized kernel of the fermionic projector, as we now explain. Our starting point is the defining equation for the fermionic signature operator (18.0.4), which we can now write in the short from

$$\tilde{k} \cdot \tilde{k} = \tilde{\mathcal{S}}_m \,|_t\, \tilde{k} .$$

Iterating this relation, we obtain for any $p \in \mathbb{N}$

$$\big(\tilde{k} \cdot\big)^p\, \tilde{k} = \big(\tilde{\mathcal{S}}_m \,|_t\big)^p\, \tilde{k} = \big(\tilde{\mathcal{S}}_m\big)^p \,\big|_t\, \tilde{k} \qquad \text{for all } p \in \mathbb{N} .$$

Consequently, this formula also holds for the functional calculus, i.e.

$$f\big(\tilde{k}\big) \cdot \tilde{k} = f\big(\tilde{\mathcal{S}}_m\big) \,\big|_t\, \tilde{k} .$$

This formula makes it possible to express the unregularized kernel $\tilde{P}_-$ in (15.4.1) by

$$\begin{aligned} \tilde{P}_- &= -\chi_{(-\infty,0)}\big(\tilde{\mathcal{S}}_m\big)\, \tilde{k}_m = -\chi_{(-\infty,0)}\big(\tilde{\mathcal{S}}_m\big) \,|_t\, \tilde{k} = \chi_{(-\infty,0)}\big(\tilde{k}\big) \cdot \tilde{k} \\ &= -\Big(\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_-} \tilde{R}_\lambda\, \mathrm{d}\lambda \Big) \cdot \tilde{k} = -\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_-} (-\lambda)\, \tilde{R}_\lambda\, \mathrm{d}\lambda \end{aligned}$$

Substituting the perturbation expansion for $\tilde{R}_\lambda$ in (18.2.16) and writing the vacuum resolvent in the form (18.2.17), one can carry out the contour integral with residues. This gives the desired perturbation expansion for P_-. More details on this method and explicit formulas can be found in [**86**, Section 3.3 and Appendix A].

18.3. Exercises

EXERCISE 18.1. *(Perturbative description of gauge transformations)* We consider the perturbation expansion for the Dirac Green's operators (18.1.1) for a perturbation by a pure gauge potential, i.e.

$$\mathcal{B}(x) = \partial\!\!\!/ \Lambda(x)$$

with a real-valued function Λ.

(a) Show that the Dirac operator with interaction can be written as

$$\mathrm{i}\partial\!\!\!/ + (\partial\!\!\!/\Lambda) - m = \mathrm{e}^{\mathrm{i}\Lambda(x)} \big(\mathrm{i}\partial\!\!\!/ - m\big)\, \mathrm{e}^{-\mathrm{i}\Lambda(x)} \,.$$

Conclude that the perturbation of the Dirac solutions amounts to multiplication by a phase function, i.e.

$$\tilde{\Psi}(x) = \mathrm{e}^{\mathrm{i}\Lambda(x)}\, \Psi(x) \,.$$

Explain why these findings are a manifestation of the local gauge freedom of electrodynamics.

(b) Show that the gauge phases also appear in the perturbation expansion (18.1.1) in the sense that

$$\tilde{s}^\wedge_m(x,y) = \mathrm{e}^{\mathrm{i}\Lambda(x)}\, s^\wedge_m(x,y)\, \mathrm{e}^{-\mathrm{i}\Lambda(y)} \,.$$

Hint: To first order, one needs to show that

$$-\big(s^\wedge_m \,\mathcal{B}\, s^\wedge_m\big)(x,y) = \mathrm{i}\big(\Lambda(x) - \Lambda(y)\big)\, s^\wedge_m(x,y) \,.$$

To this end, it is convenient to write the perturbation operator as a commutator,

$$\mathcal{B} = -\mathrm{i}\, \big[(\mathrm{i}\partial\!\!\!/ - m),\, \Lambda\big]$$

and use the defining equation of the Green's operator (16.1.1). To higher order, one can proceed inductively.

EXERCISE 18.2. Prove the identity (18.2.8). *Hint:* Use the multiplication rules derived in Exercises 16.9 and 16.10. Make use of the fact that one gets telescopic sums.

EXERCISE 18.3. Verify the identity (18.2.20) in a perturbation expansion to first and second order. To this end, compute both sides of this equation using the perturbation expansion of $\tilde{R}_\lambda$ and carrying out the contour integrals. *Hint:* Similar formulas can be found in the appendices of [**55**] and [**86**].

CHAPTER 19

Methods of Microlocal Analysis

19.1. The Hadamard Expansion in Minkowski Space

In Chapter 15, the unregularized kernel of the fermionic projector $P(x,y) = P_-(x,y)$ was constructed abstractly. In Chapter 18, we saw how this kernel can be expanded in a perturbation series in powers of the external potential. In order to gain more explicit information on the form of the unregularized kernel, it is very useful to analyze its singularity structure on the light cone. It turns out that $P(x,y)$ has singularities on the light cone, which can be described by the so-called *Hadamard expansion* of the form

$$P(x,y) = \lim_{\varepsilon \searrow 0} \mathrm{i}\partial\!\!\!/_x \left(\frac{U(x,y)}{\Gamma_\varepsilon(x,y)} + V(x,y)\, \log \Gamma_\varepsilon(x,y) + W(x,y) \right), \tag{19.1.1}$$

where

$$\Gamma_\varepsilon(x,y) := (y-x)^j\, (y-x)_j - \mathrm{i}\varepsilon\, (y-x)^0\,, \tag{19.1.2}$$

and U, V and W are smooth functions on $\mathscr{M} \times \mathscr{M}$ taking values in the 4×4-matrices acting on the spinors (we always denote spacetime indices by latin letters running from $0, \ldots, 3$). This local expansion is based on the method of *integration along characteristics* which will be explained below (see after (19.1.8) or also [**99, 88**] or [**6**]). In Minkowski space, the *light-cone expansion* [**39, 40**] (see also [**45**, Section 2.2]) gives an efficient procedure for computing an infinite number of Hadamard coefficients in one step. The Hadamard form (19.1.1) carries over to curved spacetime. Moreover, there is an interesting connection to the so-called wave front set in microlocal analysis. These generalizations will be briefly outlined in Section 19.3 below. In all the other sections of this chapter, we restrict attention to Minkowski space.

It turns out that, for an external potential in Minkowski space, the kernel of the fermionic projector is indeed of Hadamard form.

THEOREM 19.1.1. *Assume that the external potential $\mathcal{B}$ is smooth, and that its time derivatives decay at infinity in the sense that* (17.2.1) *holds and in addition that*

$$\int_{-\infty}^{\infty} |\partial_t^p \mathcal{B}(t)|_{C^0}\, \mathrm{d}t < \infty \qquad \text{for all } p \in \mathbb{N}$$

(with the C^0-norm as defined in (17.2.2)*). Moreover, assume that the potential satisfies the bound*

$$\int_{-\infty}^{\infty} |\mathcal{B}(t)|_{C^0}\, \mathrm{d}t < \sqrt{2} - 1\,. \tag{19.1.3}$$

Then the fermionic projector $P(x,y)$ is of Hadamard form.

The proof of this theorem will be given in Section 19.4 below.

We conclude this section by explaining how the expansion (19.1.1) comes about and how the involved functions U, V and W, at least in principle, can be computed iteratively

using the method of *integration along characteristics*. We begin by computing the unregularized kernel in the Minkowski vacuum. To this end, one rewrites the factor $(\not{k} + m)$ in (5.8.1) in terms of a differential operator in position space,

$$P(x,y) = (\mathrm{i}\not{\partial}_x + m)\, T_{m^2}(x,y)\,, \tag{19.1.4}$$

where T_{m^2} is the scalar bi-distribution

$$T_{m^2}(x,y) := \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \delta(k^2 - m^2)\, \Theta(-k^0)\, \mathrm{e}^{-\mathrm{i}k(x-y)}\,.$$

We remark that the distribution T_{m^2} solves the Klein-Gordon equation $(-\Box - m^2)\, T_{m^2} = 0$; in quantum field theory it is sometimes denoted by Δ_- (see for example [**147**, Section 5.2]). Away from the light cone (i.e. for $\xi^2 \neq 0$), the distribution $T_{m^2}(x,y)$ is a smooth function given by

$$T_{m^2}(x,y) = \begin{cases} \dfrac{m}{16\pi^2}\, \dfrac{Y_1\big(m\sqrt{\xi^2}\big)}{\sqrt{\xi^2}} + \dfrac{\mathrm{i}m}{16\pi^2}\, \dfrac{J_1\big(m\sqrt{\xi^2}\big)}{\sqrt{\xi^2}}\, \epsilon(\xi^0) & \text{if } \xi \text{ is timelike} \\[2ex] \dfrac{m}{8\pi^3}\, \dfrac{K_1\big(m\sqrt{-\xi^2}\big)}{\sqrt{-\xi^2}} & \text{if } \xi \text{ is spacelike}\,, \end{cases} \tag{19.1.5}$$

where we set

$$\xi := y - x\,,$$

and J_1, Y_1 and K_1 are Bessel functions. Expanding the Bessel functions in (19.1.5) in a power series, one obtains (see [**124**, (10.2.2), (10.8.1) and (10.25.2), (10.31.1)])

$$\begin{aligned} T_{m^2}(x,y) = -\frac{1}{8\pi^3} \left(\frac{\mathrm{PP}}{\xi^2} + \mathrm{i}\pi\delta\big(\xi^2\big)\, \epsilon\big(\xi^0\big) \right) \\ + \frac{m^2}{32\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\, \frac{\big(m^2\xi^2\big)^j}{4^j} \Big(\log\big|m^2\xi^2\big| + c_j + \mathrm{i}\pi\, \Theta\big(\xi^2\big)\, \epsilon\big(\xi^0\big) \Big) \end{aligned}$$

with real coefficients c_j (here Θ and ϵ are again the Heaviside and the sign function, respectively). In particular, one sees that T_{m^2} is a distribution which is singular on the light cone. These singularities can be written in a shorter form using residues as

$$\begin{aligned} T_{m^2}(x,y) = \lim_{\varepsilon \searrow 0} \Bigg(-\frac{1}{8\pi^3}\, \frac{1}{\Gamma_\varepsilon(x,y)} \\ + \frac{m^2}{32\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\, \frac{\big(m^2\Gamma_\varepsilon(x,y)\big)^j}{4^j} \Big(\log\big(m^2\, \Gamma_\varepsilon(x,y)\big) + c_j \Big) \Bigg) \end{aligned} \tag{19.1.6}$$

(where Γ_ε is again defined by (19.1.2); for the proof one uses the distributional relation $\lim_{\varepsilon \searrow 0} (r^2 + (\varepsilon + it)^2)^{-1} = -\mathrm{PP}/\xi^2 - \mathrm{i}\pi\, \delta(\xi^2)\, \epsilon(\xi^0)$ and similarly the behavior of the logarithm in the complex plane). Noting that the series converge, one obtains a function of the desired form as in the brackets in (19.1.1). This shows that the term $\mathrm{i}\not{\partial}_x\, T_{m^2}(x,y)$ in (19.1.4) is of Hadamard form. For the term $m\, T_{m^2}(x,y)$, this can be shown by pulling out one derivative and working with matrix-valued kernels. Indeed,

$$m\, T_{m^2}(x,y) = -\frac{1}{m}\, \Box_x \Big(T_{m^2}(x,y) - T_0(x,y) \Big) = \not{\partial}_x \Big\{ -\frac{1}{m}\, \not{\partial}_x \big(T_{m^2}(x,y) - T_0(x,y) \big) \Big\}\,,$$

and computing the curly brackets by differentiating (19.1.6) one obtains again an expression of the Hadamard form (19.1.1).

The summands in (19.1.6) can be understood by verifying that T_{m^2} satisfies the Klein-Gordon equation. Indeed, using the abbreviation $\xi := y - x$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x^j}\Big(\frac{1}{\Gamma_\varepsilon(x,y)}\Big) &= -\frac{\partial_j \Gamma_\varepsilon(x,y)}{\Gamma_\varepsilon(x,y)^2} = \frac{1}{\Gamma_\varepsilon(x,y)^2}\Big(2\xi_j - \mathrm{i}\varepsilon\delta_{j,0}\Big) \\
\Box_x\Big(\frac{1}{\Gamma_\varepsilon(x,y)}\Big) &= \frac{2}{\Gamma_\varepsilon(x,y)^3}\Big(2\xi_j - \mathrm{i}\varepsilon\delta_{j,0}\Big)\Big(2\xi_j - \mathrm{i}\varepsilon\delta_0^j\Big) - \frac{8}{\Gamma_\varepsilon(x,y)^2} \\
&= \frac{2}{\Gamma_\varepsilon(x,y)^3}\Big(4\xi^2 - 4\mathrm{i}\varepsilon\,\xi_0 - \varepsilon^2\Big)\Big) - \frac{8}{\Gamma_\varepsilon(x,y)^2} = -\frac{2\varepsilon^2}{\Gamma_\varepsilon(x,y)^3}\,,
\end{aligned}$$

and this tends to zero as $\varepsilon \searrow 0$. Thus the leading term in (19.1.6) satisfies the scalar wave equation. In the Klein-Gordon equation, however, the term involving the mass remains,

$$-m^2\,\frac{1}{\Gamma_\varepsilon(x,y)}\,. \tag{19.1.7}$$

This term is compensated by the next term in the expansion (19.1.6), because

$$\begin{aligned}
\frac{\partial}{\partial x^j}\log\big(\Gamma_\varepsilon(x,y)\big) &= \frac{\partial_j \Gamma_\varepsilon(x,y)}{\Gamma_\varepsilon(x,y)} = -\frac{1}{\Gamma_\varepsilon(x,y)}\Big(2\xi_j - \mathrm{i}\varepsilon\delta_{j,0}\Big) \\
\Box_x\log\big(\Gamma_\varepsilon(x,y)\big) &= -\frac{1}{\Gamma_\varepsilon(x,y)^2}\Big(2\xi_j - \mathrm{i}\varepsilon\delta_{j,0}\Big)\Big(2\xi_j - \mathrm{i}\varepsilon\delta_0^j\Big) + \frac{8}{\Gamma_\varepsilon(x,y)} \\
&= -\frac{1}{\Gamma_\varepsilon(x,y)^2}\Big(4\xi^2 - 4\mathrm{i}\varepsilon\,\xi_0 - \varepsilon^2\Big)\Big) + \frac{8}{\Gamma_\varepsilon(x,y)} \\
&= \frac{4}{\Gamma_\varepsilon(x,y)} + \frac{\varepsilon^2}{\Gamma_\varepsilon(x,y)^2}\,.
\end{aligned}$$

Now the first summand in the last line cancels the term (19.1.7) in the Klein-Gordon equation. Proceeding order by order in powers of $\Gamma_\varepsilon(x,y)$, one can verify all the coefficients in (19.1.6).

This method of applying the wave operator term by term is also useful for computing the functions U, V and W in (19.1.1) in the case that an external potential is present. In fact, these functions can be expressed in terms of line integrals along the light cone. This *method of integration along characteristics* goes back go Hadamard [**99**] and is described in the classic textbook [**88**] in curved spacetime. In order to explain the method in the simplest possible context, let us assume that we consider the wave equation with an external scalar potential $a(x)$, i.e.

$$\big(-\Box_x - a(x)\big)\tilde{T}(x,y) = 0$$

(the Dirac equation will be treated more systematically in Section 19.2). In modification of the series in (19.1.6) we make the ansatz

$$\tilde{T}(x,y) = \lim_{\varepsilon\searrow 0}\left(\frac{1}{\Gamma_\varepsilon(x,y)} + \sum_{n=1}^\infty f_n(x,y)\,\Gamma_\varepsilon(x,y)^n\,\log\big(\Gamma_\varepsilon(x,y)\big)\right) \tag{19.1.8}$$

Compared to (19.1.7), now the error term of the first summand involves the potential $a(x)$,

$$-\frac{a(x)}{\Gamma_\varepsilon(x,y)}\,. \tag{19.1.9}$$

The hope is to compensate this term by a suitable choice of $f_1(x,y)$. Indeed,

$$\frac{\partial}{\partial x^j}\Big(f_1(x,y)\;\log\Gamma_\varepsilon(x,y)\Big) = f_1(x,y)\;\frac{\partial_j\Gamma_\varepsilon(x,y)}{\Gamma_\varepsilon(x,y)} + \partial_j f_1(x,y)\;\log\Gamma_\varepsilon(x,y)$$
$$\Box_x\Big(f_1(x,y)\;\log\Gamma_\varepsilon(x,y)\Big) = f_1(x,y)\;\frac{4}{\Gamma_\varepsilon(x,y)} - 2\,\partial_j f_1(x,y)\;\frac{2\xi_j}{\Gamma_\varepsilon(x,y)} + \cdots\,,$$

where $\cdots$ stands for all terms which either have a lower order singularity on the light cone or tend to zero as $\varepsilon \searrow 0$. In order for this contribution to compensate (19.1.9), the function f_1 must satisfy the equation

$$4\,f_1(x,y) - 4\,\xi^j\,\partial_j f_1(x,y) = a(x)\,.$$

Such a differential equation of first order can be solved with the method of characteristics (see for example [**32**, Section I.3.2]). More specifically, the solution is an integral along the straight line $\xi\mathbb{R}$. In order to describe the singular behavior on the light cone, it suffices to consider the case that ξ is tangential to the light cone. Similarly, also to higher order in the expansion parameter n, we obtain transport equations along the light cone, which can be solved iteratively order by order.

19.2. The Light-Cone Expansion

We first give the basic definition of the light-cone expansion and explain it afterward.

DEFINITION 19.2.1. *A distribution $A(x,y)$ on $\mathscr{M}\times\mathscr{M}$ is of the order $\mathcal{O}((y-x)^{2p})$ for $p\in\mathbb{Z}$ if the product*

$$(y-x)^{-2p}\,A(x,y)$$

is a regular distribution (i.e. a locally integrable function). An expansion of the form

$$A(x,y) = \sum_{j=g}^{\infty} A^{[j]}(x,y) \tag{19.2.1}$$

with $g\in\mathbb{Z}$ is called **light-cone expansion** *if the $A^{[j]}(x,y)$ are distributions of the order $\mathcal{O}((y-x)^{2j})$ and if A is approximated by the partial sums in the sense that for all $p\geq g$,*

$$A(x,y) - \sum_{j=g}^{p} A^{[j]}(x,y) \qquad \text{is of the order } \mathcal{O}\big((y-x)^{2p+2}\big)\,. \tag{19.2.2}$$

The parameter g gives the leading order of the singularity of $A(x,y)$ on the light cone. We point out that we do not demand that the infinite series in (19.2.1) converges. Thus, similar to a formal Taylor series, the series in (19.2.1) is defined only via the approximation by the partial sums (19.2.2). The notion of the light-cone expansion is illustrated in Exercise 19.1.

As a concrete example, due to the factors $\Gamma_\varepsilon(x,y)$, the series (19.1.6) is a light-cone expansion. The term with the leading singularity becomes integrable after multiplying by $(y-x)^2$, showing that $g=-1$.

Our task is to perform the light-cone expansion of the unregularized kernel of the fermionic projector. Schematically, this construction consists of several steps:

(1) Perform the light-cone expansion of the causal Green's operators $\tilde{s}^\vee_m$ and $\tilde{s}^\wedge_m$. Here one proceeds inductively for each summand of the perturbation series (18.1.1).
(2) Using the relation (18.2.1), one obtains a corresponding light-cone expansion for the causal fundamental solution $\tilde{k}_m$.

(3) Finally, the so-called residual argument relates the sought-after light-cone expansion of $\tilde{P}(x,y)$ to that of $\tilde{k}_m$.

This procedure is described in detail in [**45**, Chapter 2]. In order to avoid an unnecessary overlap, we here focus on the light-cone expansion of the causal Green's operators and only introduce the concepts needed for the basics on the continuum limit in Chapter 21. Before doing so, we illustrate the light-cone expansion by a simple example.

EXAMPLE 19.2.2. Consider the massless Dirac equation in the presence of an external electromagnetic potential A,

$$\big(\mathrm{i}\partial\!\!\!/ + A\!\!\!/\big)\tilde{P}(x,y)\,.$$

For simplicity assume that A is smooth and compactly supported in spacetime. Then, to first order in perturbation theory, the light-cone expansion of the unregularized kernel $\tilde{P}(x,y)$ takes the form

$$\begin{aligned}
\tilde{P}(x,y) = \frac{\mathrm{i}}{2}\, \exp\bigg(-\mathrm{i} \int_0^1 A_j\big|_{\alpha y+(1-\alpha)x}\, \xi^j \;\mathrm{d}\alpha \bigg)\, P(x,y) \qquad & (19.2.3)\\
- \frac{1}{2}\, \not{\xi}\, \xi_i \int_0^1 (\alpha-\alpha^2)\, j^i\big|_{\alpha y+(1-\alpha)x}\; \mathrm{d}\alpha\, T^{(0)} \qquad & (19.2.4)\\
+ \frac{1}{4}\, \not{\xi} \int_0^1 F^{ij}\big|_{\alpha y+(1-\alpha)x}\, \gamma_i\gamma_j\; \mathrm{d}\alpha\, T^{(0)} \qquad & (19.2.5)\\
- \xi_i \int_0^1 (1-\alpha)\, F^{ij}\big|_{\alpha y+(1-\alpha)x}\, \gamma_j\; \mathrm{d}\alpha\, T^{(0)} \qquad & (19.2.6)\\
- \xi_i \int_0^1 (1-\alpha)(\alpha-\alpha^2)\, \partial\!\!\!/ j^i\big|_{\alpha y+(1-\alpha)x}\; \mathrm{d}\alpha\, T^{(1)} \qquad & (19.2.7)\\
- \int_0^1 (1-\alpha)^2\, j^i\big|_{\alpha y+(1-\alpha)x}\, \gamma_i\; \mathrm{d}\alpha\, T^{(1)} \qquad & (19.2.8)\\
+ \not{\xi}\,(\deg < 1) + (\deg < 0) + \mathcal{O}(A^2)\,, &
\end{aligned}$$

where $F^{jk} = \partial^j A^k - \partial^k A^j$ is the field tensor and $j^k = \partial^k_j A^j - \Box A^k$ is the corresponding Maxwell current. Moreover, the factors $T^{(0)}$ and $T^{(1)}$ are the leading summands in (19.1.6); more precisely,

$$\begin{aligned}
T^{(0)}(x,y) &= -\frac{1}{8\pi^3}\, \lim_{\varepsilon \searrow 0} \frac{1}{\Gamma_\varepsilon(x,y)} \\
T^{(1)}(x,y) &= \frac{1}{32\pi^3}\, \lim_{\varepsilon \searrow 0} \log \Gamma_\varepsilon(x,y)\,.
\end{aligned} \tag{19.2.9}$$

Each summand has the general structure of being the product of a smooth function and a distribution which is singular on the light cone. The smooth factor is an integral along the straight line segment joining the points x and y. The integrand involves the electromagnetic potential and its partial derivatives. We remark for clarity that the term (19.2.3) involves a gauge phase as needed for gauge invariance (as already mentioned in (5.8.5) in Section 5.8). All the other integrands are gauge invariant, as is obvious from the fact that they are expressed in terms of the electromagnetic field tensor and the Maxwell current.

To higher order on the light cone or to higher order in the mass or the external potentials, the formulas of the light-cone expansions have a similar structure. More

detailed formulas can be found in the original papers [**39, 40**], in [**41**, Appendix B] and [**45**, Appendix B]. ♢

We now explain how to perform the light-cone expansion of the causal Green's operators. In order to get a first idea for how to proceed, we begin by considering the free advanced Green's operator $s^\vee_m$ of the Dirac equation of mass m in position space: Similar to (19.1.4), it is again convenient to pull the Dirac matrices out of $s^\vee_m$ by setting

$$s^\vee_m(x,y) = (\mathrm{i}\partial\!\!\!/_x + m)\, S^\vee_{m^2}(x,y)\,, \tag{19.2.10}$$

where $S^\vee_{m^2}$ is the advanced Green's operator of the Klein-Gordon operator,

$$S^\vee_{m^2}(x,y) = \lim_{\nu \searrow 0} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \frac{1}{p^2 - m^2 - \mathrm{i}\nu p^0}\, \mathrm{e}^{-\mathrm{i}p(x-y)}\,. \tag{19.2.11}$$

Computing this Fourier integral and expanding the resulting Bessel function in a power series gives (for details see Exercise 19.3)

$$\begin{aligned} S^\vee_{m^2}(x,y) &= -\frac{1}{2\pi}\, \delta\big(\xi^2\big)\, \Theta\big(\xi^0\big) \\ &\quad + \frac{m^2}{4\pi}\, \frac{J_1\Big(\sqrt{m^2\, \xi^2}\Big)}{\sqrt{m^2\, \xi^2}}\, \Theta\big(\xi^2\big)\, \Theta\big(\xi^0\big) \qquad (19.2.12) \\ &= -\frac{1}{2\pi}\, \delta\big(\xi^2\big)\, \Theta\big(\xi^0\big) \\ &\quad + \frac{m^2}{8\pi} \sum_{j=0}^\infty \frac{(-1)^j}{j!\,(j+1)!}\, \frac{\big(m^2 \xi^2\big)^j}{4^j}\, \Theta\big(\xi^2\big)\, \Theta\big(\xi^0\big)\,. \qquad (19.2.13) \end{aligned}$$

This computation shows that $S^\vee_{m^2}(x,y)$ has a $\delta(\xi^2)$-like singularity on the light cone. Furthermore, one sees that $S^\vee_{m^2}$ is a power series in m^2. The important point for what follows is that the higher order contributions in m^2 contain more factors ξ^2 and are thus of higher order on the light cone. More precisely,

$$\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^n S^\vee_a(x,y)\Big|_{a=0} \qquad \text{is of the order } \mathcal{O}\big(\xi^{2n-2}\big) \tag{19.2.14}$$

(here and in what follows, we often use the abbreviation $a = m^2$). According to (19.2.10), the Dirac Green's operator is obtained by taking the first partial derivatives of (19.2.13). Therefore, $s^\vee_m(x,y)$ has a singularity on the light cone which is even $\sim \delta'(\xi^2)$. The higher order contributions in m are again of increasing order on the light cone. This means that we can view the Taylor expansion of (19.2.10) in m,

$$s^\vee_m(x,y) = \sum_{n=0}^\infty (\mathrm{i}\partial\!\!\!/ + m)\, \frac{m^{2n}}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^n S^\vee_a(x,y)\Big|_{a=0}\,,$$

as a light-cone expansion of the free Green's operator. Our idea is to generalize this formula to the case with interaction. More precisely, we want to express the perturbed Green's operator in the form

$$\tilde{s}^\vee(x,y) = \sum_{n=0}^\infty F_n(x,y)\, \left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^n S^\vee_a(x,y)\Big|_{a=0} \tag{19.2.15}$$

with factors F_n which depend on the external potential. We will see that this method is very convenient; especially, we can in this way avoid working with the rather complicated

explicit formula (19.2.13). Apart from giving a motivation for the desired form (19.2.15) of the formulas of the light-cone expansion, the mass expansion (19.2.13) leads to the conjecture that even the higher order contributions in the mass to the *perturbed* Green's operators might be of higher order on the light cone. If this conjecture were true, it would be a good idea to expand the perturbation expansion of $\tilde{s}$ with respect to the parameter m. Therefore, our strategy is to first expand (18.1.1) with respect to the mass and to try to express the contributions to the resulting expansion in a form similar to (19.2.15).

The expansion of (18.1.1) with respect to m gives a double sum over the orders in the mass parameter and in the external potential. It is convenient to combine these two expansions in a single perturbation series. To this end, we rewrite the Dirac operator as

$$\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m = \mathrm{i}\partial\!\!\!/ + B \qquad \text{with} \qquad B := \mathcal{B} - m\,.$$

For the light-cone expansion of the Green's operators, we will always view B as the perturbation of the Dirac operator. This has the advantage that the unperturbed objects are massless. Expanding in powers of B gives the mass expansion and the perturbation expansion in one step. In order to further simplify the notation, for the massless objects we usually omit the index m. Thus we write the Green's operator of the massless Dirac equation in the Minkowski vacuum as

$$s^\vee(x,y) = \mathrm{i}\partial\!\!\!/_x\, S^\vee_{m^2}(x,y)\big|_{m=0}\,, \qquad s^\wedge(x,y) = \mathrm{i}\partial\!\!\!/_x\, S^\wedge_{m^2}(x,y)\big|_{m=0}\,.$$

Then the interacting Green's operators are given by the perturbation series

$$\tilde{s}^\vee = \sum_{k=0}^\infty (-s^\vee B)^k s^\vee\,, \qquad \tilde{s}^\wedge = \sum_{k=0}^\infty (-s^\wedge B)^k s^\wedge\,. \tag{19.2.16}$$

The constructions of the following subsections are exactly the same for the advanced and retarded Green's operators. In order to treat both cases at once, in the remainder of this section we will omit all superscripts '$\vee$' and '$\wedge$'. The formulas for the advanced and retarded Green's operators are obtained by either adding '$\vee$' or '$\wedge$' to all factors s and S.

We now explain how each contribution to the perturbation expansion (19.2.16) can be written similar to the right side of (19.2.15) as a sum of terms of increasing order on the light cone. For the mass expansion of S_{m^2}, we again set $a = m^2$ and use the notation

$$S^{(l)} = \left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^l S_a\big|_{a=0}\,.$$

In preparation, we derive some computation rules for the $S^{(l)}$: S_a satisfies the defining equation of a Klein-Gordon Green's operator

$$(-\Box_x - a)\, S_a(x,y) = \delta^4(x-y)\,.$$

Differentiating with respect to a and setting $a = 0$ gives

$$-\Box_x S^{(l)}(x,y) = \delta_{l,0}\, \delta^4(x-y) + l\, S^{(l-1)}(x,y)\,, \qquad l \geq 0. \tag{19.2.17}$$

(For $l = 0$, this formula does not seem to make sense because $S^{(-1)}$ is undefined. The expression is meaningful, however, if one keeps in mind that in this case the factor l is zero, and thus the whole second summand vanishes. We will also use this convention in the following calculations.) Next, we differentiate the formulas for S_a in momentum space,

$$S^\vee_a(p) = \frac{1}{p^2 - a - \mathrm{i}\nu p^0}\,, \qquad S^\wedge_a(p) = \frac{1}{p^2 - a + \mathrm{i}\nu p^0}$$

with respect to both p and a. Comparing the results gives the relation

$$\frac{\partial}{\partial p^k} S_a(p) = -2p_k \, \frac{\mathrm{d}}{\mathrm{d}a} S_a(p) \,,$$

or, after expanding in the parameter a,

$$\frac{\partial}{\partial p^k} S^{(l)}(p) = -2p_k \, S^{(l+1)}(p) \,, \qquad l \geq 0. \tag{19.2.18}$$

This formula also determines the derivatives of $S^{(l)}$ in position space; namely

$$\begin{aligned} \frac{\partial}{\partial x^k} S^{(l)}(x,y) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, S^{(l)}(p) \, (-\mathrm{i} p_k) \, \mathrm{e}^{-\mathrm{i}p(x-y)} \\ &\overset{(19.2.18)}{=} \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, \frac{\partial}{\partial p^k} S^{(l-1)}(p) \, \mathrm{e}^{-\mathrm{i}p(x-y)} \\ &= -\frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, S^{(l-1)}(p) \, \frac{\partial}{\partial p^k} \mathrm{e}^{-\mathrm{i}p(x-y)} \\ &= \frac{1}{2} \, (y-x)_k \, S^{(l-1)}(x,y) \,, \qquad l \geq 1. \end{aligned} \tag{19.2.19}$$

We iterate this relation to calculate the Laplacian,

$$\begin{aligned} -\Box_x S^{(l)}(x,y) &= -\frac{1}{2} \, \frac{\partial}{\partial x^k} \Big((y-x)^k \, S^{(l-1)}(x,y) \Big) \\ &= 2 \, S^{(l-1)}(x,y) + \frac{1}{4} \, (y-x)^2 \, S^{(l-2)}(x,y) \,, \qquad l \geq 2 \end{aligned}$$

(in the last step we used the product rule and applied (19.2.19) for l replaced by $l-1$). After comparing with (19.2.17), we conclude that

$$(y-x)^2 \, S^{(l)}(x,y) = -4l \, S^{(l+1)}(x,y) \,, \qquad l \geq 0 \,.$$

Finally, $S^{(l)}(x,y)$ is only a function of $(y-x)$, which implies that

$$\frac{\partial}{\partial x^k} S^{(l)}(x,y) = -\frac{\partial}{\partial y^k} S^{(l)}(x,y) \,, \qquad l \geq 0 \,.$$

The following lemma gives the light-cone expansion of an operator product which is linear in the external potential. It can be used iteratively to perform the light-cone expansion of more complicated operator products; in this case, the potential is a composite expression in B and its partial derivatives. With this in mind, in the next lemma we denote the external potential by V.

LEMMA 19.2.3. **(light-cone expansion to first order)** *For any $l, r \geq 0$, the operator product $S^{(l)} \, V \, S^{(r)}$ has the light-cone expansion*

$$\begin{aligned} &(S^{(l)} \, V \, S^{(r)})(x,y) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l \, (1-\alpha)^r \, (\alpha - \alpha^2)^n \, (\Box^n V)_{|\alpha y + (1-\alpha) x} \, \mathrm{d}\alpha \, S^{(n+l+r+1)}(x,y) \,. \end{aligned} \tag{19.2.20}$$

Before coming to the proof, we briefly explain this lemma. We first recall that, according to (19.2.14), the higher a-derivatives of $S_a(x,y)$ are of higher order on the light cone. Thus the summands in (19.2.20) are of increasing order on the light cone, and the infinite sum is mathematically well-defined in the sense of Definition 19.2.1 via the approximation by the partial sums (19.2.2).

The second point which requires an explanation is related to the arbitrariness in choosing the potential V in the case $l = 0$ (and analogously in the case $r = 0$). In this case, accothe distribution $S^{(l)} = S_0$ is supported on the light cone (see (19.2.12)). Therefore, the function V enters the operator product on the left side of (19.2.20) only evaluated on the light cone $L_x = \{z \,|\, (x-z)^2 = 0\}$. This means that we may modify the function V arbitarily outside this light cone. When doing so, the argument $\Box^V$ in the integrand on the right side of (19.2.20) does in general change. Therefore, the individual summands in (19.2.20) do in general change. But, clearly, in order for the identity (19.2.20) to remain valid, the whole series must remain unchanged. This is indeed the case due to cancellations in the series (this is illustrated in Exercise 19.4). With this in mind, one can sometimes simplify the application of the above lemma in the case $l = 0$ by choosing V outside the light cone L_x in such a way that the computation of the right side simplifies.

Proof of Lemma 19.2.3. The method of proof is to compute the Laplacian of both sides of (19.2.20). The resulting formulas will have a similar structure, making it possible to proceed inductively.

On the left side of (19.2.20), we calculate the Laplacian with the help of (19.2.17) to

$$-\Box_x (S^{(l)} \, V \, S^{(r)})(x,y) = \delta_{l,0} \, V(x) \, S^{(r)}(x,y) + l \, (S^{(l-1)} \, V \, S^{(r)})(x,y) \,. \tag{19.2.21}$$

The Laplacian of the integral on the right side of (19.2.20) can be computed with the help of (19.2.19) and (19.2.17),

$$\begin{aligned} &-\Box_x \int_0^1 \alpha^l \, (1-\alpha)^r \, (\alpha - \alpha^2)^n \, (\Box^n V)_{|\alpha y + (1-\alpha) x} \, \mathrm{d}\alpha \, S^{(n+l+r+1)}(x,y) \qquad (19.2.22) \\ &\quad = -\int_0^1 \alpha^l \, (1-\alpha)^{r+2} \, (\alpha - \alpha^2)^n \, (\Box^{n+1} V)_{|\alpha y + (1-\alpha) x} \, \mathrm{d}\alpha \, S^{(n+l+r+1)}(x,y) \\ &\qquad - \int_0^1 \alpha^l \, (1-\alpha)^{r+1} \, (\alpha - \alpha^2)^n \, (\partial_k \Box^n V)_{|\alpha y + (1-\alpha) x} \, \mathrm{d}\alpha \, (y-x)^k \, S^{(n+l+r)}(x,y) \\ &\qquad + (n+l+r+1) \int_0^1 \alpha^l \, (1-\alpha)^r \, (\alpha - \alpha^2)^n \, (\Box^n V)_{|\alpha y + (1-\alpha) x} \, \mathrm{d}\alpha \, S^{(n+l+r)}(x,y) \,. \end{aligned}$$

In the second summand, we rewrite the partial derivative as a derivative with respect to α,

$$(y-x)^k (\partial_k \Box^n V)_{|\alpha y + (1-\alpha) x} = \frac{\mathrm{d}}{\mathrm{d}\alpha} (\Box^n V)_{|\alpha y + (1-\alpha) x}$$

(as is verified immediately by computing the right side with the chain rule). This makes it possible to integrate in α by parts. We thus obtain

$$\begin{aligned} &\int_0^1 \alpha^l \, (1-\alpha)^{r+1} \, (\alpha - \alpha^2)^n \, (\partial_k \Box^n V)_{|\alpha y + (1-\alpha) x} \, \mathrm{d}\alpha \, (y-x)^k \\ &\quad = \int_0^1 \alpha^l \, (1-\alpha)^{r+1} \, (\alpha - \alpha^2)^n \, \frac{\mathrm{d}}{\mathrm{d}\alpha} \Big((\Box^n V)\big|_{\alpha y + (1-\alpha) x} \Big) \, \mathrm{d}\alpha \\ &\quad = -\delta_{n,0} \, \delta_{l,0} \, V(x) - (n+l) \int_0^1 \alpha^l \, (1-\alpha)^{r+2} \, (\alpha - \alpha^2)^{n-1} \, (\Box^n V)_{|\alpha y + (1-\alpha) x} \, \mathrm{d}\alpha \\ &\qquad + (n+r+1) \int_0^1 \alpha^l \, (1-\alpha)^r \, (\alpha - \alpha^2)^n \, (\Box^n V)_{|\alpha y + (1-\alpha) x} \, \mathrm{d}\alpha \\ &\quad = -\delta_{n,0} \, \delta_{l,0} \, V(x) \end{aligned}$$

$$
\begin{aligned}
&- n \int_0^1 \alpha^l \, (1-\alpha)^{r+2} \, (\alpha-\alpha^2)^{n-1} \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, \mathrm{d}\alpha \\
&+ (n+l+r+1) \int_0^1 \alpha^l \, (1-\alpha)^r \, (\alpha-\alpha^2)^n \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, \mathrm{d}\alpha \\
&- l \int_0^1 \alpha^{l-1} \, (1-\alpha)^r \, (\alpha-\alpha^2)^n \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, \mathrm{d}\alpha \, .
\end{aligned}
$$

We substitute back into the original equation to obtain

$$
\begin{aligned}
(19.2.22) &= \delta_{n,0} \, \delta_{l,0} \, V(x) \, S^{(r)}(x,y) \\
&+ l \int_0^1 \alpha^{l-1} \, (1-\alpha)^r \, (\alpha-\alpha^2)^n \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, \mathrm{d}\alpha \, S^{(n+l+r)}(x,y) \\
&- \int_0^1 \alpha^l \, (1-\alpha)^{r+2} \, (\alpha-\alpha^2)^n \, (\Box^{n+1} V)_{|\alpha y+(1-\alpha)x} \, \mathrm{d}\alpha \, S^{(n+l+r+1)}(x,y) \\
&+ n \int_0^1 \alpha^l \, (1-\alpha)^{r+2} \, (\alpha-\alpha^2)^{n-1} \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, \mathrm{d}\alpha \, S^{(n+l+r)}(x,y) \, .
\end{aligned}
$$

After dividing by $n!$ and summation over n, the last two summands are telescopic and cancel each other. Thus one gets

$$
\begin{aligned}
&- \Box \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 \alpha^l \, (1-\alpha)^r \, (\alpha-\alpha^2)^n \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, \mathrm{d}\alpha \, S^{(n+l+r+1)}(x,y) \\
&= \delta_{l,0} \, V(x) \, S^{(r)}(x,y) \\
&\quad + l \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 \alpha^{l-1} \, (1-\alpha)^r \, (\alpha-\alpha^2)^n \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, \mathrm{d}\alpha \, S^{(n+l+r)}(x,y) \, .
\end{aligned}
\tag{19.2.23}
$$

We now compare the formulas (19.2.21) and (19.2.23) for the Laplacian of both sides of (19.2.20). In the special case $l = 0$, these formulas coincide, and we can use a uniqueness argument for the solutions of the wave equation to prove (19.2.20): We assume that we consider the advanced Green's operator (for the retarded Green's operator, the argument is analogous). For given y, we denote the difference of both sides of (19.2.20) by $F(x)$. Since the support of $F(x)$ is in the past light cone $x \in L_y^\wedge$, F vanishes in a neighborhood of the hypersurface $\mathcal{H} = \{z \in \mathbb{R}^4 \,|\, z^0 = y^0 + 1\}$. Moreover, the Laplacian of F is identically zero according to (19.2.21) and (19.2.23). We conclude that

$$
\Box F = 0 \qquad \text{and} \qquad F_{|\mathcal{H}} = \partial_k F_{|\mathcal{H}} = 0 \, .
$$

Since the wave equation has a unique solution for given initial data on the Cauchy surface $\mathcal{H}$, F vanishes identically.

The general case follows by induction in l: Suppose that (19.2.20) holds for given $\hat{l}$ (and arbitrary r). Then, according to (19.2.21), (19.2.23), and the induction hypothesis, the Laplacian of both sides of (19.2.20) coincides for $l = \hat{l} + 1$. The above uniqueness argument for the solutions of the wave equation again gives (19.2.20). □

We finally remark that the method of the previous lemma generalizes to other operator products. In particular, in [**52**, Appendix C] light-cone expansions are derived which involve unbounded line integrals.

19.3. The Hadamard Form in Curved Spacetime and the Wave Front Set

The Hadamard expansion (19.1.1) can also be formulated in curved spacetime. To this end, one simply replaces the function (19.1.2) by

$$\Gamma_\varepsilon(x,y) := \Gamma(x,y) - \mathrm{i}\varepsilon \left(\mathfrak{t}(y) - \mathfrak{t}(x)\right),$$

where $\mathfrak{t}$ is a time function and $\Gamma(x,y)$ is the geodesic distance squared, with the sign convention that Γ is positive in timelike and negative in spacelike directions. It turns out that if a bi-distribution is of Hadamard form in one chart, it is also of Hadamard form in any other chart. More details on the Hadamard expansion for Dirac fields can be found in [**138, 98**] or [**56**, Appendix A].

The Hadamard form can be formulated alternatively in terms of the wave front set, as we we now briefly mention. We work in an open subset $U \subset \mathbb{R}^n$. We denote the distributions in U by $\mathcal{D}'(U)$ (being the dual space of $C^\infty(U, \mathbb{C})$ with the topology induced by the C^k-norms). An *open conic neighborhood* of a point $\xi \in \mathbb{R}^n$ is defined to be an open neighborhood which is invariant under the action of $\mathbb{R}^+$ by multiplication. Thus an open conic neighborhood can be written in the form

$$\{\lambda x \,|\, x \in S, \lambda \in \mathbb{R}^+\},$$

where S is an open subset of $S^{n-1} \subset \mathbb{R}^n$.

Definition 19.3.1. *Let $\phi \in \mathcal{D}'(U)$. The* **wave front set** $\mathrm{WF}(\phi)$ *is the complement in $U \times \mathbb{R}^n \setminus \{0\}$ of all points $(x,\xi) \in U \times \mathbb{R}^n \setminus \{0\}$ with the following property: There exists a function $f \in C^\infty(U, \mathbb{R})$ with $f(x) = 1$ and an open conic neighborhood V of ξ such that*

$$\sup_{\zeta \in V} \left(1 + |\zeta|\right)^N \left| \big(\widehat{f\phi}\big)(\zeta) \right| < \infty \qquad \text{for all } N \in \mathbb{N}. \tag{19.3.1}$$

In simple terms, the wave front set consists of all points $x \in U$ where the distribution is singular, together with the directions ξ into which the singularity points. More precisely, the above definition can be understood as follows. First, in view of taking the complement, the condition (19.3.1) ensures that the point (x,ξ) does *not* lie in the wave front set. With the help of the cutoff function f one can disregard the behavior of ϕ away from x. In other words, the condition 19.3.1 only depends on the behavior of ϕ in an arbitrarily small neighborhood of x. This condition states that the Fourier transform has rapid decay in a cone around ξ. Since decay properties of the Fourier transform correspond to smoothness properties in position space, we obtain a smoothness statement for ϕ at x, but only along the "wave front" described by ξ.

Definition 19.3.1 readily extends to a distribution ϕ on a manifold $\mathscr{M}$, in which case the wave front set is a subset of the cotangent bundle,

$$\mathrm{WF}(\phi) \subset T^*\mathscr{M} \setminus 0$$

(where 0 is the zero section). The wave front set can also be defined for bundle-valued distributions by choosing a local trivialization and taking the wave front sets of the component functions. The unregularized kernel of the fermionic projector P is a bi-distribution on $\mathscr{M} \times \mathscr{M}$. Therefore, its wave front set takes values in the product of the cotangent bundles,

$$\mathrm{WF}(P) \subset \big(T^*\mathscr{M} \setminus 0\big) \times \big(T^*\mathscr{M} \setminus 0\big).$$

DEFINITION 19.3.2. *The unregularized kernel $P \in \mathcal{D}'(\mathscr{M} \times \mathscr{M})$ is said to be of* **Hadamard form** *if its wave front set has the property*

$$\mathrm{WF}(P) \subset \Big\{ (x_1, \xi_1, x_2, -\xi_2) \,\Big|\, \text{there is a null geodesic } \gamma : I \to \mathscr{M} \text{ with } a, b \in I, \\ \gamma(a) = x_1, \gamma(b) = x_2 \quad \text{and} \quad \xi_1 = \dot{\gamma}(a), \xi_2 = \dot{\gamma}(b) \text{ past-directed} \Big\} .$$

In words, this definition means that there are singularities only on the light cone, and that these singularities are formed only of negative frequencies. The equivalence of this definition with the local Hadamard expansion (19.1.1) has been established in [**129**]. Physically, the Hadamard condition can be understood as a microlocal formulation of an energy condition, noting that "frequencies" can also be interpreted as "energies." Good references on microlocal analysis and the wave front set are [**105**] and [**4**, Chapter 4].

19.4. Proof of the Hadamard Property in an External Potential

In this section, we give the proof of Theorem 19.1.1. We closely follow the presentation in [**79**]. In preparation, we derive so-called frequency splitting estimates which give control of the "mixing" of the positive and negative frequencies in the solutions of the Dirac equation as generated by the time-dependent external potential (Theorem 19.4.1). Based on these estimates, we will complete the proof of Theorem 19.1.1 at the end of Section 19.4.2.

19.4.1. Frequency Mixing Estimates. For the following constructions, we again choose the hypersurface $\mathscr{N} := \mathscr{N}_{t_0}$ at some given time t_0. Moreover, we always fix the mass parameter $m > 0$. Since we are no longer considering families of solutions, for ease of notation we omit the index m at the Dirac wave functions, the scalar products and the corresponding norms. We also identify the solution space $\mathcal{H}_m$ with the Hilbert space $\mathcal{H}_{t_0}$ of square integrable wave functions on $\mathscr{N}$. On $\mathcal{H}_{t_0}$, we can act with the Hamiltonian H of the vacuum, and using the above identification, the operator H becomes an operator on $\mathcal{H}_m$ (which clearly depends on the choice of t_0).

We work with a so-called *frequency splitting* with respect to the vacuum dynamics. To this end, we decompose the Hilbert space $\mathcal{H}_m$ as

$$\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^- \qquad \text{with} \qquad \mathcal{H}^\pm = \chi^\pm(H) \mathcal{H}_m ,$$

where $\chi^\pm$ are the characteristic functions

$$\chi^+ := \chi_{[0,\infty)} \qquad \text{and} \qquad \chi^- := \chi_{(-\infty,0)} . \tag{19.4.1}$$

For convenience, we write this decomposition in components and use a block matrix notation for operators, i.e.

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} A^+_+ & A^+_- \\ A^-_+ & A^-_- \end{pmatrix} ,$$

where $A^s_{s'} = \chi^s(H) A \chi^{s'}(H)$ and $s, s' \in \{\pm\}$.

The representation in Proposition 17.2.5 makes it possible to let the fermionic signature operator $\tilde{\mathcal{S}}_m$ act on the Hilbert space $\mathcal{H}_m$ (for fixed m). We decompose this operator with respect to the above frequency splitting,

$$\tilde{\mathcal{S}}_m = S^{\mathrm{D}} + \Delta\tilde{\mathcal{S}} , \qquad \text{where} \qquad S^{\mathrm{D}} := \tilde{\mathcal{S}}^+_+ + \tilde{\mathcal{S}}^-_- \quad \text{and} \quad \Delta\tilde{\mathcal{S}} := \tilde{\mathcal{S}}^+_- + \tilde{\mathcal{S}}^-_+ .$$

Thus the operator S^{D} maps positive to positive and negative to negative frequencies. The operator $\Delta\tilde{\mathcal{S}}$, on the other hand, mixes positive and negative frequencies. In the next theorem, it is shown under a suitable smallness assumption on $\mathcal{B}$ that the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$

coincide with the projections $\chi^\pm(H)$, up to smooth contributions. The main task in the proof is to control the "frequency mixing" as described by the operator $\Delta\tilde{\mathcal{S}}$.

THEOREM 19.4.1. *Under the assumptions of Theorem 19.1.1, the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$ have the representations*

$$\chi^\pm(\tilde{\mathcal{S}}_m) = \chi^\pm(H) + \frac{1}{2\pi\mathrm{i}} \oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, \Delta\tilde{\mathcal{S}}\, (S^{\mathrm{D}} - \lambda)^{-1}\, \mathrm{d}\lambda\,, \tag{19.4.2}$$

where the contour integral is an integral operator with a smooth integral kernel.

Here $B_{\frac{1}{2}}$ denotes the open ball of radius $1/2$. The operator $(\tilde{\mathcal{S}}_m - \lambda)^{-1}$ is also referred to as the *resolvent* of $\tilde{\mathcal{S}}_m$.

This theorem will be proved in several steps. We begin with a preparatory lemma.

LEMMA 19.4.2. *Under the assumptions* (17.2.1) *and* (19.1.3)*, the spectrum of S^{D} is located in the set*

$$\sigma(S^{\mathrm{D}}) \subset \left[-\frac{3}{2}, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{3}{2}\right]. \tag{19.4.3}$$

Moreover,

$$\chi^\pm(S^{\mathrm{D}}) = \chi^\pm(H)\,, \tag{19.4.4}$$

and the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$ have the representations (19.4.2)*.*

PROOF. Since the subspaces $\mathcal{H}^\pm$ are invariant under the action of S^{D}, our task is to show that the spectrum of $S^{\mathrm{D}}|_{\mathcal{H}^\pm}$ is positive and negative, respectively. This statement would certainly be true if we replaced S^{D} by $\mathcal{S}_m$, because the operator $\mathcal{S}_m$ has the eigenvalues ± 1 with $\mathcal{H}^\pm$ as the corresponding eigenspaces. Estimating the representation in Proposition 17.2.5 with the Schwarz inequality, we obtain

$$\left|(\psi|S^{\mathrm{D}}\phi) - (\psi|\mathcal{S}_m\phi)\right| \leq \left(c + \frac{c^2}{2}\right) \|\psi\|\, \|\phi\| \quad \text{with} \quad c := \int_{-\infty}^{\infty} |\mathcal{B}(\tau)|_{C^0}\, \mathrm{d}\tau\,.$$

Using the assumption (19.1.3), we conclude that

$$\left|(\psi|S^{\mathrm{D}}\phi) - (\psi|\mathcal{S}_m\phi)\right| < \frac{1}{2}\, \|\psi\|\, \|\phi\| \qquad \text{for all } \psi, \phi \in \mathcal{H}_m\,.$$

Standard estimates on the continuity of the spectrum (see for example [**108**, §IV.3]) yield that the spectrum of S^{D} differs by that of the operator $\mathcal{S}_m$ at most by $1/2$. This gives (19.4.3) and (19.4.4).

In order to prove the representation (19.4.2), we take the resolvent identity

$$(\tilde{\mathcal{S}}_m - \lambda)^{-1} = (S^{\mathrm{D}} - \lambda)^{-1} - (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, \Delta\tilde{\mathcal{S}}\, (S^{\mathrm{D}} - \lambda)^{-1}\,,$$

form the contour integral and apply (19.4.4). This gives the result. □

The next lemma relates the smoothness of an integral kernel to the boundedness of the product of the operator with powers of the vacuum Hamiltonian.

LEMMA 19.4.3. *Let $A \in \mathrm{L}(\mathcal{H}_m)$ be an operator which maps smooth functions to smooth functions and has the property that for all $p, q \in \mathbb{N}$, the operator product*

$$H^q\, A\, H^p \;:\; C_0^\infty(\mathcal{N}, S\mathscr{M}) \to C^\infty(\mathcal{N}, S\mathscr{M}) \tag{19.4.5}$$

extends to a bounded linear operator on $\mathcal{H}_m$. Then, considering A as an operator on $\mathcal{H}_m$, this operator can be represented as an integral operator with a smooth integral kernel, i.e.

$$(A\psi)(x) = \int_{\mathcal{N}} \mathcal{A}\big(x, (t_0, \vec{y})\big)\, \gamma^0\, \psi(t_0, \vec{y})\; \mathrm{d}^3y \qquad \textit{with} \qquad \mathcal{A} \in C^\infty(\mathscr{M} \times \mathscr{M})\,.$$

PROOF. Since in momentum space, the square of the Hamiltonian takes the form

$$H\big(\vec{k}\big)^2 = \Big(\gamma^0\big(\vec{\gamma}\vec{k} + m\big)\Big)^2 = \big(-\vec{\gamma}\vec{k} + m\big)\big(\vec{\gamma}\vec{k} + m\big) = |\vec{k}|^2 + m^2\,,$$

the wave function $\hat{\psi}$ defined by

$$\hat{\psi}(\vec{k}) := \frac{1}{|\vec{k}|^2 + m^2}\, \mathrm{e}^{\mathrm{i}\vec{k}\vec{x}_0}\, \Xi$$

for a constant spinor Ξ and $\vec{x}_0 \in \mathbb{R}^3$, satisfies the equation

$$H^2\, \psi(\vec{x}) = \delta^3(\vec{x} - \vec{x_0})\, \Xi\,.$$

Moreover, one verifies immediately that $\psi \in \mathcal{H}_{t_0}$ is square-integrable. Using the last equation together with (19.4.5), we conclude that

$$H^q A\big(\delta^3(\vec{x} - \vec{x_0})\, \Xi\big) = H^q A H^2 \psi \in \mathcal{H}_{t_0}\,.$$

Since q is arbitrary, it follows that A has an integral representation in the spatial variables,

$$(A\phi)(\vec{x}) = \int_{\mathcal{N}} \mathcal{A}(\vec{x}, \vec{y})\, \gamma^0\, \phi(\vec{y})\; \mathrm{d}^3y \qquad \text{with} \qquad \mathcal{A} \in C^\infty(\mathcal{N} \times \mathcal{N})\,.$$

We now extend this integral kernel to $\mathscr{M} \times \mathscr{M}$ by solving the Cauchy problem in the variables x and y. This preserves smoothness by the global existence and regularity results for linear hyperbolic equations, giving the result. □

LEMMA 19.4.4. *Under the assumptions of Theorem 19.1.1, for all $p \in \mathbb{N}$ the iterated commutator*

$$\mathcal{S}^{(p)} := \underbrace{\Big[H, \big[H, \ldots, [H,}_{p \text{ factors}} \tilde{\mathcal{S}}_m] \cdots \big]\Big]$$

is a bounded operator on $\mathcal{H}_m$.

PROOF. In the vacuum, the Hamiltonian clearly commutes with the time evolution operator,

$$\big[H, U_m^{t,t'}\big] = 0\,. \tag{19.4.6}$$

In order to derive a corresponding commutator relation in the presence of the external potential, one must take into account that $\tilde{H}$ is time-dependent. For ease of notation, we do not write out this dependence, but instead understand that the Hamiltonian is to be evaluated at the correct time, i.e.

$$\tilde{U}_m^{t,t'}\, \tilde{H} \equiv \tilde{U}_m^{t,t'}\, \tilde{H}(t') \qquad \text{and} \qquad \tilde{H}\, \tilde{U}_m^{t,t'} \equiv \tilde{H}(t)\, \tilde{U}_m^{t,t'}\,.$$

Then

$$(\mathrm{i}\partial_t - \tilde{H})\big(\tilde{H}\, \tilde{U}_m^{t,t'} - \tilde{U}_m^{t,t'}\, \tilde{H}\big) = \mathrm{i}\dot{\tilde{H}}\, \tilde{U}_m^{t,t'} \qquad \text{and} \qquad \tilde{H}\, \tilde{U}_m^{t,t'} - \tilde{U}_m^{t,t'}\, \tilde{H}\big|_{t=t'} = 0$$

(here and in what follows the dot denotes the partial derivative with respect to t). Solving the corresponding Cauchy problem gives

$$\big[\tilde{H}, \tilde{U}_m^{t,t'}\big] = \int_{t'}^{t} \tilde{U}_m^{t,\tau} \dot{\tilde{H}}\, \tilde{U}_m^{\tau,t'}\; \mathrm{d}\tau\,. \tag{19.4.7}$$

In order to compute the commutator of H with the operator products in (17.2.9) and (17.2.10), we first differentiate the expression $U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'}$ with respect to t,

$$\mathrm{i}\partial_t\big(U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'}\big) = \mathrm{i}U_m^{t'',t}\,\dot{\mathcal{V}}\,\tilde{U}_m^{t,t'} + U_m^{t'',t}\,\mathcal{V}\,\tilde{H}\,\tilde{U}_m^{t,t'} - U_m^{t'',t}\,H\,\mathcal{V}\,\tilde{U}_m^{t,t'}\,. \tag{19.4.8}$$

Moreover, using the commutation relations (19.4.6) and (19.4.7), we obtain

$$\begin{aligned}
&H\,(U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'}) - (U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'})\,\tilde{H} \\
&\quad= U_m^{t'',t}\,H\,\mathcal{V}\,\tilde{U}_m^{t,t'} - U_m^{t'',t}\,\mathcal{V}\,\tilde{H}\,\tilde{U}_m^{t,t'} + U_m^{t'',t}\,\mathcal{V}\,[\tilde{H},\tilde{U}_m^{t,t'}] \\
&\quad= \mathrm{i}U_m^{t'',t}\,\dot{\mathcal{V}}\,\tilde{U}_m^{t,t'} - \mathrm{i}\partial_t\big(U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'}\big) + \int_{t'}^{t} U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,\tau}\,\dot{\tilde{H}}\,\tilde{U}_m^{\tau,t'}\;\mathrm{d}\tau\,,
\end{aligned}$$

where in the last step we applied (19.4.8). It follows that

$$\begin{aligned}
\big[H, U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'}\big] &= H\,(U_m^{t'',t}\mathcal{V}\tilde{U}_m^{t,t'}) - (U_m^{t'',t}\mathcal{V}\tilde{U}_m^{t,t'})\,\tilde{H} + (U_m^{t'',t}\mathcal{V}\tilde{U}_m^{t,t'})\,\mathcal{V} \\
&= \mathrm{i}U_m^{t'',t}\,\dot{\mathcal{V}}\,\tilde{U}_m^{t,t'} + (U_m^{t'',t}\mathcal{V}\tilde{U}_m^{t,t'})\,\mathcal{V} - \mathrm{i}\partial_t\big(U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'}\big) + \int_{t'}^{t} U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,\tau}\,\dot{\tilde{H}}\,\tilde{U}_m^{\tau,t'}\;\mathrm{d}\tau\,.
\end{aligned}$$

Proceeding in this way, one can calculate the commutator of H with all the terms in (17.2.9) and (17.2.10). We write the result symbolically as

$$[H, \tilde{\mathcal{S}}_m] = \mathcal{S}^{(1)}\,,$$

where $\mathcal{S}^{(1)}$ is a bounded operator. Higher commutators can be computed inductively, giving the result. □

We point out that this lemma only makes a statement on the iterative commutators. Expressions like $[H^p, \tilde{\mathcal{S}}_m]$ or $H^q\,\tilde{\mathcal{S}}_m\,H^p$ will not be bounded operators in general. However, the next lemma shows that the operator $\Delta\tilde{\mathcal{S}}$ has the remarkable property that multiplying by powers of H from the left and/or right again gives a bounded operator.

LEMMA 19.4.5. *Under the assumptions of Theorem 19.1.1, for all $p, q \in \mathbb{N} \cup \{0\}$ the product $H^q\,\Delta\tilde{\mathcal{S}}\,H^p$ is a bounded operator on $\mathcal{H}_m$.*

PROOF. We only consider the products $H^q\,\mathcal{S}_+^-\,H^p$ because the operator $\mathcal{S}_-^+$ can be treated similarly. Multiplying (19.4.7) from the left and right by the resolvent of H, we obtain

$$\big[(H-\mu)^{-1}, \tilde{\mathcal{S}}_m\big] = -(H-\mu)^{-1}\,\mathcal{S}^{(1)}\,(H-\mu)^{-1}\,.$$

Writing the result of Lemma 19.4.4 as

$$[H, \mathcal{S}^{(p)}] = \mathcal{S}^{(p+1)} \qquad \text{with} \qquad \mathcal{S}^{(p+1)} \in \mathrm{L}(\mathcal{H})$$

yields more generally the commutation relations

$$\big[(H-\mu)^{-1}, \mathcal{S}^{(p)}\big] = -(H-\mu)^{-1}\,\mathcal{S}^{(p+1)}\,(H-\mu)^{-1} \qquad \text{for } p \in \mathbb{N}\,. \tag{19.4.9}$$

Choosing a contour γ which encloses the interval $(-\infty, -m]$ as shown in Figure 19.1, one finds

$$\begin{aligned}
H\mathcal{S}_+^- &= -\frac{1}{2\pi\mathrm{i}} \int_\gamma \mu\,(H-\mu)^{-1}\,\tilde{\mathcal{S}}_m\,\chi^+(H)\,\mathrm{d}\mu \\
&= \mathcal{S}\,H\,\chi^-(H)\,\chi^+(H) + \frac{1}{2\pi\mathrm{i}} \int_\gamma \mu\,(H-\mu)^{-1}\,\mathcal{S}^{(1)}(H-\mu)^{-1}\,\chi^+(H)\,\mathrm{d}\mu \\
&= \frac{1}{2\pi\mathrm{i}} \int_\gamma \mu\,(H-\mu)^{-1}\,\mathcal{S}^{(1)}(H-\mu)^{-1}\,\chi^+(H)\,\mathrm{d}\mu\,,
\end{aligned}$$

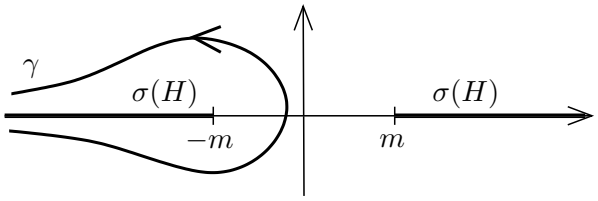


FIGURE 19.1. The contour γ.

where in the last step we used that $\chi^-(H)\,\chi^+(H) = 0$. In order to show that this operator product is bounded, it is useful to employ the spectral theorem for H, which we write as

$$f(H) = \int_{\mathbb{R}\setminus[-m,m]} f(\lambda)\;\mathrm{d}E_\lambda\,, \tag{19.4.10}$$

where dE_λ is the spectral measure of H. This gives

$$\begin{aligned} H\,\mathcal{S}_+^- &= \iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mu}{\lambda-\mu}\;\frac{1}{\lambda'-\mu}\,\chi^+(\lambda')\;\mathrm{d}E_\lambda\right) \mathcal{S}^{(1)}\;\mathrm{d}E_{\lambda'}\;\mathrm{d}\mu \\ &= -\iint_{\mathbb{R}\times\mathbb{R}} \frac{\lambda}{\lambda-\lambda'}\,\chi^-(\lambda)\,\chi^+(\lambda')\;\mathrm{d}E_\lambda\,\mathcal{S}^{(1)}\;\mathrm{d}E_{\lambda'}\,. \end{aligned} \tag{19.4.11}$$

Note that the term $\lambda - \lambda'$ is bounded away from zero. Thus the factor $\lambda/(\lambda-\lambda')$ is bounded, showing that the operator $H\mathcal{S}_+^-$ is in $\mathrm{L}(\mathcal{H}_m)$.

This method can be iterated. To this end, we first rewrite the product with commutators,

$$\begin{aligned} H^q\,\mathcal{S}_+^- &= \chi^-(H)\,\big(H^-\,\chi^-(H)\big)^p\,\tilde{\mathcal{S}}_m\,\chi^+(H) \\ &= \chi^-(H)\Big[H^-,\big[H^-,\ldots,[H^-,\mathcal{S}]\cdots\big]\Big]\,\chi^+(H)\,, \end{aligned}$$

where we used the abbreviation $H^- := H\,\chi^-(H)$. Multiplying from the right by H^p, we can commute factors $H^+ := H\,\chi^+(H)$ to the left to obtain

$$H^q\,\mathcal{S}_+^- H^p = (-1)^p\,\chi^-(H)\,\underbrace{\big[H^+,\ldots,\big[H^+}_{p\text{ factors}},\underbrace{\big[H^-,\ldots,\big[H^-}_{q\text{ factors}},\tilde{\mathcal{S}}_m\big]\cdots\big]\big]\cdots\big]\,\chi^+(H)\,.$$

Representing each factor $H^\pm$ by a contour integral, one can compute the commutators inductively with the help (19.4.9). Applying the spectral theorem (19.4.10) to the left and right of the resulting factor $\mathcal{S}^{(p+q)}$ yields a constant times the expression

$$\begin{aligned} &\iint_{\mathbb{R}\times\mathbb{R}} \chi^-(\lambda)\,\chi^+(\lambda')\;\mathrm{d}E_\lambda\,\mathcal{S}^{(p+q)}\;\mathrm{d}E_{\lambda'} \\ &\qquad\qquad \times \oint_{\gamma_1} \frac{\mu_1\,\mathrm{d}\mu_1}{(\lambda-\mu_1)(\lambda'-\mu_1)} \cdots \oint_{\gamma_{p+q}} \frac{\mu_{p+q}\,\mathrm{d}\mu_{p+q}}{(\lambda-\mu_{p+q})(\lambda'-\mu_{p+q})}\;. \end{aligned}$$

Carrying out the contour integrals with residues, we obtain similar to (19.4.11) an expression of the form

$$H^q\,\mathcal{S}_+^- H^p = \iint_{\mathbb{R}\times\mathbb{R}} f(\lambda,\lambda')\,\chi^-(\lambda)\,\chi^+(\lambda')\;\mathrm{d}E_\lambda\,\mathcal{S}^{(p+q)}\;\mathrm{d}E_{\lambda'}$$

with a bounded function f. This concludes the proof. □

PROOF OF THEOREM 19.4.1. It remains to show that the contour integral in (19.4.2) has a smooth integral kernel. To this end, we multiply the integrand from the left by H^q and from the right by H^p and commute the factors H iteratively to the inside. More precisely, we use the formula

$$H^q (\tilde{\mathcal{S}}_m - \lambda)^{-1} = \sum_{a=0}^{q} \underbrace{\big[H, \ldots, [H}_{a \text{ factors}}, (\tilde{\mathcal{S}}_m - \lambda)^{-1}] \cdots \big]\, H^{q-a}$$

(note that the sum is telescopic; here we use the convention that the summand for $a = 0$ is simply $(\tilde{\mathcal{S}}_m - \lambda)^{-1} H^q$). Hence

$$\begin{aligned} &H^q\, (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, \Delta\tilde{\mathcal{S}}\, (S^{\mathrm{D}} - \lambda)^{-1}\, H^p \\ &= \sum_{a=0}^{q} \sum_{b=0}^{p} \underbrace{\Big[H, \ldots, \big[H}_{a \text{ factors}}, (\tilde{\mathcal{S}}_m - \lambda)^{-1}\big] \cdots \Big]\, H^{q-a}\, \Delta\tilde{\mathcal{S}}\, H^{p-b} \Big[\cdots \big[(S^{\mathrm{D}} - \lambda)^{-1}, \underbrace{H\big], \ldots, H}_{b \text{ factors}} \Big]. \end{aligned}$$

According to Lemma 19.4.5, the intermediate product $H^{q-a}\, \Delta\tilde{\mathcal{S}}\, H^{p-b}$ is a bounded operator. Moreover, the commutators can be computed inductively by applying Lemma 19.4.4 and the formula

$$\big[H, (\tilde{\mathcal{S}}_m - \lambda^{-1})\big] = -(\tilde{\mathcal{S}}_m - \lambda^{-1})\, \big[H, \tilde{\mathcal{S}}_m\big]\, (\tilde{\mathcal{S}}_m - \lambda^{-1})$$

(and similarly for S^{D}). This gives operators which are all bounded for $\lambda \in \partial B_{\frac{1}{2}}(\pm 1)$. Since the integration contour is compact, the result follows. □

19.4.2. Proof of the Hadamard Form. Relying on the frequency mixing estimates of the previous section, we can now give the proof of Theorem 19.1.1. Recall that the fermionic projector is given by (see (15.4.1))

$$P = -\chi^-(\tilde{\mathcal{S}}_m)\, \tilde{k}_m\,, \tag{19.4.12}$$

where we again used the short notation (19.4.1). Here again the operator $\chi^-(\tilde{\mathcal{S}}_m)$ acts on the solution space $\mathcal{H}_m$ of the Dirac equation, which can be identified with the space $\mathcal{H}_{t_0}$ of square integrable wave functions at time t_0 (see the beginning of Section 19.4.1). For the following arguments, it is important to note that this identification can be made at any time t_0.

In order to prove that the bi-distribution corresponding to P is of Hadamard form, we compare the fermionic projectors for three different Dirac operators and use the theorem on the propagation of singularities in [**138**]. More precisely, we consider the following three fermionic projectors:

(1) The fermionic projector P^{vac} in the Minkowski vacuum.

(2) The fermionic projector $\breve{P}$ in the presence of the external potential

$$\breve{\mathcal{B}}(x) := \eta\big(x^0\big)\, \mathcal{B}(x)\,,$$

where $\eta \geq 0$ is a smooth function with $\eta|_{(-\infty,0)} \equiv 0$ and $\eta|_{(1,\infty)} \equiv 1$.

(3) The fermionic projector P in the presence of the external potential $\mathcal{B}(x)$.

The potential $\breve{\mathcal{B}}$ vanishes for negative times, whereas for times $x^0 > 1$ it coincides with $\mathcal{B}$. Thus it smoothly interpolates between the dynamics with and without external potential. The specific form of the potential $\breve{\mathcal{B}}$ in the transition region $0 \leq x^0 \leq 1$ is of no relevance for our arguments.

In the Minkowski vacuum, the relation (19.4.12) gives the usual two-point function composed of all negative-frequency solutions of the Dirac equation. It is therefore obvious that the bi-distribution $P^{\text{vac}}(x,y)$ is of Hadamard form.

We now compare P^{vac} with $\breve{P}$. To this end, we choose an arbitrary time $t_0 < 0$. Then, applying the result of Theorem 19.4.1 to (19.4.12), we get

$$P^{\text{vac}} = -\chi^-(H)\, k_m \qquad \text{and} \qquad \breve{P} = -\chi^-(H)\, \breve{k}_m + (\text{smooth})\,,$$

where $\breve{k}_m$ is the causal fundamental solution in the presence of the potential $\breve{\mathcal{B}}$. Since $\breve{\mathcal{B}}$ vanishes in a neighborhood of the Cauchy surface at time t_0, we conclude that P^{vac} and $\breve{P}$ coincide in this neighborhood up to a smooth contribution. It follows that also $\breve{P}(x,y)$ is of Hadamard form in this neighborhood. Using the theorem on the propagation of singularities [**138**, Theorem 5.5], we conclude that $\breve{P}(x,y)$ is of Hadamard form for all $x, y \in \mathscr{M}$.

Next, we compare $\breve{P}$ with P. Thus we choose an arbitrary time $t_0 > 1$. Using again the result of Theorem 19.4.1 in (19.4.12), we obtain

$$\breve{P} = -\chi^-(H)\, \breve{k}_m + (\text{smooth}) \qquad \text{and} \qquad P = -\chi^-(H)\, \tilde{k}_m + (\text{smooth})$$

(where the smooth contributions may of course be different). Since $\breve{\mathcal{B}}$ and $\mathcal{B}$ coincide in a neighborhood of the Cauchy surface at time t_0, we infer that $\breve{P}$ and P coincide in this neighborhood up to a smooth contribution. As a consequence, $P(x,y)$ is of Hadamard form in this neighborhood. Again applying [**138**, Theorem 5.5], it follows that $P(x,y)$ is of Hadamard form for all $x, y \in \mathscr{M}$. This concludes the proof of Theorem 19.1.1.

19.5. Exercises

EXERCISE 19.1. This exercise explains the notion of the *light-cone expansion* in simple examples.

(a) What is the light-cone expansion of a smooth function on $\mathscr{M} \times \mathscr{M}$? In which sense is it trivial? In which sense is it non-unique?

(b) Show that $A(x,y) = \log\big(|y-x|^2\big)$ is a well-defined distribution on $\mathscr{M} \times \mathscr{M}$. What is the order on the light cone? Write down a light-cone expansion.

(c) Now consider the distributional derivatives

$$\Big(\frac{\partial}{\partial x^0}\Big)^p A(x,y) \qquad \text{with} \qquad p \in \mathbb{N}$$

and $A(x,y)$ as in part (b). What is the order on the light cone? Write down a light-cone expansion.

(d) Consider the function

$$E(x,y) = \sin\big((y-x)^2\big)\, \log\big(|y-x|^2\big)\,.$$

Determine the order on the light cone and give a light-cone expansion.

(e) Consider the function

$$E(x,y) = \begin{cases} \mathrm{e}^{-\frac{1}{(y-x)^2}} & \text{if } (y-x)^2 \geq 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Determine the order on the light cone and give a light-cone expansion.

(f) Show that the expression

$$\lim_{\varepsilon \searrow 0} \frac{\log\big(|y-x|^2\big)}{(y-x)^4 + \mathrm{i}\varepsilon}$$

is a well-defined distribution on $\mathscr{M} \times \mathscr{M}$. Derive its light-cone expansion.

EXERCISE 19.2. *(Understanding the light-cone expansion)* This exercise aims to familiarize you with some of the particularities of the light-cone expansion.

(a) Let $A(x,y) := (x-y)^{2k_0}$ with $k_0 \in \mathbb{Z}$. Which order(s) on the light cone is this? (Prove your answer.) Construct a light-cone expansion of $A(x,y)$ and prove that it is one.

(b) Let $B(x,y) := (x-y)^{2k_0} + (x-y)^{2k_1}$, where $k_0, k_1 \in \mathbb{Z}$ and $k_0 < k_1$. Which order(s) on the light cone is this? (Prove your answer.) Construct a light-cone expansion of $B(x,y)$ and prove that it is one.

(c) Let $C(x,y) := (x-y)^{2k_0} f(x,y) + (x-y)^{2k_1} g(x,y)$, where f and g are smooth functions in x and y and $k0, k1$ as above. Construct a light-cone expansion of $C(x,y)$ and prove that it is one.

(d) Let $D(x,y) := \sin\big((x-y)^2\big)\,(x-y)^2$. Use your results from (b) and (c) to construct two different light-cone expansions of $D(x,y)$. Why might this non-uniqueness not be a problem for the scope of this book?

(e) Finally, consider the function

$$E(x,y) = \sin\big((y-x)^2\big) + \begin{cases} \mathrm{e}^{-\frac{1}{(y-x)^2}} & \text{if } (y-x)^2 \geq 0 \\ 0 & \text{else} \end{cases},$$

Determine its order on the light cone and derive a light-cone expansion.

Hint: For (d) and (e): Expand the sine function.

EXERCISE 19.3. This exercise is devoted to computing the Fourier transform of the *advanced Green's operator* (19.2.11) and deriving the series expansion (19.2.13).

(a) We again set $\xi = y - x$ and $\xi = (t, \vec{\xi})$ with $t > 0$. Moreover, we choose polar coordinates $r = (|\vec{\xi}|, \vartheta, \varphi)$. Carry out the ω-integration with residues and compute the angular integrals to obtain

$$S^\vee_{m^2}(x,y) = \frac{\mathrm{i}}{8\pi r} \int_0^\infty \frac{p}{\omega(p)} \big(\mathrm{e}^{-\mathrm{i}pr} - \mathrm{e}^{\mathrm{i}pr}\big)\big(\mathrm{e}^{\mathrm{i}\omega(p)\,t} - \mathrm{e}^{-\mathrm{i}\omega(p)\,t}\big)\, \mathrm{d}p\,,$$

where $p = |\vec{p}|$ and $\omega(p) := \sqrt{|\vec{p}^2| + m^2}$. Justify this integral as the Fourier transform of a distribution and show that

$$S^\vee_{m^2}(x,y) = \frac{\mathrm{i}}{8\pi r} \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{e}^{-\varepsilon p} \frac{p}{\omega(p)} \big(\mathrm{e}^{-\mathrm{i}pr} - \mathrm{e}^{\mathrm{i}pr}\big)\big(\mathrm{e}^{\mathrm{i}\omega(p)\,t} - \mathrm{e}^{-\mathrm{i}\omega(p)\,t}\big)\, \mathrm{d}p$$

with convergence as a distribution.

(b) Verify (19.2.12) in the case $m = 0$ by setting $\omega(p) = p$ and using (16.5.2).

(c) In order to analyze the behavior away from the light cone, it is most convenient to take the limit $r \searrow 0$ and use Lorentz invariance. Show that in this limit,

$$S^\vee_{m^2}(x,y) = \frac{1}{4\pi} \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{e}^{-\varepsilon p} \frac{p^2}{\omega(p)} \big(\mathrm{e}^{\mathrm{i}\omega(p)\,t} - \mathrm{e}^{-\mathrm{i}\omega(p)\,t}\big)\, \mathrm{d}p \tag{19.5.1}$$

$$= \frac{1}{4\pi} \lim_{\varepsilon \searrow \omega} \int_m^\infty \mathrm{e}^{-\varepsilon p} \sqrt{\omega^2 - m^2}\, \big(\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}\big)\, \mathrm{d}\omega\,. \tag{19.5.2}$$

Compute this integral using [**96**, formula (3.961.1)]. Use the relations between Bessel functions [**124**, (10.27.6), (10.27.11)] to obtain (19.2.12) away from the light cone.

As an alternative method for computing the Fourier integral, one can begin from the integral representation for J_0 in [**124**, (10.9.12)], differentiate with respect to x and use [**124**, (10.6.3)].

(d) Combine the results of (b) and (c) to prove (19.2.12). Why is there no additional contribution at $\xi = 0$?
(e) Use the series expansion [**124**, (10.2.2)] to derive (19.2.13).
(f) The series expansion (19.2.13) can also be derived without using Bessel functions. To this end, one expands (19.5.1) in powers of m^2 and computes the Fourier transform term by term. Verify explicitly that this procedure really gives (19.2.13).

EXERCISE 19.4. In this exercise we illustrate the dependence of the light-cone expansion (19.2.20) on the the function V. We choose $l = 0$, $r \in \mathbb{N}_0$ arbitary, $x = 0$ and $V(z) = z^2$.

(a) Show that the left side of (19.2.20) vanishes for all $y \in M$. *Hint:* Use the causal structure of S_a as given in (19.2.12) in the massless case.
(b) Show that all the summands in (19.2.20) for $n \geq 2$ vanish.
(c) Show that the summands in (19.2.20) for $n = 0$ and $n = 1$ are both non-zero and cancel each other. *Hint:* Compute $\Box_z z^2$. Moreover, make use of the relation between the coefficients of the power series (19.2.13) for $j = r + 1$ and $j = r + 2$. It might be a good idea to begin with the case $r = 0$.

Part 4

Applications and Outlook

CHAPTER 20

A Few Explicit Examples of Causal Variational Principles

In this chapter we introduce a few examples of causal variational principles and analyze them in detail. These examples are too simple for being of direct physical interest. Instead, they are chosen in order to illustrate the different mathematical structures introduced previously. It is a specific feature of these examples that a minimizing measure can be given in closed form, making it possible to analyze the system explicitly. Similar examples were first given in [**77**].

When constructing simple explicit examples, it is often convenient to choose *non-smooth* Lagrangians, which involve for example characteristic functions or are even distributional. In order to treat this non-smooth setting in a mathematically convincing way, one needs to work with additional jet spaces, which we now introduce (for more details see for example [**71, 61**]).

Clearly, the fact that a jet $\mathfrak{u}$ is smooth does not imply that the functions ℓ or $\mathcal{L}$ are differentiable in the direction of $\mathfrak{u}$. This must be ensured by additional conditions which are satisfied by suitable subspaces of $\mathfrak{J}$ which we now define. First, we let Γ^{diff} be those vector fields for which the directional derivative of the function ℓ exists,

$$\Gamma^{\text{diff}} = \big\{ u \in C^\infty(M, T\mathcal{F}) \;\big|\; D_u \ell(x) \text{ exists for all } x \in M \big\} .$$

This gives rise to the jet space

$$\mathfrak{J}^{\text{diff}} := C^\infty(M, \mathbb{R}) \oplus \Gamma^{\text{diff}} \;\subset\; \mathfrak{J} .$$

For the jets in $\mathfrak{J}^{\text{diff}}$, the combination of multiplication and directional derivative in (7.2.2) is well-defined. We choose a linear subspace $\mathfrak{J}^{\text{test}} \subset \mathfrak{J}^{\text{diff}}$ with the property that its scalar and vector components are both vector spaces, i.e.

$$\mathfrak{J}^{\text{test}} = C^{\text{test}}(M, \mathbb{R}) \oplus \Gamma^{\text{test}} \;\subseteq\; \mathfrak{J}^{\text{diff}}$$

for suitable subspaces $C^{\text{test}}(M, \mathbb{R}) \subset C^\infty(M, \mathbb{R})$ and $\Gamma^{\text{test}} \subset \Gamma^{\text{diff}}$. We then write the *restricted EL equations* (7.2.4) in the weaker form

$$\nabla_{\mathfrak{u}} \ell|_M = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}^{\text{test}} .$$

Finally, when considering weak solutions of the linearized field equations, it is sometimes useful to restrict attention to jets in a suitably chosen subspace of $\mathfrak{J}^{\text{test}}$, which in agreement with (14.2.6) we denote by

$$\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}^{\text{test}} .$$

To summarize, we have the inclusions

$$\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}^{\text{test}} \subset \mathfrak{J}^{\text{diff}} \subset \mathfrak{J} .$$

The compactly supported jets are always denoted by an additional subscript zero.

20.1. A One-Dimensional Gaussian

We let $\mathcal{F} = \mathbb{R}$ and choose the Lagrangian as the Gaussian

$$\mathcal{L}(x,y) = \frac{1}{\sqrt{\pi}}\, \mathrm{e}^{-(x-y)^2} \,. \tag{20.1.1}$$

LEMMA 20.1.1. *The Lebesgue measure*

$$\mathrm{d}\rho = \mathrm{d}x$$

is a minimizer of the causal action principle for the Lagrangian (20.1.1) *in the class of variations of finite volume (see* (6.3.2) *and* (6.3.1)*). It is the unique minimizer within this class of variations.*

PROOF. Writing the difference of the actions as in (6.3.2), we can carry out the integrals over ρ using that the Gaussian is normalized (see Exercise 20.1),

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) = 1 \,.$$

We thus obtain

$$\begin{aligned} \mathcal{S}(\rho) - \mathcal{S}(\tilde{\rho}) &= 2 \int_N \mathrm{d}(\rho-\tilde{\rho})(x) + \int_N \mathrm{d}(\rho-\tilde{\rho})(x) \int_N \mathrm{d}(\rho-\tilde{\rho})(y)\, \mathcal{L}(x,y) \\ &= \int_N \mathrm{d}(\rho-\tilde{\rho})(x) \int_N \mathrm{d}(\rho-\tilde{\rho})(y)\, \mathcal{L}(x,y) \,, \end{aligned}$$

where in the last line we used the volume constraint (6.3.1). In order to show that the last double integral is positive, we take the Fourier transform and use that the Fourier transform of a Gaussian is again a Gaussian. More precisely,

$$\int_N \mathrm{e}^{-\mathrm{i}p(x-y)}\, \mathcal{L}(x,y)\; \mathrm{d}y = \mathrm{e}^{-\frac{p^2}{4}} =: f(p) \,. \tag{20.1.2}$$

Moreover, the estimate

$$\Big| \int_N \mathrm{e}^{\mathrm{i}px}\; \mathrm{d}(\rho-\tilde{\rho})(x) \Big| \leq \big|\tilde{\rho}-\rho\big|(\mathcal{F}) < \infty$$

shows that the Fourier transform of the signed measure $\tilde{\rho}-\rho$ is a bounded function $g \in L^\infty(\mathbb{R})$. Approximating this function in $L^2(\mathbb{R})$, we can apply Plancherel's theorem and use the fact that convolution in position space corresponds to multiplication in momentum space. We thus obtain

$$\begin{aligned} &\int_N \mathrm{d}(\rho-\tilde{\rho})(x) \int_N \mathrm{d}(\rho-\tilde{\rho})(y)\, \mathcal{L}(x,y) \\ &\quad = \int_N \big(\mathcal{F}^{-1}(fg)\big)(x)\; \mathrm{d}(\rho-\tilde{\rho})(x) = \int_{-\infty}^\infty \overline{g(p)}\, \mathrm{e}^{-\frac{p^2}{4}}\; g(p)\; \mathrm{d}p \geq 0 \,, \end{aligned} \tag{20.1.3}$$

and the inequality is strict unless $\tilde{\rho} = \rho$. This concludes the proof. □

The EL equations read

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) = 1 \qquad \text{for all } x \in \mathbb{R} \,. \tag{20.1.4}$$

We now specify the jet spaces. Since the Lagrangian is smooth, it is obvious that

$$\mathfrak{J}^{\text{diff}} = \mathfrak{J} = C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R})$$

(where we identify a vector field $a(x)\partial_x$ on $\mathbb{R}$ with the function $a(x)$). The choice of $\mathfrak{J}^{\text{test}}$ is less obvious. For simplicity, we restrict attention to functions which are bounded together with all their derivatives, denoted by

$$C^\infty_{\text{b}} := \big\{ f \in C^\infty(\mathbb{R}) \,\big|\, f^{(n)} \in L^\infty(\mathbb{R}) \text{ for all } n \in \mathbb{N}_0 \big\} .$$

Now different choices are possible. Our first choice is to consider jets whose scalar components are compactly supported,

$$\mathfrak{J}^{\text{test}} = C^\infty_0(\mathbb{R}) \oplus C^\infty_{\text{b}}(\mathbb{R}) . \tag{20.1.5}$$

The linearized field equations (8.1.7) reduce to the scalar equation

$$\int_N \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}}\, 1 = 0 \qquad \text{for all } x \in \mathbb{R} , \tag{20.1.6}$$

because if this equation holds, then the x-derivative of the left side is also zero. Differentiating the EL equations (20.1.4) with respect to x, we find that

$$\int_N \nabla_{1,\mathfrak{v}}\mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}}\, 1 = 0 \qquad \text{for all } x \in \mathbb{R} .$$

Subtracting this equation from (20.1.6), the linearized field simplify to

$$\int_N \nabla_{2,\mathfrak{v}}\mathcal{L}(x,y)\, \mathrm{d}\rho(y) = 0 \qquad \text{for all } x \in \mathbb{R} .$$

A specific class of solutions can be given explicitly. Indeed, choosing

$$\mathfrak{u} = (a, A) \qquad \text{with} \qquad a \in C^\infty_0(\mathbb{R}) \text{ and } A(x) := \int_\infty^x a(t)\, \mathrm{d}t \in C^\infty_{\text{b}}(\mathbb{R}) , \tag{20.1.7}$$

integration by parts yields

$$\int_N \nabla_{2,\mathfrak{u}}\mathcal{L}(x,y)\, \mathrm{d}\rho(y) = \int_N \big(A'(y) + A(y)\, \partial_y\big)\mathcal{L}(x,y)\, \mathrm{d}y = 0 . \tag{20.1.8}$$

These linearized solutions are referred to as *inner solutions*, as introduced in a more general context in Section 8.3 and [**61**]. Inner solutions can be regarded as infinitesimal generators of transformations of M which leave the measure ρ unchanged. Therefore, inner solutions do not change the causal fermion system, but merely describe symmetry transformations of the measure. With this in mind, inner solutions are not of interest by themselves. But they can be used in order to simplify the form of the jet spaces. For example, by adding suitable inner solutions one can arrange that the test jets have vanishing scalar components. Indeed, given a jet $\mathfrak{v} = (b, v) \in \mathfrak{J}^{\text{test}}$ (with $\mathfrak{J}^{\text{test}}$ according to (20.1.5)), taking an indefinite integral of b,

$$B(t) := \int_{-\infty}^t b(\tau)\, \mathrm{d}\tau \;\in\; C^\infty_{\text{b}}(\mathbb{R}) ,$$

the resulting jet $\mathfrak{u} := (-b, -B)$ is an inner solution (20.1.7). Adding this jet to $\mathfrak{v}$ gives

$$\tilde{\mathfrak{v}} := \mathfrak{v} + \mathfrak{u} = (0, v - B) \in \mathfrak{J}^{\text{test}} ,$$

which is physically equivalent to $\mathfrak{v}$ and, as desired, has a vanishing scalar component.

In our example, we can use the inner solutions alternatively in order to eliminate the vector component of the test jets. To this end, it is preferable to choose the space of test jets as

$$\mathfrak{J}^{\text{test}} = C^\infty_{\text{b}}(\mathbb{R}) \oplus C^\infty_{\text{b}}(\mathbb{R}) . \tag{20.1.9}$$

Now the vector component disappears under the transformation

$$\mathfrak{v} = (b, v) \mapsto \tilde{\mathfrak{v}} := \mathfrak{v} + \mathfrak{u} \qquad \text{with} \qquad \mathfrak{u} = (-v', -v) \in \mathfrak{J}^{\text{test}} .$$

Therefore, it remains to consider the scalar components of jets. For technical simplicity, we restrict attention to compactly supported functions. Thus we choose the jet space $\mathfrak{J}^{\text{vary}}$ as

$$\mathfrak{J}^{\text{vary}} = C_0^\infty(\mathbb{R}) \oplus \{0\} .$$

Then the linearized field operator in (8.1.7) reduces to the integral operator with kernel $\mathcal{L}(x, y)$,

$$\big(\Delta(b, 0)\big)(x) = \int_{\mathcal{F}} \mathcal{L}(x, y)\, b(y)\ \mathrm{d}y .$$

20.2. A Minimizing Measure Supported on a Hyperplane

In the previous example, the support of the minimizing measure was the whole space $\mathcal{F}$. In most examples motivated from the physical applications, however, the minimizing measure will be supported on a low-dimensional subset of $\mathcal{F}$ (see for instance the minimizers with singular support for the causal variational principle on the sphere in [**84, 10**] discussed in Section 6.1). We now give a simple example where the minimizing measure is supported on a hyperplane of $\mathcal{F}$. We let $\mathcal{F} = \mathbb{R}^2$ and choose the Lagrangian as

$$\mathcal{L}(x, y; x', y') = \frac{1}{\sqrt{\pi}}\, \mathrm{e}^{-(x-x')^2} \big(1 + y^2\big)\big(1 + y'^2\big) , \tag{20.2.1}$$

where $(x, y), (x', y') \in \mathcal{F}$.

Lemma 20.2.1. *The measure*

$$\mathrm{d}\rho = \mathrm{d}x \times \delta_y \tag{20.2.2}$$

(where δ_y is the Dirac measure) is the unique minimizer of the causal action principle for the Lagrangian (20.2.1) *under variations of finite volume (see* (6.3.2) *and* (6.3.1)*).*

Note that this measure is supported on the x-axis,

$$M := \operatorname{supp} \rho = \mathbb{R} \times \{0\} .$$

Proof of Lemma 20.2.1. Let $\tilde{\rho}$ be a regular Borel measure on $\mathcal{F}$ satisfying (6.3.1). Then the difference of actions (6.3.2) is computed by

$$\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho) = \frac{2}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x, y) \int_N \mathrm{d}x'\, \mathrm{e}^{-(x-x')^2}\, (1 + y^2) \tag{20.2.3}$$

$$+ \frac{1}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x, y) \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x', y')\, \mathrm{e}^{-(x-x')^2}\, \big(1 + y^2\big)\big(1 + y'^2\big) . \tag{20.2.4}$$

Using that the negative part of the measure $\tilde{\rho} - \rho$ is supported on the x-axis, the first term (20.2.3) can be estimated by

$$\begin{aligned} &\frac{2}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x, y) \int_N \mathrm{d}x'\, \mathrm{e}^{-(x-x')^2}\, (1 + y^2) \\ &\overset{(*)}{\geq} \frac{2}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x, y) \int_N \mathrm{d}x'\, \mathrm{e}^{-(x-x')^2} = \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x, y) = 0 , \end{aligned}$$

where in the last step we used the volume constraint. The second term (20.2.4), on the other hand, can be rewritten as

$$\frac{1}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}\mu(x, y) \int_{\mathcal{F}} \mathrm{d}\mu(x', y')\, \mathrm{e}^{-(x-x')^2}$$

with the signed measure ρ defined by

$$\mathrm{d}\mu(x,y) := \big(1+y^2\big)\ \mathrm{d}(\tilde{\rho}-\rho)(x,y)\,.$$

Now we can proceed as in the proof of Lemma 20.1.1 and use that the Fourier transform of the integral kernel is strictly positive. For the uniqueness statement one uses that the inequality in $(*)$ is strict unless $\tilde{\rho}$ is supported on the x-axis. Then one can argue as in the proof of Lemma 20.1.1. □

For the minimizing measure (20.2.2), the function ℓ takes the form

$$\ell(x,y) = \int_{\mathcal{F}} \mathcal{L}(x,y;x',y')\,\mathrm{d}\rho(x',y') - 1 = y^2\,,$$

showing that the EL equations (7.1.2) are indeed satisfied. We now specify the jet spaces. Since the Lagrangian is smooth, it is obvious that

$$\mathfrak{J}^{\text{diff}} = \mathfrak{J} = C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R}, \mathbb{R}^2)\,,$$

where $C^\infty(\mathbb{R}, \mathbb{R}^2)$ should be regarded as the space of two-dimensional vector fields along the x-axis. Similar as explained after (20.1.9), we want to use the inner solutions for simplifying the vector components of the jets. To this end, in analogy to (20.1.9) we choose

$$\mathfrak{J}^{\text{test}} = C^\infty_{\text{b}}(\mathbb{R}) \oplus C^\infty_{\text{b}}(\mathbb{R}, \mathbb{R}^2)\,. \tag{20.2.5}$$

The linearized field equations (8.1.7) read

$$\begin{aligned} 0 &= \nabla_{\mathfrak{u}} \bigg(\int_{-\infty}^\infty \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathrm{e}^{-(x-x')^2} \big(1+y^2\big)\big(1+y'^2\big)\ \mathrm{d}\rho(x',y') - \nabla_{\mathfrak{v}} \sqrt{\pi} \bigg) \Big|_{y=y'=0} \\ &= \nabla_{\mathfrak{u}} \bigg(\big(1+y^2\big) \int_{-\infty}^\infty \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathrm{e}^{-(x-x')^2}\ \mathrm{d}x' - \nabla_{\mathfrak{v}} \sqrt{\pi} \bigg) \Big|_{y=y'=0}\,. \end{aligned}$$

Now the inner solutions are generated by the vector fields tangential to the x-axis. More precisely, in analogy to (20.1.7), we consider the jet

$$\mathfrak{v} = \big(b, (B, 0)\big) \qquad \text{with} \qquad b \in C^\infty_0(\mathbb{R}) \text{ and } B(x) := \int_\infty^x b(t)\ \mathrm{d}t \in C^\infty_{\text{b}}(\mathbb{R})\,.$$

Exactly as in the example of the one-dimensional Gaussian, integrating by parts as in (20.1.8) one sees that the jet $\mathfrak{v}$ indeed satisfies the linearized field equations.

By suitably subtracting inner solutions, we can compensate the tangential components of the jets. This leads us to choose

$$\mathfrak{J}^{\text{vary}} = C^\infty_0(\mathbb{R}) \oplus \big(\{0\} \oplus C^\infty_0(\mathbb{R})\big)\,.$$

Then the Laplacian simplifies as follows,

$$\begin{aligned}
&\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) \\
&= \frac{1}{\sqrt{\pi}}\, \nabla_{\mathfrak{u}} \Big(\int_{-\infty}^{\infty} \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathrm{e}^{-(x-x')^2} \,\big(1+y^2\big)\big(1+y'^2\big)\, \mathrm{d}x' - \nabla_{\mathfrak{v}}\, \sqrt{\pi} \Big) \Big|_{y=y'=0} \\
&= \frac{2}{\sqrt{\pi}}\, u(x)\, v(x) \int_{-\infty}^{\infty} \mathrm{e}^{-(x-x')^2}\, \mathrm{d}x' + \frac{1}{\sqrt{\pi}}\, a(x) \int_{-\infty}^{\infty} \mathrm{e}^{-(x-x')^2}\, b(x')\, \mathrm{d}x' \\
&\quad + a(x) \Big(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} b(x)\, \mathrm{e}^{-(x-x')^2}\, \mathrm{d}x' - b(x) \Big) \\
&= 2\, u(x)\, v(x) + \frac{1}{\sqrt{\pi}}\, a(x) \int_{-\infty}^{\infty} \mathrm{e}^{-(x-x')^2}\, b(x')\, \mathrm{d}x' \,,
\end{aligned}$$

where $\mathfrak{u} = (a, (0, u))$ and $\mathfrak{v} = (b, (0, v))$. Hence the inhomogeneous linearized field equations (8.1.8) with $\mathfrak{w} = (e, w)$ give rise to separate equations for the scalar and vector components,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(x-x')^2}\, b(x')\, \mathrm{d}x' = e(x) \,, \qquad v(x) = \frac{w(x)}{2} \,.$$

20.3. A Non-Homogeneous Minimizing Measure

In the previous examples, the minimizing measure ρ was translation invariant in the direction of the x-axis. We now give a general procedure for constructing examples of causal variational principles where the minimizing measure has no translational symmetry. In order to work in a concrete example, our starting point is again the one-dimensional Gaussian (20.1.1). But our method can be adapted to other kernels in a straightforward way. In view of these generalizations, we begin with the following abstract result.

LEMMA 20.3.1. *Let μ be a measure on the m-dimensional manifold $\mathcal{F}$ whose support is the whole manifold,*

$$\operatorname{supp} \mu = \mathcal{F} \,.$$

Moreover, let $\mathcal{L}(x,y) \in L^1_{\text{loc}}(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0)$ be a symmetric, non-negative kernel on $\mathcal{F} \times \mathcal{F}$. Next, let $h \in C^0(\mathcal{F}, \mathbb{R}^+)$ be a strictly positive, continuous function on $\mathcal{F}$. Assume that:

(i) $\displaystyle\int_{\mathcal{F}} \mathcal{L}(x,y)\, h(y)\, \mathrm{d}\mu(y) = 1$ *for all $x \in \mathcal{F}$.*

(ii) *For all compactly supported bounded functions with zero mean,*

$$g \in L^\infty_0(\mathcal{F}, \mathbb{R}^+) \qquad \text{and} \qquad \int_{\mathcal{F}} g\, \mathrm{d}\mu = 0 \,,$$

the following inequality holds,

$$\int_{\mathcal{F}} \mathrm{d}\mu(x) \int_{\mathcal{F}} \mathrm{d}\mu(y)\, \mathcal{L}(x,y)\, g(x)\, g(y) \geq 0 \,. \tag{20.3.1}$$

Then the measure $\mathrm{d}\rho := h\, \mathrm{d}\mu$ is a minimizer of the causal action principle under variations of finite volume (see (6.3.2) and (6.3.1)). If the inequality (20.3.1) is strict for any non-zero g, then the minimizing measure is unique within the class of such variations.

PROOF. We consider the variation

$$\tilde{\rho}_\tau = \rho + \tau\, g\, \mathrm{d}\mu = (h + \tau g)\, \mathrm{d}\mu \,. \tag{20.3.2}$$

Since h is continuous and strictly positive and g is bounded and compactly supported, the function $h + \tau g$ is non-negative for sufficiently small $|\tau|$. Furthermore, using that g has mean zero, we conclude that (20.3.2) is an admissible variation of finite volume (6.3.1). Moreover, the difference of the actions (6.3.2) is well-defined and computed by

$$\begin{aligned}
&\mathcal{S}(\tilde{\mu}_\tau) - \mathcal{S}(\rho) \\
&= 2\tau \int \mathrm{d}\rho(x)\, g(x) \int_N \mathrm{d}\rho(y)\, h(y)\, \mathcal{L}(x,y) + \tau^2 \int_N \mathrm{d}\rho(x) \int_N \mathrm{d}\rho(x)\, \mathcal{L}(x,y)\, g(x)\, g(y) \\
&\geq 2\tau \int_N g(y)\, \mathrm{d}\rho(y) = 0\,, \qquad (20.3.3)
\end{aligned}$$

where in the second step we used the above assumptions (i) and (ii). The last step follows from the fact that g has mean zero. If the inequality (20.3.1) is strict, so is the inequality in (20.3.3), showing that the minimizer ρ is unique.

We conclude that the measure ρ is a minimizer under variations of the form (20.3.2). In order to treat a general variation of finite volume (6.3.1), we approximate $\tilde{\rho}$ by a sequence of functions g_n with the property that the measures $g_n \rho$ converge to $\tilde{\rho}$ (here one can work with the notion of vague convergence; for details see [**8**, Definition 30.1] or [**31**]). □

Our goal is to apply this lemma to kernels of the form

$$\mathcal{L}(x,y) = f(x)\, \mathrm{e}^{-(x-y)^2} f(y) \qquad (20.3.4)$$

with a strictly positive function f, which for convenience we again choose as a Gaussian,

$$f(x) = \mathrm{e}^{\alpha x^2} \qquad \text{with } \alpha \in \mathbb{R}\,. \qquad (20.3.5)$$

This kernel has the property (ii) because for all non-trivial $g \in L^\infty_0(\mathcal{F}, \mathbb{R}^+)$,

$$\int_{\mathcal{F}} \mathrm{d}\mu(x) \int_{\mathcal{F}} \mathrm{d}\mu(y)\, \mathcal{L}(x,y)\, g(x)\, g(y) = \int_{\mathcal{F}} \mathrm{d}\mu(x) \int_{\mathcal{F}} \mathrm{d}\mu(y)\, \mathrm{e}^{-(x-y)^2}\, (fg)(x)\, (fg)(y) > 0\,,$$

where the last step is proved exactly as in the example of the Gaussian (see (20.1.3)). In order to arrange (i), for h we make an ansatz again with a Gaussian,

$$h(x) = c\, \mathrm{e}^{\beta x^2}\,. \qquad (20.3.6)$$

Then

$$\begin{aligned}
&\int_{\mathcal{F}} \mathcal{L}(x,y)\, h(y)\, \mathrm{d}\mu(y) = c \int_{-\infty}^{\infty} \mathrm{e}^{\alpha x^2}\, \mathrm{e}^{-(x-y)^2}\, \mathrm{e}^{(\alpha+\beta)y^2}\, \mathrm{d}y \\
&= c \exp\Big(\alpha x^2 - x^2 - \frac{x^2}{\alpha+\beta-1}\Big) \int_{-\infty}^{\infty} \exp\Big\{(\alpha+\beta-1)\Big(y - \frac{x}{\alpha+\beta-1}\Big)^2\Big\}\, \mathrm{d}y \\
&= c \sqrt{\frac{\pi}{1-\alpha-\beta}}\, \exp\Big(\alpha x^2 - x^2 - \frac{x^2}{\alpha+\beta-1}\Big)\,.
\end{aligned}$$

In order to arrange that this function is constant one, we choose

$$c = \sqrt{\frac{1-\alpha-\beta}{\pi}} \qquad \text{and} \qquad \beta = -\frac{\alpha(2-\alpha)}{1-\alpha}\,. \qquad (20.3.7)$$

For the above Gaussian integral to converge, we need to ensure that $1 - \alpha - \beta > 0$. In view of the formula

$$1 - \alpha - \beta = \frac{1}{1-\alpha}\,,$$

this can be arranged simply by choosing $\alpha < 1$. Our finding is summarized as follows.

PROPOSITION 20.3.2. *For any $\alpha < 1$, we let f and h be the Gaussians* (20.3.5) *and* (20.3.6) *with c and β according to* (20.3.7). *Then the measure $\mathrm{d}\rho = h\,dx$ is the unique minimizer of the causal action corresponding to the Lagrangian* (20.3.4) *within the class of variations of finite volume.*

As a concrete example, we consider the well-known *Mehler kernel* (see for example [**93**, Section 1.5])

$$E(x,y) = \frac{1}{\sqrt{1-\mu^2}}\, \exp\Big(-\frac{\mu^2(x^2+y^2) - 2\mu xy}{(1-\mu^2)} \Big)$$

with $\mu > 0$. Rescaling x and y according to

$$x, y \to \sqrt{\frac{1-\mu^2}{\mu}}\, x, y\,,$$

the Mehler kernel becomes

$$E(x,y) = \frac{1}{\sqrt{1-\mu^2}}\, \exp\Big(-\mu(x^2+y^2) - 2xy \Big)\,.$$

This kernel is of the desired form (20.3.4) if we choose

$$\alpha = 1-\mu < 1\,, \qquad \beta = \frac{\mu^2-1}{\mu}\,.$$

We finally remark that this non-homogeneous example can be used as the starting point for the construction of higher-dimensional examples with minimizing measures supported on lower-dimensional subsets, exactly as explained for the Gaussian in Section 20.2.

20.4. A Minimizing Measure in Two-Dimensional Minkowski Space

In the previous examples, the Lagrangian was strictly positive (see (20.1.1), (20.2.1), (20.3.4)). Therefore, the causal structure of the resulting spacetime was trivial, because all pairs or points were timelike separated. We now give examples where the minimizing measure gives rise to nontrivial causal relations in spacetime. We let $\mathcal{F} = \mathbb{R}^2$, denote the coordinates by (t,x) and choose the Lagrangian

$$\mathcal{L}\big(t,x;t',x\big) = \mathrm{e}^{-(t-t')^2} \Big(\delta\big((t-t') - (x-x')\big) + \delta\big((t-t') + (x-x')\big) \Big)\,. \tag{20.4.1}$$

The Lagrangian is non-negative, and it is strictly positive on the "light rays" $(t-t') = \pm(x-x')$.

LEMMA 20.4.1. *The Lebesgue measure*

$$\mathrm{d}\rho = \mathrm{d}t\; \mathrm{d}x$$

is a minimizer of the causal action principle for the Lagrangian (20.4.1) *in the class of variations of finite volume (see* (6.3.2) *and* (6.3.1)*). It is the unique minimizer within this class of variations.*

PROOF. Proceeding as in the proof of Lemma 20.1.1, our task is to show that the Fourier transform of the Lagrangian is strictly positive. To this end, we note that

$$\int_{\mathbb{R}^2} \delta(t-x)\, \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kx}\; \mathrm{d}t\; \mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega x - \mathrm{i}kx}\, dx = 2\pi\, \delta(\omega - k)\,.$$

We thus obtain

$$\int_{\mathbb{R}^2} \Big(\delta\big((t-t')-(x-x')\big)+\delta\big((t-t')+(x-x')\big)\Big)\, \mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}kx}\, \mathrm{d}t\, \mathrm{d}x$$
$$= 2\pi \left(\delta(\omega+k)+\delta(\omega-k)\right).$$

Multiplying by the Gaussian in (20.4.1) corresponds to a convolution in momentum space again by a Gaussian. This convolution gives a strictly positive function, as desired. □

The Lagrangian (20.4.1) has the shortcoming that it is supported only on the boundary of the light cone. In order to improve the situation, we next consider the example

$$\begin{aligned}\mathcal{L}\big(t,x;t',x\big) &= \mathrm{e}^{-(t-t')^2}\, \Big(\delta\big((t-t')-(x-x')\big)+\delta\big((t-t')+(x-x')\big)\Big) \\ &\quad + a\, \mathrm{e}^{-\frac{(t-t')^2}{2}}\, \Theta\big((t-t')^2-(x-x')^2\big)\,.\end{aligned} \tag{20.4.2}$$

LEMMA 20.4.2. *Choosing $|\alpha|<1$, the Lebesgue measure*

$$\mathrm{d}\rho = \mathrm{d}t\, \mathrm{d}x$$

is a minimizer of the causal action principle for the Lagrangian (20.4.2) *in the class of variations of finite volume (see* (6.3.2) *and* (6.3.1)*). It is the unique minimizer within this class of variations.*

PROOF. We compute the Fourier transform of the Heaviside function.

$$\begin{aligned}&\int_{\mathbb{R}^2} \Theta\big(t^2-x^2\big)\, \mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}kx}\, \mathrm{e}^{-\varepsilon|t|}\, \mathrm{d}t\, \mathrm{d}x \\ &= 4\int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}t\, \Theta(t-x)\, \cos(\omega t)\, \cos(kx)\, \mathrm{e}^{-\varepsilon t}\, \mathrm{d}t\, \mathrm{d}x \\ &= 2\int_0^\infty \left(-\frac{\mathrm{e}^{\mathrm{i}\omega x-\varepsilon x}}{\mathrm{i}\omega-\varepsilon}-\frac{\mathrm{e}^{-\mathrm{i}\omega x-\varepsilon x}}{-\mathrm{i}\omega-\varepsilon}\right) \cos(kx)\, \mathrm{d}x \\ &= -\frac{1}{\mathrm{i}\omega-\varepsilon}\left(\frac{1}{\mathrm{i}\omega+k-\varepsilon}+\frac{1}{\mathrm{i}\omega-k-\varepsilon}\right)-\frac{1}{-\mathrm{i}\omega-\varepsilon}\left(\frac{1}{-\mathrm{i}\omega+k-\varepsilon}+\frac{1}{-\mathrm{i}\omega-k-\varepsilon}\right).\end{aligned}$$

In the limit $\varepsilon \searrow 0$, this converges to a tempered distribution which is singular on the light cone. Taking the convolution with the Gaussian and choosing a sufficiently small, the resulting function is dominated near the light cone by the Fourier transform computed in the proof of Lemma 20.4.1. Moreover, due to its decay properties at infinity, the same is true away from the light cone. This concludes the proof. □

20.5. A Nonlinear Wave Equation in Two-Dimensional Minkowski Space

In the previous examples the minimizing measures were unique. This means in particular that the systems had no dynamical degrees of freedom, and the linearized field equations only admitted trivial solutions. We now explain how one can build in dynamical degrees of freedom. For simplicity, we consider the example of a nonlinear wave equation on a spacetime lattice, but the method can be generalized to many other situations. We choose $\mathcal{F} = \mathbb{R}^2 \times S^1$ and denote the coordinates by $(t,x) \in \mathbb{R}^2$ and $\mathrm{e}^{\mathrm{i}\alpha} \in S^1$. We choose

$$\begin{aligned}\mathcal{L}\big(t,x,\alpha;t',x',\alpha'\big) &= \mathrm{e}^{-(t-t')^2}\, \mathrm{e}^{-(x-x')^2} + \delta(t-t')\, \delta(x-x')\, (\sin\alpha-\sin\alpha')^2 \\ &\quad + g\big(t-t',x-x'\big)\, \sin\alpha\, \sin\alpha'\,,\end{aligned}$$

where g is the convolution $g = h * h$ with

$$h(t,x) := \delta(t-1)\,\delta(x) + \delta(t+1)\,\delta(x) - \delta(t)\,\delta(x+1) - \delta(t)\,\delta(x-1)$$

(thus h is the kernel of a discretized wave operator). We remark that this Lagrangian violates our usual positivity assumption $\mathcal{L}\big(t,x,\alpha;t',x',\alpha'\big) \geq 0$. However, this inequality could be arranged without changing the qualitative properties of the example by mollifying the δ distributions and adding a constant.

PROPOSITION 20.5.1. *Every minimizing measure ρ has the form*

$$\mathrm{d}\rho(t,x,\alpha) = \mathrm{d}t\;\mathrm{d}x\,\delta\big(\alpha - \phi(t,x)\big)\;\mathrm{d}\alpha\,,$$

where $\phi(t,x)$ solves the nonlinear discrete wave equation

$$\sin\big(\phi(t+1,x)\big) + \sin\big(\phi(t-1,x)\big) - \sin\big(\phi(t,x+1)\big) - \sin\big(\phi(t,x-1)\big) = 0\,.$$

We begin with a preparatory lemma.

LEMMA 20.5.2. *Every minimizing measure has the form*

$$\mathrm{d}\rho(t,x,\alpha) = \mathrm{d}\mu(t,x)\,\delta\big(\alpha - \phi(t,x)\big)\;\mathrm{d}\alpha\,, \tag{20.5.1}$$

with μ the push-forward to the first two variables, i.e.

$$\mu = \pi_*\rho \qquad \text{with} \qquad \pi : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2\,, \quad (t,x,\alpha) \mapsto (t,x)\,,$$

and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a μ-measurable function.

PROOF. Let ρ be a measure on $\mathcal{F}$. We introduce the function $\phi(t,x)$ by

$$\sin\phi(t,x)\,\mathrm{d}\mu(t,x) = \int_0^{2\pi} \sin\alpha\,\mathrm{d}\rho(t,x,\alpha)\,.$$

In words, $\sin\phi(t,x)$ coincides with the mean of $\sin\alpha$ integrated over the circle. The function $\phi(t,x)$ exists because this mean lies in the interval $[-1,1]$ and because the sine takes all values in this interval. Denoting the resulting measure of the form (20.5.1) by $\tilde{\rho}$, we obtain Then

$$\mathcal{S}(\rho) - \mathcal{S}(\tilde{\rho}) = \int_{\mathcal{F}} \mathrm{d}\rho(t,x,\alpha) \int_{\mathcal{F}} \mathrm{d}\rho(t',x',\alpha)\,\delta(t-t')\,\delta(x-x')\,\big(\sin\alpha - \phi(t,x)\big)^2\,.$$

Therefore, ρ is a minimizer if and only if $\rho = \tilde{\rho}$. □

PROOF OF PROPOSITION 20.5.1. For measures of the form (20.5.1), the action takes the form

$$\begin{aligned}\mathcal{S} = &\int_{\mathbb{R}^2} \mathrm{d}\mu(t,x) \int_{\mathbb{R}^2} \mathrm{d}\mu(t',x')\,\mathrm{e}^{-(t-t')^2}\,\mathrm{e}^{-(x-x')^2} \\ &+ \int_{\mathbb{R}^2} \mathrm{d}\mu(t,x) \int_{\mathbb{R}^2} \mathrm{d}\mu(t',x')\,g\big(t-t',x-x'\big)\,\sin\phi(t,x)\,\sin\phi(t',x')\,.\end{aligned}$$

Using that g is a convolution,

$$g\big(t-t',x-x'\big) = \int_{\mathbb{R}^2} h(t-\tau,x-z)\,h(t'-\tau,x'-z)\;\mathrm{d}\tau\;\mathrm{d}z\,,$$

the action can be rewritten as

$$\mathcal{S} = \int_{\mathbb{R}^2} \mathrm{d}\mu(t,x) \int_{\mathbb{R}^2} \mathrm{d}\mu(t',x')\, \mathrm{e}^{-(t-t')^2}\, \mathrm{e}^{-(x-x')^2} \tag{20.5.2}$$

$$+ \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} h(t-\tau, x-z)\, \sin\phi(t,x)\, \mathrm{d}\mu(t,x) \right)^2 \mathrm{d}\tau\; \mathrm{d}z\,. \tag{20.5.3}$$

Exactly as shown in Section 20.1, the minimizer of (20.5.2) is given by the Lebesgue measure. The contribution (20.5.3), on the other hand, is minimal if $\sin\phi(t,x)$ satisfies the discrete wave equation. This concludes the proof. □

20.6. Exercises

EXERCISE 20.1. *(Functions with self-similar Fourier transform)* The example of Lemma 20.1.1 was based on the fact that the Fourier transform of a Gaussian is again Gaussian (20.1.2).

(a) Prove (20.1.2) by direct computation.

(b) Another example of a function which is self-similar under Fourier transforms is the distribution in Minkowski space

$$K_0(p) = \delta(k^2)\, \epsilon(k^0)\,.$$

Show that its Fourier transform indeed give, up to a constant, the same distribution back. *Hint:* The distribution $K_0(p)$ is the analog of the causal fundamental solution (13.6.2) for the scalar wave equation (see also (16.3.8)). Using this fact, one can make use of the explicit form of the causal Green's operators for the scalar wave equation.

(c) Can you think of other functions which are self-similar under the Fourier transform in the above sense? Is there a systematic way to characterize them all?

EXERCISE 20.2. *(Non-negative functions with non-negative Fourier transforms)* Another specific feature of the Gaussian in (20.1.1) which was used in Lemma 20.1.1 is that it is a positive function whose Fourier transform is again positive.

(a) Show that the same is true for the δ distribution. Can you come up with other functions with this property.

(b) The Lagrangian (20.4.1) involves of a function of two variables with the properties that it is non-negative and has a non-negative Fourier transform. How can this idea be used to construct other Lagrangians with the property that the Lebesgue measure is a minimizer?

CHAPTER 21

Basics on the Continuum Limit

In the *continuum limit* one analyzes the EL equations of the causal action principle for systems of Dirac seas in the presence of classical bosonic fields. As worked out in detail in [**45**, Chapters 3-5], this limiting case yields the interactions of the standard model and gravity on the level of second-quantized fermionic fields interacting with classical bosonic fields. In this chapter we explain schematically how the analysis of the continuum limit works and give an overview of the obtained results.

21.1. Causal Fermion Systems in the Presence of External Potentials

In Chapters 15–19 it was explained how to construct and analyze the unregularized kernel of the fermionic projector $\tilde{P}(x,y)$ in Minkowski space in the presence of an external potential $\mathcal{B}$. The general question is whether the causal fermion system corresponding to this kernel satisfies the EL equations corresponding to the causal action principle. Thus we would like to evaluate the EL equations as stated abstractly in Theorem 7.1.1 for $\tilde{P}(x,y)$. The basic procedure is to form the closed chain (see (5.7.4)) and to compute its eigenvalues $\lambda_1^{xy}, \ldots \lambda_{2n}^{xy} \in \mathbb{C}$. This, in turn, makes it possible to compute the causal action and the constraints (see (5.6.1)–(5.6.5)). Considering first variations of $P(x,y)$, one then obtains the EL equations.

The main obstacle before one can carry out this program is that, in order to obtain mathematically well-defined quantities, one needs to introduce an *ultraviolet regularization*. As explained in detail in Chapter 5, this regularization is not merely a technical procedure, but it corresponds to implementing a specific microscopic structure of spacetime. In the vacuum, the regularization was introduced with the help of a regularization operator $\mathfrak{R}_\varepsilon$ (see (5.5.1)). Different choices of regularization operators correspond to different microscopic structures of spacetime. Since the structure of our physical spacetime on the Planck scale is largely unknown, the strategy is to allow for a general class of regularization operators, making it possible to analyze later on how the results depend on the regularization (for more details on this so-called *method of variable regularization* see [**45**, §1.2.1]).

In more detail, we proceed as follows. In the vacuum, we can follow the procedure explained in Section 5, choosing $\mathcal{H}$ as the subspace of all negative-frequency solutions of the Dirac equation. In preparation of extending this construction to the interacting situation, it is useful to note that the causal action principle can be formulated in terms of the kernel of the fermionic projector given abstractly by (5.7.3). Therefore, our task is to compute this kernel. It can be obtained alternatively by starting from unregularized kernel of the fermionic projector constructed in Section 15.4 and introducing a regularization. In the simplest case, working with a regularization which preserves the Dirac equation, i.e.

$$\mathfrak{R}_\varepsilon \,:\, \mathcal{H}_m \to \mathcal{H}_m \cap C^0(\mathscr{M}, S\mathscr{M})\,,$$

the regularization can be introduced similar to (15.4.1) by

$$P^\varepsilon := -\mathfrak{R}_\varepsilon\, \pi_{\mathcal{H}}\, \mathfrak{R}_\varepsilon^*\, k_m \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \to \mathcal{H}_m\,.$$

For more general regularization operators which do *not* preserve the Dirac equation, one can introduce the regularization by modifying the right side of (5.7.14) to

$$P^\varepsilon(x,y) := -\big(\mathfrak{R}_\varepsilon \Psi\big)(x)\, \big(\mathfrak{R}_\varepsilon \Psi\big)(y)^*\,.$$

where $\Psi : \mathcal{H}_m \to L^2_{\text{loc}}(\mathscr{M}, S\mathscr{M})$ is the unregularized wave evaluation operator, and regularization operator $\mathfrak{R}_\varepsilon \;:\; \mathcal{H}_m \to C^0(\mathscr{M}, S\mathscr{M})$ now maps more generally to continuous wave functions (not necessarily Dirac solutions).

The latter construction has the advantage that it also applies in the presence of an external potential. In a perturbative treatment, it gives rise to the causal perturbation expansion developed in Section 18.2. In this way, we obtain the *regularized kernel* $\tilde{P}^\varepsilon(x,y)$ in the presence of an external potential. Following the procedure explained in Chapter 5, we obtain a corresponding causal fermion system. After suitable identifications (as worked out in [**45**, Section 1.2]), this regularized kernel coincides with the kernel of the fermionic projector as defined abstractly in (5.7.3).

The subtle question is whether a chosen regularization of the vacuum also determines the regularization of the kernel $\tilde{P}^\varepsilon(x,y)$ in the presence of an external potential. The general answer to this question is no, simply because the interaction introduces additional freedoms for regularizing. Moreover, it is not clear a-priori whether the regularized objects should still satisfy the Dirac equation. But at least, in [**45**, Appendix F] and [**41**, Appendix D] a canonical procedure is given for *regularizing the light-cone expansion* (see [**73**] for related constructions in curved spacetime). It consists in taking the formulas of the (unregularized) light-cone expansion (like for example (19.2.3)–(19.2.8) in Example 19.2.2) and replacing the singular factors $T^{(n)}$ (like for example (19.2.9)) by corresponding functions where the singularities on the light cone have been regularized on the scale ε. The precise procedure will be explained in the next section.

21.2. The Formalism of the Continuum Limit

We now give a brief summary of the formalism of the continuum limit. More details can be found in [**45**, Section 2.4]. The reader interested in the derivation of this formalism is referred to [**41**, Chapter 4].

Having introduced the regularized kernel of the fermionic projector $\tilde{P}^\varepsilon(x,y)$, we can form the closed chain

$$A^\varepsilon_{xy} := \tilde{P}^\varepsilon(x,y)\, \tilde{P}^\varepsilon(y,x)\,,$$

compute its eigenvalues and proceed by analyzing the EL equations. In the continuum limit, one focuses on the limiting case $\varepsilon \searrow 0$ when the ultraviolet regularization is removed. This limiting case is comparatively easy to analyze. This can be understood from the fact that, in the limit $\varepsilon \searrow 0$, the closed chain A^ε_{xy} becomes singular on the light cone. Therefore, asymptotically for small ε, it suffices to take into account the contributions to A^ε_{xy} on the light cone. These contributions, on the other hand, are captured precisely by the light-cone expansion of the unregularized kernel $\tilde{P}(x,y)$ (see Section 19.2 or the explicit formulas in Example 19.2.2). This is the basic reason why, in the continuum limit, the EL equations can be rewritten as field equations involving fermionic wave functions as well as derivatives of the bosonic potentials.

More specifically, the asymptotics $\varepsilon \searrow 0$ is captured by the *formalism of the continuum limit*, which we now outline (for more details see [**45**, Section 2.4] or the derivation

of the formalism in [**41**, Chapter 4]). In the first step, one regularizes the light-cone expansion symbolically by leaving all smooth contributions unchanged, whereas to the singular factors $T^{(n)}$ we employ the replacement rule

$$m^p\, T^{(n)} \to m^p\, T^{(n)}_{[p]}\,.$$

Thus for the formulas of Example 19.2.2, the factors $T^{(n)}$ get an additional index [0]. If the light-cone expansion involves powers of the rest mass, these powers are taken into account in the lower index. The resulting factors $T^{(n)}_{[p]}$ are smooth functions, making all the subsequent computations well-defined. The detailed form of these functions does not need to be specified, because all we need are the following computation rules. In computations one may treat the factors $T^{(n)}_{[p]}$ as complex functions. However, one must be careful when tensor indices of factors $\not{\xi}$ are contracted with each other. Naively, this gives a factor ξ^2 which vanishes on the light cone and thus changes the singular behavior on the light cone. In order to describe this effect correctly, we first write every summand of the light-cone expansion such that it involves at most one factor $\not{\xi}$ (this can always be arranged using the anti-commutation relations of the Dirac matrices). We now associate every factor $\not{\xi}$ to the corresponding factor $T^{(n)}_{[p]}$. In short calculations, this can be indicated by putting brackets around the two factors, whereas in the general situation we add corresponding indices to the factor $\not{\xi}$, giving rise to the replacement rule

$$m^p\, \not{\xi} T^{(n)} \to m^p\, \not{\xi}^{(n)}_{[p]}\, T^{(n)}_{[p]}\,.$$

For example, we write the regularized fermionic projector of the vacuum as

$$P^\varepsilon = \frac{\mathrm{i}}{2} \sum_{n=0}^\infty \frac{m^{2n}}{n!}\, \not{\xi}^{(-1+n)}_{[2n]}\, T^{(-1+n)}_{[2n]} + \sum_{n=0}^\infty \frac{m^{2n+1}}{n!}\, T^{(n)}_{[2n+1]}\,.$$

The kernel $P(y,x)$ is obtained by taking the conjugate (see (5.7.7)). The conjugates of the factors $T^{(n)}_{[p]}$ and $\xi^{(n)}_{[p]}$ are the complex conjugates,

$$\overline{T^{(n)}_{[p]}} := \big(T^{(n)}_{[p]}\big)^* \qquad \text{and} \qquad \overline{\xi^{(n)}_{[p]}} := \big(\xi^{(n)}_{[p]}\big)^*\,.$$

One must carefully distinguish between the factors with and without complex conjugation. In particular, the factors $\not{\xi}^{(n)}_{[p]}$ need not be symmetric, i.e., in general,

$$\big(\not{\xi}^{(n)}_{[p]}\big)^* \neq \not{\xi}^{(n)}_{[p]}\,.$$

When forming composite expressions, the tensor indices of the factors ξ are contracted to other tensor indices. The factors ξ which are contracted to other factors ξ are called *inner factors*. The contractions of the inner factors are handled with the so-called *contraction rules*

$$(\xi^{(n)}_{[p]})^j\, (\xi^{(n')}_{[p']})_j = \frac{1}{2} \left(z^{(n)}_{[p]} + z^{(n')}_{[p']} \right) \tag{21.2.1}$$

$$(\xi^{(n)}_{[p]})^j\, \overline{(\xi^{(n')}_{[p']})_j} = \frac{1}{2} \left(z^{(n)}_{[p]} + \overline{z^{(n')}_{[p']}} \right) \tag{21.2.2}$$

$$z^{(n)}_{[p]}\, T^{(n)}_{[p]} = -4 \left(n\, T^{(n+1)}_{[p]} + T^{(n+2)}_{\{p\}} \right), \tag{21.2.3}$$

which are to be complemented by the complex conjugates of these equations. Here the factors $z_{[p]}^{(n)}$ can be regarded simply as a book-keeping device to ensure the correct application of the rule (21.2.3). The factors $T_{\{p\}}^{(n)}$ have the same scaling behavior as the $T_{[p]}^{(n)}$, but their detailed form is somewhat different; we simply treat them as a new class of symbols. In cases where the lower index does not need to be specified we write $T_\circ^{(n)}$. After applying the contraction rules, all inner factors ξ have disappeared. The remaining so-called *outer factors* ξ need no special attention and are treated like smooth functions.

Next, to any factor $T_\circ^{(n)}$ we associate the *degree* $\deg T_\circ^{(n)}$ by

$$\deg T_\circ^{(n)} = 1 - n \,.$$

The degree is additive in products, whereas the degree of a quotient is defined as the difference of the degrees of numerator and denominator. The degree of an expression can be thought of as describing the order of its singularity on the light cone, in the sense that a larger degree corresponds to a stronger singularity (for example, the contraction rule (21.2.3) increments n and thus decrements the degree, in agreement with the naive observation that the function $z = \xi^2$ vanishes on the light cone). Using formal Taylor series, we can expand in the degree. In all our applications, this will give rise to terms of the form

$$\eta(x,y)\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \qquad \text{with } \eta(x,y) \text{ smooth} \,. \tag{21.2.4}$$

The quotient of the two monomials in this equation is referred to as a *simple fraction*.

A simple fraction can be given a quantitative meaning by considering one-dimensional integrals along curves which cross the light cone transversely away from the origin $\xi = 0$. This procedure is called *weak evaluation on the light cone*. For our purpose, it suffices to integrate over the time coordinate $t = \xi^0$ for fixed $\vec{\xi} \neq 0$. Moreover, using the symmetry under reflections $\xi \to -\xi$, it suffices to consider the upper light cone $t \approx |\vec{\xi}|$. The resulting integrals diverge if the regularization is removed. The leading contribution for small ε can be written as

$$\int_{|\vec{\xi}|-\varepsilon}^{|\vec{\xi}|+\varepsilon} \mathrm{d}t\; \eta(t,\vec{\xi})\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \;\approx\; \eta(|\vec{\xi}|,\vec{\xi})\, \frac{c_{\text{reg}}}{(\mathrm{i}|\vec{\xi}|)^L}\, \frac{\log^r(\varepsilon|\vec{\xi}|)}{\varepsilon^{L-1}} \,, \tag{21.2.5}$$

where L is the degree of the simple fraction and c_{reg}, the so-called *regularization parameter*, is a real-valued function of the spatial direction $\vec{\xi}/|\vec{\xi}|$ which also depends on the simple fraction and on the regularization details (the error of the approximation will be specified below). The integer r describes a possible logarithmic divergence. Apart from this logarithmic divergence, the scalings in (21.2.5) in both ξ and ε are described by the degree.

When analyzing a sum of expressions of the form (21.2.4), one must know if the corresponding regularization parameters are related to each other. In this respect, the *integration-by-parts rules* are important, which are described symbolically as follows. On the factors $T_\circ^{(n)}$ we introduce a derivation ∇ by

$$\nabla T_\circ^{(n)} = T_\circ^{(n-1)} \,.$$

Extending this derivation with the product and quotient rules to simple fractions, the integration-by-parts rules state that

$$\nabla \left(\frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \right) = 0 \,. \tag{21.2.6}$$

Carrying out the derivative with the product rule, one gets relations between simple fractions. Simple fractions which are not related to each other by the integration-by-parts rules are called *basic fractions*. As shown in [**41**, Appendix E], there are no further relations between the basic fractions. Thus the corresponding *basic regularization parameters* are independent.

The above symbolic computation rules give a convenient procedure to evaluate composite expressions in the fermionic projector, referred to as the *analysis in the continuum limit*: After applying the contraction rules and expanding in the degree, the EL equations can be rewritten as equations involving a finite number of terms of the form (21.2.4). By applying the integration-by-parts rules, we can arrange that all simple fractions are basic fractions. We evaluate weakly on the light cone (21.2.5) and collect the terms according to their scaling in ξ. Taking for every given scaling in ξ only the leading pole in ε, we obtain equations which involve linear combinations of smooth functions and basic regularization parameters. We consider the basic regularization parameters as empirical parameters describing the unknown microscopic structure of spacetime. We thus end up with equations involving smooth functions and a finite number of free parameters.

We finally specify the error of the above expansions. By not regularizing the bosonic potentials and fermionic wave functions, we clearly disregard the

$$\text{higher orders in } \varepsilon/\ell_{\text{macro}} \,. \tag{21.2.7}$$

Furthermore, in (21.2.5) we must stay away from the origin, meaning that we neglect the

$$\text{higher orders in } \varepsilon/|\vec{\xi}| \,. \tag{21.2.8}$$

The higher order corrections in $\varepsilon/|\vec{\xi}|$ depend on the fine structure of the regularization and thus seem unknown for principal reasons. Neglecting the terms in (21.2.7) and (21.2.8) also justifies the formal Taylor expansion in the degree. Clearly, leaving out the terms (21.2.8) is justified only if $|\vec{\xi}| \gg \varepsilon$. Therefore, whenever using the above formalism, we must always ensure that $|\vec{\xi}|$ is much larger than ε.

We finally remark that, when working out the Einstein equations, one must go beyond error terms of the form (21.2.7) and (21.2.8). The reason is that the gravitational scales like $\kappa \sim \delta^2 \approx \varepsilon^2$. In order not to loose the relevant terms in the error terms, one must take certain higher order contributions into account. This is done by using the so-called ι-formalism. Here we do not enter the details but merely refer the interested reader to [**45**, §4.2.7].

21.3. Overview of Results of the Continuum Limit Analysis

The formalism of the continuum limit makes it possible to evaluate the EL equations of the causal action for the regularized kernel $\tilde{P}^\varepsilon(x,y)$ in the presence of an external potential $\mathcal{B}$. In order to avoid confusion, we point out that, a-priori, the external potential can be chosen arbitrarily; in particular, it does need to satisfy any field equations. We find that the EL equations of the causal action are satisfied in the continuum limit if and only if the potential $\mathcal{B}$ has a specific structure and satisfies dynamical equations. Restricting

attention to potentials of this form and complementing the Dirac equation (1.3.14) by the dynamical equations for $\mathcal{B}$, the potentials are no longer given as external potentials, but instead one gets a coupled system of equations describing a mutual interaction of the Dirac wave functions with classical bosonic fields. The dynamical equations for $\mathcal{B}$ are referred to as the *classical field equations*. In this way, the classical field equations are *derived* from the causal action principle.

We now outline the main results of the continuum limit analysis as obtained in [**45**, Chapters 3-5]. The main input is to specify the regularized kernel $P^\varepsilon(x,y))$ of the *vacuum*. This involves:

- The fermion configuration in the vacuum, including the masses of the leptons and quarks. Moreover, it is built in that the neutrinos break the chiral symmetry.
- The vacuum kernel should satisfy the EL equations. This poses a few constraints on the regularization operator.

The output of the continuum limit are the following results:

- The structure of the interaction on the level of classical gauge theory.
- The gauge groups and their coupling to the fermions.
- The equations of linearized gravity.

In [**45**] the continuum limit is worked out in three steps for systems of increasing complexity. In Chapter 3, a system formed of a sum of three Dirac seas is considered. This configuration, referred to as a *sector*, can be thought of as a simplified model describing the three generations of charged leptons (e, μ, τ). In the continuum limit, we obtain the following results for the interaction as described by the causal action principle:

- The fermions interact via an *axial gauge field*.
- This axial gauge field is massive, with the mass determined by the masses of the fermions and the regularization.
- We find that the field equations for the axial gauge field arise in the continuum limit only if the number of generations equals three. For one or two generations, the resulting equations are overdetermined, whereas for more than three generations, the equations are under-determined (which means in particular that there is no well-posed Cauchy problem).
- We obtain nonlocal corrections to the classical field equations described by integral kernels which decay on the Compton scale. It seems that these nonlocal corrections capture certain features of the underlying quantum field theory. But the detailed connection has not been worked out.
- There is no gravitational field and no Higgs field.

In Chapter 4, a system formed as a direct sum of two sectors is considered. This system is referred to as a *block*. The first sector looks as in Chapter 3. In the second sector, however, the chiral symmetry is broken. This system can be regarded as a model for the leptons, including the three generations of neutrinos. In the continuum limit, we obtain the following results for the interaction as described by the causal action principle:

- The fermions interact via an SU(2) gauge field, which couples only to one chirality (say, the left-handed fermions).
- The corresponding gauge field is again massive.
- Moreover, the fermions interact linearly via the linearized Einstein equations, where the coupling constant is related to the regularization length.

Finally, in Chapter 5 a realistic system involving leptons and quarks is considered. To this end, one considers a direct sum of eight sectors, one of which with broken chiral symmetry (the neutrino sector). These eight sectors form pairs, referred to as blocks. The block containing the neutrino sector describes the leptons, whereas the other three blocks describe the quarks. Moreover, we obtain the following results:

- The fermions interact via the gauge group $\mathrm{U}(1) \times \mathrm{SU}(2)_L \times \mathrm{SU}(3)$. The corresponding gauge fields couple to the fermions as in the standard model. The SU(2)-field couples only to the left-handed component and is massive. The other gauge fields are massless.
- Moreover, the fermions interact linearly via the linearized Einstein equations.
- The EL equations corresponding to the causal action principle coincide with those of the standard model after spontaneous symmetry breaking, plus linearized gravity.
- There are scalar degrees of freedom which can be identified with the Higgs potential. However, the corresponding dynamical equations have not yet been worked out.
- Again, the fermions interact linearly via the linearized Einstein equations, where the coupling constant is related to the regularization length. Taking into account that the causal action principle is diffeomorphism invariant, we obtain the Einstein equations, up to possible higher order corrections in curvature (which scale in powers of $(\delta^2 \,\mathrm{Riem})$, where δ is the Planck length and Riem is the curvature tensor). Thus, including error terms, the derived Einstein equations take the form (see [**45**, Theorems 4.9.3 and 5.4.4])

$$R_{jk} - \frac{1}{2}\, R\, g_{jk} + \Lambda\, g_{jk} = G\, T_{jk} + \mathcal{O}\big(\delta^4\, \mathrm{Riem}^2\big)$$

(where T_{jk} is the energy-momentum tensor and G is the gravitational coupling constant).

We conclude this section by discussing a few aspects of the derivation of these results. We begin with the system of one sector as considered in [**45**, Chapter 3]. In this case, the kernel of the fermionic projector is the sum of $g \in \mathbb{N}$ Dirac seas of masses $m_1, \ldots, m_g$, i.e.

$$P(x,y) = \sum_{\beta=1}^{g} P_{m_\alpha}(x,y)\,, \tag{21.3.1}$$

where again

$$P_m(x,y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, (\not{k} + m)\, \delta\big(k^2 - m^2\big)\, \Theta(-k^0)\, \mathrm{e}^{-\mathrm{i}k(x-y)}\,.$$

In order to perturb the system by gauge potentials, we first introduce the kernel of the *auxiliary fermionic projector* $P^{\mathrm{aux}}(x,y)$, which is obtained from $P(x,y)$ by replacing the sums by direct sums,

$$P^{\mathrm{aux}}(x,y) = \bigoplus_{\beta=1}^{g} P_{m_\alpha}(x,y)$$

(this means that $P^{\mathrm{aux}}(x,y)$ is represented by a $(4g \times 4g)$-matrix). The auxiliary kernel satisfies the Dirac equation

$$\Big(\mathrm{i}\partial\!\!\!/_x - \begin{pmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_g \end{pmatrix}\Big) P^{\mathrm{aux}}(x,y) = 0\,.$$

Therefore, it can be perturbed as usual by inserting a potential $\mathcal{B}$ into the Dirac equation

$$\Big(i\partial\!\!\!/_x + \mathcal{B}(x) - \begin{pmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_g \end{pmatrix}\Big) \tilde{P}^{\text{aux}}(x,y) = 0$$

(where $\mathcal{B}(x)$ is a matrix potential acting on $\mathbb{C}^{4g}$). The perturbed kernel $\tilde{P}^{\text{aux}}$ can be computed with the methods explained in Chapters 18 and 19. Finally, we obtain the perturbed kernel of the fermionic projector by summing over the generation indices in an operation referred to as the *sectorial projection*,

$$\tilde{P}(x,y) := \sum_{\alpha,\beta=1}^{g} \tilde{P}^{\alpha}_{\beta}(x,y)\,.$$

After introducing an ultraviolet regularization, this kernel can be analyzed in the EL equations of the causal action principle, exactly as outlined in Section 21.2 above.

In order to gain the largest possible freedom in perturbing the system, the operator $\mathcal{B}$ should be chosen as general as possible. For this reason, in [**45**, Chapter 4] a general class of potential was considered, including nonlocal potentials (i.e. integral operators). A general conclusion of the analysis is that, in order to satisfy the EL equations, the potential $\mathcal{B}$ must be local, i.e. a differential operator or a multiplication operator by a potential which may involve left- and right-handed potentials, but also bilinear, scalar or pseudo-scalar potentials,

$$\mathcal{B}(x) = \chi_L A\!\!\!/_R(x) + \chi_R A\!\!\!/_L(x) + \sigma^{ij}\,\Lambda_{ij}(x) + \Phi(x) + \mathrm{i}\Gamma\,\Xi(x) \tag{21.3.2}$$

(where each of the potentials is a $g \times g$-matrix acting on the generations, and Γ is the pseudo-scalar matrix, which in physics textbooks is often denoted by γ^5). Analyzing the continuum limit for such multiplication operators, one gets the above-mentioned results.

One feature which at first sight might be surprising is that, despite local gauge symmetry, we get *massive gauge fields*. In order to understand how this comes about, we need to consider local gauge symmetries in connection with the chiral gauge potentials in (21.3.2). On the fundamental level of the causal fermion system, *local gauge transformations* arise from the freedom in choosing bases of the spin spaces (see (5.9.1) and (5.9.2) in Section 5.9). In the present setting with four-component Dirac spinors, the local gauge transformations take the form

$$\psi(x) \to U(x)\,\psi(x) \qquad \text{with} \qquad U(x) \in \mathrm{U}(2,2)\,, \tag{21.3.3}$$

where $\mathrm{U}(2,2)$ is the group of unitary transformations of the spinors at the spacetime point x. The causal action principle is gauge invariant in the sense that the causal action is invariant under such gauge transformations. The group $\mathrm{U}(2,2)$ can be used to describe gravity as a gauge theory (for details see Section 4.2 or [**38**]). Restricting attention to flat spacetime, the main interest is that $\mathrm{U}(2,2)$ contains the gauge group $\mathrm{U}(1)$ of electrodynamics as a subgroup. In other words, the causal action principle is gauge invariant under local phase transformations

$$\psi(x) \to \mathrm{e}^{-\mathrm{i}\Lambda(x)}\,\psi(x)$$

with a real-valued function Λ.

The chiral potentials in (21.3.2) also give rise to generalized phase transformations. This can be seen for example by working out the leading term to the light-cone expansion (similar to (19.2.3) for the electromagnetic potential). One finds that the chiral gauge

potentials lead to phase transformations of the left- and right-handed components of the wave functions, i.e.

$$\psi(x) \to U(x)\, \psi(x) \qquad \text{with} \qquad U(x) := \chi_L\, \mathrm{e}^{-\mathrm{i}\Lambda_L(x)} + \chi_R\, \mathrm{e}^{-\mathrm{i}\Lambda_R(x)} \tag{21.3.4}$$

(again with real-valued functions Λ_L and Λ_R). The point is that this transformation is *not unitary* with respect to the spin inner product, because the chirality flips when taking the adjoint

$$U^* = \chi_R\, \mathrm{e}^{\mathrm{i}\Lambda_L(x)} + \chi_L\, \mathrm{e}^{-\mathrm{i}\Lambda_R(x)} \quad \text{but} \quad U^{-1} = \chi_L\, \mathrm{e}^{\mathrm{i}\Lambda_L(x)} + \chi_R\, \mathrm{e}^{-\mathrm{i}\Lambda_R(x)}$$

(note that $\chi_L^* = (1-\Gamma)^*/2 = (1+\Gamma)/2 = \chi_R$ because $\Gamma^* = -\Gamma$). Therefore, as soon as $A_L \neq A_R$, the generalized phase transformation $U(x)$ in (21.3.4) is *not* a local transformation of the form (21.3.3). Consequently, the local transformation in (21.3.4) does *not* correspond to a symmetry of the causal action principle. Therefore, it is not a contradiction if these gauge potentials arise in the effective field equations as mass terms.

More specifically, the relative phases between left- and right-handed potentials do come up in the closed chain $A_{xy} = P(x,y)P(y,x)$, as one sees immediately from the fact that, if $P(x,y)$ is vectorial, then the chirality flips at the corresponding factor, i.e.

$$\begin{aligned} \chi_L A_{xy} &= \chi_L P(x,y)\, \chi_R P(y,x) \\ &\to \exp\Big(-\mathrm{i}\big(\Lambda_L(x) - \Lambda_R(x)\big)\Big) \exp\Big(\mathrm{i}\big(\Lambda_L(y) - \Lambda_R(y)\big)\Big)\, A_{xy}\,. \end{aligned} \tag{21.3.5}$$

Working out the corresponding contribution to the EL equations in the continuum limit, one finds that the axial current and a corresponding axial mass term come up in the effective field equations. The coupling constant and the bosonic mass depend on the detailed form of the regularization. But they can be computed for specific choices of the regularization, as is exemplified in [**45**, Chapter 3] for a hard cutoff in momentum space and the $i\varepsilon$-regularization.

We now move on to the system of two sectors as analyzed in [**45**, Chapter 4]. The vacuum is described by a kernel of the fermionic projector $P(x,y)$ being a direct sum of two summands, each of which is of the form (21.3.1), where we choose the number of generations as $g = 3$. Hence $P(x,y)$ is a 8×8-matrix. Replacing the sums by direct sums, one obtains the corresponding auxiliary kernel $P^{\text{aux}}(x,y)$ (being represented by a 24×24-matrix). In order to account for the observational fact that neutrinos are left-handed particles, one must *break the chiral symmetry* of one of the sectors (the *neutrino sector*). To this end, we assume that the regularization of the neutrino sectors is different from that of the other sector (the *charged sector*) by contributions which are not left-right invariant. The relevant length scale is denoted by $\delta \gtrsim \varepsilon$. This procedure is very general and seems the right thing to do, because the regularization effects on the scale δ are also needed in order to obtain the correct form of the curvature term in the Einstein equations. In fact, the obtained linearized Einstein equations involve the coupling constant $G \sim \delta^2$. As briefly mentioned at the end of Section 10.2, the derivation of the Einstein equations uses the ι-formalism, which goes beyond the standard formalism of the continuum limit.

The system analyzed in [**45**, Chapter 5] is obtained similarly by adding direct summands to $P(x,y)$ describing the three generations of quarks. We begin with eight sectors. These eight sectors form pairs, giving rise to four blocks. We conclude by outlining how this mechanism of *spontaneous block formation* comes about. For this purpose, we return to the gauge phases as already mentioned in (21.3.3) and (21.3.4). We already saw in (21.3.5) that, if the kernel of the fermionic projector is vectorial, then the relative

phases (i.e. the difference of left- and right-handed gauge phases) show up in the eigenvalues of the closed chain. Such phase factors drop out of the causal Lagrangian because of the absolute values in (5.6.1). However, the situation becomes more involved if the kernel of the fermionic projector is *not* vectorial. Indeed, expanding the vacuum kernel in powers of the rest mass, the zero order contribution to $P(x,y)$ is vectorial, whereas the first order contribution is scalar (more generally, one sees from (19.1.1) that the even orders in the mass are vectorial, whereas the odd orders are scalar). As a consequence, the absolute values of the eigenvalues $|\lambda_i^{xy}|$ depend in a rather complicated way on the chiral gauge phases. Moreover, considering a direct sum of Dirac seas, one must keep into account that the gauge phases in the above formulas must be replaced by generalized phases which can be described in terms of ordered exponentials of the gauge potentials. Evaluating the causal Lagrangian (5.6.1), one gets conditions for the chiral gauge phases. In simple terms, these conditions can be stated by demanding that matrices formed of ordered exponentials of the gauge potentials must have degeneracies. Qualitatively speaking these degeneracies mean that the left-handed gauge potential must be the same in each block, and this condition even makes it possible to explain why such blocks form. A more detailed and more precise explanation can be found in [**45**, Chapter 5].

21.4. Exercises

EXERCISE 21.1. This exercise explains in a simple example how the *regularization of the Hadamard expansion* works.

(a) Consider the singular term of the first summand of the Hadamard expansion (19.1.1) in Minkowski space,

$$\lim_{\nu \searrow 0} \frac{1}{\xi^2 - \mathrm{i}\nu\, \xi^0} \tag{21.4.1}$$

(where again $\xi := y - x$). A simple method to remove the pole is not to take the limit $\nu \searrow 0$, but instead to set $\nu = 2\varepsilon$,

$$\frac{1}{\xi^2 - 2\mathrm{i}\varepsilon\, \xi^0}\,. \tag{21.4.2}$$

Show that this regularization can be realized by the replacement

$$\xi^0 \to \xi^0 - \mathrm{i}\varepsilon\,,$$

up to a multiplicative error of the order

$$\left(1 + \mathcal{O}\Big(\frac{\varepsilon^2}{\xi^2}\Big)\right). \tag{21.4.3}$$

The basic concept behind the regularized Hadamard expansion is to regularize all singular terms in this way, leaving all smooth functions unchanged. This gives a consistent formalism is one works throughout with error terms of the form (21.4.3). *Hint:* This is the so-called iε-regularization introduced in [**45**, Section 2.4]. For details in curved spacetime see [**73**].

(b) Show that for kernels written as Fourier transforms

$$K(x,y) = \int_M \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \hat{K}(p)\, \mathrm{e}^{-\mathrm{i}p(y-x)}$$

(with $\hat{K}$ supported in say the lower half plane $\{p^0 < 0\}$), the replacement rule (21.4.2) amounts to inserting a convergence-generating factor $\mathrm{e}^{\varepsilon p^0}$ into the integrand.

EXERCISE 21.2. The goal of this exercise is to explore *weak evaluation on the light cone* in a simple example.

(a) Show that, setting $t = \xi^0$ and choosing polar coordinates with $r = |\vec{\xi}|$, regularizing the pole in (21.4.1) according to (21.4.2) gives the function

$$\frac{1}{(t - \mathrm{i}\varepsilon)^2 - r^2} .$$

(b) As a simple example of a composite expression, we take the absolute square of the regularized function

$$\frac{1}{\left|(t - \mathrm{i}\varepsilon)^2 - r^2\right|^2} . \tag{21.4.4}$$

Show that this expression is ill-defined in the limit $\varepsilon \searrow 0$ even as a distribution.

(c) Use the identity

$$\frac{1}{(t - \mathrm{i}\varepsilon)^2 - r^2} = \frac{1}{(t - \mathrm{i}\varepsilon - r)(t - i\varepsilon + r)} = \frac{1}{2r} \left(\frac{1}{t - \mathrm{i}\varepsilon - r} - \frac{1}{t - \mathrm{i}\varepsilon + r} \right)$$

to rewrite the integrand in (21.4.4) in the form

$$\sum_{p,q=0}^{1} \frac{\eta_{p,q}(t, r, \varepsilon)}{(t - \mathrm{i}\varepsilon - r)^p \, (t + \mathrm{i}\varepsilon - r)^q} ,$$

with functions $\eta_{p,q}(t, r, \varepsilon)$ which in the limit $\varepsilon \searrow 0$ converge to smooth functions. Compute the functions $\eta_{p,q}$.

(d) We now compute the leading contributions and specify what we mean by "leading." First compute the following integrals with residues:

$$I_0(\varepsilon) := \int_{-\infty}^{\infty} \frac{1}{(t - \mathrm{i}\varepsilon - r) \, (t + \mathrm{i}\varepsilon - r)} \, \mathrm{d}t .$$

Show that

$$\int_{-\infty}^{\infty} \frac{\eta_{1,1}(t, r)}{(t - \mathrm{i}\varepsilon - r)^2 \, (t + \mathrm{i}\varepsilon - r)^2} \, \mathrm{d}t = I_0(\varepsilon) \, \eta_{2,2}(r, r) + \mathcal{O}(\varepsilon) .$$

Explain in which sense this formula is a special case of the weak evaluation formula (21.2.5).

CHAPTER 22

Connection to Quantum Field Theory

In this chapter we give an outlook on how to get a connection between the causal action principle and the dynamics of quantum fields. Since this direction of research is very recent and partly work in progress, we do not enter any details but instead try to explain a few basic concepts and ideas. Our presentation is based on the recent research papers [**61, 62, 65, 23**]. Partly, our methods were already explored in the alternative approach in [**44**], which is more closely tied to the analysis of the continuum limit (as outlined in Chapter 21).

22.1. Convex Combinations of Measures and Dephasing Effects

Before beginning, we point out that in most examples of causal fermion systems considered in this book, the measure ρ was the push-forward of the volume measure on Minkowski space or a Lorentzian manifold. Thus we first constructed a local correlation map (see (5.5.4))

$$F^\varepsilon \,:\, \mathscr{M} \to \mathcal{F}\,,$$

and then introduced the measure ρ on $\mathcal{F}$ by (see (5.5.5))

$$\rho = (F^\varepsilon)_* \mu_{\mathscr{M}}\,, \tag{22.1.1}$$

where $\mu_{\mathscr{M}}$ is the four-dimensional volume measure on $\mathscr{M}$. In all these examples, the measure ρ had the special property that it was supported on a smooth four-dimensional subset of $\mathcal{F}$ given by (for details see Exercise 8.1)

$$M := \operatorname{supp} \rho = \overline{F^\varepsilon(\mathscr{M})}\,.$$

Also when varying the measure in the derivation of the linearized field equations or in the study of interacting systems in the continuum limit, we always restricted attention to measures having this property (see (8.1.3) in Section 8.1 or Chapter 21). While this procedure seems a good starting point for the analysis of the causal action principle and gives good approximate solutions of the EL equations, we cannot expect that true minimizers are of this particular form.

With this in mind, our strategy is to allow for more general measures on $\mathcal{F}$ and to analyze the causal action principle for these general measures. As we will see, this analysis gives rise to close connections to quantum field theory. We proceed step by step and begin by explaining a construction which explains why going beyond push-forward measures of the form (22.1.1) makes it possible to further decrease the causal action. In other words, the following argument shows that minimizers of the causal action will *not* have the form of a push-forward measures (22.1.1), but will have a more complicated structure. This argument is given in more detail in [**45**, §1.5.3]. Assume that we are

given L measures $\rho_1, \ldots, \rho_L$ on $\mathcal{F}$. Then their *convex combination* $\tilde{\rho}$ given by

$$\tilde{\rho} := \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \rho_{\mathfrak{a}} \tag{22.1.2}$$

is again a positive measure on $\mathcal{F}$. Moreover, if the $\rho_{\mathfrak{a}}$ satisfy the linear constraints (i.e. the volume constraint (5.6.3) and the trace constraint (5.6.4)), then these constraints are again respected by $\tilde{\rho}$.

Next, we let ρ be a minimizing measure (describing for example the vacuum). Choosing unitary transformations $U_1, \ldots, U_L$, we introduce the measures $\rho_{\mathfrak{a}}$ in (22.1.2) as

$$\rho_{\mathfrak{a}}(\Omega) := \rho\big(U^{-1} \Omega U\big) .$$

Thus, in words, the measures $\rho_{\mathfrak{a}}$ are obtained from ρ by taking the unitary transformations by $U_{\mathfrak{a}}$. Since the causal action and the constraints are unitarily invariant, each of the measures $\rho_{\mathfrak{a}}$ is again minimizing. Let us compute the action of the convex combination (22.1.2). First, by (5.6.2),

$$\mathcal{S}(\tilde{\rho}) = \frac{1}{L^2} \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho_{\mathfrak{a}}(x)\, \mathrm{d}\rho_{\mathfrak{b}}(y) .$$

If $\mathfrak{a} = \mathfrak{b}$, we obtain the action of the measure $\rho_{\mathfrak{a}}$ which, due to unitary invariance, is equal to the action of ρ. We thus obtain

$$\mathcal{S}(\tilde{\rho}) = \frac{\mathcal{S}(\rho)}{L} + \frac{1}{L^2} \sum_{\mathfrak{a} \neq \mathfrak{b}} \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho_{\mathfrak{a}}(x)\, \mathrm{d}\rho_{\mathfrak{b}}(y) . \tag{22.1.3}$$

Let us consider the contributions for $\mathfrak{a} \neq \mathfrak{b}$ in more detail. In order to simplify the explanations, it is convenient to assume that the measures $\rho_{\mathfrak{a}}$ have mutually disjoint supports (this can typically be arranged by a suitable choice of the unitary transformations $U_{\mathfrak{a}}$). Then the spacetime $\tilde{M} := \operatorname{supp} \tilde{\rho}$ can be decomposed into L "sub-spacetimes" $M_{\mathfrak{a}} := \operatorname{supp} \rho_{\mathfrak{a}}$,

$$\tilde{M} = M_1 \cup \cdots \cup M_L \qquad \text{and} \qquad M_{\mathfrak{a}} \cap M_{\mathfrak{b}} = \varnothing \quad \text{if } \mathfrak{a} \neq \mathfrak{b} .$$

The Lagrangian of the last summand in (22.1.3) is computed from the fermionic projector $P_{\mathfrak{a},\mathfrak{b}}(x,y)$, where $x \in M_{\mathfrak{a}}$ and $y \in M_{\mathfrak{b}}$ are in different sub-spacetimes. Similar to (5.7.10), it can be expressed in terms of the physical wave functions by (for details see [**45**, Lemma 1.5.2])

$$P_{\mathfrak{a},\mathfrak{b}}(x,y) = -\sum_{i,j} |\psi^{e_i}(x) \succ (U_{\mathfrak{a}}\, U_{\mathfrak{b}}^*)^i_j \prec \psi^{e_j}(y)| . \tag{22.1.4}$$

The point is that this fermionic projector involves the operator product $U_{\mathfrak{a}} U_{\mathfrak{b}}^*$. By choosing the unitary operators $U_{\mathfrak{a}}$ and $U_{\mathfrak{b}}$ suitably, one can arrange that this operator product involves many phase factors. Moreover, one can arrange that, when carrying out the sums in (22.1.4), these phases cancel each other due to destructive interference. In this way, the kernel $P(x,y)$ can be made small if x and y lie in different sub-spacetimes. As a consequence, the last summand in (22.1.3) can be arranged to be very small. Taking into account the factor $1/N$ in the first summand in (22.1.3), also the causal action of $\tilde{\rho}$ becomes small. Clearly, this argument applies only if the number L of sub-spacetimes is not too large, because otherwise it becomes more and more difficult to arrange destructive interference for all summands of the sum in (22.1.3) (estimating the optimal number L of subsystems is a difficult problem which we do not enter here). Also, we

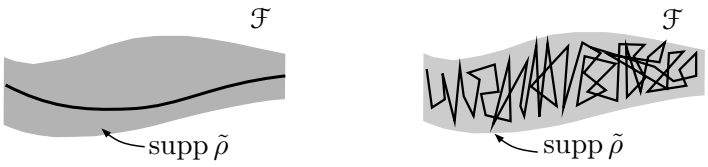


FIGURE 22.1. A measure obtained by fragmentation (left) and by holographic mixing with fluctuations (right).

cannot expect that the simple ansatz (22.1.2) will already give a minimizer. But at least, the above argument explains why it is too naive to think of a minimizing measure as being the push-forward measure of a volume measure under a smooth local correlation map. Instead, a minimizing measure could be composed of a large number of sub-spacetimes.

In the above consideration, it is crucial that the kernels $P_{\mathfrak{a},\mathfrak{b}}(x,y)$ for $\mathfrak{a} \neq \mathfrak{b}$ are very small due to decoherence effects. It is a subtle point how small these kernels are. If they are so small that we may assume that they vanishes, then this means that the sub-spacetimes do not interact with each other. Therefore, one can take the point of view that, in order to describe all physical phenomena, it suffices to restrict attention to one sub-spacetime. The appearance of many sub-spacetimes which are completely decoherent is an intriguing mathematical effect which may have interesting philosophical implications, but it is of no relevance as far as physical predictions are concerned. For this reason, here we shall not discuss these decoherent sub-spacetimes further. Also, we leave the question open whether they really occur for minimizing measures. Instead, we take the point of view that, in case our minimizing measure consists of several decoherent sub-spacetimes, we restrict it to one sub-spacetime and denote the resulting measure by ρ.

In order to understand the dynamics of a causal fermion system, it is more interesting to consider convex combinations of measures which are *not* completely decoherent. In order to explain the resulting effects in a simple example, suppose we choose electromagnetic potentials $A_1, \ldots, A_L$ in Minkowski space (which do not need to satisfy Maxwell's equations). Constructing the regularized kernels $P^\varepsilon_\mathfrak{a}(x,y)$ (as explained in Chapters 18 and 21), one gets corresponding causal fermion systems described by measures $\rho_\mathfrak{a}$. Abstractly, these measures can be written similar as explained in the context of the linearized field equations (see (8.1.9) in Section 8.1) as

$$\tilde{\rho} = \sum_{\mathfrak{a}=1}^{L} (F_\mathfrak{a})_* \big(f_\mathfrak{a}\, \rho\big)\,, \tag{22.1.5}$$

where $F_\mathfrak{a}$ is the corresponding local correlation map, and $f_\mathfrak{a}$ is a weight function. Since these measure are obtained from each other by small perturbations, it seems a good idea to depict the corresponding supports $M_\mathfrak{a} := \operatorname{supp} \rho_\mathfrak{a}$ as being close together (see Figure 8.1 (b)). The convex combination of these measures (22.1.2) is referred to as a measure with *fragmentation* (see [**49**, Sections 1 and 5] or [**51**, Section 5]). The reason why we consider convex combinations (rather than general linear combinations) is that we need to preserve the positivity of the measure. In the limit when N gets large, the fragmented measure $\tilde{\rho}$ goes over to a measure with enlarged support (see Figure 8.1 (c)). Integrating with respect to this measure also involves an integration over the "internal degrees of freedom" corresponding to the directions which are transverse to $M := \operatorname{supp} \rho$ (see the left of Figure 22.1). This integral with respect to $\tilde{\rho}$ bears similarity to the path

integral formulation of quantum field theory if one identifies the above "internal degrees of freedom" with field configurations.

22.2. The Mechanism of Holographic Mixing

For the mathematical description of the interacting measure $\tilde{\rho}$, working with fragmented measures as introduced in the previous section does not seem to be the best method. One difficulty is that it is a-priori not clear how large the number L of fragments is to be chosen. Moreover, mechanisms where L changes dynamically are difficult to implement, at least perturbatively. For these reasons, it seems preferable to describe $\tilde{\rho}$ with a different method, referred to as *holographic mixing*. At first sight, this method seems very different from fragmentation. However, as we will explain at the very end of this section, fragmentation also allows for the description of fragmented measures, at least if the construction is carried out in sufficiently large generality. We now explain the general idea and a few related constructions.

Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system (for example describing the Minkowski vacuum). The *wave evaluation operator* Ψ introduced in (5.7.11) is a mapping which to every vector in $\mathcal{H}$ associates the corresponding physical wave function (for more details see for example [**45**, §1.1.4])

$$\Psi \;:\; \mathcal{H} \to C^0(M, SM)\,, \qquad u \mapsto \psi^u\,,$$

where the physical wave function ψ^u is again given by (5.7.9). Evaluating at a fixed spacetime point gives the mapping

$$\Psi(x) \;:\; \mathcal{H} \to S_x M\,, \qquad u \mapsto \psi^u(x)\,.$$

Working with the wave evaluation operator makes it possible to write the kernel of the fermionic projector (5.7.10) in the short form (for a detailed proof see [**45**, Lemma 1.1.3])

$$P(x,y) = -\Psi(x)\,\Psi(y)^*\,.$$

The general procedure of holographic mixing is to replace the wave evaluation operator by a linear combination of wave evaluation operators $\Psi_{\mathfrak{a}}$,

$$\tilde{\Psi} := \sum_{\mathfrak{a}=1}^{L} \tilde{\Psi}_{\mathfrak{a}}\,, \tag{22.2.1}$$

which in turn are all obtained by perturbing Ψ (more details see below). Now we form the corresponding local correlation map,

$$\tilde{F} \;:\; M \to \mathcal{F}\,, \qquad \tilde{F}(x) := -\tilde{\Psi}(x)^*\,\tilde{\Psi}(x)\,,$$

and take the corresponding push-forward measure,

$$\tilde{\rho} := \tilde{F}_* \rho\,. \tag{22.2.2}$$

In this way, we have constructed a new measure ρ which incorporates the perturbations described all the wave evaluation operators $\tilde{\Psi}_1, \ldots, \tilde{\Psi}_L$. However, in contrast to the convex combination of measures (22.1.5), the support of the measure (22.2.2) in general does not decompose into several fragments. In fact, if the mapping $\tilde{F}$ is continuous, injective and closed, the support of $\tilde{\rho}$ will again be homeomorphic to M. In other words, the topological structure of spacetime remains unchanged by the above procedure.

More concretely, the perturbed wave evaluation operators $\Psi_{\mathfrak{a}}$ can be obtained as follows. Suppose that the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ was constructed similar as

in Section 5.5 from a system of Dirac wave functions satisfying for example the Dirac equation

$$\big(\mathcal{D} - m\big)\psi = 0\,.$$

Then one can perturb the system by considering the Dirac equation in the presence of classical potentials $\mathcal{B}_1, \ldots, \mathcal{B}_L$,

$$\big(\mathcal{D} + \mathcal{B}_\mathfrak{a} - m\big)\tilde{\psi}_\mathfrak{a} = 0\,. \tag{22.2.3}$$

The corresponding wave evaluation operators $\tilde{\Psi}_\mathfrak{a}$ are built up of all these Dirac solutions. In this way, the resulting wave evaluation operator (22.2.1) involves all the classical potentials $\mathcal{B}_\mathfrak{a}$. Qualitatively speaking, the resulting spacetime $\tilde{M}$ can be thought of as being in a "superposition" of all these potentials. But this analogy does not carry over to a more technical level.

As already mentioned after (22.2.2), taking the push-forward with respect to a mapping F does not change the topological structure of spacetime. Even more, if F is smooth and varies only on macroscopic scales, then all microscopic structures of spacetime remain unchanged. This does not account for the picture of a measure $\tilde{\rho}$ which accounts for additional "internal degrees of freedom" as shown in Figure 8.1 (c) and the left of Figure 22.1. In order to allow the description of such measures, one needs to consider mappings F which are *not smooth* but instead "fluctuate" on a microscopic scale (as is shown symbolically on the right of Figure 22.1). If we allow for such fluctuations even on the Planck scale, then the procedure (22.2.1) does allow for the description of all measures described previously with fragmentation (22.1.5). This considerations explains why the wave evaluation operators $\tilde{\Psi}_\mathfrak{a}$ should be constructed not only by introducing classical potentials (22.2.3), but in addition by introducing small-scale fluctuations. This can be realized as follows. We choose operators $A_\mathfrak{a}$ on $\mathcal{H}$ which add up to the identity,

$$\sum_{\mathfrak{a}=1}^{N} A_\mathfrak{a} = \mathbb{1}\,,$$

and then decompose the local correlation operator by multiplying from the right by $A_\mathfrak{a}$,

$$\Psi_\mathfrak{a} := \Psi A_\mathfrak{a}\,. \tag{22.2.4}$$

In the second step, the physical wave functions in $\Psi_\mathfrak{a}$ are perturbed by classical potentials $A_\mathfrak{a}$, again by considering the Dirac equation (22.2.3). In the last step, we again take the sum of the wave evaluation operators (22.2.1) and form the push-forward measure (22.2.2). This procedure is referred to as *holographic mixing*.

The resulting wave evaluation operator $\tilde{\Psi}$ involves both the operators $A_\mathfrak{a}$ and the potentials $\mathcal{B}_\mathfrak{a}$. Similar as explained in (22.1.4) in the context of fragmentation, the operators $A_\mathfrak{a}$ enter the kernel of the fermionic projector,

$$P(x,y) = -\sum_{\mathfrak{a},\mathfrak{b}=1}^{N} |\psi^{e_i}(x)\succ (A_\mathfrak{a}\, A_\mathfrak{b}^*)^i_j \prec \psi^{e_j}(y)|\,.$$

In this way, one can build phase factors into this kernel, possibly giving rise to destructive interference. In other words, the wave evaluation operator $\tilde{\Psi}$ is a sum of many, partly decoherent components. The name "holographic mixing" is inspired by the similarity to a hologram in which several pictures are stored, each of which becomes visible only when looking at the hologram in the corresponding coherent light.

The above ideas and constructions are implemented in the recent paper [**23**] in an enhanced way. The main improvement compared to the above description is to build in *current conservation*. Indeed, forming the wave evaluation as sum of terms (22.2.1), each being a solution of a different Dirac equation (22.2.3) has the disadvantage that the conservation of the Dirac current (which holds for each wave function ψ_a) no longer holds for the sum. This is not satisfying, also because we know from our general setup that, even in the setting of general quantum spacetimes, there should be a conserved inner product (namely the commutator inner product introduced in Section 9.4). In order to resolve this shortcoming, it is preferable to work with a single Dirac equation of the form

$$\big(\mathcal{D} + \mathcal{B} - m\big)\tilde{\Psi} = 0\,.$$

This is indeed possible if the operator $\mathcal{B}$ is chosen as an integral operator with integral kernel of the form

$$\mathcal{B}(x,y) = \sum_{\mathfrak{a}=1}^{N} \mathcal{B}_{\mathfrak{a}}\Big(\frac{x+y}{2}\Big)\, L_{\mathfrak{a}}(y-x)\,, \tag{22.2.5}$$

where $\mathcal{B}_{\mathfrak{a}}$ are again classical potentials and $L_{\mathfrak{a}}$ are certain symmetric kernels. In this description, there is a conserved current and a corresponding conserved inner product on the Dirac solutions which has a similar structure as the commutator inner product (9.4.8). We refer the interested reader for detailed explanations to [**23**]. We finally remark that the nonlocal operator $\mathcal{B}$ of the form (22.2.5) composed of many potentials $\mathcal{B}_{\mathfrak{a}}$ was also derived in [**52**] by a thorough analysis of the linearized field equations for causal fermion systems describing Minkowski space.

22.3. A Distinguished Quantum State

The constructions outlined in the previous sections make it possible to construct general measures $\tilde{\rho}$ which go beyond measures describing a classical spacetime with classical bosonic fields. The EL equations for these measures can be understood as equations describing the dynamics in these generalized spacetimes. With this in mind, the remaining question is how to interpret the resulting measure $\tilde{\rho}$. Can it be understood in terms of an interaction via quantum fields? Or, in more physical terms, what does the measure $\tilde{\rho}$ tell us about measurements performed in the corresponding spacetime? In order to address these questions in a systematic way, in [**62**] a *distinguished quantum state* is constructed. It describes how the interacting measure $\tilde{\rho}$ looks like if measurements are performed at a given time using the objects of a causal fermion describing the vacuum. This "measurement" can also be understood more generally as a "comparison" of the measures $\tilde{\rho}$ and ρ at time t. In technical terms, the quantum state, denoted by ω^t, is a positive linear functional on the algebra of fields $\mathscr{A}$ of the non-interacting spacetime,

$$\omega : \mathscr{A} \to \mathbb{C} \qquad \text{with} \qquad \omega(A^* A) \geq 0 \quad \text{for all } A \in \mathscr{A}\,. \tag{22.3.1}$$

Here we use the language of algebraic quantum field theory (as introduced for example in the textbooks [**4, 19, 132**]) which seems most suitable for describing quantum fields in the needed generality. This notion of quantum state is illustrated in Exercise 22.1.

We now outline the construction of the quantum state as given in [**62**]. We are given two causal fermion systems $(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\rho})$ and $(\mathcal{H}, \mathcal{F}, \rho)$ describing the interacting system and the vacuum, respectively. Our goal is to "compare" these causal fermion systems at a given time. In order to specify the time, we choose sets $\tilde{\Omega} \subset \tilde{M} := \operatorname{supp} \tilde{\rho}$ and $\Omega \subset M := \operatorname{supp} \rho$ which can be thought of as the past of this time in the respective spacetime. We want to relate the two causal fermions systems with the help of the nonlinear surface layer

integral (9.6.1) introduced in Section 9.6. However, we need to take into account that the causal fermion systems are defined on two different Hilbert spaces $\tilde{\mathcal{H}}$ and $\mathcal{H}$. Therefore, in order to make sense of the nonlinear surface layer integral, we need to identify the Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ by a unitary transformation denoted by V,

$$V : \mathcal{H} \to \tilde{\mathcal{H}} \qquad \text{unitary} .$$

Then the operators in $\tilde{F}$ can be identified with operators in $\mathcal{F}$ by the unitary transformation,

$$\mathcal{F} = V^{-1}\, \tilde{\mathcal{F}}\, V .$$

An important point to keep in mind is that this identification is not canonical, but it leaves the freedom to transform the operator V according to

$$V \to V \mathcal{U} \qquad \text{with} \qquad \mathcal{U} \in \mathrm{L}(\mathcal{H}) \text{ unitary} . \tag{22.3.2}$$

The freedom in choosing $\mathcal{U}$ must be taken into account in the nonlinear surface layer integral, which now takes the form

$$\begin{aligned} &\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \mathcal{U}\rho) \\ &= \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x, \mathcal{U} y \mathcal{U}^{-1}) - \int_{\Omega} \mathrm{d}\rho(x) \int_{\tilde{M} \setminus \tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(\mathcal{U} y \mathcal{U}^{-1}, y) . \end{aligned}$$

The method for dealing with the freedom in choosing $\mathcal{U}$ is to integrate over the unitary group. Moreover, it is preferable to consider the exponential of the nonlinear surface layer integral. This leads us to introduce the *partition function* $Z^{\tilde{\Omega},\Omega}$ by

$$Z^{\tilde{\Omega},\Omega}(\beta, \tilde{\rho}) = \fint_{\mathcal{G}} \exp\Big(\beta\, \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \mathcal{U}\rho)\Big)\, \mathrm{d}\mu_{\mathcal{G}}(\mathcal{U}) , \tag{22.3.3}$$

where $\mu_{\mathcal{G}}$ is the normalized Haar measure on the unitary group (in order for this Haar measure to be well-defined, one needs to assume that the Hilbert space $\mathcal{H}$ is finite-dimensional, or else one must exhaust $\mathcal{H}$ by finite-dimensional subspaces).

In analogy to the path integral formulation of quantum field theory, the quantum state is obtained by introducing insertions into the integrand of the partition function, i.e. symbolically,

$$\omega(\cdots) = \frac{1}{Z^{\tilde{\Omega},\Omega}(\beta, \tilde{\rho})} \fint_{\mathcal{G}} (\cdots) \exp\Big(\beta\, \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \mathcal{U}\rho)\Big)\, \mathrm{d}\mu_{\mathcal{G}}(\mathcal{U}) . \tag{22.3.4}$$

These insertions have the structure of surface layer integrals involving linearized solutions in the vacuum spacetime. Likewise, the argument of the state on the left side is formed of operators which are parametrized by the same linearized solutions which enter the insertions on the right side. More precisely, they are operators of the field algebra $\mathscr{A}$, being defined as the $*$-algebra generated by the linearized solutions, subject to the canonical commutation and anti-commutation relations. The commutation relations involve the causal fundamental solution of the linearized solutions which can be constructed with energy methods as outlined in Section 14 (for details see [**22**]). Likewise, for the anti-commutation relations, we use the causal fundamental solutions of the dynamical wave equation mentioned at the end of Section 9.4 in (9.4.7) (for more details see [**64**]). The positivity property of the state is ensured by the specific form of the insertions. We refer the interested reader to [**62**]. We remark that, as is worked out in [**65**], the above quantum state allows for the description of general entanglement. Moreover, the dynamics of the quantum state is studied in [**23**].

We finally note that the definition of the partition function (22.3.3) and of the insertions in the definition of the state (22.3.4) bears a similarity with the path integral formulation of quantum theory (see for example [**110, 93**]). However, this similarity does not seem to go beyond formal analogies. In particular, one should keep in mind that, in contrast to the integral over field configurations in the path integral formulation, in (22.3.3) one integrates over the unitary transformations arising from the freedom in identifying the Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ (see (22.3.2)). This is a major conceptual difference which, at least at present, prevents us from getting a tighter connection to path integrals and the functional integral approach.

22.4. Exercises

EXERCISE 22.1. The purpose of this exercise is to get familiar with the notion of a *quantum state* as defined by (22.3.1). In quantum mechanics, the system is usually described by a unit vector ψ in a Hilbert space $(\mathcal{H}, \langle .|. \rangle)$. An observable corresponds to a symmetric operator $A \in \mathrm{L}(\mathcal{H})$ on this Hilbert space (for simplicity, we here restrict attention to bounded operators). The expectation value of a measurement is given by the expectation $\langle \psi | A | \psi \rangle$.

(a) Show that the linear operator $W \in \mathrm{L}(\mathcal{H})$ defined by

$$W\phi = \langle \phi | \psi \rangle \, \psi \qquad \text{or, in bra/ket notation,} \qquad W = |\psi\rangle\langle\psi| \tag{22.4.1}$$

is a projection operator (i.e. it is symmetric and idempotent). Show that the expectation value of a measurement can be written as

$$\langle \psi | A | \psi \rangle = \mathrm{tr}_{\mathcal{H}} \left(WA \right) .$$

(b) Show that the mapping

$$\omega \;:\; A \mapsto \mathrm{tr}_{\mathcal{H}} \left(WA \right) \tag{22.4.2}$$

is a quantum state in the sense (22.3.1) (here for the algebra $\mathscr{A}$ we take the *algebra of observables*, i.e. the set of all operators obtained from all observables by taking products and linear combinations).

(c) Let ψ_1 and ψ_2 be two distinct, non-zero vectors of $\mathcal{H}$. Show that, choosing

$$W := |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| \,, \tag{22.4.3}$$

the mapping (22.4.2) again defines a quantum state in the sense (22.3.1). Show that this quantum state cannot be written in the form (22.4.1). One refers to (22.4.1) as a *pure state*, whereas (22.4.3) is a *mixed state*.

(d) Is the quantum state in (c) properly normalized in the sense that $\omega(\mathbb{1}) = 1$? If not, how can this normalization be arranged?

APPENDIX A

The Spin Coefficients

In this appendix, we verify by explicit computation that the matrices E_j containing the spin coefficients as given by (4.2.26),

$$E_j = \frac{\mathrm{i}}{2}\,\Gamma\,\partial_j\Gamma - \frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right) G_m G_n + \frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma G_j\,\nabla_m G^m\right)\Gamma\,, \tag{A.0.1}$$

have the following behavior under gauge transformations:

$$E_j \to U E_j U^{-1} \qquad \text{for U(1) gauge transformations} \tag{A.0.2}$$

$$E_j \to U E_j U^{-1} + \mathrm{i} U(\partial_j U^{-1}) \qquad \text{for SU(2,2) gauge transformations} \tag{A.0.3}$$

Under $U(1)$ gauge transformations, all the terms in (A.0.1) remain unchanged because U and its partial derivatives commute with Γ as well as with the G^j. Therefore, the relation (A.0.2) is obvious. Thus it remains to consider $\mathrm{SU}(2,2)$ gauge transformations. Our goal is to verify (A.0.3) for a fixed spacetime point p.

We decompose the gauge transformation U as $U = U_2\,U_1$ with

$$U_1(x) = U(p)$$
$$U_2(x) = U(x)\,U^{-1}(p)\,.$$

Being constant, the first transformation clearly satisfies the transformation law (A.0.3). Therefore, it suffices to consider a gauge transformation U with $U(p) = \mathbb{1}$. Then (A.0.3) can be written as

$$\tilde{E}_j = E_j - \mathrm{i}\partial_j U\,. \tag{A.0.4}$$

We now compute the transformation law of each of the summands in (A.0.1) after each other:

(1) $\dfrac{\mathrm{i}}{2}\,\Gamma\,\partial_j\Gamma$

(i) odd transformations:

$$\begin{aligned}\frac{\mathrm{i}}{2}\,\tilde{\Gamma}\,\partial_j\tilde{\Gamma} &= \frac{\mathrm{i}}{2}\,\Gamma\left(\partial_j\Gamma + [\partial_j U, \Gamma]\right)\\ &= \frac{\mathrm{i}}{2}\,\Gamma\,\partial_j\Gamma + \frac{\mathrm{i}}{2}\,\Gamma\Big(\left(\partial_j U\right)\Gamma - \Gamma\left(\partial_j U\right)\Big)\\ &= \frac{\mathrm{i}}{2}\,\Gamma\,\partial_j\Gamma - i\Gamma\Gamma\,\partial_j U\\ &= \frac{\mathrm{i}}{2}\,\Gamma\,\partial_j\Gamma - \mathrm{i}\partial_j U\end{aligned}$$

(ii) even transformations:

$$\frac{\mathrm{i}}{2}\,\tilde{\Gamma}\,\partial_j\tilde{\Gamma} = \frac{\mathrm{i}}{2}\,\Gamma\Big(\partial_j\Gamma + \big[\partial_j U, \Gamma\big]\Big) = \frac{\mathrm{i}}{2}\,\Gamma\,\partial_j\Gamma$$

Thus for odd transformations, we get the correct transformation law, whereas for even transformations the desired term $\mathrm{i}\partial_j U$ is still missing.

(2) $-\frac{\mathrm{i}}{16}\ \mathrm{Tr}(G^m\,\nabla_j G^n)\,G_m G_n$

(i) odd transformations:

$$
\begin{aligned}
&-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(\tilde{G}^m\,\nabla_j\tilde{G}^n\right)\tilde{G}_m\tilde{G}_n\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\left[\partial_j U,G^n\right]\right)G_m G_n \qquad \text{(A.0.5)}\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n\,, \qquad \text{(A.0.6)}
\end{aligned}
$$

where we used that G^m, G^n and $\partial_j U$ are odd, implying that the trace in the last summand in (A.0.5) vanishes.

(ii) $\partial_j U=\mathrm{i}\sigma_{kl}$ for for arbitrary indices k,l:

$$
\begin{aligned}
&-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(\tilde{G}^m\,\nabla_j\tilde{G}^n\right)\tilde{G}_m\tilde{G}_n\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\left[i\sigma_{kl},G^n\right]\right)G_m G_n\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(\left[G^n,G^m\right]i\sigma_{kl}\right)G_m G_n\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n-\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\sigma^{mn}\,i\sigma_{kl}\right)\sigma_{mn}\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n-\mathrm{i}\partial_j U
\end{aligned}
$$

(iii) $\partial_j U=\Gamma$:

$$
\begin{aligned}
&-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(\tilde{G}^m\,\nabla_j\tilde{G}^n\right)\tilde{G}_m\tilde{G}_n\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\left[\Gamma,G^n\right]\right)G_m G_n\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(\Gamma\left[G^n,G^m\right]\right)G_m G_n\\
&\quad=-\frac{\mathrm{i}}{16}\,\mathrm{Tr}\left(G^m\,\nabla_j G^n\right)G_m G_n\,,
\end{aligned}
$$

because $\mathrm{Tr}\,(\Gamma\sigma^{mn})=0$ for all m,n.

Thus we get the correct transformation law for bilinear transformations $\partial_j U=\mathrm{i}\sigma_{kl}$.

(3) $\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma\,G_j\,\nabla_m G^m\right)\Gamma$

$$
\begin{aligned}
\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\tilde{\Gamma}\,\tilde{G}_j\,\nabla_m\tilde{G}^m\right)\tilde{\Gamma}&=\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma\,G_j\,\nabla_m G^m\right)\Gamma+\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma\,G_j\left[\partial_m U,G^m\right]\right)\Gamma\\
&=\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma\,G_j\,\nabla_m G^m\right)\Gamma+\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\partial_m U\left[G^m,\Gamma G_j\right]\right)\Gamma\\
&=\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma\,G_j\,\nabla_m G^m\right)\Gamma-\frac{\mathrm{i}}{4}\,\mathrm{Tr}\left(\partial_m U\,\Gamma\,\delta^m_j\right)\Gamma\\
&=\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma\,G_j\,\nabla_m G^m\right)\Gamma-\frac{\mathrm{i}}{4}\,\mathrm{Tr}\left(\left(\partial_j U\right)\Gamma\right)\Gamma\,,
\end{aligned}
$$

where we used the relations

$$\left[G^j,\Gamma G^k\right]=-\Gamma\left\{G^j,G^k\right\}$$

as well as the anti-commutation relations for Dirac matrices. We again distinguish different cases:

(i) $\partial_j U$ is odd or $\partial_j U = i\sigma_{kl}$:

$$\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\tilde{\Gamma}\,\tilde{G}_j\,\nabla_m\tilde{G}^m\right)\tilde{\Gamma} = \frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma\,G_j\,\nabla_m G^m\right)\Gamma$$

(ii) $\partial_j U = \Gamma$:

$$\frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\tilde{\Gamma}\,\tilde{G}_j\,\nabla_m\tilde{G}^m\right)\tilde{\Gamma} = \frac{\mathrm{i}}{8}\,\mathrm{Tr}\left(\Gamma\,G_j\,\nabla_m G^m\right)\Gamma - \mathrm{i}\partial_j U$$

Hence we get the correct transformation law if $\partial_j U = \Gamma$.

Adding all the terms gives the desired transformation law (A.0.4).

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About this book

This textbook introduces the basic concepts of the theory of causal fermion systems, a recent approach to the description of fundamental physics. The theory yields quantum mechanics, general relativity and quantum field theory as limiting cases and is therefore a candidate for a unified physical theory. From the mathematical perspective, causal fermion systems provide a general framework for describing and analyzing non-smooth geometries and "quantum geometries". The dynamics is described by a novel variational principle, the causal action principle.

The book includes a detailed summary of the mathematical and physical preliminaries. It explains the physical concepts behind the causal fermion system approach from the basics. Moreover, all the mathematical objects and structures are introduced step by step. The mathematical methods used for the analysis of causal fermion systems and the causal action principle are introduced in depth. Many examples and applications are worked out.

The textbook is addressed to master and graduate students in mathematics or physics. Furthermore, it serves as a reference work for researchers working in the field.

About the authors

Felix Finster studied physics and mathematics at the University of Heidelberg, where he graduated in 1992 with Claus Gerhardt and Franz Wegner. In 1992-1995 he wrote his PhD thesis at ETH Zürich with Konrad Osterwalder. In 1996-1998 he was a postdoc with Shing-Tung Yau at Harvard University. From 1998-2002 he was member of the Max Planck Institute for Mathematics in the Sciences in Leipzig in the group of Eberhard Zeidler. Since 2002 he has been full professor of mathematics at the University of Regensburg. He works on problems in general relativity and quantum field theory.

Sebastian Kindermann studied physics and mathematics at the University of Regensburg, where he graduated in 2020. He is teacher at the Comenius Gymnasium in Deggendorf.

Jan-Hendrik Treude studied physics and mathematics at the University of Freiburg and the University of Vienna. He wrote his PhD thesis at the University of Regensburg, graduating in 2015. Currently, he belongs to the permanent academic staff and is department manager (Fachbereichsreferent) at the Department of Mathematics and Statistics of the University of Konstanz.