An Introduction to the Fermionic Projector and Causal Fermion Systems

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Comments are welcome!
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Preface

This book is intended as an easily accessible introduction to the theory of causal fermion systems. After giving the necessary physical and mathematical preliminaries (Part 1), the theory of causal fermion systems is introduced (Part 2). We proceed by providing mathematical methods which can be regarded as a toolbox for analyzing causal fermion systems (Part 3). We conclude with an outlook on the applications (Part 4).

In order to address as large an audience as possible, the book contains extensive preliminaries which cover both physical and mathematical aspects. We have two prototypical audiences in mind when writing these preliminaries: Physicists who only had a basic course in mathematics and mathematicians who only had a basic course in physics.

The book is based on two main resources: First, the lecture notes of the spring school “Relativistic Fermion Systems” held in Regensburg in April 2013, adapted for the spring school “Causal Fermion Systems” held in Regensburg in March 2016. Second, the lecture “Causal Variational Principles” given at the University of Regensburg in the summer semester 2017.

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Part 1

Physical and Mathematical Preliminaries
CHAPTER 1

Physical Preliminaries

In this chapter we recall the necessary physical background. Clearly, our presentation cannot replace the standard physics textbooks, which will be cited along the way. We also fix the conventions and introduce the notation which will be used consistently throughout this book.

1.1. The Schrödinger Equation

A quantum mechanical particle is described by its wave function \( \psi(t, \vec{x}) \), where \( t \in \mathbb{R} \) is time and \( \vec{x} \in \mathbb{R}^3 \) is the position vector. We begin with nonrelativistic quantum mechanics without spin. In this case, the wave function \( \psi \) is complex-valued. Its absolute square \( |\psi(t, \vec{x})|^2 \) has the interpretation as the probability density of the particle to be located at the position \( \vec{x} \). For this interpretation to be sensible, the integral over the probability density must be equal to one,

\[
\int_{\mathbb{R}^3} |\psi(t, \vec{x})|^2 \, d^3x = 1. \tag{1.1.1}
\]

This equation must hold for all times. This entails that the dynamical equations must preserve the integral \( (1.1.1) \).

A basic ingredient to quantum mechanics is the superposition principle. It states that for any wave functions \( \psi \) and \( \phi \), also their complex linear combination

\[
\tilde{\psi} = \alpha \psi + \beta \phi \quad \text{with} \; \alpha, \beta \in \mathbb{C} \tag{1.1.2}
\]

(defined by \( \tilde{\psi}(t, \vec{x}) = \alpha \psi(t, \vec{x}) + \beta \phi(t, \vec{x}) \)) is a physically admissible wave function. In more mathematical terms, the physical wave functions form a complex vector space. Evaluating \( (1.1.1) \) for the wave function \( \tilde{\psi} \) and using that the probability integral must be preserved in time for all \( \alpha \) and \( \beta \), one concludes that the integral

\[
\int_{\mathbb{R}^3} \overline{\phi(t, \vec{x})} \psi(t, \vec{x}) \, d^3x \tag{1.1.3}
\]

must be time independent for any wave functions \( \psi \) and \( \phi \). The procedure to deduce \( (1.1.3) \) from \( (1.1.1) \) is sometimes referred to as polarization. The integral \( (1.1.3) \) defines a scalar product on the wave functions, which we denote by

\[
\langle \phi | \psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^3} \overline{\phi(t, \vec{x})} \psi(t, \vec{x}) \, d^3x. \tag{1.1.4}
\]

The complex vector space of wave functions endowed with this scalar product forms the Hilbert space \( L^2(\mathbb{R}^3, \mathbb{C}) \) of square-integrable functions, which we also denote by \( (\mathcal{H}, \langle ., . \rangle_{\mathcal{H}}) \). (For basics on Hilbert spaces see Section 2.3.2 below).

The dynamics of the wave function is described by a linear evolution equation on \( \mathcal{H} \), the Schrödinger equation, which we write as

\[
i \partial_t \psi = H \psi, \tag{1.1.5}
\]
where \( H \), the so-called Hamiltonian, is a linear operator acting on the Hilbert space \( \mathcal{H} \). The Schrödinger equation is linear, and this is essential in order to ensure that the evolution is compatible with the superposition principle. The requirement that the scalar (1.1.4) must be time independent implies that

\[
0 = \partial_t \langle \phi | \psi \rangle_{\mathcal{H}} = -i \left( \langle H \phi | \psi \rangle_{\mathcal{H}} - \langle \phi | H \psi \rangle_{\mathcal{H}} \right) \tag{1.1.6}
\]

for all wave functions \( \psi, \phi \). In other words, Hamiltonian must be a selfadjoint operator on the Hilbert space \( \mathcal{H} \) (mathematical issues like domains and the distinction between symmetric and selfadjoint operators are postponed to Section 2.3.9 below).

In the simplest setting without spin, the Hamiltonian is chosen as

\[
H = -\frac{1}{2m} \Delta + V,
\]

where we have set \( \hbar = c = 1 \) (we will do so throughout this book). Here, \( \Delta \) is the Laplacian on \( \mathbb{R}^3 \), and \( V(t, \vec{x}) \) is a real-valued potential. The parameter \( m > 0 \) is the rest mass of the particle.

The Schrödinger equation can be analyzed with various methods. If the Hamiltonian is time independent, the Schrödinger equation can be solved by exponentiating (for details on the spectral theorem see again Section 2.3.9 below),

\[
\psi(t) = e^{-iHt} \psi(0).
\]

Then the dynamics of \( \psi \) can then be related to spectral properties of the Hamiltonian. Another method, which has the advantage that it also applies if the Hamiltonian depends on time, is to analyze the Schrödinger equation as a parabolic partial differential equation. Since our focus are the relativistic equations, we do not explain these methods here, but refer to the textbooks [\( \text{[F]} \)].

### 1.2. Special Relativity and Minkowski Space

We now give a brief introduction to special relativity and the Dirac equation. We closely follow the presentation in [\( \text{[28]} \) §1.1 and §1.2]. For more details and the physical background we refer for example to the textbooks [\( \text{[11, 78, 84]} \)].

In special relativity, space-time is described by Minkowski space \((\mathcal{M}, \langle \cdot, \cdot \rangle)\), a real four-dimensional vector space endowed with an inner product \( \langle \cdot, \cdot \rangle \) of signature '(+−−−)'. Choosing a pseudo-orthonormal basis \((e_i)_{i=0,...,3}\) and representing the vectors of \( \mathcal{M} \) in this basis, \( \xi = \sum_{i=0}^{3} \xi^i e_i \), the inner product takes the form

\[
\langle \xi, \eta \rangle = \sum_{j,k=0}^{3} g_{jk} \xi^j \eta^k, \tag{1.2.1}
\]

where \( g_{ij} \), the Minkowski metric, is the diagonal matrix \( g = \text{diag}(1, -1, -1, -1) \). In what follows we usually omit the sums using Einstein’s summation convention (i.e. we sum over all indices which appear twice, once as an upper and once as a lower index). Also, we sometimes abbreviate the Minkowski scalar product by writing \( \xi \eta := \langle \xi, \eta \rangle \) and \( \xi^2 := \langle \xi, \xi \rangle \). A pseudo-orthonormal basis \((e_i)_{i=0,...,3}\) is also referred to as a reference frame, because the corresponding coordinate system \((x^i)\) of Minkowski space gives the time and space coordinates for an observer in a system of inertia. We also refer to \( t := x^0 \) as time and denote the spatial coordinates by \( \vec{x} = (x^1, x^2, x^3) \).
The sign of the Minkowski metric encodes the causal structure of space-time. Namely, a vector \( \xi \in \mathcal{M} \) is said to be
\[
\begin{align*}
timelike & \quad \text{if } \langle \xi, \xi \rangle > 0 \\
spacelike & \quad \text{if } \langle \xi, \xi \rangle < 0 \\
lightlike & \quad \text{if } \langle \xi, \xi \rangle = 0
\end{align*}
\]
(1.2.2)

Lightlike vectors are also referred to as null vectors and the term \( \text{non-spacelike} \) refers to timelike or lightlike vectors. The null vectors form the double cone \( L = \{ \xi \in \mathcal{M} \mid \langle \xi, \xi \rangle = 0 \} \), referred to as the \textit{light cone}. Physically, the light cone is formed out of all light rays through the origin of \( \mathcal{M} \) which propagate with the speed of light. Similarly, the timelike vectors correspond to velocities slower than the speed of light; they form the \textit{interior light cone} \( I = \{ \xi \in \mathcal{M} \mid \langle \xi, \xi \rangle > 0 \} \). Finally, we introduce the \textit{closed light cone} \( J = \{ \xi \in \mathcal{M} \mid \langle \xi, \xi \rangle \geq 0 \} \).

The space-time trajectory of a moving object is described by a curve \( q(\tau) \) in Minkowski space (with \( \tau \) an arbitrary parameter). We say that the space-time curve \( q \) is timelike if the tangent vector to \( q \) is everywhere timelike. Spacelike, null, and non-spacelike curves are defined analogously. The usual statement of causality that no information can travel faster than the speed of light can then be expressed as follows:

\textit{Causality}: Information can be transmitted only along non-spacelike curves.

The set of all points which can be joined with a given space-time point \( x \) by a non-spacelike curve is precisely the closed light cone centered at \( x \), denoted by \( J_x := J - x \). It is the union of the two single cones
\[
\begin{align*}
J^\vee_x & = \{ y \in \mathcal{M} \mid (y - x)^2 \geq 0, \ (y^0 - x^0) \geq 0 \} \\
J^\wedge_x & = \{ y \in \mathcal{M} \mid (y - x)^2 \geq 0, \ (y^0 - x^0) \leq 0 \}
\end{align*}
\]
interpreted as the points in the causal future and past of \( x \), respectively. Therefore, we refer to \( J^\vee_x \) and \( J^\wedge_x \) as the closed future and past light cones centered at \( x \), respectively. The sets \( I^\vee_x, I^\wedge_x \) and \( L^\vee_x, L^\wedge_x \) are introduced similarly.

Special relativity demands that physical equations be \textit{Lorentz invariant}, which means that they must be formulated in Minkowski space and in particular be independent of the choice of reference frame. The simplest relativistic wave equation is the \textit{Klein-Gordon equation}
\[
(-\Box - m^2) \psi = 0 ,
\]
(1.2.3)
where \( \Box \equiv \partial_j \partial^j \) is the wave operator. This equation describes a scalar particle (i.e. a particle without spin) of mass \( m \). If the particle has electric charge, one needs to suitably insert the electromagnetic potential \( A \) into the Klein-Gordon equation. One finds empirically that the equation
\[
-(\partial_k - ieA_k)(\partial^k - ieA^k) \psi = m^2 \psi
\]
(1.2.4)
describes a scalar particle of mass \( m \) and charge \( e \) in the presence of an electromagnetic field.

1.3. The Dirac Equation

We now give a brief introduction to the Dirac equation, the relativistic generalization of the Schrödinger equation. Our presentation closely follows the presentation in [28], §1.1 and §1.2].
In order to describe a relativistic particle with spin, Dirac had the idea to work with a first order differential operator whose square is the wave operator. This operator is constructed by introducing the Dirac matrices $\gamma^j$ as $4 \times 4$-matrices which satisfy the anti-commutation relations

$$2 g^{jk} 1 = \{\gamma^j, \gamma^k\} \equiv \gamma^j \gamma^k + \gamma^k \gamma^j.$$  \hfill (1.3.1)

Then taking the square of the operator $\gamma^j \partial_j$ indeed gives the wave operator,

$$(\gamma^j \partial_j)^2 = \gamma^j \gamma^k \partial_j \partial_k = \frac{1}{2} \{\gamma^j, \gamma^k\} \partial_{jk} = \Box.$$ \hfill (1.3.2)

For convenience, we shall always work in the Dirac representation, i.e. we set

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$ \hfill (1.3.3)

where $\sigma^i$ are the three Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Including mass, the Dirac equation in vacuum (i.e. without any interaction) reads

$$\left( i \gamma^k \frac{\partial}{\partial x^k} - m \right) \psi(x) = 0,$$ \hfill (1.3.4)

where $\psi(x)$, the Dirac spinor, has four complex components. If we multiply (1.3.4) by the operator $(i \gamma^j \partial_j + m)$ and use Eq. (1.3.2), we finds that each component of $\psi$ satisfies the Klein-Gordon equation $\Box \psi = 0$. Following the standard conventions in physics, we also denote the contraction with Dirac matrices by a slash, i.e. $\hat{u} = \gamma^j u_j$ for $u$ a vector of Minkowski space and $\hat{\partial} = \gamma^j \partial_j$. The action of the matrix $\hat{u}$ upon a spinor $\psi$, i.e. $\hat{u} \psi$, is often referred to as Clifford multiplication by the vector $u$. The leptons and quarks in the standard model are Dirac particles, and thus one can say that on the fundamental, all matter is level described by the Dirac equation.

In the presence of an electromagnetic field, the Dirac equation is modified to

$$i \gamma^k (\partial_k - i A_k) \psi = m \psi.$$ \hfill (1.3.5)

Multiplying by the operator $(i \gamma^j (\partial_j - i A_j) + m)$ and using the anti-commutation relations, we obtain the equation

$$\left( - (\partial_k - i A_k) (\partial^k - i A^k) + \frac{i}{2} F_{jk} \gamma^j \gamma^k - m^2 \right) \psi = 0,$$

where $F_{jk} = \partial_j A_k - \partial_k A_j$. This differs from the Klein-Gordon equation (1.2.4) by the extra term $\frac{i}{2} F_{jk} \gamma^j \gamma^k$, which describes the coupling of the spin to the electromagnetic field.

The Dirac spinors at every space-time point are endowed with an indefinite scalar product of signature $(2, 2)$, which we call spin scalar product and denote by

$$\langle \psi | \phi \rangle(x) = \sum_{\alpha=1}^4 s_\alpha \psi^\alpha(x)^\dagger \phi^\alpha(x), \quad s_1 = s_2 = 1, \ s_3 = s_4 = -1,$$ \hfill (1.3.6)

where $\psi^\dagger$ is the complex conjugate wave function (this scalar product is often written as $\bar{\psi} \phi$ with the so-called adjoint spinor $\bar{\psi} \equiv \psi^\dagger \gamma^0$). By the adjoint $A^*$ of an operator
acting on spinors we always mean the adjoint with respect to the spin scalar product $\langle .| . \rangle$ defined by
$$\langle A^* \psi | \phi \rangle = \langle \psi | A \phi \rangle \quad \text{for all } \psi, \phi.$$ 
In an obvious way, this definition of the adjoint gives rise to the notions selfadjoint, anti-selfadjoint and unitary. With these notions, the Dirac matrices are selfadjoint, meaning that
$$\langle \gamma_l^! \psi | \phi \rangle = \langle \psi | \gamma_l^! \phi \rangle \quad \text{for all } \psi, \phi.$$ 
To every solution $\psi$ of the Dirac equation we can associate a vector field $J$ by
$$J^k = \langle \psi | \gamma^k \psi \rangle,$$ 
which is called the Dirac current. It is either timelike or lightlike (see Exercise 1.2). Moreover, it is divergence-free, as the following computation shows:
$$\partial_k J^k = \partial_k \langle \psi | \gamma^k \psi \rangle = \langle \partial_k \psi | \gamma^k \psi \rangle + \langle \psi | \gamma^k \partial_k \psi \rangle$$
$$= i \left( \langle i \partial \psi | \psi \rangle - \langle \psi | i \partial \psi \rangle \right)$$
$$= i \left( \langle (i \partial + A - m) \psi | \psi \rangle - \langle \psi | (i \partial + A - m) \psi \rangle \right) = 0.$$ 
This is referred to as current conservation.

Current conservation is closely related to the probabilistic interpretation of the Dirac wave function, as we now explain. Suppose that $\psi$ is a smooth solution of the Dirac equation with suitable decay at spatial infinity. Then current conservation allows us to apply the Gauß divergence theorem to obtain
$$0 = \int_{t_1}^{t_2} dt \int_{\mathbb{R}^3} d^3x \partial_k \langle \psi | \gamma^k \psi \rangle (t, \vec{x})$$
$$= \int_{\mathbb{R}^3} \langle \psi | \gamma^0 \psi \rangle (t_2, \vec{x}) d^3x - \int_{\mathbb{R}^3} \langle \psi | \gamma^0 \psi \rangle (t_1, \vec{x}) d^3x$$
(this argument works similarly on a region $\Omega \subset M$ whose boundary consists of two spacelike hypersurfaces). Polarizing (similar as explained after (1.1.2)), we conclude that for any two solutions $\phi, \psi$ of the Dirac equation, the spatial integral
$$\langle \phi | \psi \rangle := 2\pi \int_{\mathbb{R}^3} \langle \phi | \gamma^0 \psi \rangle (t, \vec{x}) d^3x$$
(1.3.11)
is time independent. Since the inner product $\langle . | \gamma^0 . \rangle$ is positive definite, the integral (1.3.11) defines a scalar product. We denote the Hilbert space corresponding to this scalar product by $H = L^2(\mathbb{R}^3)$. In analogy to the integrand in (1.1.4) in non-relativistic quantum mechanics, the quantity $\langle \psi | \gamma^0 \psi \rangle$ can be interpreted as the probability density. Current conservation (1.3.10) ensures that the probability integral is time independent.

The previous considerations generalize immediately to the situation in the presence of a more general external potential. To this end, we replace the operator $A$ in the Dirac equation (1.3.5) by a multiplication operator $B(x)$, which we assume to be smooth and to be symmetric with respect to the spin scalar product, i.e.
$$\langle B \psi | \phi \rangle = \langle \psi | B \phi \rangle \quad \text{for all } \psi, \phi.$$ 
We write the Dirac equation with a Dirac operator $D$ as
$$(D - m) \psi = 0 \quad \text{where} \quad D := i\partial + B.$$ 
(1.3.13)
The symmetry assumption (1.3.12) is needed for current conservation to hold (as one sees immediately if in (1.3.9) one replaces $A$ by $B$).
1. PHYSICAL PRELIMINARIES

Similar to (1.1.5), we can again rewrite the dynamics with a symmetric operator $H$. To this end, we multiply the Dirac equation (1.3.13) by $\gamma^0$ and bring the $t$-derivative on a separate side of the equation,

$$i\partial_t \psi = H \psi \quad \text{where} \quad H = -\gamma^0 (i\vec{\gamma} \vec{\nabla} + B - m) \quad (1.3.14)$$

(note that $\gamma^j \partial_j = \gamma^0 \partial_0 + \vec{\gamma} \vec{\nabla}$). We refer to (1.3.14) as the Dirac equation in the Hamiltonian form. Now we can again apply (1.1.6) to conclude that the Hamiltonian is a symmetric operator on $\mathcal{H}$.

We remark that in the Hamiltonian formulation, one often combines the prefactor $\gamma^0$ in (1.3.14) with the other Dirac matrices and works with the new matrices

$$\beta := \gamma^0 \quad \text{and} \quad \vec{\alpha} := \gamma^0 \vec{\gamma}.$$

This is convenient because these new matrices are Hermitian with respect to the standard scalar product on $\mathbb{C}^4$. In this book, we shall not work with $\alpha$ and $\vec{\beta}$. We prefer the notation (1.3.14), because it becomes clearer which parts of the operators are Lorentz invariant. For calculations using $\beta$ and $\vec{\alpha}$ we refer for example to the monograph \[96\].

In addition to integrating over space (1.3.11), one can also introduce an inner product on Dirac wave functions by integrating the spin scalar product over all of space-time,

$$\langle \psi | \phi \rangle = \int_{\mathcal{M}} \langle \psi | \phi \rangle_x \, d\mu_{\mathcal{M}}. \quad (1.3.15)$$

This inner product will in general not be well-defined on solutions of the Dirac equation, because the time integral may diverge. But it can be considered for example on spinorial wave functions which are compactly supported in space-time (but are no solutions of the Dirac equation). This space-time inner product will be important for the constructions in Chapter 13. In this context, it is very useful that the Dirac operator is symmetric w.r.t. the space-time inner product, meaning that that

$$\langle D \psi | \phi \rangle = \langle \psi | D \phi \rangle \quad (1.3.16)$$

for all spinorial wave functions which decay sufficiently fast at spatial infinity and for large times. Indeed, the symmetry property (1.3.16) holds in curved space-time as well (see the explanation after (3.2.31) below).

So far, Dirac spinors were introduced in a given reference frame. Let us verify that our definitions are coordinate independent. A linear transformation of Minkowski space which leaves the form of the Minkowski metric (1.2.1) invariant is called a Lorentz transformation. All Lorentz transformations form a group, the so-called Lorentz group. The Lorentz transformations which preserve both the time direction and the space orientation form a subgroup of the Lorentz group, the orthonormal proper Lorentz group. We consider two reference frames $(x^j)$ and $(\tilde{x}^l)$ with the same orientation of time and space. Then the reference frames are related to each other by an orthonormal proper Lorentz transformation $\Lambda$, i.e. in components

$$\tilde{x}^l = \Lambda^l_j \, x^j, \quad \frac{\partial}{\partial x^j} = \frac{\partial \tilde{x}^l}{\partial x^j} \frac{\partial}{\partial \tilde{x}^l} = \Lambda^l_j \frac{\partial}{\partial \tilde{x}^l},$$

and $\Lambda$ leaves the Minkowski metric invariant,

$$\Lambda^l_j \Lambda^m_k \, g_{lm} = g_{jk}. \quad (1.3.17)$$
Under this change of space-time coordinates, the Dirac operator \( i\gamma^j \left( \frac{\partial}{\partial \tilde{x}^j} - iA_j \right) \) transforms to

\[
i\tilde{\gamma}^l \left( \frac{\partial}{\partial \tilde{x}^l} - i\tilde{A}_l \right) \quad \text{with} \quad \tilde{\gamma}^l = \Lambda^l_j \gamma^j. \tag{1.3.18}
\]

This transformed Dirac operator does not coincide with the Dirac operator \( i\gamma^j \left( \frac{\partial}{\partial \tilde{x}^j} - i\tilde{A}_j \right) \) as defined in the reference frame \( (\tilde{x}^l) \) because the new Dirac matrices have a different form. However, the next lemma shows that the two Dirac operators do coincide after a suitable unitary transformation of the spinors (unitary w.r.to the spin scalar product (1.3.6)).

**Lemma 1.3.1.** For any orthochronous proper Lorentz transformation \( \Lambda \) there is a unitary matrix \( U(\Lambda) \) such that

\[
U(\Lambda) \Lambda^l_j \gamma^j U(\Lambda)^{-1} = \gamma^l.
\]

**Proof.** Since \( \Lambda \) is orthochronous and proper, we can write it in the form \( \Lambda = \exp(\lambda) \), where \( \lambda \) is a suitable generator of a rotation and/or a Lorentz boost. Then \( \Lambda(t) := \exp(t\lambda), \ t \in \mathbb{R}, \) is a family of Lorentz transformations, and differentiating (1.3.17) with respect to \( t \) at \( t = 0 \), we find that

\[
\lambda^l_j g_{lk} = -g_{jm} \lambda^m_k
\]

(note that \( \Lambda(t)^l_j = \delta^n_l + \lambda^n_j t + \ldots \)). Using this identity together with the fact that the Dirac matrices are selfadjoint, it is straightforward to verify that the matrix

\[
u := \frac{1}{4} \lambda^l_k \gamma_l \gamma^k
\]

is anti-selfadjoint. As a consequence, the family of matrices

\[
U(t) := \exp(tu)
\]

is unitary. We now consider for a fixed index \( l \) the family of matrices

\[
A(t) := U(t) \Lambda(t)^l_j \gamma^j U(t)^{-1}.
\]

Clearly, \( A(0) = \gamma^l \). Furthermore, differentiating with respect to \( t \) gives

\[
\frac{d}{dt} A(t) = U \Lambda^l_j \left\{ u \gamma^j - \gamma^j u + \lambda^l_j \gamma^k \right\} U^{-1},
\]

and a short calculation using the commutation relations

\[
\left[ \gamma_l \gamma^k, \gamma^j \right] = 2 \left( \gamma_l g^{kj} - \delta_l^j \gamma^k \right)
\]

shows that the curly brackets vanish. We conclude that \( A(1) = A(0) \), proving the lemma. \( \square \)

Applying this lemma to the Dirac operator in (1.3.18), one sees that the Dirac operator is invariant under the joint transformation of the space-time coordinates and the spinors

\[
x^j \longrightarrow \Lambda^l_j x^k, \quad \psi \longrightarrow U(\Lambda) \psi.
\]

Moreover, since the matrix \( U(\Lambda) \) is unitary, the representation of the spin scalar product (1.3.6) is valid in any reference frame. We conclude that our definition of spinors is indeed Lorentz invariant.

For what follows, it is important to keep in mind that, in contrast to the spin scalar product, the combination \( \psi^\dagger \phi = \langle \psi | \gamma^0 \phi \rangle \) is *not* Lorentz invariant. Instead, it is the zero component of a Minkowski vector. Consequently, the integrand in (1.3.11) is not a
1. PHYSICAL PRELIMINARIES

Scalar, but a density. Its integral, on the other hand, is again Lorentz invariant due to current conservation.

Out of the Dirac matrices one can form the pseudoscalar matrix $\Gamma$ by

$$\Gamma = \frac{i}{4!} \epsilon_{jklm} \gamma^j \gamma^k \gamma^l \gamma^m = i \gamma^0 \gamma^2 \gamma^3 \quad (1.3.19)$$

(in the physics literature, this matrix is usually denoted by $\gamma^5$). Here $\epsilon_{jklm}$ is the totally antisymmetric symbol (i.e. $\epsilon_{jklm}$ is equal to $\pm 1$ if $(j,k,l,m)$ is an even and odd permutation of $(0,1,2,3)$, respectively, and vanishes otherwise). A short calculation shows that the pseudoscalar matrix is anti-selfadjoint and $\Gamma^2 = 1$. As a consequence, the matrices

$$\chi_L = \frac{1}{2} (\mathbf{1} - \Gamma), \quad \chi_R = \frac{1}{2} (\mathbf{1} + \Gamma) \quad (1.3.20)$$

satisfy the relations

$$\chi^2_{L/R} = \mathbf{1} \quad (1.3.21)$$

They can be regarded as the spectral projectors of the matrix $\Gamma$ and are called the chiral projectors. The projections $\chi_L \psi$ and $\chi_R \psi$ are referred to as the left- and right-handed components of the spinor, respectively. A matrix is said to be even and odd if it commutes respectively anti-commutes with $\Gamma$. It is straightforward to verify that the Dirac matrices are odd, and therefore

$$\gamma^j \chi_{L/R} = \chi_{R/L} \gamma^j.$$ 

By multiplying the Dirac equation (1.3.5) from the left by $\chi_{L/R}$, one can rewrite it as a system of equations for the left- and right-handed components of $\psi$,

$$i \gamma^k (\partial_k - i A_k) \chi_L \psi = m \chi_R \psi, \quad i \gamma^k (\partial_k - i A_k) \chi_R \psi = m \chi_L \psi.$$ 

If $m = 0$, these two equations decouple, and we get separate equations for the left- and right-handed components of $\psi$. This observation is the starting point of the 2-component Weyl spinor formalism. We shall not use this formalism here, but will instead describe chiral massless particles (like massless neutrinos) by the left- or right-handed component of a Dirac spinor.

1.4. The Hilbert Space of Dirac Solutions

We now formulate the above structures with a convenient notation, which has the advantage of generalizing to curved space-time. The Dirac wave functions are four-component complex wave functions in Minkowski space. We can consider them as sections of the vector bundle

$$S.\mathcal{M} := \mathcal{M} \times \mathbb{C}^4,$$

referred to as the spinor bundle. Thus, denoting the fibre of this vector bundle at a space-time point $x \in \mathcal{M}$ by $S_x.\mathcal{M}$ (the spinor space at $x$), the wave functions take values in the corresponding spinor spaces, i.e.

$$\psi(x) \in S_x.\mathcal{M} \simeq \mathbb{C}^4.$$ 

Moreover, the spin scalar product (1.3.6) is an inner product on the fibre, as we often clarify by an additional subscript $x$,

$$\langle . , . \rangle_x : S_x.\mathcal{M} \times S_x.\mathcal{M} \to \mathbb{C}.$$ 

Next, the scalar product (1.3.11) on the Dirac solutions gives rise to a Hilbert space structure. In order to construct the correct class of solutions, one can begin by solving the Cauchy problem for smooth initial data with compact support given for example on
the hypersurface \{t = 0\}. Rewriting the Dirac equation as a linear symmetric hyperbolic system (see Chapter 11 below), one sees that this Cauchy problem has a unique global solution in Minkowski space. Moreover, we know that this solution is smooth and, due to finite propagation speed, has compact support on any other hypersurface \{t = \text{const}\}. Indeed, the solution has compact support when restricted to any Cauchy surface. One says that the solution has \textit{spatially compact support}. The smooth and spatially compact sections of the spinor bundle (not necessarily solutions of the Dirac equation) are denoted by \(C_{\infty}^{sc}(\mathcal{M}, S\mathcal{M})\). Clearly, for spatially compact solutions, the scalar product (1.3.11) is well-defined and finite. Taking the abstract completion, one obtains a Hilbert space denoted by \((H_{m}, \langle \cdot, \cdot \rangle)\). By construction, we know that

\[ C_{\infty}^{sc}(\mathcal{M}, S\mathcal{M}) \cap H_{m} \text{ is dense in } H_{m}. \]

We note that the solutions in \(H_{m}\) can also be characterized in terms of Sobolev spaces (see [23]). Indeed, the solutions in \(H_{m}\) are in \(H^{1,2}(\mathcal{M}, S\mathcal{M})\). By the trace theorem, their restriction to a hypersurface \(\{t = \text{const}\}\) is in \(L_{\text{loc}}^{2}(\mathbb{R}^{3}, \mathbb{C}^{4})\), so that the integrand of the spatial integral in (1.3.11) is locally integrable. The solutions in \(H_{m}\) have the additional property that their restriction to the hypersurfaces are even in \(L^{2}(\mathbb{R}^{3}, \mathbb{C}^{4})\), so that the integral in (1.3.11) exists and is finite.

**Exercises**

**Exercise 1.1.** Show that, multiplying by the operator \((i\gamma^{j}(\partial_{j} - iA_{j}) + m)\) and using the anti-commutation relations, we obtain the equation

\[ \left( - (\partial_{k} - iA_{k})(\partial^{k} - iA^{k}) + \frac{i}{2} F_{jk}\gamma^{j}\gamma^{k} - m^{2} \right) \psi = 0. \]

This differs from the Klein-Gordon equation (1.2.4) by the extra term \(\frac{i}{2} F_{jk}\gamma^{j}\gamma^{k}\), which describes the coupling of the spin to the electromagnetic field.

**Exercise 1.2.** In this exercise, we shall verify that for any non-zero spinor \(\psi\), the corresponding Dirac current vector \(J^{k} = \langle \psi | \gamma^{k} \psi \rangle\) is non-spacelike.

(a) Show that the matrix \(\gamma^{0}\gamma^{1}\) is Hermitian and has eigenvalues \(\pm 1\). Deduce that

\[ \langle \psi, \gamma^{0}\gamma^{1}\psi \rangle_{\mathbb{C}^{4}} \leq \|\psi\|_{\mathbb{C}^{4}}^{2}. \]

(b) Show that the last inequality implies that \(|J^{1}| \leq J^{0}\).

(c) Use the rotational symmetry of the Dirac equation to conclude that \(J^{0} \geq |\vec{J}|\) (where \(\vec{J} = (J^{1}, J^{2}, J^{3}) \in \mathbb{R}^{3}\)).
CHAPTER 2
Mathematical Preliminaries

2.1. A Few Basic Notions from Abstract Measure Theory

Let \( \mathcal{F} \) be a topological space.

**Definition 2.1.1.** A system of subsets \( \mathcal{M} \) of \( \mathcal{F} \) is a \( \sigma \)-algebra if it has the following properties:

(i) \( \emptyset \in \mathcal{M} \)
(ii) \( \mathcal{M} \) is closed under taking complements, i.e.
\[
A \in \mathcal{M} \implies \mathcal{F} \setminus A \in \mathcal{M}.
\]
(iii) \( \mathcal{M} \) is closed under at most countable unions, i.e. for any sequence \( (A_n)_{n \in \mathbb{N}} \),
\[
A_n \in \mathcal{M} \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}.
\]

The sets in \( \mathcal{M} \) are also referred to as the measurable sets.

In this book, we will mainly consider the Borel algebra, which is defined as the smallest \( \sigma \)-algebra which contains all open sets (see Exercise 2.52).

**Definition 2.1.2.** A measure \( \rho \) is a mapping from a \( \sigma \)-algebra to the non-negative reals,
\[
\rho : \mathcal{M} \to \mathbb{R}_0^+
\]
with the following properties:

(i) \( \rho(\emptyset) = 0 \)
(ii) \( \sigma \)-additivity holds, i.e. for every sequence \( (A_n)_{n \in \mathbb{N}} \) of pairwise disjoint measurable sets,
\[
\rho \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \sum_{n=1}^\infty \rho(A_n).
\]

**Definition 2.1.3.** The support of a measure is defined as the complement of the largest open set of measure zero, i.e.
\[
supp \rho := \mathcal{F} \setminus \bigcup \left\{ \Omega \subset \mathcal{F} \mid \Omega \text{ is open and } \rho(\Omega) = 0 \right\}.
\]

Note that the support is by definition a closed subset of \( \mathcal{F} \).

A measure on the Borel algebra is referred to as a Borel measure.

**Definition 2.1.4.** A Borel measure \( \rho \) on \( \mathcal{F} \) is called regular if for any measurable set \( A \),
\[
\rho(A) = \sup_{K \subset A \text{ compact}} \rho(K) = \inf_{\Omega \supseteq A \text{ open}} \rho(\Omega).
\]

It is locally finite if every point of \( \mathcal{F} \) has an open neighborhood \( \Omega \) with \( \rho(\Omega) < \infty \). Regular, locally finite Borel measures are also referred to as Radon measures.
Suppose that we want to compare two Radon measures $\rho$ and $\tilde{\rho}$ on $\mathcal{F}$. A natural idea is to take the difference $\rho - \tilde{\rho}$ and to look at its size. The difficulty is that for a measurable set $A \subset \mathcal{F}$, its measures $\rho(A)$ and $\tilde{\rho}(A)$ could both take the value $+\infty$, in which case their difference would be ill-defined. In order to avoid this problem, we use the regularity to and exhaust by compact sets. Assuming that $\mathcal{F}$ is locally compact, the fact that the measures are locally finite implies that the measures of compact sets are always finite. This leads us to the following definition:

**Definition 2.1.5.** Given Radon measures $\rho$ and $\tilde{\rho}$ on a locally compact topological space $\mathcal{F}$, we define the measures $\mu^\pm$ by

$$
\mu^+(A) = \sup_{K \subset A \text{ compact}} (\rho(K) - \tilde{\rho}(K))
$$

$$
\mu^-(A) = \sup_{K \subset A \text{ compact}} (\tilde{\rho}(K) - \rho(K)).
$$

The difference or measures $\rho - \tilde{\rho}$ is said to have **bounded total variation** the measures $\mu^\pm$ are bounded, i.e. if

$$
\mu_+(\mathcal{F}), \mu_-(\mathcal{F}) < \infty.
$$

If this is the case, the **total variation measure** $|\rho - \tilde{\rho}|$ is defined by

$$
|\rho - \tilde{\rho}| = \mu^+ + \mu^-.
$$

We remark that $\rho - \tilde{\rho}$ can also be defined in the context of signed measures; we refer the interested reader to [71, §28] or [90, Section 6.1].

Another notion of measure theory which we will use frequently is the push-forward measure, which we now define (for more details see for example [12, Section 3.6]). To this end, let $(\mathcal{F}, \mathcal{M}, \rho)$ be a measure space. Moreover, we consider a mapping $f : \mathcal{F} \to X$, where $X$ is a point set. Then $f$ induces a measure on $X$. Indeed, we let $\mathcal{M}_X$ be the set of all subsets $\Omega \subset X$ whose preimage $f^{-1}(\Omega)$ is $\rho$-measurable. Using the elementary identities

$$
f^{-1}\left(\bigcup_{n \in \mathbb{N}} \Omega_n\right) = \bigcup_{n \in \mathbb{N}} f^{-1}(\Omega_n) \quad \text{and} \quad f^{-1}(X \setminus \Omega) = \mathcal{F} \setminus f^{-1}(\Omega),
$$

one verifies that $\mathcal{M}_X$ is $\sigma$-algebra on $X$. The **push-forward measure**, denoted by $f_\ast \rho$ is a measure on $\mathcal{M}_X$ defined by

$$
(f_\ast \rho)(\Omega) := \rho(f^{-1}(\Omega)).
$$

### 2.2. Distributions and Fourier Transformation

Consider the following process of “concentration” of a mass density of fixed total mass:

**Figure 2.1.** A sequence of functions concentrating in one point.

Viewed as functions, the values converge to zero everywhere except at the origin, where they diverge to infinity. Hence the limiting object cannot sensibly be described as a function. The mathematical theory of distributions, also called “singular functions”, provides a generalization of the usual concept of a function of $n$ real variables which is capable to encapture also limiting objects of processes as the one illustrated above. This framework has proven to be extremely useful in the study of partial differential equations as well in the last century up to today.
2.2. DISTRIBUTIONS AND FOURIER TRANSFORMATION

2.2.1. Test function spaces. Let $\Omega \subset \mathbb{R}^n$ be an open subset. All functions are assumed to be complex-valued in the following.

We set $\mathcal{E}(\Omega) := C^\infty(\Omega)$ and $\mathcal{D}(\Omega) := C^\infty_0(\Omega) := \{f \in C^\infty(\Omega) \mid \text{supp } f \text{ is compact}\}$.

Recall here that the support of a function $f \in C^\infty(\Omega)$ is defined as

$$\text{supp } f := \{x \in \Omega \mid f(x) \neq 0\}, \quad (2.2.1)$$

where the overline denotes the closure in $\Omega$. We refer to elements of $\mathcal{E}(\Omega)$ as test functions, and elements of $\mathcal{D}(\Omega)$ as compactly supported test functions. Polynomials and functions defined by convergent power series are examples of test functions. Concerning compactly supported smooth functions, exercise $2.53$ contains a common basic construction of such a function, a so-called smooth cutoff function. Note that both $\mathcal{E}(\Omega)$ and $\mathcal{D}(\Omega)$ are complex vector spaces with respect to pointwise addition and scalar multiplication.

The definition of distributions also requires a notion of convergence of sequence of test functions.

**Definition 2.2.1 (Convergence of test functions).**

1. We say a sequence $(u_n)_{n \in \mathbb{N}}$ of test functions $\mathcal{E}$-converges to $u \in \mathcal{E}(\Omega)$ if all partial derivatives of the sequence converge uniformly on all compact subsets of $\Omega$ to the corresponding partial derivatives of $u$.

2. We say a sequence $(u_n)_{n \in \mathbb{N}}$ of compactly supported test functions $\mathcal{D}$-converges to $u \in \mathcal{D}(\Omega)$ if there exists a compact subset $K \subset \Omega$ with $\text{supp } u, \text{supp } u_n \subset K$ for all $n \in \mathbb{N}$ and if all partial derivatives of the sequence converge uniformly to the corresponding partial derivatives of $u$.

Written out explicitly, one has $u_n \to u$ in $\mathcal{E}(\Omega)$ if

$$\forall K \subset \Omega \text{ compact: } \forall \alpha \in \mathbb{N}^n : \sup_{x \in K} |\partial^\alpha u_n(x) - \partial^\alpha u(x)| \to 0. \quad (2.2.2)$$

For a sequence of compactly supported test functions clearly $\mathcal{D}$-convergence implies $\mathcal{E}$-convergence. The converse is false, since the supports of a merely $\mathcal{E}$-convergent sequence may “expand to infinity” in the limit. A concrete, simple example of this phenomenon is contained in exercise $2.53$.

Apart from test functions and compactly supported test functions, there is a third class of test functions which is important in particular in the context of the Fourier transform. It lies somewhat “in between” the previously introduced spaces in the sense of decay properties at infinity.

**Definition 2.2.2 (Schwartz functions).** A smooth function $u : \mathbb{R}^n \to \mathbb{C}$ is said to be rapidly decaying or a Schwartz function if

$$\forall \alpha, \beta \in \mathbb{N}^n : \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < \infty. \quad (2.2.3)$$

The vector space of all Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^n)$.

The notion “rapidly decaying” is perhaps best understood if one rewrites $(2.2.3)$ as follows: For any $\alpha, \beta \in \mathbb{N}^n$ there exists $C > 0$ such that

$$\forall x \in \mathbb{R}^n \setminus \{0\} : |\partial^\beta u(x)| \leq \frac{C}{|x^\alpha|}.$$ 

This shows that any derivative of $u$ is required to decay fast than any polynomial at infinity. Since this is a condition at infinity, we restrict to functions defined on all of $\mathbb{R}^n$. In exercise $2.55$ you may explore some further reformulations of this condition.
Concerning sequential continuity, let $u : \mathbb{R}^n \to \mathbb{C}$ defined by $u(x) = e^{-|x|^2}$ lies in $\mathcal{S}(\mathbb{R}^n)$. For any $\alpha \in \mathbb{N}^n$ one has $\partial^\alpha u = p_\alpha u$ for some polynomial $p_\alpha$, as can be shown by induction on $|\alpha|$. Using this, (2.2.3) is easily established by l’Hopitals rule for $n = 1$, the case $n > 1$ can be traced back to the one-dimensional case.

More generally, for any polynomial $p$ and any positive definite, symmetric matrix $A \in \text{Mat}_n(\mathbb{R})$, the function $v : \mathbb{R}^n \to \mathbb{C}$ defined by $v(x) = p(x)e^{-x^T Ax}$ lies in $\mathcal{S}(\mathbb{R}^n)$. The details are left to the reader as exercise 2.56.

As before we now specify a notion of convergence for Schwartz functions which will be needed for the definition of the corresponding space of distributions.

**Definition 2.2.4.** We say that a sequence $(u_n)_n$ of Schwartz functions $\text{S}$-converges to some $u \in \mathcal{S}(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}^n$ the functions $x^\alpha \partial^\beta u_n$ converge uniformly to $x^\alpha \partial^\beta u$.

Note the following obvious inclusions of the three test function spaces:

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n). \quad (2.2.4)$$

Note further that a $\mathcal{D}$-convergent sequence also $\mathcal{S}$-converges to the same limit, and an $\mathcal{S}$-convergent sequence also $\mathcal{E}$-converges to the same limit. Therefore the inclusions (2.2.4) are both sequentially continuous. Clearly for any $\Omega \subset \mathbb{R}^n$ open the inclusion $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ still holds and is sequentially continuous.

Since Schwartz functions decay rapidly at infinity, they are integrable on all of $\mathbb{R}^n$. They even include continuously into all $L^p$-spaces, as proven by the following Lemma. Its proof contains a simple but very useful argument which is worth remembering.

**Lemma 2.2.5.** If $u \in \mathcal{S}(\mathbb{R}^n)$, then $u \in L^p(\mathbb{R}^n)$ for all $p \in [1, \infty]$. Moreover, the inclusion $\mathcal{S}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is sequentially continuous.

**Proof.** We focus on the cases $p \in [1, \infty)$ and leave the case $p = \infty$ as exercise. Let $u \in \mathcal{S}(\mathbb{R}^n)$ be given. For $p \in [1, \infty)$ given, choose $m \in \mathbb{N}$ with $pm > n$. Then the function $x \mapsto (1 + |x|^2)^{-\frac{pm}{2}}$ is integrable on $\mathbb{R}^n$ (use polar coordinates). Therefore

$$\|u\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |u(x)|^p \, dx$$

$$\quad = \int_{\mathbb{R}^n} |u(x)|^p (1 + |x|^2)^{-\frac{pm}{2}} (1 + |x|^2)^{-\frac{pm}{2}} \, dx$$

$$\quad = \int_{\mathbb{R}^n} \left(1 + |x|^2\right)^{-\frac{pm}{2}} u(x)^p \left(1 + |x|^2\right)^{-\frac{pm}{2}} \, dx$$

$$\leq \sup_{x \in \mathbb{R}^n} \left(1 + |x|^2\right)^{-\frac{pm}{2}} u(x)^p \cdot \int_{\mathbb{R}^n} \left(1 + |x|^2\right)^{-\frac{pm}{2}} \, dx$$

$$\quad < \infty.$$  

Concerning sequential continuity, let $(u_n)_n$ be a sequence in $\mathcal{S}(\mathbb{R}^n)$ with $u_n \to u \in \mathcal{S}(\mathbb{R}^n)$. Applying the previous computation to $u_n \to u$ shows that

$$\|u_n - u\|_{L^p(\mathbb{R}^n)}^p \leq C \sup_{x \in \mathbb{R}^n} \left| (1 + |x|^2)^{\frac{pm}{2}} (u_n - u) \right|$$

for some $C > 0$. By definition 2.2.4 this tends to zero for $n \to \infty$ since $u_n \to u$ in $\mathcal{S}$. \qed
The special function
\[ \mathbb{R}^n \ni x \mapsto \langle x \rangle := (1 + |x|^2)^{1/2} \in \mathbb{R} \] (2.2.5)
used in the proof is called *Japanese bracket*. For large $|x|$, its value is comparable to $|x|$, whereas for small $|x|$ its value is comparable to 1. The latter has the advantage that one can take negative powers of it without having to worry about creating singularities.

**Remark 2.2.6.** It is possible to equip each test function space with a topology such that the corresponding notion of convergence coincides with the one introduced above. Since for most practical purposes this does not give any real benefits, we chose to not include it here. The interested reader is referred to [Friedlander].

### 2.2.2. Definition and examples of distributions

Now we are set up for the definition of a distribution. To each of the three test function spaces corresponds a space of distributions.

**Definition 2.2.7 (Distributions).** Let $\Omega \subset \mathbb{R}^n$ be an open subset.

1. A **distribution** on $\Omega$ is a sequentially continuous linear map $T : D(\Omega) \to \mathbb{C}$. The vector space of all distributions on $\Omega$ is denoted by $D'(\Omega)$.

2. A **compactly supported distribution** on $\Omega$ is a sequentially continuous linear map $T : E(\Omega) \to \mathbb{C}$. The vector space of all distributions on $\Omega$ is denoted by $E'(\Omega)$.

3. A **tempered distribution** is a sequentially continuous linear map $T : S(\mathbb{R}^n) \to \mathbb{C}$. The vector space of all tempered distributions is denoted by $S'(\mathbb{R}^n)$.

For a distribution $T$ and a test function $u$, we will often write $\langle T, u \rangle$ instead of $T(u)$.

To be precise, the requirement of sequential continuity of a distribution $T$ means that whenever $u_n \to u$ holds in the corresponding space of test functions, then $\langle T, u_n \rangle \to \langle T, u \rangle$ holds in $\mathbb{C}$. Due to the sequentially continuous inclusion (2.2.4) of test function spaces one has the reverse inclusion of distribution spaces:

\[ \mathcal{E}'(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n). \] (2.2.6)

Again, for a domain $\Omega \subset \mathbb{R}^n$ one still has the inclusion $\mathcal{E}'(\Omega) \subset D'(\Omega)$.

We now go over some examples of distributions, further examples can be found in the exercises.

**Example 2.2.8 (Dirac $\delta$-distribution).** For $\Omega \subset \mathbb{R}^n$ open and $x_0 \in \Omega$, the **(Dirac) $\delta$-distribution** at $x_0$ is the distribution $\delta_{x_0} \in \mathcal{E}'(\Omega)$ defined by

\[ \delta_{x_0}(u) := u(x_0) \] (2.2.7)

for any $u \in \mathcal{E}(\Omega)$. Clearly $\delta_{x_0}$ is a linear function. Concerning sequential continuity, suppose that we have $u_n \to u$ in $\mathcal{E}(\Omega)$. This means in particular that $u_n \to u$ uniformly on any compact subset of $\Omega$ which implies pointwise convergence as well. Therefore

\[ \delta_{x_0}(u_n) = u_n(x_0) \to u(x_0) = \delta_{x_0}(u). \]

Since test functions are differentiable and also the notions of convergence of test functions takes into account derivatives, one may also use derivatives of test functions for the construction of distributions. For instance, for any $\Omega \subset \mathbb{R}^n$ open, any $x_0 \in \Omega$, and any $\alpha \in \mathbb{N}^n$, the map

\[ \mathcal{E}(\Omega) \ni u \mapsto (\partial^\alpha u)(x_0) \in \mathbb{C} \]

defines a distribution. This can be checked similarly as for the $\delta$-distribution.
Example 2.2.9 (Integration as a distribution). Consider integration over \( \mathbb{R} \) as a map \( I : \mathcal{D}(\mathbb{R}) \to \mathbb{C}, I[u] := \int_{\mathbb{R}} u(x) \, dx \). Since any \( u \in \mathcal{D}(\mathbb{R}) \) has compact support and is continuous, \( I(u) \) is well-defined. Clearly \( I \) is linear as well. Concerning sequential continuity, suppose that \( u_n \to u \) in \( \mathcal{D}(\mathbb{R}) \). This means that there exists some \( R > 0 \) such that \( \text{supp}\, u_n, \text{supp}\, u \subset [-R, R] \) and (in particular) \( u_n \to u \) uniformly on \([-R, R]\). Hence

\[
I(u_n) = \int_{\mathbb{R}} u_n(x) \, dx = \int_{-R}^{R} u_n(x) \, dx \to \int_{-R}^{R} u(x) \, dx = \int_{\mathbb{R}} u(x) \, dx = I(u),
\]

convergence of the integrals following for instance from uniform convergence.

The next example is of particular importance since it describes in which sense a function may be understood as a distribution as well, and hence why distributions indeed generalize the notion of functions.

Example 2.2.10 (Functions as distributions). A function \( f : \Omega \to \mathbb{C} \) is said to be \textbf{locally integrable} if \( \int_K |f(x)| \, dx < \infty \) for any compact subset \( K \subset \Omega \). If \( f \) is such a function, then for any \( u \in \mathcal{D}(\Omega) \) the integral

\[
T_f(u) := \int_{\Omega} f(x)u(x) \, dx \tag{2.2.8}
\]
is well-defined. Clearly the map \( T_f : \mathcal{D}(\Omega) \to \mathbb{C} \) is linear. It is also sequentially continuous: If \( u_n \to u \) in \( \mathcal{D}(\Omega) \), there exists \( K \subset \Omega \) compact with \( \text{supp}\, u_n, \text{supp}\, u \subset K \) for all \( n \in \mathbb{N} \) and \( u_n \to u \) uniformly. By uniform convergence, we have \( |u_n| \leq C + |u| \) for some \( C > 0 \) and all \( n \in \mathbb{N} \). Since all these functions are supported in the compact set \( K \), we also have \( |u_n|f \leq (C + |u|) |f| \in L^1(\Omega) \). It now follows from the dominated convergence theorem that \( T_f(u_n) \to T_f(u) \) holds.

Distributions of the form (2.2.8) are called \textit{regular distributions}. By the so-called \textbf{fundamental theorem in the calculus of variations} (see [ ] for \( f \) and \( g \) locally integrable, \( T_f = T_g \) implies that \( f = g \) almost everywhere (with respect to Lebesgue measure). For continuous functions you are asked to prove this in exercise \[2.61\]. Hence, if as usual in integration theory we identify functions which agree almost everywhere and denote by \( L^1_{\text{loc}}(\Omega) \) the space of the corresponding equivalence classes of all locally integrable functions on \( \Omega \) (all with respect to Lebesgue measure), then the map

\[
L^1_{\text{loc}}(\Omega) \ni f \mapsto T_f \in \mathcal{D}'(\Omega) \tag{2.2.9}
\]
is injective. Therefore one can view \( L^1_{\text{loc}}(\Omega) \) as a subspace of \( \mathcal{D}'(\Omega) \). For this reason, we often simply write \( f \) instead of \( T_f \) if it is clear from the context that we want to understand \( f \) as a distribution. Note that in particular all continuous functions and all \( L^p \)-functions for \( p \in [1, \infty] \) are contained in \( L^1_{\text{loc}} \). Be aware that, just as \( L^p \)-functions, locally integrable functions may also be unbounded even on compact sets, i.e. “have poles”, as long as these poles are integrable, see the exercises \[2.63\] and \[2.64\].

The \( \delta \)-distribution is \textbf{not} a regular distribution. To see this in the case of \( \Omega = \mathbb{R} \), fix some \( u \in \mathcal{D}(\mathbb{R}) \) with \( 0 \leq f \leq 1, f(0) = 1, \text{supp}\, u \subset [-1, 1] \). Set \( u_n(x) := u(nx) \), then we have \( \delta_0(u_n) = u_n(0) = u(0) = 1 \) for all \( n \in \mathbb{N} \). However, for any locally integrable function \( f : \mathbb{R} \to \mathbb{C} \) we have \( u_n(x)f(x) \to 0 \) for all \( x \neq 0 \) and \( |u_n f| \leq 1_{[-1,1]}f \in L^1(\mathbb{R}) \), so by dominated convergence it follows that

\[
T_f(u_n) = \int_{\mathbb{R}} u_n(x)f(x) \, dx \to 0.
\]
Hence \( T_f \neq \delta_0 \) for all \( f \in L^1_{\text{loc}}(\mathbb{R}) \).
The following exercise connects distributions with another possible generalization of the notion of a function, namely that of a measure. In a certain sense (which is made precise by the so-called Riesz representation theorem), these are all distributions which only use the values of a test function but not if its derivatives.

Example 2.2.11 (Locally finite measures as distributions). Let $\mu$ be a Borel measure on $\Omega$, i.e. a measure on the Borel $\sigma$-algebra of $\Omega$. Then $\mu$ is called **locally finite** if every point $x \in \Omega$ possesses an open neighborhood $U_x \subset \Omega$ such that $\mu(U_x) < \infty$. Using that any open cover of a compact set has a finite subcover, any locally finite Borel $\mu$ measure defines a distribution $T_\mu \in D'(\Omega)$ by integration, i.e. by $u \mapsto \int_{\Omega} u(x) \, d\mu(x)$. You are asked to fill in the details in exercise 2.66.

In contrast to the previous example, in exercise 2.67 you find a distribution which uses arbitrarily many derivatives (i.e., of arbitrary high order).

We finish this section with a convenient reformulation of sequential continuity of a linear functional on a test function space in terms of an estimate. If you are familiar with basic functional analysis you may compare this to the result that a linear map on a Banach space is continuous if and only if it is bounded.

**Proposition 2.2.12 (Sequential continuity of linear functionals on $D$).**

For a linear map $A : D(\Omega) \to \mathbb{C}$ the following are equivalent:

1. $A$ is sequentially continuous.
2. $A$ is sequentially continuous at 0.
3. For every compact set $K \subset \Omega$ there exist $C > 0$ and $m \in \mathbb{N}$ such that
   \[ |Au| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^\infty(K)} \] (2.2.10)
   for all $u \in D(\Omega)$ with $\text{supp } u \subset K$.

**Proof.** Clearly (1) implies (2). Conversely, if (2) holds and if $u_k \to u$, then we have $v_k := u_k - u \to 0$, so $Av_k \to 0$ by (2). By linearity it follows that $Au_k = Av_k + Au \to Au$.

Next we show that (3) implies (2). To this end, suppose that $u_k \to 0$. Then there exists $K \subset \Omega$ compact such that $\text{supp } u_k \subset K$ for all $k \in \mathbb{N}$ and $\|\partial^\alpha u_k\|_{L^\infty(K)} \to 0$ for all $\alpha \in \mathbb{N}^n$. Let $C > 0$ and $m \in \mathbb{N}$ be such that (2.2.10) holds for this $K$. Then we have

\[ |Au_k| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha u_k\|_{L^\infty(K)} \to 0. \]

This shows that (2) holds.

Conversely, suppose that (2) holds. Suppose for the sake of a contradiction that (3) does not hold. Then there exist $K \subset \Omega$ compact and a sequence $(u_k)_k$ in $D(\Omega)$ such that

\[ |Au_k| > k \sum_{|\alpha| \leq k} \|\partial^\alpha u_k\|_{L^\infty(K)} \] (*)&

holds for all $k \in \mathbb{N}$. This implies that $\|u_k\|_{L^\infty(K)} > 0$ for all $k \in \mathbb{N}$ since otherwise $u_k = 0$ and so $Au_k = 0$ by linearity, which contradicts (*). Therefore we can rescale and set

\[ v_k := \frac{u_k}{k \sum_{|\alpha| \leq k} \|\partial^\alpha u_k\|_{L^\infty(K)}} \in D(\Omega). \]
Clearly supp $v_k = \text{supp} \, u_k \subset K$. We claim that $v_k \to 0$ in $\mathcal{D}$. To see this, let $\alpha \in \mathbb{N}^n$. For every $k \in \mathbb{N}$ with $k > |\alpha|$ we have
\[
\|\partial^\alpha v_k\|_{L^\infty(K)} = \frac{1}{k} \sum_{|\beta| \leq k} \|\partial^\beta u_k\|_{L^\infty(K)} < \frac{1}{k} \|\partial^\alpha u\|_{L^\infty(K)}.
\]
Hence $\|\partial^\alpha v_k\|_{L^\infty(K)} \to 0$ and so $v_k \to 0$ in $\mathcal{D}$. However, by the choice of $v_k$ and ($\ast$) it follows that $|Av_k| > 1$ holds for all $k \in \mathbb{N}$. This contradicts $Av_k \to 0$. Hence (3) must hold after all.

Similar results of course also hold for linear functions on $\mathcal{E}$ and $\mathcal{S}$, as you are asked to prove in the exercises 2.68 and 2.69. Let us illustrate the use of this result in an example.

Example 2.2.13. The function $\frac{1}{x}$ is not locally integrable on $\mathbb{R}$, so it cannot directly be interpreted as a distribution. However, exploiting the fact that it is an odd function, we can define a distribution v.p. $(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R})$, called Cauchy principal value of $\frac{1}{x}$ by
\[
\langle \text{v.p.} \left(\frac{1}{x}\right), u \rangle := \text{V.P.} \int_{-\infty}^{\infty} \frac{u(x)}{x} \, dx := \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{u(x)}{x} \, dx \tag{2.2.11}
\]
for any $u \in \mathcal{D}(\mathbb{R})$. To see that this integral is well-defined, note that $\frac{1}{x}$ being odd we have
\[
\int_{|x| \geq \epsilon} \frac{u(x)}{x} \, dx = \frac{1}{2} \int_{|x| \geq \epsilon} \frac{u(x) - u(-x)}{x} \, dx,
\]
and that the function $\frac{u(x) - u(-x)}{x}$ has a continuous extension to $x = 0$ by $2u'(0)$ while still being compactly supported. It is clear that (2.2.11) is linear in $u$, and we are going to use Proposition 2.2.12 to show that it is also sequentially continuous. To this end, let $K \subset \mathbb{R}$ be compact and $u \in \mathcal{D}(\mathbb{R})$ with $\text{supp} \, u \subset K$. Pick $N > 0$ with $K \subset [-N,N]$. By the mean value theorem, for any $x \in \mathbb{R} \setminus \{0\}$ there exists $\theta$ between $-x$ and $x$ such that $\frac{u(x) - u(-x)}{2x} = u'(\theta)$. Therefore the function $\frac{u(x) - u(-x)}{2x}$ is bounded in absolute value by $\|u'\|_{L^\infty(K)}$. It follows that for any $\epsilon > 0$ we have
\[
\left| \int_{|x| \geq \epsilon} \frac{u(x)}{x} \, dx \right| = \int_{N \geq |x| \geq \epsilon} \frac{u(x)}{x} \, dx = \int_{N \geq |x| \geq \epsilon} \frac{u(x) - u(-x)}{2x} \, dx \leq 2N \|u'\|_{L^\infty(K)}.
\]
It follows that also $\langle \text{v.p.} \left(\frac{1}{x}\right), u \rangle \leq 2N \|u'\|_{L^\infty(K)}$ holds which by Proposition 2.2.12 implies sequential continuity of v.p. $(\frac{1}{x})$.

2.2.3. Convergence of distributions. We started with the example of a “collapsing” sequence of functions as motivation for the notion of distributions. We now introduce a notion of convergence of distributions with respect to which such a sequence of functions has a distributional limit.

Definition 2.2.14 (Convergence of distributions). A sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathcal{D}'(\Omega)$ is said to $\mathcal{D}'$-converge to some $T \in \mathcal{D}'(\Omega)$ if
\[
\forall u \in \mathcal{D}(\Omega) : \quad T_n(u) \to T(u). \tag{2.2.12}
\]
Convergence of compactly supported and tempered distributions is defined analogously.

In physics one often says that one first has to “smear with a test function” and then take the limit. Mathematically, (2.2.12) is of course nothing but pointwise convergence of the sequence when viewed as functions from $\mathcal{D}(\Omega)$ to $\mathbb{C}$. With respect to these notions of convergence, the inclusions $\mathcal{D}' \subset \mathcal{S}' \subset \mathcal{E}'$ become sequentially continuous.
To illustrate convergence in the sense of distributions, we describe a common way of approximating the δ-distribution by regular (even smooth) distributions. This will also be important later for showing that any distribution can be approximated by smooth functions.

**Definition 2.2.15 (Dirac δ-sequences).** A sequence \((\psi_k)_{k \in \mathbb{N}}\) in \(C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\) is called (Dirac) δ-sequence or approximate identity if it satisfies the following properties:

1. L₁-equiboundedness: \(\sup_{k \in \mathbb{N}} \|\psi_k\|_{L^1(\mathbb{R}^n)} < \infty\).
2. Normalization: For every \(k \in \mathbb{N}\) one has \(\int_{\mathbb{R}^n} \psi_k(x) \, dx = 1\).
3. Concentration at 0: For every \(R > 0\) one has \(\lim_{k \to \infty} \left( \int_{\{|x| > R\}} |\psi_k(x)| \, dx \right) = 0\).

If additionally \(\psi_k \in C^\infty(\mathbb{R}^n)\) for all \(k \in \mathbb{N}\) and \(\operatorname{supp}(\psi_k) \to \{0\}\) in the sense that for every \(R > 0\) there exists \(N \in \mathbb{N}\) such that \(\operatorname{supp}(\psi_k) \subset B_R(0)\) for all \(k \geq N\), then one speaks of a strict δ-sequence.

We now show that any such sequence converges to the δ-distribution. In exercise [2.70](#) and [2.71](#) you find concrete examples of δ-sequences.

**Theorem 2.2.16 (Smooth approximation of the δ-distribution).** Let \((\psi_k)\) be a δ-sequence. Then \(\psi_k \to \delta_0\) in \(D'\) and in \(S'\). If \((\psi_k)\) is a strict δ-sequence, then also \(\psi_k \to \delta_0\) in \(E'\).

**Proof.** Let \(u \in D(\mathbb{R}^n)\) be given. We wish to show that \(\int_{\mathbb{R}^n} \psi_k(x)f(x) \, dx \to f(0)\). By the normalization condition we can write

\[
\int_{\mathbb{R}^n} \psi_k(x)f(x) \, dx - f(0) = \int_{\mathbb{R}^n} \psi_k(x)(f(x) - f(0)) \, dx.
\]

Now we split this integral into large and small \(|x|\) and use that \(|f(x) - f(0)|\) will be small for small \(|x|\), whereas \(|\psi_k(x)|\) will be small for large \(|x|\) (and large \(k\)). More precisely, let \(\epsilon > 0\) be given. Choose \(\delta > 0\) such that \(|f(x) - f(0)| < \epsilon\) for all \(|x| < \delta\). Moreover, choose \(N \in \mathbb{N}\) such that \(\int_{\mathbb{R}^n \setminus B_\delta(0)} |\psi_k(x)| \, dx < \epsilon\) for all \(k \geq N\). Then for all \(k \geq N\) we have

\[
\left| \int_{\mathbb{R}^n} \psi_k(x)(f(x) - f(0)) \, dx \right| \leq \int_{B_\delta(0)} |\psi_k(x)| \cdot |f(x) - f(0)| \, dx + \int_{\mathbb{R}^n \setminus B_\delta(0)} |\psi_k(x)| \cdot |f(x) - f(0)| \, dx \\
= \epsilon (\|\psi_k\|_{L^1} + \|f\|_\infty).
\]

As \(\epsilon > 0\) was arbitrary, this shows convergence of the left-hand side to 0.

For \(u \in \mathcal{S}(\mathbb{R}^n)\) the exact same argument works as well. For \(u \in \mathcal{E}(\mathbb{R}^n)\) and \((\psi_k)\) a strict δ-sequence, one replaces \(u\) by \(\chi u\) for some \(\chi \in C_0^\infty(\mathbb{R}^n)\) with \(\chi(0) = 1\) and \(\operatorname{supp}(\psi_k) \subset \operatorname{supp}(\chi)\) for all sufficiently large \(k\). \(\square\)

Dirac δ-sequences illustrate how a sequence of functions which concentrate more and more “mass” into a small space may have a limit in the sense of distributions. The following example shows another way in which a sequence of functions, which clearly do not converge as functions, may have a distributional limit.

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[Note: The text is a summary of the content of the document. The original content is not included due to the limitations of the system.]
Example 2.2.17. For \( k \in \mathbb{N} \), let \( \psi_k \in C^\infty(\mathbb{R}) \) be given by \( \psi_k(x) = \sin(kx) \). For \( u \in \mathcal{D}'(\mathbb{R}) \), by integration by parts we have

\[
\langle \psi_k, u(x) \rangle = \int_\mathbb{R} \sin(kx)u(x) \, dx = \int_\mathbb{R} -\frac{1}{k}(\cos(kx))'u(x) \, dx = \frac{1}{k} \int_\mathbb{R} \cos(kx)u'(x) \, dx \to 0.
\]

This shows that \( \psi_k \to 0 \) in \( \mathcal{D}'(\mathbb{R}^n) \) for \( k \to \infty \). In intuitive terms, the ever faster oscillations of \( \psi_k \) as \( k \) increases lead to cancellations in the integral which results in the convergence to 0.

Finally, as yet another illustration for computing distributional limits, we prove the following famous formula which relates two distributional “renormalizations” of \( \frac{1}{x} \).

Theorem 2.2.18 (Sokhotski-Plemelj formula). The family of functions \( \left\{ \frac{1}{x+\epsilon} \right\}_{\epsilon > 0} \) has a distributional limit in \( \mathcal{D}'(\mathbb{R}) \) for \( \epsilon \to 0 \), denoted by \( \frac{1}{x} \). This limit satisfies

\[
\frac{1}{x \pm i0} := \lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = \text{v.p.} \left( \frac{1}{x} \right) = \mp i\pi \delta, \quad \text{(2.2.13)}
\]

where v.p. \( \left( \frac{1}{x} \right) \) denotes the Cauchy principal value of \( \frac{1}{x} \) defined in Example 2.2.13.

Proof. Let \( u \in \mathcal{D}(\mathbb{R}) \) be given. As in example 2.2.13, we are going to cleverly exploit the fact that \( \frac{1}{x} \) is odd. Observe that as a consequence we have \( \langle \text{v.p.} \left( \frac{1}{x} \right), \varphi \rangle = 0 \) for any even function \( \varphi \in \mathcal{D}(\mathbb{R}) \). Since the function \( \varphi(x) := \frac{u(0)}{1+x^2} \) is even we have \( \langle \text{v.p.} \left( \frac{1}{x} \right), u \rangle = \langle \text{v.p.} \left( \frac{1}{x} \right), u - \varphi \rangle \).

On the other hand,

\[
\int_\mathbb{R} \frac{1}{x \pm i\epsilon} u(x) \, dx = \int_\mathbb{R} \frac{u(x) - \varphi(x)}{x \pm i\epsilon} \, dx + u(0) \int_\mathbb{R} \frac{1}{(x \pm i\epsilon)(1 + x^2)} \, dx. \quad \text{(\ast)}
\]

Regarding the first integral, note that by l’Hopital’s rule

\[
\lim_{x \to 0} \frac{u(x) - \varphi(x)}{x} = \lim_{x \to 0} \frac{u'(x) - \varphi'(x)}{1} = u'(0),
\]

so the function \( \frac{u(x) - \varphi(x)}{x} \) extends continuously to all of \( \mathbb{R} \). Moreover, since \( u \) has compact support and \( \varphi \) decays as \( \frac{1}{x^2} \) at infinity, it is also integrable. Therefore, an application of dominated convergence yields that

\[
\lim_{\epsilon \to 0} \int_\mathbb{R} \frac{u(x) - \varphi(x)}{x \pm i\epsilon} \, dx = \int_\mathbb{R} \frac{u(x) - \varphi(x)}{x} \, dx = \lim_{\delta \to 0} \int_{|x| \geq \delta} \frac{u(x) - \varphi(x)}{x} \, dx = \langle \text{v.p.} \left( \frac{1}{x} \right), u - \varphi \rangle = \langle \text{v.p.} \left( \frac{1}{x} \right), u \rangle.
\]

For any \( \epsilon > 0 \), the second integral in (\ast) can easily be computed by the residue theorem, closing the contour in the upper half-plane. In the limit \( \epsilon \to 0 \) this leaves \( \mp i\pi \), just as desired.

Next we show that the inclusions of various functions into distributions (as regular distributions) are sequentially continuous. Concretely this simply means that convergence as (test) functions implies convergence as distributions.
Theorem 2.2.19. All of the following maps, which map a function $f$ to the regular distribution $T_f$, are linear, injective and sequentially continuous:

$$\mathcal{E}(\Omega) \to \mathcal{D}'(\Omega), \quad \mathcal{D}(\Omega) \to \mathcal{E}'(\Omega), \quad L^p(\Omega) \to \mathcal{D}'(\Omega), \quad L^p(\mathbb{R}^n) \to \mathcal{S}'(\Omega).$$

Proof. Linearity is obvious and injectivity follows from the fundamental theorem in the calculus of variations. We show sequential continuity in the first case, the other cases can be handled similarly and are left as exercise. Suppose that $f_n \to f$ in $\mathcal{E}(\Omega)$. By linearity we may assume $f = 0$, otherwise we consider the difference $f_n - f$. Let $u \in \mathcal{D}(\Omega)$ be given. Since supp $u \subset \Omega$ is compact, by definition of $\mathcal{E}$-convergence the functions $f_n$ converge uniformly to zero on supp $f$. This implies that there exists a constant $C > 0$ such that $|f_n| < C$ on supp $u$ for all $n \in \mathbb{N}$, and thus $|f_n u| \leq C|u| \in C^0(\Omega) \subset L^1(\Omega)$. Since clearly also $f_n(x) \to 0$ for every $x \in \Omega$ it now follows by dominated convergence that $\int_{\Omega} f_n(x) u(x) \, dx \to 0$. Hence $f_n \to 0$ also holds in $\mathcal{D}'(\Omega)$. \hfill \Box

2.2.4. Calculus with distributions. One motivation for studying distributions is to solve (linear) differential equations. To this end, one first needs to specify how a (linear) partial differential operator acts on a distribution, in particular how to differentiate a distribution and how to multiply it by a smooth function. The guideline for this is that on smooth functions, viewed as regular distributions, differentiation and multiplication by smooth functions should work in the same way one is used to. To explain what this means, let $f \in C^\infty(\mathbb{R})$ be a smooth function and let $T_f \in \mathcal{D}'(\mathbb{R})$ be the corresponding (regular) distribution. Concerning differentiation, compatibility means that we would like to define $(T_f)'$ in such a way that $(T_f)' = T_{f'}$ holds. Since for any test function $u \in \mathcal{D}(\mathbb{R})$ integration by parts gives

$$\langle T_f', u \rangle = \int_{\mathbb{R}} f'(x) u(x) \, dx = - \int_{\mathbb{R}} f(x) u'(x) \, dx = - \langle T_f, u' \rangle,$$

we should define $(T_f)'$ by the very right-hand side in this equation. Note that in this right-hand side we can replace $T_f$ by any, not necessarily regular, distribution $T$.

Definition 2.2.20. Let $T \in \mathcal{D}'(\Omega)$. For $\alpha \in \mathbb{N}^n$ and for $f \in C^\infty(\Omega)$ we define $\partial^\alpha T \in \mathcal{D}'(\Omega)$ and $fT \in \mathcal{D}'(\Omega)$ as the distributions acting on a test function $u \in \mathcal{D}(\Omega)$ by

$$\langle \partial^\alpha T, u \rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha u \rangle,$$

$$\langle fT, u \rangle := \langle T, fu \rangle.$$

(2.2.14) \quad (2.2.15)

For $\mathcal{E}'(\Omega)$ and $\mathcal{S}'(\mathbb{R}^n)$ one uses the same definitions, except that tempered distributions may only be multiplied by functions which grow at most polynomially at infinity (so-called functions of “moderate growth”, see exercise 2.58).

Of course one has to verify that (2.2.14) and (2.2.15) indeed define distributions, i.e., sequentially continuous linear functionals. This is not difficult and left as exercise.

Note that any distribution can be infinitely often differentiated. Therefore, if viewed as distribution, also a function which in the classical sense is not differentiable, may be differentiated as a distribution. As before, we illustrate these new notions by concrete examples. More examples can be found in the exercises.

Example 2.2.21. The Heaviside function $\Theta : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Theta(x) = \begin{cases} 
0 & x \leq 0 \\
1 & x > 0 
\end{cases}.$$

(2.2.16)
Clearly $\Theta \in L^1_{\text{loc}}(\mathbb{R})$ since it is bounded, so we can view $\theta$ as a distribution. Computing the derivative $\Theta'$ as a distribution is actually quite easily: For any $u \in \mathcal{D}(\mathbb{R})$ we have
\[
\langle \Theta', u \rangle \overset{\text{Def}}{=} -\langle \Theta, u' \rangle = -\int_{\mathbb{R}} \Theta(x)u'(x) \, dx = -\int_0^\infty u'(x) \, dx = u(0) = \langle \delta_0, u \rangle.
\]
Hence $\Theta' = \delta_0$, as intuition might have suggested.

**Example 2.2.22.** For $x_0 \in \mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$, we compute $f \delta_{x_0} \in \mathcal{D}'(\mathbb{R}^n)$. To this end, note that for any $u \in \mathcal{D}(\mathbb{R}^n)$ we have
\[
\langle f \delta_{x_0}, u \rangle \overset{\text{Def}}{=} \langle \delta_{x_0}, fu \rangle = f(x_0)u(x_0) = \langle f(x_0)\delta_{x_0}, u \rangle.
\]
This shows that $f \delta_{x_0} = f(x_0)\delta_{x_0}$.

The usual rules of differentiation also apply to distributions as far as they make sense. They are easy to prove by simply going back to the definitions, therefore the verifications are left as exercise.

**Proposition 2.2.23** (Calculus rules for distributions). Let $T \in \mathcal{D}'(\Omega)$.

1. For any $\alpha, \beta \in \mathbb{N}^n$ one has $\partial^\alpha (\partial^\beta T) = \partial^{\alpha+\beta} T$.
2. For any $f, g \in C^\infty(\Omega)$ one has $f(gT) = (fg)T$.
3. For any $\alpha \in \mathbb{N}^n$ and $f \in C^\infty(\Omega)$ the Leibniz rule holds:
\[
\partial^\alpha (fT) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)(\partial^\beta T) \tag{2.2.17}
\]

By the fundamental theorem of calculus, any (continuously) differentiable function can be reconstructed from its derivative up to an additive constant. We now prove the same statement for distributions, the proof may also serve as further illustration of how to work with distributions.

**Proposition 2.2.24.** Let $S, T \in \mathcal{D}'(\mathbb{R})$ and let $f \in C(\mathbb{R})$.

1. If $T' = Tf$, then $T = TF$ for some antiderivative $F \in C^1(\mathbb{R})$ of $f$.
2. If $S' = T'$, then $S$ and $T$ coincide “up to a constant”, i.e., there exists $c \in \mathbb{R}$ such that $\langle S, u \rangle = \langle T, u \rangle + \int_\mathbb{R} cu(x) \, dx$ for all $u \in \mathcal{D}(\mathbb{R})$.

**Proof.** Hint: It might be helpful to show first that $v \in \mathcal{D}(\mathbb{R})$ satisfies $v = v'$ for some other test function $w \in \mathcal{D}(\mathbb{R})$ if and only if $\int_\mathbb{R} v(x) \, dx = 0$. Then decompose a general $u \in \mathcal{D}(\mathbb{R})$ as $u = u_1 + u_2$ where $\int_\mathbb{R} u_1(x) \, dx = 0$ by using a cutoff function $\chi$ with $\int_\mathbb{R} \chi(x) \, dx = 1$. \(\Box\)

Note that the first part of the previous proposition may be viewed as a (very simple) “regularity result”: If a distribution $T$ satisfies the differential equation $T' = w$ with a regular (continuous) right-hand side, then $T$ itself is regular (actually one degree more regular than the right-hand side).

Having defined differentiation and multiplication by smooth functions, we can now define how a linear differential operator (with smooth coefficients) acts on distributions. Concretely, if $A = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ is a linear differential operator with certain coefficients $a_\alpha \in C^\infty(\Omega)$, then for $T \in \mathcal{D}'(\Omega)$ we define
\[
AT := \sum_{|\alpha| \leq m} a_\alpha (\partial^\alpha T). \tag{2.2.18}
\]

With this definition it now makes sense to say that a distribution $T$ satisfies the Laplace equation $\Delta T = 0$ or the wave equation $(\partial_t^2 - D)T = 0$, or any other linear partial
differential equation, also replacing \(0\) on the right-hand side by an inhomogeneity. Be careful however that at this point it is not yet clear how one might impose initial or boundary values. In concrete calculations one should also be careful that when unwinding \(\langle AT, u \rangle\) for some \(u \in D(\Omega)\), one obtains
\[
\langle AT, u \rangle = \sum_{\alpha \leq m} \langle a_\alpha(\partial^\alpha T), u \rangle = \sum_{\alpha \leq m} \langle \partial^\alpha T, a_\alpha u \rangle = \sum_{\alpha \leq m} (-1)^{\alpha} \langle T, \partial^\alpha (a_\alpha u) \rangle,
\]
where on the right-hand side also the coefficients of \(A\) are differentiated. This is of course just as it would happen in an integration by parts computation.

Particularly important distributional solutions of linear partial differential equations are obtained by setting the right-hand side equal to the \(\delta\)-distribution. 

**Definition 2.2.25 (Green’s function).** If \(A\) is linear differential operator on some open set \(\Omega \subset \mathbb{R}^n\), and \(x_0 \in \Omega\), then a distribution \(G_{x_0} \in D'(\Omega)\) satisfying \(AG_{x_0} = \delta_{x_0}\), where \(\delta_{x_0}\) is the \(\delta\)-distribution at \(x_0\), is called **Green’s function in** \(x_0\) **for** \(A\).

In the exercises \(2.83\) - \(2.88\) you find examples of Green’s functions for various equations. Note that Green’s functions are in general only unique up to addition of solutions of the homogeneous equation \(AT = 0\).

We close this section with the remark that that differentiation and multiplication are sequentially continuous linear maps on test function spaces and distribution spaces. The proofs, which are left to the reader in exercise \(2.82\), provides an excellent opportunity to consolidate the various previous definitions.

### 2.2.5. Fourier transform.

The Fourier transform is an important tool, for instance in the study of differential equations, and it is very useful that it can also be generalized to (tempered) distributions. Before doing so, we recall the definition and some important properties about the Fourier transform of functions. For further reading one may for instance consult [Stein and Shakarchi, Fourier Analysis].

For \(f \in L^1(\mathbb{R}^n)\), the **Fourier transform** is the function \(\hat{f} : \mathbb{R}^n \to \mathbb{C}\) defined by
\[
\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix \cdot \xi} \, dx,
\]
where \(x \cdot \xi = x^1 \xi^1 + \ldots + x^n \xi^n\). Instead of \(\hat{f}\) we will also write \(\mathcal{F}(f)\) for the Fourier transform of \(f\). The basic properties of the Fourier transform are most easily stated for Schwartz functions, but many of the following properties however also hold for more general classes of functions. Most of these properties can be proved by straight-forward manipulations of integrals, and are left to the reader in exercise \(2.90\).

**Theorem 2.2.26 (Basic properties of the Fourier transform on \(S\)).** Let \(f, g \in S(\mathbb{R}^n)\). Then the following hold:

1. We have \(\hat{f} \in S(\mathbb{R}^n)\), and for any \(\alpha \in \mathbb{N}^n\) it holds that
\[
(\partial^\alpha \hat{f})(\xi) = (i\xi)^\alpha \hat{f}(\xi) \quad \text{and} \quad (x^\alpha \hat{f})(\xi) = i^{\alpha} \partial^\alpha \hat{f}(\xi).
\]

2. It holds that
\[
\int_{\mathbb{R}^n} f(x) \, g(x) \, dx = \int_{\mathbb{R}^n} \hat{f}(\xi) \, g(\xi) \, d\xi.
\]
(3) As a map $F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$, the Fourier transform is a sequentially continuous isomorphism. The inverse is given explicitly by the Fourier inversion formula:

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{ix\cdot\xi} \, d\xi. \quad (2.2.22)$$

(4) Parseval’s relation holds:

$$\int_{\mathbb{R}^n} f(x) g(x) \, dx = \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\xi) \, d\xi. \quad (2.2.23)$$

(5) Plancharel’s formula holds:

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}. \quad (2.2.24)$$

(6) The convolution theorem holds:

$$\hat{f} \ast g = \hat{f} \cdot \hat{g}. \quad (2.2.25)$$

Now we move on to the Fourier transform of a distribution. For this purpose, there are actually two possible definitions: On the one hand the definition (2.2.19) of the Fourier transform may be read as the action of the (regular) distribution $T_f$ on the function $x \mapsto e^{-ix\cdot\xi}$. On the other hand, the identity (2.2.21) suggests to define the Fourier transform of a distribution by “moving the Fourier transform to the test function” (similar as for differentiation). We will take the second approach since this allows to define the Fourier transform for a larger class of distributions, namely all tempered distributions, whereas the first approach is restricted to compactly supported distributions (since the imaginary exponential function does not decay). In the end we are going to show that for compactly supported distributions both approaches coincide.

**Definition 2.2.27 (Fourier transform of tempered distributions).** For a tempered distribution $T \in S'(\mathbb{R}^n)$, its **Fourier transform** $\hat{T} \in S'(\mathbb{R}^n)$ is defined by

$$\langle \hat{T}, u \rangle := \langle T, \hat{u} \rangle \quad (2.2.26)$$

for all $u \in S(\mathbb{R}^n)$. Instead of $\hat{T}$ we will also write $F(T)$.

Note that $\hat{T}$ is indeed linear and sequentially continuous again since it is simply the composition $\hat{T} = T \circ F$ of two linear and sequentially continuous maps on $S$. Even more, since $F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is an isomorphism, the same holds for the Fourier transform on $S'(\mathbb{R}^n)$. It is also not difficult to verify that the “calculus rules” (2.2.27) continue to hold, we leave the proof to the reader in exercise 2.91.

**Theorem 2.2.28 (Fourier transform on tempered distributions).** The Fourier transform is a sequentially continuous linear isomorphism $F : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$. Moreover, for any $T \in S'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$ we have

$$\langle \partial_\alpha^T, \hat{u} \rangle = (i\xi)^\alpha \langle \hat{T}, \xi^\alpha \hat{u} \rangle \quad \text{and} \quad \langle x^\alpha T, \hat{u} \rangle = i^|\alpha| \langle \hat{T}, \partial_{\xi)^\alpha \hat{u} \rangle. \quad (2.2.27)$$

Concerning (2.2.27), note that the multiplication of a tempered distribution by a monomial such as $x^\alpha$ or $\xi^\alpha$ is well-defined since monomials are clearly of moderate growth (cf. exercise 2.58).

Let us make some examples. The second one shows that the extension of the Fourier transform to distributions can also help to compute the Fourier transforms of functions.
Example 2.2.29. Let \( \delta \in \mathcal{S}'(\mathbb{R}^n) \) be the \( \delta \)-distribution (at the origin). For \( u \in \mathcal{S}(\mathbb{R}^n) \) we have
\[
\langle \hat{\delta}, u \rangle \overset{\text{Def.}}{=} \langle \delta, \hat{u} \rangle = \int_{\mathbb{R}^n} e^{0 \cdot x} u(x) \, dx = \int_{\mathbb{R}^n} 1 \cdot u(x) \, dx = \langle 1, u \rangle .
\]
This shows that \( \hat{\delta} = 1 \). In particular, the Fourier transform of \( \hat{\delta} \) is a regular distribution.

Example 2.2.30. Consider the function \( f \in L^1(\mathbb{R}) \) defined by \( f(t) = -e^t \) for \( t \leq 0 \) and \( f(t) = e^{-t} \) for \( t > 0 \). It is easy to verify that \( f'' = 2\delta' + f \) holds in \( \mathcal{S}' \). From (2.2.27) and the previous example it follows that \( -\xi^2 \hat{f} = 2i\xi \hat{f} \) holds, so \( \hat{f} = -\frac{2i}{1+\xi^2} \).

We close this section by connecting our definition of the distributional Fourier transform to the alternative approach of defining the distributional Fourier transform by viewing the definition of the Fourier transform of a function as application of a regular distribution to the test function \( e^{-ix \cdot \xi} \). While this only works for compactly supported distributions, this connection proves that the Fourier transform of such a distribution is in fact a smooth (even analytic) function.

Theorem 2.2.31. Let \( T \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \). Then its Fourier transform \( \hat{T} \) is actually a smooth function of moderate growth, and is explicitly given by
\[
\hat{T}(\xi) = \langle T(x), e^{-ix \cdot \xi} \rangle .
\]
Moreover, \( \hat{T} \) can be extended from \( \mathbb{R}^n \) to \( \mathbb{C}^n \) as an entire function by this formula.

Proof. Can probably be done by first approximating \( T \) and then controlling everything in the limit. \( \square \)

It is a general and important theme that smoothness properties of a function or distribution and decay properties of its Fourier transform are related. This already shines through from how the Fourier transform interchanges differentiation and multiplication by monomials, cf. (2.2.27).

2.2.6. Support and singular support of a distribution. Intuitively the \( \delta \)-distribution (at the origin) is supported in the origin since it is “zero everywhere but at the origin (where it is infinite)”. Note however that it is of course not sensible to speak about the value of a distribution “at a point”, rather one always has to “smear with a test function”. It is, however, still possible to define the notion of support of a distribution.

Definition 2.2.32 (Localization and support of distributions). Let \( S, T \in \mathcal{D}'(\Omega) \).

1. For an open subset \( U \subset \Omega \) we say that \( S \) and \( T \) coincide on \( U \) and write \( S|_U = T|_U \) if \( \langle S, u \rangle = \langle T, u \rangle \) for all \( u \in \mathcal{D}(\Omega) \) with \( \text{supp} \, u \subset U \).

2. We say that \( T \) vanishes in a point \( x \in \Omega \) if there exists an open neighborhood \( U \subset \Omega \) of \( x \) such that \( T|_U = 0 \).

3. The set
\[
\text{supp} \, T := \Omega \setminus \{ x \in \Omega \mid T \text{ vanishes at } x \}
\]
is called the support of \( T \).

As first example we show that for a continuous function, its support as a function and as a distribution coincide. In the example we further illustrate an important technique for proving local properties of distributions by suitable choices of cutoff functions.
Example 2.2.33. For \( f \in C(\Omega) \) we show that \( \text{supp} T_f = \text{supp} f \). Recall that
\[
\text{supp}\ f = \{ x \in \Omega \mid f(x) \neq 0 \},
\]
where the overline denotes closure in \( \Omega \). One the one hand, if \( x \notin \text{supp}\ f \), then since \( \text{supp}\ f \subset \Omega \) is closed there exists an open neighborhood \( U \subset \Omega \) of \( x \) such that \( f|_U = 0 \). Now clearly \( \langle T_f, u \rangle = 0 \) for any \( u \in \mathcal{D}(\Omega) \) with \( \text{supp}\ u \subset U \), hence \( x \notin \text{supp} T_f \). This shows that \( \text{supp} T_f \subset \text{supp} f \) holds. Conversely, let \( x \in \text{supp} f \). Choose any open neighborhood \( U \subset \Omega \) of \( x \) and \( x' \in U \) with \( f(x') \neq 0 \) (note that \( f(x) \) may be zero). After possibly rescaling \( f \), we may assume that \( f(x') = 2 \). By continuity, there exists \( \varepsilon > 0 \) sufficiently small such that \( B_\varepsilon(x') \subset U \) and \( f|_{B_\varepsilon(x')} > 1 \). Now pick a cutoff function \( \chi \in \mathcal{D}(\Omega) \) with \( \text{supp}\ \chi \subset B_\varepsilon(x) \subset U \), \( 0 \leq \chi \leq 1 \), and \( \chi|_{B_{\varepsilon/2}(x)} = 1 \). Then \( \langle T_f, \chi \rangle \geq |B_{\varepsilon/2}(x)| > 0 \). Therefore \( x \in \text{supp} T_f \).

Exercise 2.1. Compute the support of the \( \delta \)-distribution at the origin.

Exercise 2.2. Show that the support of a distribution on \( \Omega \) is a closed subset of \( \Omega \).

Exercise 2.3. Show that for \( f \in L^1_{\text{loc}}(\Omega) \), the support of the regular distribution \( T_f \) coincides with the essential support of \( f \), which is defined by
\[
\text{esssup} f := \Omega \setminus \bigcup \{ U \subset \Omega \text{ open: } f|_U = 0 \text{ almost everywhere} \}.
\]

Exercise 2.4. Let \( T \in \mathcal{D}'(\Omega) \) and \( \chi \in C^\infty(\Omega) \) with \( \chi|_{\text{supp}\ T} = 1 \). Show that \( \chi T = T \).

Exercise 2.5. Let \( T \in \mathcal{D}'(\Omega) \) and \( \alpha \in \mathbb{N}^n \). Show that \( \text{supp}(\partial^\alpha T) \subset \text{supp}(T) \). Find a simple example where the inclusion is strict.

The following exercise explains the terminology “compactly supported distribution” for the elements of \( \mathcal{E}' \): These are in fact precisely those elements of \( \mathcal{D}' \) whose support is compact.

Exercise 2.6. Let \( T \in \mathcal{D}'(\Omega) \). Show that \( \text{supp} T \subset \Omega \) is compact if and only if \( T \) extends to a continuous linear functional on \( \mathcal{E}(\Omega) \).

Hint: Recall the characterization of \( \mathcal{E} \)-continuity given in Exercise 2.68.

As another illustration of how to use the concept of support of a distribution we completely characterize all those distributions whose support consists of a single point.

Theorem 2.2.34 (Distributions supported at a point). Let \( T \in \mathcal{D}'(\Omega) \) with \( \text{supp}\ T = \{ x_0 \} \). Then there exist (unique) \( m \in \mathbb{N} \) and \( c_\alpha \in \mathbb{C} \) for all \( \alpha \in \mathbb{N} \) with \( |\alpha| \leq m \) such that
\[
T = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \delta_{x_0}. \tag{2.2.30}
\]

Proof. Reference, or exercise. \( \square \)

Besides the support, another local concept is that of the singular support which describes where a distribution is not smooth. Similar as the support, this is expressed via the action on test functions.

Definition 2.2.35. Let \( T \in \mathcal{D}'(\Omega) \). We say that \( T \) is smooth at a point \( x \in \Omega \) if there exists an open neighborhood \( U \subset \Omega \) of \( x \) and \( f \in C^\infty(U) \) with \( f|_U = T_f \). The set
\[
\text{singsupp} u := \Omega \setminus \{ x \in \Omega \mid u \text{ is smooth at } x \} \tag{2.2.31}
\]
is called the singular support of \( u \).
Be careful that even a regular distribution can have nonempty singular support since regular distributions need only be locally integrable functions whereas the singular support is about (non-)smoothness (cf. exercise 2.7).

**Example 2.2.36.** We show that $$\text{singsupp} \delta_0 = \{0\}$$. Clearly $$\text{singsupp} \delta_0 \subset \{0\}$$ since for any $$x \neq 0$$ and any $$f \in \mathcal{D}(\mathbb{R})$$ with $$\text{supp } f \subset B_{|x|/2}(x)$$ we have $$\langle \delta, f \rangle = f(0) = 0$$, hence $$\delta_0$$ coincides on $$B_{|x|/2}(x)$$ with the smooth constant function $$0$$. On the other hand, let $$U \subset \mathbb{R}$$ be any open neighborhood of $$0$$. Pick $$\epsilon > 0$$ with $$[\epsilon, \epsilon] \subset U$$ and a sequence $$(u_n)_n$$ of test functions with $$\text{supp } u_n \subset (-\epsilon, \epsilon)$$, $$\|u_n\|_{L^1} \leq 1$$ and $$u_n(0) = n$$. Then $$\langle \delta_0, u_n \rangle = u_n(0) = n \to \infty$$, whereas for any $$f \in C^\infty(U)$$ we have

$$\left| \langle T_f, u_n \rangle \right| = \left| \int_{-\epsilon}^{\epsilon} f(x) u_n(x) \, dx \right| \leq \|f\|_{C^0([-\epsilon, \epsilon])} \|u_n\|_{L^1(\mathbb{R})} \leq \|f\|_{C^0([-\epsilon, \epsilon])} \cdot$$

Since this remains bounded for $$n \to \infty$$, it follows that $$\delta_0|_U \neq T_f$$. Hence $$0 \in \text{singsupp} \delta_0$$.

**Exercise 2.7.** Let $$f \in C(\mathbb{R}) \subset L^1_{\text{loc}}(\mathbb{R})$$ be given by $$f(x) = |x|$$. Compute $$\text{singsupp } T_f$$.

**Exercise 2.8.** Show that the singular support of a distribution is always closed and contained in the support of the distribution.

**Exercise 2.9.** Let $$T \in \mathcal{D}'(\Omega)$$ and $$\alpha \in \mathbb{N}^n$$. Show that $$\text{singsupp}(T) = \text{singsupp}(\partial^\alpha T)$$.

**Exercise 2.10.** Let $$T \in \mathcal{D}'(\Omega)$$ and $$f \in C^\infty(\Omega)$$. Show that $$\text{singsupp}(T) \subset \text{singsupp}(fT)$$. Give a simple example, where equality fails.

**Exercise 2.11.** Let $$S, T \in \mathcal{D}'(\Omega)$$. Show that $$\text{singsupp}(S + T) \subset \text{singsupp } S + \text{singsupp } T$$. Give a simple example where equality fails.

**Exercise 2.12.** Compute the singular support of regular distribution $$T_H \in \mathcal{D}'(\mathbb{R})$$ where $$H \in L^1_{\text{loc}}(\mathbb{R})$$ is the Heaviside function (cf. Example [2.2.21]).

**Exercise 2.13.** Compute the singular support of the Cauchy principal value of $$\frac{1}{x}$$ (cf. Exercise ??).

**Exercise 2.14.** Let $$T \in \mathcal{D}'(\Omega)$$. Show that $$\text{singsupp } T = \emptyset$$ if and only if $$T = T_f$$ for a smooth function $$f \in C^\infty(\Omega)$$.

### 2.2.7. Convolution and regularization of distributions.

We have already seen in Theorem [2.2.10] that the $$\delta$$-distribution can be approximated (in the sense of distributions) by smooth functions. The same is actually true for any distribution. Similarly as the approximation of non-smooth functions by smooth ones, this can be shown by the technique of convolution.

We start by recalling some important results about convolution of functions. First of all, the convolution of two functions $$f$$ and $$g$$ is the function $$f \ast g$$ defined by

$$(f \ast g)(x) := \int_{\mathbb{R}^n} f(x - y) g(y) \, dy.$$ (2.2.32)

In order for this integral to be well-defined, one of course needs to assume that $$f$$ and $$g$$ satisfy suitable integrability properties. One obvious sufficient condition is for instance that $$f \in L^1_{\text{loc}}(\mathbb{R}^n)$$ and $$g \in C_0(\mathbb{R}^n)$$. We will focus on this case in the following, but also other conditions are sufficient (see ??).

One of the most interesting features of convolution is that $$f \ast g$$ is as smooth as the smoother one of the two functions $$f$$ and $$g$$. Therefore convolution can be used to “smoothen” non-smooth functions. This and other important properties of the convolution of functions are summarized in the following statement.
The expression in the middle makes more explicit on which variable the distribution can be viewed as a sort of generalization of “differentiating under the integral”. The proof we leave it as an exercise.

.proof proceeds by working out difference quotients explicitly without using any special tricks.

\( \psi \) and \( T \) see how this may be reasonably defined, note that we can view (2.2.32) as action of the \( \langle \cdot, \psi \rangle \) by a sequence of smooth functions. To do so, we need to study the dependence of \( \langle T, \psi \rangle \) on the “parameter” \( x \). Here the following more general result is useful, which can be viewed as a sort of generalization of “differentiating under the integral”. The proof proceeds by working out difference quotients explicitly without using any special tricks. We leave it as an exercise.

Exercise 2.15. Give (or look up) a proof of Theorem 2.2.37.

Hint: The proof of Theorem 2.2.16 might give you some idea for estimating the integrals.

We now extend the process of convolution from test functions to distributions. To see how this may be reasonably defined, note that we can view (2.2.32) as action of the distribution \( T_g \) on the “shifted test function” \( y \mapsto f(x-y) \).

Definition 2.2.38 (Convolution of distribution with test function). For \( T \in \mathcal{D}(\mathbb{R}^n) \) and \( \psi \in C_0^\infty(\mathbb{R}^n) \), the convolution of \( u \) with \( \psi \) is the function \( T * \psi : \mathbb{R}^n \to \mathbb{C} \) defined by

\[
(T * \psi)(x) := \langle T(y), \psi(x-y) \rangle := \langle T, \psi(x-\cdot) \rangle.
\]

(2.2.36)

The expression in the middle makes more explicit on which variable the distribution \( T \) acts.

Note that the convolution of a distribution with a smooth function is a function. Note also that by this definition we cannot take the convolution of two distributions (this is possible but requires further thought).

Exercise 2.16. Show that for \( T \in \mathcal{D}'(\mathbb{R}^n) \) and \( \psi \in C_0^\infty(\mathbb{R}^n) \) one has

\[
\text{supp}(T * \psi) \subset \text{supp} T + \text{supp} \psi.
\]

We now want to show that the convolution of a distribution with a smooth function is again a smooth function, and use this to show that any distribution can be approximated by a sequence of smooth functions. To do so, we need to study the dependence of \( \langle T, \psi(x-\cdot) \rangle \) on the “parameter” \( x \). Here the following more general result is useful, which can be viewed as a sort of generalization of “differentiating under the integral”. The proof proceeds by working out difference quotients explicitly without using any special tricks. We leave it as an exercise.

We now apply Theorem 2.2.47 to the convolution of a distribution with a test function.

Corollary 2.2.39. For any \( T \in \mathcal{D}(\mathbb{R}^n) \) and \( \psi \in C_0^\infty(\mathbb{R}^n) \), we have \( T * \psi \in C^\infty(\mathbb{R}^n) \).

Proof. If \( \psi \in C_0^\infty(\mathbb{R}^n) \), then the function \( \varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) given by \( \varphi(x,y) = \psi(x-y) \) satisfies the assumptions of Theorem 2.2.47. Therefore \( T * \psi \) is smooth. \( \square \)
Our next aim is to show that if $T \in \mathcal{D}'(\Omega)$ and if $(\psi_k)_k$ is a (model) delta sequence, then $T \ast \psi_k \to T$ in $\mathcal{D}'$. This shows that every distribution can be approximated by smooth functions. By definition of $\mathcal{D}'$-convergence, we have to show that

$$\forall u \in \mathcal{D}(\Omega) : \langle T * \psi_k, u \rangle \to \langle T, u \rangle$$

The idea is simply to “move the convolution to the other side” and use that $u * \psi_k \to u$ in $\mathcal{D}(\Omega)$ by Theorem [2.2.37]. To this end, a sort of generalization of Fubini’s theorem is required. To see this, consider first the case that $T = T_f$ is a regular distribution. Then using Fubini’s Theorem one has

$$\langle T_f * \psi, u \rangle = \int_{\mathbb{R}^n} (T_f * \psi)(x)u(x) \, dx$$

$$= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y)\psi(x-y) \, dy \right) u(x) \, dx$$

$$= \left( \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} u(x)\psi(x-y) \, dx \right) \, dy \right)$$

$$= \langle T_f, \tilde{\psi} * u \rangle,$$

where in the last line we used the notation $\tilde{\psi}(x) := \psi(-x)$ for the “reflected function” (note the order of variables in the inner integral in the third line). We now show that this still holds true if $u$ is a distribution.

**Proposition 2.2.40.** Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then for all $\psi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\langle T * \varphi, \psi \rangle = \langle T, \tilde{\varphi} * \psi \rangle,$$

(2.2.38)

where $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^n)$ denotes the “reflected function” with $\tilde{\varphi}(x) = \varphi(-x)$.

**Proof.** Choose $R > 0$ sufficiently large such that $\text{supp} \, \psi \subseteq [-R,R]^n$. Then $(T * \varphi)\psi \in C^\infty_0([-R,R]^n)$. Making a Riemann sum approximation we have **have to say more precisely how $[R,R]^n$ is cut apart and what the $x_j$ are**

$$\langle T * \varphi, \psi \rangle = \int_{[-R,R]^n} (T * \varphi)(x)\psi(x) \, dx$$

$$= \int_{[-R,R]^n} \langle T(y), \varphi(x-y) \rangle \psi(x) \, dx$$

$$= \lim_{N \to \infty} \sum_{j=1}^N \langle T(y), \varphi(x_j-y) \rangle \psi(x_j) \left( \frac{2R}{N} \right)^n$$

$$= \lim_{N \to \infty} \langle T(y), \sum_{j=1}^N \tilde{\varphi}(y-x_j)\psi(x_j) \left( \frac{2R}{N} \right)^n \rangle. \quad (*)$$

The sum in the second factor converges to $\tilde{\varphi} * \psi$ (pointwise, i.e., for fixed $y$). If we can argue that we can take the limit inside the second factor, then we are done. For this we need to show that the sum converges in $\mathcal{D}$ (cf. Definition [2.2.1] for $\mathcal{D}$-convergence). To this end, note first that for any $y \in \mathbb{R}^n$ the support of the function $x \mapsto \tilde{\varphi}(y-x)\psi(x)$ is contained in the fixed compact set $\text{supp} \, \varphi + \text{supp} \, \psi$. Moreover, since $x \mapsto \tilde{\varphi}(y-x)\psi(x)$ and also all of its derivatives are continuous and compactly supported, all of them are actually uniformly continuous functions. Using this one can show that the Riemann sum
converges uniformly to \( \hat{\varphi} \ast \psi \), and equally for all derivatives. We leave the details as an exercise. \( \square \)

**Exercise 2.17.** Let \( f, g \in C_0(\mathbb{R}^n) \) with \( \text{supp}(fg) \subset [-R, R]^n \) for some \( R > 0 \). For any \( N \in \mathbb{N} \), decompose \([-R, R]^n\) into \( N^n \) cubes of same size and let \( x_1, \ldots, x_{N^n} \) denote their middle points. Show that the functions \( x \mapsto \sum_{j=1}^{N^n} f(x - x_j)g(x_j) \left( \frac{R}{N} \right)^n \) converges uniformly (in \( x \)) to \( x \mapsto (f \ast g)(x) \) as \( N \to \infty \).

It now follows immediately that any distribution can be approximated by a sequence of smooth functions, even compactly supported ones by using a simple cutoff trick.

**Corollary 2.2.41** (Distributions can be approximated by smooth functions). \( \mathcal{D}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n) \) is dense, i.e., for any \( T \in \mathcal{D}'(\mathbb{R}^n) \) there exists a sequence \((f_k)_k \subset \mathcal{D}(\mathbb{R}^n)\) such that \( T_{f_k} \to T \) in \( \mathcal{D}' \).

**Proof.** Let \( T \in \mathcal{D}'(\mathbb{R}^n) \). Choose a strict delta sequence \((\psi_k)_k\) and a bump function \( \chi \in C_0^\infty(\mathbb{R}^n) \) with \( \chi|_{B_1(0)} = 1 \). Set

\[
T_k := \langle \chi(\frac{x}{k})T, \psi_k \rangle \in \mathcal{D}(\mathbb{R}^n),
\]

then for any \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) we have

\[
\langle T_k, \varphi \rangle = \langle \chi(\frac{x}{k})T, \hat{\psi}_k \ast \varphi \rangle = \langle T, \chi(\frac{x}{k})(\hat{\psi}_k \ast \varphi) \rangle.
\]

It is easy to verify that \( \chi(\frac{x}{k})(\hat{\psi}_k \ast \varphi) \to \varphi \) in \( \mathcal{D} \), so by continuity of \( T \) the right-hand side converges to \( \langle T, \varphi \rangle \). This shows that \( T \ast \psi_k \to T \) in \( \mathcal{D}'(\mathbb{R}^n) \) as desired. \( \square \)

**The following can probably also be left out.** So far we have only defined the convolution of a distribution with a smooth function. Building on this it is not difficult to define also the convolution of two distributions. The definition for this can again be motivated by regarding smooth functions as (regular) distributions. So let \( f \in C^\infty(\mathbb{R}^n) \) and \( g \in C_0^\infty(\mathbb{R}^n) \), and let \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) be a test function. Then the action of the distribution \( T_{f \ast g} \) on \( \varphi \) can be rewritten using Fubinis theorem as

\[
\langle T_{f \ast g}, \varphi \rangle = \int_{\mathbb{R}^n} (f \ast g)(x) \varphi(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y)g(x - y) \, dy \right) \varphi(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} g(x - y) \varphi(x) \, dx \right) \, dy
\]

\[
= \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} \hat{g}(y - x) \varphi(x) \, dx \right) \, dy
\]

\[
= \int_{\mathbb{R}^n} f(y)(T \hat{g} \ast \varphi)(y) \, dy
\]

\[
= \langle T_f, T \hat{g} \ast \varphi \rangle.
\]

Here we again used the “check-notation” for the reflection of a function. With the last line we have almost arrived at an expression where the regular distributions \( T_f \) and \( T \hat{g} \) can be replaced by general distributions, we only have to explain first how to extend the “check-operation” to distributions. We leave this as a simple exercise to the reader.

**Exercise 2.18.** Let \( T \in \mathcal{D}'(\mathbb{R}^n) \). Show that \( \varphi \mapsto \langle \hat{T}, \varphi \rangle := \langle T, \hat{\varphi} \rangle \) defines a distribution \( \hat{T} \in \mathcal{D}'(\mathbb{R}^n) \). Show further that \( T_f = T_f \) for any \( f \in C^\infty(\mathbb{R}^n) \).
2.2. DISTRIBUTIONS AND FOURIER TRANSFORMATION

**Definition 2.2.42 (Convolution of two distributions).** For \( S \in \mathcal{D}'(\mathbb{R}^n) \) and \( T \in \mathcal{E}'(\mathbb{R}^n) \), the convolution of \( S \) with \( T \) is the distribution \( S * T \in \mathcal{D}'(\mathbb{R}^n) \) defined by

\[
\langle S * T, \varphi \rangle := \langle S, \hat{T} * \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{2.2.39}
\]

Here the distribution \( \hat{T} \) is defined by \( \langle \hat{T}, \varphi \rangle := \langle T, \hat{\varphi} \rangle \) for all \( \varphi \in \mathcal{D}(\mathbb{R}^n) \), cf. Exercise 2.11.

The verification that (2.2.39) indeed defines a distribution is left as an exercise.

**Exercise 2.19.** Let \( S \in \mathcal{D}'(\mathbb{R}^n) \) and \( T \in \mathcal{E}'(\mathbb{R}^n) \). Show that (2.2.39) really defines a distribution, i.e., a (sequentially) continuous linear functional on \( \mathcal{D}(\mathbb{R}^n) \).

Using convolution of two distributions one can show an interesting structural result about distributions, namely that locally, any distribution is the (distributional) derivative of a continuous function.

**Theorem 2.2.43 (Local structure of distributions).** Let \( T \in \mathcal{D}'(\mathbb{R}^n) \). Then for any bounded domain \( \Omega \subset \mathbb{R}^n \) there exists \( f \in C(\mathbb{R}^n) \) and \( \alpha \in \mathbb{N}^n \) such that

\[
T|_\Omega = (\partial^n T_f)|_\Omega. \tag{2.2.40}
\]

Here the right-hand side has to be understood in the distributional sense.

**Proof.** The idea is to write \( u = u * \delta \) and then write \( \delta = \partial^n E \) with some continuous \( E \) such that \( u * E \) is continuous. \( \square \)

**Corollary 2.2.44 (Global structure of compactly supported distributions).** Let \( u \in \mathcal{E}'(\mathbb{R}^n) \), and let \( \Omega \subset \mathbb{R}^n \) be an open neighborhood of \( \text{supp} \, u \). Then there exists \( m \in \mathbb{N} \) and functions \( f_\alpha \in C_0(\Omega) \) for all \( \alpha \in \mathbb{N}^n \) with \( |\alpha| \leq m \) such that

\[
u = \sum_{|\alpha| \leq m} \partial^n f_\alpha. \tag{2.2.41}\]

The right-hand side again has to be understood in the distributional sense.

- convolution of two distributions
- some examples, or nice formulae
- maybe solve some evolution equations by convolution as motivation

The following only makes sense if the convolution of two distributions has been defined before. We finish this section by showing that the convolution theorem (2.2.25) about the Fourier transform of a convolution still holds for the convolution of a tempered distribution and a compactly supported one. Note that without the compact support assumption the multiplication of the two Fourier transforms wouldn’t even be defined.

**Theorem 2.2.45.** Let \( u \in \mathcal{S}'(\mathbb{R}^n) \) and \( v \in \mathcal{E}'(\mathbb{R}^n) \). Then \( u * v \in \mathcal{S}'(\mathbb{R}^n) \) and

\[
\hat{u} \ast \hat{v} = \hat{u} \cdot \hat{v}. \tag{2.2.42}
\]

in the sense of tempered distributions.

**Proof.** To be done! \( \square \)

Compact support seems like a somewhat brutal decay assumption, and one can indeed relax it. For instance, the convolution theorem also holds for \( u \in \mathcal{S}'(\mathbb{R}^n) \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \).\footnote{Note that \( \mathcal{S}(\mathbb{R}^n) \not\subset \mathcal{E}'(\mathbb{R}^n) \).} In this case one can show that \( u * \varphi \in \mathcal{O}_M(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \) and \( \hat{u} \ast \hat{\varphi} = \hat{u} \cdot \hat{\varphi} \) again holds in the sense of tempered distributions.

The convolution theorem is important for instance because one can use it to produce solution formulas for certain (linear) evolution equations. We illustrate this by one example and give some further exercises for you to work out.
Example 2.2.46. We use the convolution theorem to solve the Cauchy problem for the free Schrödinger equation, i.e.
\[
\begin{cases}
\mathcal{I} \partial_t \psi(t, x) = -\Delta \psi(t, x) \\
\psi(0, x) = \psi_0(x)
\end{cases}
\]
Hereby we assume that $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$. Moreover, we use that . . . Taking the Fourier transform in $x$, we obtain the equation
\[
\mathcal{I} \partial_t \widehat{\psi}(t, \xi) = |\xi|^2 \widehat{\psi}(t, \xi).
\]
This implies that
\[
\widehat{\psi}(t, \xi) = e^{-t|\xi|^2} \widehat{\psi_0}(\xi).
\]
Now notice that $\widehat{\psi_0} \in \mathcal{S}(\mathbb{R}^n)$, whereas the smooth and bounded function $e^{-t|\xi|^2}$ may be viewed as tempered distribution. Therefore it follows from the convolution theorem that
\[
\psi(t, x) = \mathcal{F}_{x \to \xi}^{-1} \left( e^{-t|\xi|^2} \right) * \psi_0(x) = \frac{1}{(4\pi t)^n} \int_{\mathbb{R}^n} e^{-\frac{|t| \xi^2}{4t}} \psi_0(y) \, dy. \quad (2.2.43)
\]
Here we used that the inverse Fourier transform of the complex Gaussian $\exp(-t|\xi|^2)$ is the complex Gaussian $(4\pi t)^{-n} \exp(-\frac{|\xi|^2}{4t})$.

Exercise 2.20. Use the convolution theorem to solve the Cauchy problem for the heat equation
\[
\begin{cases}
\partial_t u(t, x) = \Delta u(t, x) \\
u(0, x) = u_0(x)
\end{cases}
\]
Assume that $u_0 \in C^\infty(\mathbb{R}^n)$. Why does this only work for positive times?

Exercise 2.21. Let $P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha$ be some polynomial in $\xi$. We write $P(D)$ for the differential operator arising from $P$ by formally replacing every $\xi$ by $\mathcal{I} \partial$. How would $P(\xi)$ look like in order that $P(D) = \Delta$? Consider the first-order initial value problem
\[
\begin{cases}
\partial_t u(t, x) = P(D)u(t, x) \\
u(0, x) = u_0(x)
\end{cases}
\]
Using the convolution theorem formally, i.e. without worrying about any convergence or differentiability issues, find a representation formula for the solution. Can you find some conditions on $P(\xi)$ such that this formula makes rigorous sense for $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and yields a true solution?

Exercise 2.22. Use the convolution theorem to solve the Cauchy problem for the wave equation
\[
\begin{cases}
\Box u = 0 \\
u(0, x) = u_0(x) \\
\partial_t u(0, x) = u_1(x)
\end{cases}
\]
Here $\Box = -\partial_t^2 + \Delta$ is the wave operator. You do not need to compute the inverse Fourier transforms explicitly this time.

2.2.8. The technical toolbox.

Theorem 2.2.47 ("Differentiation under the integral"). Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ be open and $u \in C^\infty(\Omega_1 \times \Omega_2)$. Suppose that $u$ is compactly supported in the first variable with 
"locally uniformly supports in the second variable", i.e., suppose that for every $y_0 \in \Omega_2$ there exists an open neighborhood $U \subset \Omega_2$ of $y_0$ and a compact set $K \subset \Omega_1$ such that
\[
\forall y \in U : \text{ supp } u(\cdot, y) \subset K. \quad (2.2.44)
\]
Then for every distribution $T \in \mathcal{D}'(\Omega_1)$, the function
\[
\Omega_2 \ni y \mapsto \langle T, u(\cdot, y) \rangle \in \mathbb{C} \quad (2.2.45)
\]
is smooth, and for every $\alpha \in \mathbb{N}^n$ one has
\[
\partial_y^\alpha \langle T, u(\cdot, y) \rangle = \langle T, \partial_y^\alpha u(\cdot, y) \rangle. \tag{2.2.46}
\]

**Proof.** One can explicitly refer to difference quotients and estimate everything appropriately. The mean value theorem is often useful. \hfill \Box

In the previous theorem, the support assumption (2.2.44) is automatically satisfied if the test function $u$ has compact support. If on the other hand the distribution $T$ has compact support, one can in fact drop the support assumption on the test function altogether. You may prove this in exercise 2.99.

**Proposition 2.2.48.** Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ be open. Then
\[
\text{span}\{f \otimes g \mid f \in \mathcal{D}(\Omega_1), g \in \mathcal{D}(\Omega_2)\} \subset \mathcal{D}(\Omega_1 \times \Omega_2)
\]
is dense.

**Theorem 2.2.49 (Tensor product of distributions).** Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ be open. For any $S \in \mathcal{D}'(\Omega_1)$ and $T \in \mathcal{D}'(\Omega_2)$ there exists a unique $S \otimes T \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ such that
\[
\langle S \otimes T, u \otimes v \rangle = \langle S, u \rangle \langle T, v \rangle \tag{2.2.47}
\]
holds for all $u \in \mathcal{D}(\Omega_1)$ and $v \in \mathcal{D}(\Omega_2)$. Moreover, the following properties hold.

1. For any $u \in \mathcal{D}(\Omega_1 \times \Omega_2)$ we have “Fubini’s theorem”:
\[
\langle S \otimes T, u \rangle = \langle S(x), \langle T(y), u(x,y) \rangle \rangle = \langle \langle S(x), u(x,y) \rangle, T(y) \rangle \tag{2.2.48}
\]

2. $\text{supp}(S \otimes T) = \text{supp}(S) \times \text{supp}(T)$

3. $\partial_x^\alpha \partial_y^\beta (S \otimes T) = (\partial_x^\alpha S) \otimes (\partial_y^\beta T)$

4. The map $(S, T) \mapsto S \otimes T$ is bilinear and (jointly) sequentially continuous.

**2.2.9. Further topics about distributions.** There is some material here which I commented out.

**2.2.10. Further exercises.**

**Exercise 2.23.** For any $n \in \mathbb{N}$ define $G_n : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by
\[
G_1(x) = \begin{cases} x & x > 0, \\ 0 & x \leq 0, \end{cases} \quad G_2(x) = \frac{1}{2\pi} \ln(|x|) \quad \text{and} \quad G_n(x) = -\frac{|x|^{2-n}}{(n-2)c_n} \text{ for } n \geq 3.
\]

Interpret $G_n$ as distribution (how?) and show that $\Delta_n G_n = \delta$ in the sense of distributions, where $\Delta_n = \sum_{j=1}^n \partial_j^2$ denotes the Laplace operator and $\delta$ the $\delta$-distribution at the origin.

**Exercise 2.24.** For $k > 0$, define $G_H : \mathbb{R}^3 \setminus \{0\} \to \mathbb{C}$ by
\[
G_H(x) = \frac{1}{4\pi|x|} e^{ik|x|}.
\]

Interpret $G_H$ as distribution (how?) and show that $(\Delta_3 + k^2)G_H = -\delta$, where $\Delta_3 = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator and $\delta$ the $\delta$-distribution at the origin. Show further that $G_H \to G_3$ for $k \to \infty$ in the sense of distributions, where $G_3$ is the Green’s function of the Laplace equation in three dimensions from Exercise 2.23.

JH: what else do we need?
2. MATHEMATICAL PRELIMINARIES

Exercise 2.25. For some parameter $\lambda > 0$ define $G_W : \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ by

$$G_W(x,t) := \frac{1}{(4\pi\lambda t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\lambda t}} \Theta(t).$$

Interpret $G_W$ as distribution on $\mathbb{R}^{n+1}$ and show that $(\partial_t - \lambda \Delta)G_W = \delta$. Compute the singular support of $G_W$.

Exercise 2.26. For some constant $c \in \mathbb{R}$ define $G_S : \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \to \mathbb{C}$ by

$$G_S(x,t) = \frac{c}{t^\frac{n}{2}} e^{-\frac{|x|^2}{4i t}} \Theta(t).$$

(1) Show that $G_S$ is locally integrable if and only if $n = 1$.
(2) Show that

$$\langle G_S, \phi \rangle := \lim_{\epsilon \to 0} \int_\epsilon^\infty \int_{\mathbb{R}^n} \frac{c}{t^\frac{n}{2}} e^{-\frac{|x|^2}{4i t}} \phi(x,t) \, dx \, dt$$

defines a distribution.
(3) Determine $c$ such that $(i\partial_t + \Delta)G_S = \delta$ in the sense of distributions.
(4) Compute the singular support of $G_S$.

Exercise 2.27. Define $I : \mathbb{R}^2 \setminus \{0\} \to \mathbb{C}$ by

$$I(x,y) = \frac{1}{\pi(x + iy)}.$$ 

Explain how $I$ may be understood as distribution on $\mathbb{R}^2$. Show that $\frac{1}{2}(\partial_x + i\partial_y)I = \delta$ holds in the sense of distributions.

Exercise 2.28. Show that

$$\langle G^\pm, \phi \rangle := \int_{\mathbb{R}^3} \frac{1}{4\pi|x|} \phi(\pm|x|, x) \, d^3x$$

defines distributions $G^\pm \in \mathcal{D}'(\mathbb{R}^4)$ (pay attention to dimensions here). Compute the support and singular support of $G^\pm$. Show that $(\partial_t^2 - \Delta)G^\pm = \delta$ holds in the sense of distributions.

2.3. Spectral Theory in Hilbert Spaces

The main objective of this chapter is to explain to you the so-called spectral theorem for selfadjoint operators on Hilbert spaces.

Theorem 2.3.1 (Spectral theorem). Let $\mathcal{H}$ be a complex Hilbert space and let $A \in \mathcal{L}(\mathcal{H})$ be a (bounded) selfadjoint linear operator on $\mathcal{H}$. Then there exists a unique spectral measure $E : \mathcal{B}(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ on the Borel $\sigma$-algebra over the spectrum of $A$ such that

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda).$$

(2.3.1)

Moreover, for any bounded measurable function $f : \sigma(A) \to \mathbb{C}$ integration against $E$ defines a bounded operator $f(A) \in \mathcal{L}(\mathcal{H})$ via $f(A) := \int_{\sigma(A)} f(\lambda) \, dE(\lambda)$. These operators satisfy the following properties:

This important result is a generalization of the well-known fact that a Hermitian matrix may be diagonalized to the case of infinite-dimensional (Hilbert) spaces. All the notions appearing in this theorem will be explained in the following sections.
2.3.1. The finite-dimensional case. Let \( n \in \mathbb{N} \) and \( A \in \mathbb{C}^{n \times n} \) be given, and assume that \( A \) is hermitian, i.e., \( A^* = A \), where \( A^* = \overline{A} \) is the hermitian transpose, i.e., complex conjugate and transpose of \( A \). The following facts should be familiar from linear algebra:

- All eigenvalues of \( A \) are real numbers,
- eigenvectors belonging to different eigenvalues are always orthogonal,
- there exists an orthonormal basis of \( \mathbb{C}^n \) consisting of eigenvectors of \( A \).

Counting with multiplicities, we enumerate the eigenvalues of \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) and choose a corresponding orthonormal basis of eigenvectors \( v_1, \ldots, v_n \in \mathbb{C}^n \). If we denote by \( U \in \mathbb{C}^{n \times n} \) the matrix whose columns are precisely \( v_1, \ldots, v_n \), then \( U \) is unitary and the matrix \( U^*AU \) is the diagonal matrix with entries \( \lambda_1, \ldots, \lambda_n \) on the diagonal.

We now reformulate this in such a way that it becomes apparent why this is precisely the same statement as the spectral theorem from above. To this end, we write \( A = UDU^* \) and symbolically compute with matrices as follows:

\[
A = UDU^* = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1^* \\ \vdots \\ v_n^* \end{pmatrix} = \begin{pmatrix} v_1^* \\ \vdots \\ v_n^* \end{pmatrix} = \lambda_1 v_1 v_1^* + \ldots + \lambda_n v_n v_n^*.
\]

(2.3.2)

To make the connection to the spectral theorem we observe that this is exactly the same as (2.3.1) if we define the spectral measure \( E : \mathcal{B}(\sigma(A)) \to \mathbb{C}^{n \times n} \) by

\[
E(\Omega) := \sum_{i \in \{1, \ldots, n\} / \lambda_i \in \Omega} v_i v_i^*.
\]

(2.3.3)

Here the sum is over all eigenvalues (with multiplicities) lying in \( \Omega \). In other words, \( E(\Omega) \) is the sum of the projection on all those eigenspaces whose eigenvalue lies in the set \( \Omega \). We leave it as an exercise to write out the integral (2.3.1) for this spectral measure concretely (one should first read the part about spectral measures).

Exercise 2.29 (Spectral measure of a hermitian matrix). Verify that the function defined by (2.3.3) satisfies the properties of a spectral measure. Then verify that for this measure, the integral (2.3.1) coincides with (2.3.2).

2.3.2. Hilbert spaces.

2.3.3. Bounded operators and their spectra.

Definition 2.3.2. Let \( \mathcal{H} \) be a Hilbert space. A bounded operator on \( \mathcal{H} \) is a linear continuous map from \( \mathcal{H} \) to \( \mathcal{H} \). The set of all bounded operators on \( \mathcal{H} \) is denoted by \( L(\mathcal{H}) \).

Since (pointwise) sums and multiples of linear, continuous maps are again linear and continuous again, the set \( L(\mathcal{H}) \) is in fact a vector space with respect to (pointwise) addition and scaling. The terminology “bounded” is due to the fact that a linear map \( A : \mathcal{H} \to \mathcal{H} \) is continuous if and only there exists a constant \( C > 0 \) such that

\[
\forall u \in \mathcal{H} : \quad \|Au\| \leq C \cdot \|u\|.
\]

(2.3.4)

You are asked to show this in exercise 2.30. Assigning to each bounded operator the optimal constant \( C > 0 \) for which (2.3.4) holds yields a norm on \( L(\mathcal{H}) \) which turns this space into a Banach space.

\footnote{In BraKet-notation the last line of this calculation would read \( A = \lambda_1 |v_1> <v_1| + \ldots + \lambda_n |v_n> <v_n| \).}
Theorem 2.3.3 (Operator norm). For $A \in L(\mathcal{H})$, define

$$
\|A\|_\infty := \sup_{x \in \mathcal{H}, \|x\|=1} \|Ax\|.
$$

(2.3.5)

Then $\|A\|_\infty < \infty$, and this number is the smallest possible constant $C \geq 0$ such that (2.3.4) holds. Moreover, $L(\mathcal{H})$, $(L(\mathcal{H}), \|\cdot\|_\infty)$ is a Banach space.

The proof of this result is straightforward and you are asked for it in Exercise 2.31.

Exercise 2.30. Let $A : \mathcal{H} \to \mathcal{H}$ be linear. Show that the following are equivalent:

1. $A$ is continuous.
2. $A$ is continuous in $0$.
3. $\exists C > 0 : \forall x \in \mathcal{H} : \|Ax\| \leq C \cdot \|x\|$.

Exercise 2.31. Prove Theorem 2.3.3.

Now we turn towards the spectrum of a bounded operator which generalizes the set of all eigenvalues of a matrix.

Definition 2.3.4. Let $A \in L(\mathcal{H})$ be given. Set

$$
\rho(A) := \{ \lambda \in \mathbb{C} : \lambda \text{id}_\mathcal{H} - A \text{ is invertible} \},
$$

(2.3.6)

$$
\sigma(A) := \mathbb{C} \setminus \rho(A).
$$

(2.3.7)

One calls $\rho(A)$ the resolvent set and $\sigma(A)$ the spectrum of $A$.

Let us make some remarks about the spectrum of an operator. For given $A \in L(\mathcal{H})$ and $\lambda \in \mathbb{C}$, there are a priori two different reasons why one might have $\lambda \in \sigma(A)$:

1. The map $\lambda \text{id}_\mathcal{H} - A$ is not injective. Due to linearity, in this case there exists $x \in \mathcal{H}$ with $(\lambda \text{id}_\mathcal{H} - A)x = 0$, i.e., $Ax = \lambda x$. Hence $\lambda$ is an eigenvalue of $A$.
2. The map $\lambda \text{id}_\mathcal{H} - A$ is injective but not surjective. In this case, $\lambda$ is not an eigenvalue of $A$.

For $\dim \mathcal{H} < \infty$ the second case is excluded by the rank-nullity-theorem. For $\dim \mathcal{H} = \infty$ the second case is possible, however, so $\sigma(A)$ may consist of more then just eigenvalues.

Exercise 2.32. Construct a bounded operator $A$ on $\ell^2$ which is injective but not surjective. Conclude that not every element of $\sigma(A)$ is an eigenvalue of $A$.

Let us remark that as a consequence of the so-called open mapping theorem (see ??), the inverse of a bounded operator on a Hilbert space is automatically bounded again. Hence one has $(\lambda \text{id}_\mathcal{H} - A)^{-1} \in L(\mathcal{H})$ for any $\lambda \in \rho(A)$. One calls the family of operators $\mathcal{R}(\lambda) := (\lambda \text{id}_\mathcal{H} - A)^{-1}$ with $\lambda \in \rho(A)$ the resolvent of $A$. The resolvent plays an important role in the proof of some of the following properties of the spectrum of a bounded operator, most notably the fact that by a geometric series argument it can be expressed as a power series in $\lambda$ around any point of $\rho(A)$.

Theorem 2.3.5 (Basic facts about the spectrum). Let $A \in L(\mathcal{H})$ be given. The following properties are satisfied:

1. The resolvent set $\rho(A) \subseteq \mathbb{C}$ is open.
2. The spectrum $\sigma(A) \subseteq \mathbb{C}$ is compact and $|\lambda| \leq \|A\|_\infty$ for any $\lambda \in \sigma(T)$.
3. If $\mathcal{H}$ is a complex Hilbert space, then $\sigma(A) \neq \emptyset$. 


Proof. Let $\lambda_0 \in \rho(A)$ be given. For $\lambda \in \mathbb{C}$ we have (“common denominator”)  
\[
\text{id}_\mathcal{H} - (\lambda_0 - \lambda) \cdot (\lambda_0 \text{id}_\mathcal{H} - A)^{-1} = (\lambda \text{id}_\mathcal{H} - A) \cdot (\lambda_0 \text{id}_\mathcal{H} - A)^{-1},
\]
and thus  
\[
\lambda \text{id}_\mathcal{H} - A = (\text{id}_\mathcal{H} - (\lambda - \lambda_0)(\lambda_0 \text{id}_\mathcal{H} - A)^{-1}) \cdot (\lambda_0 \text{id}_\mathcal{H} - A).
\]
If $|\lambda - \lambda_0| < \|(\lambda_0 \text{id}_\mathcal{H} - A)^{-1}\|$, then the Neumann series (cf. Exercise 2.33) provides the inverse to the first factor on the right-hand side. Hence for $|\lambda - \lambda_0| < \|(\lambda_0 \text{id}_\mathcal{H} - A)^{-1}\|$ we have $\lambda \in \rho(A)$ as well. This shows the first part, and of course also that $\sigma(A) \subseteq \mathbb{C}$ is open. By a similar argument one can show that for any $\lambda \in \mathbb{C}$, if $|\lambda| > \|A\|$, then $\lambda \in \rho(A)$. This concludes the proof of the second part. For a proof of the third part, see ??.

Exercise 2.33 (Neumann series). Let $A \in L(\mathcal{H})$ be given. Show that if $\sum_{k=0}^{\infty} A^k$ converges with respect to the operator norm on $L(\mathcal{H})$, then the limit is the inverse to $\text{id}_\mathcal{H} - A$. Moreover, show that for $\|A\| < 1$ the series always converges.

Exercise 2.34. Let $A \in L(\mathcal{H})$ be given. Show that the resolvent $R(\lambda) = (\lambda \text{id}_\mathcal{H} - A)$ can be expressed as a power series in $\lambda$ around any point $\lambda_0 \in \rho(A)$.

Hint: Use the Neumann series (cf. Exercise 2.33).

Theorem 2.3.6. Let $T \in L(\mathcal{H})$ be given. Then $T$ is invertible if and only if $T(\mathcal{H}) \subseteq \mathcal{H}$ is dense and there exists $\epsilon > 0$ such that $\|Tu\| \geq \epsilon \cdot \|u\|$ holds for all $u \in \mathcal{H}$.

Proof. If $T$ is invertible, then clearly its range is dense in $\mathcal{H}$. Setting $\epsilon := \|A^{-1}\|^{-1}$ it follows from boundedness that for any $u \in \mathcal{H}$ one has  
\[
\|u\| = \|A^{-1}Au\| \leq \|A^{-1}\| \cdot \|Au\| \quad \implies \quad \epsilon \cdot \|u\| \leq \|Au\|.
\]
Conversely, we first show that $T$ is bijective. Injectivity follows immediately from the given estimate. Concerning surjectivity, it suffices to show that the range of $T$ is closed. To this end, let $w_n \in \text{Ran}(T_n)$ be a sequence with $w_n \to w \in \mathcal{H}$. Choose $u_n \in \mathcal{H}$ with $w_n = Tu_n$. Note that by the above estimate  
\[
\|u_n - u_m\| \leq \frac{1}{\epsilon} \|Tu_n - Tu_m\| = \frac{1}{\epsilon} \|w_n - w_m\|.
\]
Since $(w_n)_n$ converges, it is a Cauchy sequence. The previous estimate shows that then also $(u_n)_n$ is a Cauchy sequence, hence $u_n \to u$ for some $u \in \mathcal{H}$. By continuity of $T$ it follows that $Tu = \lim_{n \to \infty} Tu_n = \lim_{n \to \infty} w_n = w$, so $w \in \text{Ran}(T)$. Continuity of the inverse map is also an immediate consequence of the given estimate.

An extremely useful concept for linear operators on Hilbert spaces is the notion of the adjoint of an operator.

Theorem 2.3.7 (The adjoint of an operator). Let $A \in L(\mathcal{H})$ be given. There exists a unique operator $A^* \in L(\mathcal{H})$ such that  
\[
\forall u, v \in \mathcal{H} : \quad \langle Au, v \rangle = \langle u, A^*v \rangle.
\]
This operator is called the adjoint operator of $A$ or, in short, the adjoint of $A$.

Proof. For fixed $v \in \mathcal{H}$, by Cauchy-Schwarz the map $\mathcal{H} \ni u \mapsto \langle Au, v \rangle \in \mathbb{C}$ is linear and bounded. Therefore, by the Riesz representation theorem ?? there exists a unique vector $A^*v \in \mathcal{H}$ such that $\langle Au, v \rangle = \langle u, A^*v \rangle$ for any $u \in \mathcal{H}$. As you may check, the map $v \mapsto A^*v$ defines a bounded operator on $\mathcal{H}$. 

JH: Not needed. Maybe as exercise.
Exercise 2.35. Fill in the details in the proof of Theorem 2.3.7.

Exercise 2.36. Let $A, B \in L(H)$ and $\alpha, \beta \in \mathbb{C}$.

1. Show that $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$. Moreover, show that $(A^*)^* = A$.

2. Show that $\|A^*\|_\infty = \|A\|_\infty$ and $\|AA^*\|_\infty = \|A\|_2^2$.

Exercise 2.36 shows that taking the adjoint of an operator has certain similarities to taking the complex conjugate of a complex number.

Definition 2.3.8. Let $A \in L(H)$.

1. $A$ is called selfadjoint if $A = A^*$.
2. $A$ is called unitary if $A^* A = \text{id}_H$ and $A A^* = \text{id}_H$.
3. $A$ is called normal if $A A^* = A^* A$.

Note that any selfadjoint and any unitary operator is a normal operator. Note also that any unitary operator $A$ is invertible and $A^{-1} = A^*$.

Exercise 2.37. Let $H$ be a complex Hilbert space and $A \in L(H)$. Show that $A$ is selfadjoint if and only if $(Ax, x) \in \mathbb{R}$ holds for all $x \in H$.

Normal operators are precisely those for which the spectral theorem holds. Using the following result it will suffice to focus on selfadjoint operators first.

Proposition 2.3.9. An operator $A \in L(H)$ is normal if and only if the (selfadjoint) operators $S, T \in L(H)$ defined by

\[ S := \frac{1}{2}(A + A^*), \quad T := \frac{1}{2i}(A - A^*) \]  

(2.3.9)

commute. One always has $A = S + iT$.

Exercise 2.38. Prove Proposition 2.3.9.

Being selfadjoint, unitary or normal has certain consequences for the spectrum of an operator. The “algebraic” consequences are probably familiar from Linear Algebra.

Theorem 2.3.10 (Spectral properties of selfadjoint, unitary and normal operators). Let $A \in L(H)$ be given.

1. If $A$ is selfadjoint, then $\sigma(A) \subseteq \mathbb{R}$.
2. If $A$ is unitary, then $\sigma(A) \subseteq S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.
3. If $A$ is normal, then eigenvectors corresponding to different eigenvalues are orthogonal.
4. If $A$ is normal, then the norm of $A$ is equal to the spectral radius, i.e., we have $\|A\| = \sup\{|z| : z \in \sigma(A)\}$. Moreover, if $H$ is a complex Hilbert space then there exists $\lambda \in \sigma(A)$ such that $|\lambda| = \|A\|$.

2.3.4. Spectral measures. Having covered the basics about operators, we now discuss the integration part of the spectral theorem, i.e., what a spectral measure is and how one may integrate with respect to such a measure. In general, all the notions are very similar as in ordinary (Lebesgue) measure and integration theory.

Definition 2.3.11 (Spectral measure). Let $\Omega$ be a set and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ a $\sigma$-algebra on $\Omega$. A spectral measure or also projection-valued measure on $(\Omega, \mathcal{A})$ is a map $E : \mathcal{A} \to L(H)$ satisfying the following properties:

1. $E(A)$ is an orthogonal (selfadjoint) projection for every $A \in \mathcal{A}$,
(2) \( E(\Omega) = \text{id}_H \),

(3) If \( A_1, A_2, \ldots \in \mathcal{A} \) are pairwise disjoint, and \( A := \bigcup_{j \in \mathbb{N}} A_j \), then it holds that

\[
\forall u \in H : \quad E(A)u = \sum_{j=1}^{\infty} E(A_j)u . \tag{2.3.10}
\]

Thus a spectral measure is just like an ordinary (positive) measure except for the fact that its values are orthogonal projections of some Hilbert space. There is also a very simple way in which one can extract “ordinary”, i.e., real- or complex-valued measures from a given spectral measure \( E \). Namely, one may form “expectation-values” and “transition probabilities”, i.e., for given \( u, v \in H \) one defines

\[
E_u : \mathcal{A} \to [0, \infty), \quad A \mapsto E_u(A) := \langle E(A)u, u \rangle , \tag{2.3.11}
\]

\[
E_{u,v} : \mathcal{A} \to \mathbb{C}, \quad A \mapsto E_{u,v}(A) := \langle E(A)u, v \rangle . \tag{2.3.12}
\]

In Exercise 2.39 you are asked to show that these functions are (positive, respectively complex) measures. Since \( E(A) \) is an orthogonal projection, for any \( u \in H \) one has

\[
E_u(A) = \langle E(A)u, u \rangle = \langle E(A)u, E(A)u \rangle = \|E(A)u\|^2 .
\]

**Exercise 2.39.** Let \( E : \mathcal{A} \to L(\mathcal{H}) \) is a spectral measure. Verify that \( E_u \) and \( E_{u,v} \) defined by (2.3.11) and (2.3.12) are positive, respectively complex measures for any given \( u, v \in H \). Moreover, verify the following properties:

(1) \( 4E_{u,v} = E_{u+v} + iE_{u+iw} - E_{u-v} - iE_{u-iv} \)

(2) \( E_{u,v} = \overline{E_{v,u}} \)

(3) \( |E_{u,v}(A)|^2 \leq E_u(A)E_v(A) \)

(4) If \( A_1, A_2, \ldots \) are pairwise disjoint, then \( \sum_{n=1}^{\infty} |E_{u,v}(A_n)| \leq \sqrt{E_u(A)E_v(A)} \)

**Exercise 2.40.** Let \((\Omega, \mathcal{A})\) be a measurable space and let \( E : \mathcal{A} \to L(\mathbb{C}) \) be a spectral measure. Show that \( E \) is a multiple of the Dirac measure at some point.

The fact that the values of a spectral measure are orthogonal projections has certain very useful consequences.

**Theorem 2.3.12 (Properties of spectral measures).** Let \( E : \mathcal{A} \to L(\mathcal{H}) \) be a spectral measure, and let \( A, B \in \mathcal{A} \) as well as \( A_1, A_2, \ldots \in \mathcal{A} \) be given. Then the following hold:

(1) \( E(A \cap B) = E(A)E(B) = E(B)E(A) \)

(2) If \( A \cap B = \emptyset \), then \( E(A)E(B) = 0 \) and therefore \( \text{Ran}(E(A)) \perp \text{Ran}(E(B)) \).

(3) If \( A \subseteq B \), then \( E(A)E(B) = E(A) \) and therefore \( \text{Ran}(E(A)) \subseteq \text{Ran}(E(B)) \).

(4) If \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \) and \( A := \bigcup_{j=1}^{\infty} A_j \), then \( E(A_n)u \to E(A)u \) for all \( u \in H \).

(5) If \( A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \) and \( A := \bigcap_{j=1}^{\infty} A_j \), then \( E(A_n)u \to E(A)u \) for all \( u \in H \).

**Exercise 2.41.** Prove Theorem 2.3.12

Now we come to integration with respect to a given spectral measure. This is done via the usual “Lebesgue-approach” starting from simple functions. In the following, let \( E : \mathcal{A} \to L(\mathcal{H}) \) denote a given spectral measure on a measurable space \( \Omega \).

**Step 1: Integration of simple functions**

Recall that a simple function is a measurable function \( f : \Omega \to \mathbb{C} \) which only takes finitely many distinct values. Such a function \( f \) can always be written as

\[
f = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j} \tag{2.3.13}
\]
for certain $a_1, \ldots, a_n \in \mathbb{C}$ and $A_1, \ldots, A_n \in \mathcal{A}$. For instance, one may take $a_1, \ldots, a_n$ to be the distinct values of $f$ and set $A_1 := f^{-1}({a_1}), \ldots, A_n := f^{-1}({a_n})$. For this choice the sets $A_1, \ldots, A_n$ are also disjoint. We now define the integral of $f$ over $\Omega$ as

$$T_E(f) := \int_{\Omega} f(x) dE(x) := \sum_{j=1}^{n} a_j E(A_j) \in L(H),$$

(2.3.14)

where $f = \sum_{j=1}^{n} a_j 1_{A_j}$ is any such representation of $f$ (the outcome in (2.3.14) is independent of this choice). Note that the integral (2.3.14) yields an operator.

**Step 2: Integration of bounded, measurable functions**

The extension of the integral to more general functions will be done by means of approximation, using the fact that every bounded, measurable function can be uniformly approximated by a sequence of simple functions. We start with some terminology.

**Definition 2.3.13.** Let $(\Omega, \mathcal{A})$ be a measurable space and $E : \mathcal{A} \to L(H)$ a spectral measure. For a bounded, measurable function $f : \Omega \to \mathbb{C}$ one calls

$$\text{ess sup}|f| := \inf \{ M \geq 0 : E(\{x \in \Omega : |f(x)| > M\}) = 0 \}$$

(2.3.15)

the essential supremum of $f$ (with respect to $E$). We denote the space of bounded, measurable functions by $L^\infty(\Omega)$, and set $L^\infty(\Omega) := L^\infty(\Omega)/\{ f \in L^\infty(\Omega) | \text{ess sup}|f| = 0 \}$.

As in usual Lebesgue measure theory, the space $L^\infty(\Omega)$ is a Banach space. Note that it depends on the measure $E$ through (2.3.15), although this is not depicted in the notation $L^\infty(\Omega)$. The space $L^\infty(\Omega)$ does not depend on $E$, but is not a Banach space since $\text{ess sup}|\cdot|$ is not a norm on $L^\infty(\Omega)$.

**Lemma 2.3.14.** Let $f : \Omega \to \mathbb{C}$ be a simple function and let $T_E(f) \in L(H)$ be defined by (2.3.14). Then

$$\|T_E(f)\|_\infty = \text{ess sup}|f|.$$  

(2.3.16)

**Proof.** Write $f = \sum_{j=1}^{n} a_j 1_{A_j}$ for disjoint $A_1, \ldots, A_n$. For the sake of this proof we may assume that $E(A_1), \ldots, E(A_n) \neq 0$, otherwise we can alter $f$ in such way that this holds and both $T_E(f)$ and $\text{ess sup}|f|$ remain unchanged. Then $\text{ess sup}|f| = \max\{|a_1|, \ldots, |a_n|\}$, and without loss of generality we may assume that $\text{ess sup}|f| = |a_1|$.

Since $E(A_1) \neq 0$ we may choose $u \in E(A_1)(H)$ with $\|u\| = 1$. Since $A_1, \ldots, A_n$ are disjoint, $E(A_1), \ldots, E(A_n)$ project onto pairwise orthogonal subspaces. Therefore

$$T_E(f)u = \sum_{j=1}^{n} a_j E(A_j)u = a_1 u,$$

and thus

$$\|T_E(f)u\|^2 = \langle T_E(f)u, T_E(f)u \rangle = |a_1|^2 \cdot \|u\|^2 = \text{ess sup}|f|^2.$$

This shows that $\|T_E(f)\|_\infty \geq \text{ess sup}|f|$. 

Concerning the reverse estimate, note that for any $u \in \mathcal{H}$ the vectors $E(A_1)u, \ldots, E(A_n)u$ are orthonormal. Therefore, setting $A := \bigcup A_j$ one has
\[
\|T_E(f)u\|^2 = \left\| \sum_{j=1}^{n} a_j E(A_j)u \right\|^2 = \sum_{j=1}^{n} \|a_j E(A_j)u\|^2 = \sum_{j=1}^{n} |a_j|^2 \|E(A_j)u\|^2 \\
\leq \|f\|_\infty^2 \cdot \sum_{j=1}^{n} \|E(A_j)u\|^2 = \|f\|_\infty^2 \cdot \left\| \sum_{j=1}^{n} E(A_j)u \right\|^2 = \|f\|_\infty^2 \cdot \|E(A)u\|^2 \\
\leq \|f\|_\infty^2 \cdot \|u\|^2.
\]
Note for the last step that $u = E(A)u + E(A^c)u$ is an orthogonal decomposition of $u$, and hence $\|u\|^2 = \|E(A)u\|^2 + \|E(A^c)u\|^2 \geq \|E(A)u\|^2$. □

By the obvious inequality $\text{ess sup}|f| \leq \sup |f|$ which holds for all $f \in L^\infty(\Omega)$ we have $\|T_E(f)\|_\infty \leq \sup |f|$ for all simple functions $f \in L^\infty(\Omega)$.

Combining (2.3.16) with the standard fact that any bounded measurable function can be uniformly approximated by simple functions (cf. Folland, Real Analysis, Theorem 2.10), it is easy to extend integration from simple functions to bounded measurable functions.

**Theorem 2.3.15** (Integration with respect to a projection-valued measures).

Let $(\Omega, \mathcal{A})$ be a measurable space and let $E : \mathcal{A} \to L(\mathcal{H})$ be a projection-valued measure. Then there exists a unique map $T_E : L^\infty(\Omega) \to L(\mathcal{H})$, assigning to each bounded measurable function a bounded operator, with the following properties:

1. The assignment $T_E : L^\infty(\Omega) \to L(\mathcal{H})$ is an isometric, involutive homomorphism of unital $*$-algebras, i.e., for any $f, g \in L^\infty(\Omega)$ and $\alpha, \beta \in \mathbb{C}$ one has
   \[
   T_E(\alpha f + \beta g) = \alpha T_E(f) + \beta T_E(g),
   \]
   \[
   T_E(fg) = T_E(f)T_E(g),
   \]
   \[
   T_E(f^*) = T_E(f)^*,
   \]
   \[
   T_E(1) = \text{id}_\mathcal{H},
   \]
   \[
   \|T_E(f)\|_\infty = \text{ess sup}|f|.
   \]

2. For any $f \in L^\infty(\Omega)$ and any $u, v \in \mathcal{H}$ it holds that
   \[
   \langle T_E(f)u, v \rangle = \int_\Omega f(x) \text{d}E_{u,v}(x),
   \]
   \[
   \langle T_E(f)u, u \rangle = \int_\Omega f(x) \text{d}E_u(x),
   \]
   \[
   \|T_E(f)u\|^2 = \int_\Omega |f(x)|^2 \text{d}E_u(x).
   \]

One also writes $\int_\Omega f(x) \text{d}E(x)$ for the operator $T_E(f)$.

**Proof.** We first verify all the claimed properties for simple functions. Then we will define $T_E(f)$ also for general bounded, measurable functions, and verify the claimed properties also in this case. Uniqueness of the map $T_E$ is left as exercise.

For simple functions, define $T_E$ as in (2.3.16). All properties stated in (1) are then easily verified, the last one has already been shown in Lemma 2.3.14. Concerning the
properties in (2), we verify the first one exemplary and leave the others as exercises. Given a simple function \( f : \Omega \to \mathbb{C} \) and \( u, v \in \mathcal{H} \), write \( f = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j} \) and calculate

\[
\langle T_E(f)u, v \rangle = \sum_{j=1}^{n} a_j \langle E(A_j)u, v \rangle = \sum_{j=1}^{n} a_j E_{u,v}(A_j)
= \sum_{j=1}^{n} a_j \int_{\Omega} \mathbb{1}_{A_j}(x) \, dE_{u,v}(x) = \int_{\Omega} f(x) \, dE_{u,v}(x),
\]

where the last step uses linearity of integration with respect to a complex measure.

Next we define \( T_E \) also for general bounded, measurable function. To this end, let \( f : \Omega \to \mathbb{C} \) be bounded and measurable. By [Folland, Real Analysis, Theorem 2.10] we can choose a sequence \( (f_n)_n \) of simple functions \( f_n : \Omega \to \mathbb{C} \) converging uniformly to \( f \). In particular, we have \( \text{ess sup} |f_n - f| \to 0 \), so by Lemma 2.3.14 it follows that

\[
\|T_E(f_n) - T_E(f_m)\|_{\infty} = \|T_E(f_n - f_m)\|_{\infty} = \text{ess sup} |f_n - f_m|.
\]

Since \( \text{ess sup} |f_n - f| \to 0 \), the sequence \( (f_n)_n \) is a Cauchy sequence with respect to \( \text{ess sup} \cdot \). By (\ast), so is the sequence \( (T_E(f_n))_n \). Since \( L(H) \) is complete, it thus converges to some operator which we denote by \( T_E(f) \).

That all the properties in (1) and (2) remain valid for general bounded, measurable functions can be shown by uniform approximation by step functions. Concerning the integrals in (2), note that the measures \( E_n \) and \( E_{u,v} \) are all finite and a uniformly convergent sequence of functions \( (f_n)_n \) is uniformly bounded in \( n \). This allows to use the dominated convergence theorem.

\[\text{Exercise 2.42. Deduce uniqueness of the map } T_E \text{ in Theorem 2.3.15 from part (2) and the Riesz representation theorem for Hilbert spaces.}\]

\[\text{Exercise 2.43. Let } E : \mathcal{A} \to L(H) \text{ be a spectral measure and } T := E(x) \in L(H). \text{ For a polynomial } p(x) = \sum_{j=0}^{n} a_n x^n \text{ with coefficients } a_0, \ldots, a_n \in \mathbb{C} \text{ define the operator } p(T) \in L(H) \text{ by } p(T) := \sum_{j=0}^{n} a_n T^n. \text{ Show that } p(T) = T_E(p) = \int_{\Omega} p(x) \, dE(x).\]

The following properties of operators obtained by integrating a spectral measure can be easily deduced from the properties of spectral integrals.

\[\text{Corollary 2.3.16. Let } E : \mathcal{A} \to L(H) \text{ be a spectral measure and } f \in L^\infty(\Omega). \text{ Then the following holds:}\]

\[(1) \ T_E(f) \text{ is normal.}\]
\[(2) \ T_E(f) \text{ is selfadjoint if and only if } f \text{ is real-valued } (f = f) \text{ almost everywhere.}\]
\[(3) \ T_E(f) \text{ is unitary if and only if } |f| = 1 \text{ almost everywhere.}\]

\[\text{Exercise 2.44. Prove Corollary 2.3.16.}\]

\[\text{Definition 2.3.17. Let } X \text{ be a topological space and } E : \mathcal{B}(X) \to L(H) \text{ a spectral measure on the Borel-\sigma-algebra of } X. \text{ Then the support of } E \text{ is defined as the complement of the largest open set of measure 0.}\]

Clearly

\[
\int_{X} f(x) \, dE(x) = \int_{\text{supp}(E)} f(x) \, dE(x).
\]

Therefore one may in fact extend the map \( T_E \) further to those measurable functions which are bounded on \( \text{supp}(E) \), i.e. \( L^\infty(\text{supp}(E), E) \).
**Theorem 2.3.18.** Let $E : \mathcal{B}(\mathbb{C}) \to L(\mathcal{H})$ be a spectral measure on the Borel-$\sigma$-algebra of $\mathbb{C}$. Assume that $\text{supp} \ E \subseteq \mathbb{C}$ is compact, and set $T := T_E(\text{id})$. Then $\text{supp} \ E = \sigma(T)$.

**Proof.** First let $\lambda \in \text{supp} \ E$. Then for any $\epsilon > 0$ we have $E(B_\epsilon(\lambda)) \neq 0$. For any $u \in \text{Ran}(E(B_\epsilon(\lambda))) \neq \{0\}$ we have

$$E_{u,u}(B_\epsilon(\lambda)) = \langle E(B_\epsilon(\lambda)) u, u \rangle = \langle u, u \rangle = \|u\|^2 = \|E_{u,u}\|,$$

thus $\text{supp} \ E_{u,u} \subseteq B_\epsilon(\lambda)$. It follows that

$$\|(T - \lambda)u\|^2 = \int_{B_\epsilon(\lambda)} |z - \lambda|^2 \, dE_{u,u}(z) \leq \epsilon^2 \cdot \|u\|^2.$$

Since $\epsilon > 0$ was arbitrary, it follows that $(T - \lambda)|_{E(B_\epsilon(\lambda))} = 0$. But then $T - \lambda$ cannot be invertible, hence $\lambda \in \sigma(T)$.

Conversely, let $\lambda \notin \text{supp}(E)$. Since $\text{supp}(E) \subseteq \mathbb{C}$ is closed, there exists $\epsilon > 0$ such that $B_\epsilon(\lambda) \subseteq \text{supp}(E)^c$, hence $E(B_\epsilon(\lambda)) = 0$. So $B_\epsilon(\lambda)$ has $E$-measure zero and therefore the function $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = (z - \lambda)^{-1}$ for $z \neq \lambda$ and $f(\lambda) = 0$ is essentially bounded with respect to $E$. Thus we may form the operator $T_E(f) \in L(\mathcal{H})$. But clearly $T_E(f)$ is an inverse to $T - \lambda = T_E(z - \lambda)$ by multiplicity of $T_E$ and $T_E(1) = \text{id}_H$. So $\lambda \notin \sigma(T)$.

**2.3.5. Proof of the Spectral Theorem for selfadjoint bounded operators.**

In this section we give a proof of the Spectral Theorem 2.3.1 which states that any selfadjoint operator $A \in L(\mathcal{H})$ determines a unique spectral measure $E$ on $\sigma(A)$ such that $A = \int_{\sigma(A)} \lambda \, dE(\lambda)$.

The basic idea of the followig approach is to use a “duality argument”, meaning that instead of directly constructing a spectral measure we construct the map which is supposed to be the integral with respect to this measure. Such “integrals”, which have to be operators, will be defined as functions of the given operator $T$. If we manage to define this map for a sufficiently large class of functions containing characteristic functions, then the desired measure $E$ may be reconstructed from the integral by the identity $E(A) = \mathbf{1}_A(T)(= \int_{\sigma(T)} \mathbf{1}_A(x) \, dE(x))$.

The starting point is to define $f(T)$ when $f$ is a polynomial function in the usual way.

**Definition 2.3.19.** Let $T \in L(\mathcal{H})$ be given. For a polynomial $p(x) = \sum_{j=0}^n a_j x^j$ with coefficients $a_0, \ldots, a_n \in \mathbb{C}$ we define the operator $p(T) \in L(\mathcal{H})$ by $p(T) := \sum_{j=0}^n a_j T^j$.

**Exercise 2.45.** Let $T \in L(\mathcal{H})$ be selfadjoint, and let $p$ be a polynomial with real coefficients. Show that $p(T)$ is selfadjoint again.

Next we want to extend this definition to continuous functions defined on $\sigma(T)$ in the case where $T$ is selfadjoint. This will be done by approximating continuous functions by polynomials. The following statement is crucial for this.

**Theorem 2.3.20.** Let $T \in L(\mathcal{H})$ and let $p(x) = \sum_{j=0}^n a_j x^j$ be a complex polynomial. Then $\sigma(p(T)) = p(\sigma(T))$. Moreover, if $T$ is selfadjoint, then also $\|p(T)\|_\infty = \|p|_{\sigma(T)}\|_\infty$.

**Proof.** For a constant polynomial $p = a_0$ both $\sigma(p(T))$ and $p(\sigma(T))$ simply equal $\{a_0\}$. So assume in the following that $p$ is nonconstant.

First let $\lambda \in \sigma(T)$ be given. We need to show that $p(\lambda) \in \sigma(p(T))$, i.e., that $p(\lambda)\text{id} - p(T)$ is not invertible. Using polynomial division we may write $p(\lambda) - p(x) = (\lambda - x)q(x)$ for some other polynomial $q$ since the left-hand side vanishes for $x = \lambda$. It follows that
$p(\lambda)\text{id} - p(T) = (\lambda\text{id} - T)q(T)$. Since $\lambda \in \sigma(T)$, the operator $\lambda\text{id} - T$ is not invertible, hence neither can be $p(\lambda)\text{id} - p(T)$. This shows $p(\lambda) \in \sigma(p(T))$.

Conversely, let $\lambda \in \sigma(p(T))$. We need to show that $\lambda \in p(\sigma(T))$. Using the fundamental theorem of algebra, we write $\lambda - p(x) = a(z_1 - x) \cdots (z_n - x)$ for certain $a, z_1, \ldots, z_n \in \mathbb{C}$. It follows that

$$\lambda - p(T) = a(z_1\text{id} - T) \cdots (z_n\text{id} - T).$$

Since $\lambda \in \sigma(p(T))$, the left-hand side is non-invertible. Therefore at least one of the factors on the right-hand side is also not invertible. Without loss $z_1\text{id} - T$ is not invertible, thus $z_1 \in \sigma(T)$. Since on the other hand $z_1$ is (by definition) a zero of the polynomial $\lambda - p(z)$ it follows that $\lambda = p(z_1) \in p(\sigma(T))$.

Now assume moreover that $T$ is selfadjoint. Then $p(T)^* = \overline{p}(T)$, where $\overline{p}$ is the polynomial with complex-conjugated coefficients, i.e., $\overline{p}(x) = \sum_{j=0}^{n} \overline{a_j}x^j$ (nothing happens to the argument $x$). It follows that $p(T)p(T)^* = \|p\|^2(T)$, where $\|p\|^2 = \|p\|_{\infty}^2$ is again a polynomial (since in $\overline{p}$ only the coefficients are complex conjugated). Hence

$$\|p(T)\|_{\infty}^2 \overset{\text{Ex. 2.3.19}}{=} \|p(T)p(T)^*\|_{\infty} = \|\|p\|^2(T)\|_{\infty} = \sup\{|\lambda| : \lambda \in \sigma(\|p\|^2(T))\} = \|p\|_{\sigma(T)}^2.$$ 

\[\square\]

The norm equality $\|p(T)\|_{\infty} = \|p|_{\sigma(T)}\|_{\infty}$ for $T$ selfadjoint and $p$ a polynomial now allows to define $f(T)$ for continuous functions $f$ via uniform approximation of $f$ by polynomials. Such an approximation is possible by the Stone-Weierstrass theorem (note that $\sigma(T) \subseteq \mathbb{C}$ is compact since $T$ is bounded).

**Theorem 2.3.21 (Continuous functional calculus).** Let $T \in L(H)$ be selfadjoint. The map assigning to each complex polynomial $p$ the operator $p(T)$ as defined in Definition 2.3.19 has a unique continuous extension to a map $C(\sigma(T)) \rightarrow L(H)$, $f \mapsto f(A)$ with respect to the supremum norm on $C(\sigma(T))$ and the operator norm on $L(H)$. Moreover, the following properties are satisfied:

1. For all $f, g \in C(\sigma(T))$ and $\alpha, \beta \in \mathbb{C}$ it holds that
   \[ (\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T), \]
   \[ (fg)(T) = f(T)g(T), \]
   \[ f(T)^* = \overline{f(T)}, \]
   \[ [f(T), g(T)] = 0. \]

   In particular, the operator $f(A)$ is selfadjoint if and only if $f$ is real-valued.

2. For every $f \in C(\sigma(T))$ with $f \geq 0$ also $f(T) \geq 0$.

3. If $Au = \lambda u$ for $u \in H$ and $\lambda \in \mathbb{C}$, then $f(A)u = f(\lambda)u$ for every $f \in C(\sigma(T))$.

4. For every $f \in C(\sigma(T))$ one has $\sigma(f(A)) = f(\sigma(A))$.

5. For every $f \in C(\sigma(T))$ one has $\|f(A)\| = \|f\|_{\infty}$.

**Proof.** Let $f \in C(\sigma(T))$ be given. Since $\sigma(T) \subseteq \mathbb{R}$ is compact, by the Stone-Weierstrass theorem (see for instance [Rudin, Principles of Mathematical Analysis, Ch. 7] []) there exists a sequence $(p_n)_n$ of polynomials such that $\|p_n - f\|_{\infty} \rightarrow 0$. In particular
$$(p_n)_n$$ is a Cauchy sequence with respect to $$\| \cdot \|_\infty$$ on $$C(\sigma(T))$$. By Theorem 2.3.20 we have

$$\|p_n(T) - p_m(T)\|_\infty = \|(p_n - p_m)(T)\|_\infty = \|p_n|_{\sigma(T)} - p_m|_{\sigma(T)}\|_\infty.$$  

Therefore also $$(p_n(T))_n$$ is a Cauchy sequence in $$L(H)$$. Since $$L(H)$$ is complete, there exists a limit and we set $$f(T) := \lim_{n \to \infty} p_n(T)$$. It is not difficult to check that this limit is independent of the choice of the sequence $$(p_n)_n$$ with $$p_n \to f$$ in $$C(\sigma(T))$$. Moreover, it follows immediately that $$\|f(T)\|_\infty = \text{ess sup}|f|$$ holds, which proves (5). This also shows continuity of the map $$f \mapsto f(T)$$, from which uniqueness follows as well.

The properties in (1) and (3) are obviously satisfied for polynomial functions, and are verified for other continuous functions by approximation by polynomials. The claim about selfadjointness follows from (2.3.27).

Concerning (2), let $$f \in C(\sigma(T))$$ with $$f \geq 0$$ be given. Define $$g := \sqrt{f} \in C(\sigma(T))$$. Then $$g^2 = f$$ and so also $$g(A)^2 = f(A)$$ by (2.3.26). It follows that for $$u \in H$$ we have

$$\langle f(A)u, u \rangle = \langle g(A)^2u, u \rangle = \langle g(A)u, g(A)u \rangle = \|g(A)u\|^2 \geq 0.$$  

Note here that $$g(A)$$ is selfadjoint by (1) since $$g$$ is real-valued.

It remains to show (4). To this end, let $$f \in C(\sigma(T))$$ be given.

We prove $$\sigma(f(T)) \subseteq \sigma(f(T))$$ by contradiction. Assume that $$\lambda \notin \sigma(f(T))$$. Then we may define $$g \in C(\sigma(T))$$ by $$g(x) := (\lambda - f(x))^{-1}$$. By (2.3.26) it follows that $$g(T)$$ is an inverse to $$\text{Id} - f(T)$$. Hence $$\lambda \notin \sigma(f(T))$$.

To prove $$\sigma(f(T)) \subseteq \sigma(f(T))$$, let $$\lambda \in \sigma(T)$$ be given. Suppose that $$\lambda \notin \sigma(f(T))$$. Again define $$g \in C(\sigma(T))$$ by $$g(x) := \lambda - f(x)$$, then by assumption $$g(T) = \text{Id} - f(T)$$ is invertible. Since $$g(\lambda) = 0$$ it is possible to choose a sequence $$(\varphi_n)_n$$ of functions $$\varphi_n \in C(\sigma(T))$$ with $$\|\varphi_n\|_\infty = 1$$ for all $$n \in \mathbb{N}$$ but whose supports shrink towards $$\{\lambda\}$$ sufficiently fast such that $$\|\varphi_n g\|_\infty \to 0$$. Using invertibility of $$g(T)$$ it now follows that

$$1 = \|\varphi_n\|_\infty = \|\varphi_n(T)\|_\infty = \|\varphi_n(T)g(T)g(T)^{-1}\|_\infty \leq \|\varphi_n(T)g(T)\|_\infty \cdot \|g(T)^{-1}\|_\infty \to 0.$$  

This is a contradiction, thus $$\lambda \notin \sigma(f(T))$$ must hold. $\square$

**Exercise 2.46.** Carry out the details of those steps in the proof of Theorem 2.3.21 where above it was simply stated that they work by “approximating by polynomials”.

With the continuous functional calculus in hand we can now construct the spectral measure of a selfadjoint operator by using “duality arguments”. Let $$T \in L(H)$$ be selfadjoint. By the previous theorem we have a continuous linear map

$$C(\sigma(T)) \to L(H), \quad f \mapsto f(T).$$  

(*)

We want to show that this map is in fact given by integration with respect to a spectral measure (which is to be found). To find this measure we will make use of the Riesz representation theorem which states that any continuous (or positive) linear map from $$L(H)$$ to $$\mathbb{C}$$ is given by integration with respect to some complex (or positive) measure. More precisely, the following holds.

**Theorem 2.3.22** (Riesz representation theorem for the dual of $$C(X)$$).

Let $$X$$ be a compact Hausdorff space (e.g., $$X \subseteq \mathbb{R}^n$$ compact) and let $$I : C(X) \to \mathbb{C}$$ be a linear map.

1. If $$I$$ is continuous with respect to the supremum norm on $$C(X)$$, then there exists a unique complex measure $$\mu$$ on the Borel-$$\sigma$$-algebra of $$X$$ such that

$$\forall f \in C(X) : \quad I(f) = \int_X f(x) \, d\mu(x).$$  

(2.3.29)
(2) If $I$ is positive, i.e., $I(f) \geq 0$ for all $f \in C(X)$ with $f \geq 0$, then there exists a unique (positive) measure $\mu$ on the Borel-$\sigma$-algebra of $X$ such that (2.3.29) holds.

**Idea of the proof.** The idea is that one would like to define the measure $\mu$ by setting $\mu(A) = I(1_A)$. However, this is not possible since characteristic functions are not continuous (for most sets). Therefore one proceeds by approximating characteristic functions by continuous functions. Verifying that this really works involves a number of technical details which the interested reader can find in [[Folland, Real Analysis, Ch.7] for instance.

We cannot directly apply this result to the map $(\ast)$ since this map takes values in $L(H)$ instead of $\mathbb{C}$. However, the Riesz representation theorem can be applied to “expectation values” of $(\ast)$, i.e., for fixed $u, v \in H$ it can be applied to the linear function

$$C(\sigma(T)) \rightarrow \mathbb{C}, \quad f \mapsto \langle f(T)u, v \rangle.$$  

Note here that this function is continuous since

$$|\langle f(T)u, v \rangle| \leq \|f(T)u\| \cdot \|v\| \leq \|f(T)\|_{\infty} \cdot \|u\| \cdot \|v\| = \|f\|_{\infty} \cdot \|u\| \cdot \|v\|. \quad (\ast)$$

Applying the Riesz representation theorem (2.3.22) (note again that $\sigma(T)$ is compact since $T$ is bounded) we get a complex measure $E_{u,v}$, which depends on $u$ and $v$, such that

$$\forall f \in C(\sigma(T)) : \langle f(T)u, v \rangle = \int_{\sigma(T)} f(x) dE_{u,v}(x). \quad (2.3.30)$$

All these complex measures $E_{u,v}$, where $u$ and $v$ range over all of $H$ can finally be combined into one spectral measure $E$ by the Riesz representation theorem for jointly continuous bilinear maps on Hilbert spaces (see ??). Namely, for fixed $A \in \sigma(T)$ consider the map

$$B_A : H \times H \rightarrow \mathbb{C}, \quad (u, v) \mapsto E_{u,v}(A) = \int_{\sigma(A)} 1_A \ dE_{u,v}. \quad (\ast\ast)$$

It follows immediately from (2.3.30) that the map $B_A$ is sesquilinear. Moreover, for any $u, v \in H$ we have

$$|B_A(u, v)| = \left| \int_{\sigma(A)} 1_A(x) dE_{u,v}(x) \right| \leq \|1_A\|_{\infty} \cdot \|E_{u,v}\| = \|1\| \cdot \|E_{u,v}\|,$$

where $\|E_{u,v}\|$ denotes the total variation of the complex measure $E_{u,v}$ (see Section 2.1). This shows that $B_A$ is jointly continuous. By the Riesz representation theorem for jointly continuous bilinear maps on Hilbert spaces (see ??) there thus exists $E(U) \in L(H)$ such that

$$\forall u, v \in H : \langle E(A)u, v \rangle = B_A(u, v) = \int_{\sigma(A)} 1_A \ dE_{u,v}. \quad (2.3.31)$$

We now show that this define a spectral measure from which $T$, and also the whole continuous functional calculus of $T$ can be recovered by integration.

**Theorem 2.3.23** (Spectral theorem for selfadjoint operators). Let $T \in H$ be selfadjoint. The map $E : \mathcal{B}(\sigma(T)) \rightarrow L(H)$ constructed above is a spectral measure. Moreover,
for every \( f \in C(\sigma(A)) \) one has
\[
\int_{\sigma(A)} f(x) \, dE(x) = \int_{\sigma(T)} f(T) \, dE(x),
\]
were the left-hand side is defined as in Theorem \ref{thm:2.3.21} and the right-hand side as in Theorem \ref{thm:2.3.15}. In particular we have
\[
A = \int_{\sigma(A)} x \, dE(x).
\]

**Proof.** The proof in fact becomes simpler if we first extend the functional calculus to all bounded, measurable functions, and only afterwards verify that \( E \) is a spectral measure.

To define \( f(T) \) also for \( f \in \mathcal{L}^\infty(\sigma(T)) \), similarly as in the construction of \( E \) we start with the function \( B_f : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) defined by
\[
B_f(u, v) := \int_{\sigma(T)} f(x) \, dE_{u,v}(x).
\]
Just as the map \( B_A \) above, also \( B_f \) is sesquilinear and jointly continuous. So by the Riesz representation theorem for jointly continuous sesquilinear maps in Hilbert spaces there exists a unique operator \( f(T) \in L(\mathcal{H}) \) such that
\[
\langle f(T)u, v \rangle = B_f(u, v) \quad \text{for all } u, v \in \mathcal{H}^2.
\]

Note that in particular \( \mathbf{1}_A(T) = E(A) \) holds for any \( A \in \mathcal{B}(\sigma(T)) \). That \( E \) maps into the orthogonal (selfadjoint) projections will follow easily from this once we have established that this new functional calculus is still multiplicative and involutive.

Let us first show that multiplicity remains to hold. Fix \( u, v \in \mathcal{H} \). Due to multiplicativity of the continuous functional calculus, for any \( f, g \in C(\sigma(T)) \) we have
\[
\int_{\sigma(T)} f(x)g(x) \, dE_{u,v}(x) = \langle (fg)(T)u, v \rangle = \langle f(T)g(T)u, v \rangle = \int_{\sigma(T)} f(x) \, dE_{g(T)u,v}(x).
\]

Thinking of \( g \in C(\sigma(T)) \) as being fixed, this shows that integration with respect to the two measures \( g \, dE_{u,v} \) and \( E_{f(T)u,v} \) coincides on \( C(\sigma(T)) \). From the uniqueness statement in Riesz representation theorem for the dual of \( C(\sigma(T)) \) it follows that actually these two measures coincide. Therefore it follows that for any \( f \in \mathcal{L}^\infty(\Omega) \) we have
\[
\int_{\sigma(T)} f(x)g(x) \, dE_{u,v}(x) = \int_{\sigma(T)} f(x) \, dE_{g(T)u,v}(x).
\]

The right-hand side can be rewritten as
\[
\int_{\sigma(T)} f(x) \, dE_{g(T)u,v}(x) = \langle f(T)g(T)u, v \rangle = \langle g(T)u, f(T)^*v \rangle = \int_{\sigma(T)} g(x) \, dE_{u,f(T)^*v}(x).
\]

Together with the previous identity, we see that
\[
\int_{\sigma(T)} f(x)g(x) \, dE_{u,v}(x) = \int_{\sigma(T)} g(x) \, dE_{u,f(T)^*v}(x) \quad \text{(*)}
\]
holds for all \( g \in C(\sigma(T)) \) and \( f \in \mathcal{L}^\infty(\sigma(T)) \). Analogously as before, it follows from the Riesz representation theorem for the dual of \( C(\sigma(T)) \) that the measures \( f \, dE_{u,v} \) and \( dE_{u,f(T)^*v} \) coincide. Therefore (*) remains valid also for \( g \in \mathcal{L}^\infty(\sigma(T)) \). Thus, in summary, we have seen that for any \( f, g \in \mathcal{L}^\infty(\sigma(T)) \) we have
\[
\langle (fg)(T)u, v \rangle = \int_{\sigma(T)} f(x)g(x) \, dE_{u,v}(x) \equiv \int_{\sigma(T)} g(x) \, dE_{u,f(T)^*v}(x) = \langle g(T)u, f(T)^*v \rangle = \langle f(T)g(T)u, v \rangle.
\]

\(^3\)For \( f \) continuous, by \ref{thm:2.3.20} this definition of \( f(T) \) coincides with the previous one of Theorem \ref{thm:2.3.21} by \ref{thm:2.3.30}.\]
Since also \( u, v \in \mathcal{H} \) were arbitrary, this shows multiplicativity. As an immediate consequence, for any \( A \in \mathcal{B}(\sigma(T)) \) and \( u, v \in \mathcal{H} \) we have

\[
\langle (E(A)^2)u, v \rangle = \langle 1_A(T)1_A(T)u, v \rangle = \langle (1_A)^2(T)u, v \rangle = \langle 1_A(T)u, v \rangle = \langle E(A)u, v \rangle.
\]

This shows that \( E(A) \) is a projection. To show that \( E(A) \) is an orthogonal projection, we have to show that \( E(A) \) is selfadjoint. To this end, note that for any \( u, v \in \mathcal{H} \) and \( f \in C(\sigma(T)) \) we have

\[
\int_{\sigma(T)} f(x) \, dE_{u,v}(x) = \langle f(T)u, v \rangle = \langle v, f(T)u \rangle = \langle (f(T))^*v, u \rangle = \langle \overline{f(T)}v, u \rangle.
\]

As before, since this holds for all \( f \in C(\sigma(T)) \), it follows that \( E_{u,v} = \overline{E_{v,u}} \). But this precisely means that for any \( A \in \mathcal{B}(\sigma(T)) \) we have

\[
\langle E(A)u, v \rangle = E_{u,v}(A) = \overline{E_{v,u}(A)} = \langle E(A)v, u \rangle = \langle u, E(A)v \rangle.
\]

This shows that \( E(A)^* = E(A) \). We conclude that \( E \) indeed maps into the orthogonal projections of \( \mathcal{H} \).

By a similar computation as above (Exercise 2.47), using that \( E_{u,v} = \overline{E_{v,u}} \) holds for any \( u, v \in \mathcal{H} \), it follows that \( f(T)^* = \overline{f(T)} \) holds for any \( f \in l^\infty(\sigma(T)) \).

That \( E(\sigma(T)) = \text{id}_\mathcal{H} \) holds follows since \( E(\sigma(T)) = \mathbf{1}_{\sigma(T)}(T) \) and since the constant function 1 is mapped to \( \text{id}_\mathcal{H} \) already by the continuous functional calculus.

It remains to show \( \sigma \)-additivity. To this end, let \( A_1, A_2, \ldots, \sigma(T) \) be disjoint and measurable. Set \( A := \bigcup_{i=1}^n A_i \). Further set \( f_n := \sum_{i=1}^n 1_A \), then \( f_n \to 1_A \) pointwise and \( \|f_n\|_\infty \leq 1 \) for all \( n \in \mathbb{N} \). Now let \( u \in \mathcal{H} \) be given. To show that \( \sum_{i=1}^n E(A_i)u \) converges to \( E(A)u \), by Exercise ?? it suffices to show that \( \langle \sum_{i=1}^n E(A_i)u, v \rangle \to \langle E(A)u, v \rangle \) for all \( v \in \mathcal{H} \) and \( \|\sum_{i=1}^n E(A_i)u\| \to \|E(A)u\| \). Concerning weak convergence, for every \( v \in \mathcal{H} \) we have by dominated convergence (\( \|f_n\|_\infty \leq 1 \))

\[
\left\langle \sum_{i=1}^n E(A_i)u, v \right\rangle = \int_{\sigma(T)} f_n(x) \, dE_{u,v}(x) \underset{\text{dom. conv.}}{\to} \int_{\sigma(T)} 1_A(x) \, dE_{u,v}(x) = \langle E(A)u, v \rangle.
\]

Concerning convergence of the norms, we have

\[
\left\| \sum_{i=1}^n E(A_i)u \right\|^2 = \left\| f_n(A)u \right\|^2 = \langle f_n(A), f_n(A)u \rangle = \langle f_n(A)^*f_n(A)u, u \rangle = \left\| f_n \right\|^2(A)u, u \rangle
\]

\[
= \int_{\sigma(T)} |f_n(x)|^2 \, dE_{u,u}(x) \underset{\text{dom. conv.}}{\to} \int_{\sigma(T)} |1_A(x)|^2 \, dE_{u,u}(x)
\]

\[
= \int_{\sigma(T)} 1_A(x) \, dE_{u,u}(x) = \langle E(A)u, u \rangle = \|E(A)u\|^2.
\]

This concludes the proof that \( E \) is a spectral measure. The other claims have already been shown.

**Exercise 2.47.** Show that the measurable functional calculus is also involutive, i.e., show that \( f(T)^* = \overline{f(T)} \) holds for any \( f \in l^\infty(\sigma(T)) \).

We restate the possibility to define bounded, measurable functions of a selfadjoint operator in a definition. For continuous functions, respectively polynomials, this definition coincides with the previous ones.
Definition 2.3.24 (\( \mathcal{L}^\infty \)-functional calculus for selfadjoint operators). Let \( T \in L(H) \) be selfadjoint, and let \( E : B(\sigma(T)) \to L(H) \) be the spectral measure of \( T \) as in Theorem 2.3.23. For any \( f \in \mathcal{L}^\infty(\sigma(T)) \) we define the operator \( f(T) \in L(H) \) by
\[
f(T) = T_E(f) = \int_{\sigma(T)} f(x) \, dE(x),
\]
with the integral defined as in Theorem 2.3.17.

The properties of this functional calculus, most of which have already been shown, are summarized in the following theorem.

Theorem 2.3.25 (Properties of the functional calculus). Let \( T \in L(H) \) be selfadjoint.

1. For all \( f, g \in \mathcal{L}^\infty(\sigma(T)) \) and \( \alpha, \beta \in \mathbb{C} \) it holds that
\[
(\alpha f + \beta g)(T) = \alpha f(T) + \beta f(T),
\]
\[
(fg)(T) = f(T)g(T),
\]
\[
f(T)^* = \overline{f}(T),
\]
\[
[f(T), g(T)] = 0.
\]

In particular, the operator \( f(A) \) is selfadjoint if and only if \( f \) is real-valued.

2. For every \( f \in \mathcal{L}^\infty(\sigma(T)) \) with \( f \geq 0 \) also \( f(T) \geq 0 \).

3. If \( Tu = \lambda u \) for \( u \in H \) and \( \lambda \in \mathbb{C} \), then \( f(T)u = f(\lambda)u \) for every \( f \in \mathcal{L}^\infty(\sigma(T)) \).

4. For every \( f \in \mathcal{L}^\infty(\sigma(T)) \) one has \( \sigma(f(T)) \subseteq f(\sigma(T)) \).

5. For every \( f \in \mathcal{L}^\infty(\sigma(T)) \) one has \( \|f(T)\|_\infty = \text{ess}\sup |f| \).

Proof. The first part has already been shown in the proof of the previous theorem. The second part follows from multiplicativity precisely as for continuous functions. The third part can be shown by the same trick that was used several times in the proof of the previous theorem. The fourth part can again be proven precisely as for continuous functions. The last estimate can be directly established using the definition of \( f(T) \).

That the fourth statement has to be changed compared to the continuous functional calculus can be expected since one may change measurable function \( f \) on a set of measure zero (with respect to \( E \)) without changing the operator \( f(T) \), also see Exercise 2.49.

Exercise 2.48. Fill in the details of the previous proof.

Exercise 2.49. Give an example of a selfadjoint operator \( T \) and a bounded, measurable function \( f : \sigma(T) \to \mathbb{C} \) with \( \sigma(f(T)) \neq f(\sigma(T)) \).

Exercise 2.50. Give an example of a selfadjoint operator \( T \) and a bounded, measurable function \( f : \sigma(T) \to \mathbb{C} \) with \( \|f(T)\|_\infty \neq \|f\|_\infty \), where \( \|f\|_\infty \) denotes the (actual) supremum (not the essential supremum).

Exercise 2.51. Let \( T \in L(H) \) be selfadjoint and let \( E : B(\sigma(T)) \to L(H) \) be the corresponding spectral measure. Show that for any \( B \in L(H) \) one has \([T, B] = 0\) if and only if \([E(U), B] = 0\) for all \( U \in B(\sigma(T)) \). Show moreover that in this case also \([f(T), B] = 0\) for all bounded measurable functions \( f : B(\sigma(T)) \to \mathbb{C} \).
2.3.6. The spectral theorem for normal and several commuting operators.

The spectral theorem can be extended to normal operators by noting that similar to a complex number, any normal operator \( T \in L(\mathcal{H}) \) can be written as \( T = A + iB \) for operators \( A, B \in L(\mathcal{H}) \) which are selfadjoint (“real”) and commute. Simply set

\[
A := \frac{T + T^*}{2}, \quad B := \frac{T - T^*}{2i}, \tag{2.3.37}
\]

then obviously \( T = A + iB \) holds. It is easy to verify that \( A \) and \( B \) are selfadjoint. They commute since \( T \) is normal. The idea for constructing a spectral measure for \( T \) is now that similarly to how the product measure \( dx dy \) is the Lebesgue measure on \( \mathbb{C} \cong \mathbb{R}^2 \), the spectral measure of \( T \) should be the product of the spectral measures of \( A \) and \( B \).

Therefore we now explain in which sense a product of spectral measures can indeed be defined. This works nicely, if the different spectral measures commute. In the following explanation we focus on the product of two commuting spectral measures, but everything can be extended in a straight-forward sense to any finite number of commuting spectral measures.

**Theorem 2.3.26.** Let \( X_1, \ldots, X_n \) be complete, separable metric spaces, \( \mathcal{H} \) a Hilbert space, and let \( E_1 : \mathcal{B}(X_1) \to L(\mathcal{H}) \), \( \ldots, E_n : \mathcal{B}(X_n) \to L(\mathcal{H}) \) be spectral measures. Suppose that \( E_1, \ldots, E_n \) are pairwise commuting, i.e., \( [E_i(A), E_j(B)] = 0 \) holds for all \( i, j \in \{1, \ldots, n\} \) and \( A \in \mathcal{B}(X_i), B \in \mathcal{B}(X_j) \). Then there exists a unique spectral measure \( E : \mathcal{B}(X_1 \times \cdots \times X_n) \to L(\mathcal{H}) \) satisfying

\[
\forall A_1 \in \mathcal{B}(X_1), \ldots, A_n \in \mathcal{B}(X_n) : \quad E(A_1 \times \cdots \times A_n) = E_1(A_1) \cdots E_n(A_n). \tag{2.3.38}
\]

(Note that due to the commutation assumption the ordering of the operators \( E_1(A_1), \ldots, E_n(A_n) \) on the right-hand side plays no role.)

We will not prove this theorem, although the proof is not very hard, but would take us too far into measure-theoretic arguments. However, let us make some brief remarks. The idea is of course to take (2.3.38) as definition for \( E \) on the subset of \( \mathcal{B}(X_1 \times X_2) \) which consists of all product sets. By “standard” measure-theoretic arguments one can extend \( E \) to the product-\( \sigma \)-algebra of \( \mathcal{B}(X_1) \) and \( \mathcal{B}(X_2) \) (which is the \( \sigma \)-algebra generated by all products). The separability assumption gives that this product-\( \sigma \)-algebra coincides with \( \mathcal{B}(X_1 \times X_2) \), whereas in general it might be smaller (see [Folland, Real Analysis, Sec.1.2]). For details we refer to [Birman, Chapter 5, Section 2][].

2.3.7. The joint spectral theorem for commuting selfadjoint operators.

In this section we show that given several selfadjoint operators which commute with each other, there exists a “joint spectral measure” on the product of the spectra of the operators, from which all of these operators as well as functions of several of them can be reproduced via integration.

**Theorem 2.3.27 (Joint spectral theorem).** Let \( T_1, \ldots, T_n \in L(\mathcal{H}) \) be selfadjoint, pairwise commuting operators, and denote by \( E_1, \ldots, E_n \) their respective spectral measures. There exists a unique spectral measure \( E \) on \( \mathcal{B}(\sigma(T_1) \times \cdots \times \sigma(T_n)) \) such that \( E(A_1 \times \cdots \times A_n) = E_1(A_1) \cdots E_n(A_n) \) holds for all \( A_1 \in \mathcal{B}(\sigma(T_1)), \ldots, A_n \in \mathcal{B}(\sigma(T_n)) \). Moreover, for any \( f_1 \in L^\infty(\sigma(T_1)), \ldots, f_n \in L^\infty(\sigma(T_n)) \) one has

\[
\int f_1(x_1) \cdots f_n(x_n) \, dE(x_1, \ldots, x_n) = f_1(T_1) \cdots f_n(T_n).
\]
In particular,
\[ A_1 = \int x_1 \, dE(x_1, \ldots, x_n), \ldots, A_n = \int x_n \, dE(x_1, \ldots, x_n). \]
The spectral measure \( E \) is also the unique spectral measure on \( \mathcal{B}(\sigma(T_1) \times \cdots \times \sigma(T_n)) \) with the property that the last formulas hold.

**Proof.** The proof for existence of such a measure is similar to the scalar case and can be found in []. The idea is to approximate general measurable subsets of the product by products of measurable sets. The identity about the integral of a product of functions is easily verified for simple functions, the proof for general bounded, measurable functions works can afterwards be done by approximation. \( \square \)

One calls the support of the joint spectral measure of operators \( T_1, \ldots, T_n \) their **joint spectrum**, and denotes it by \( \sigma(T_1, \ldots, T_n) \). In general, \( \sigma(T_1, \ldots, T_n) \) may only be a subset of \( \sigma(T_1) \times \cdots \times \sigma(T_n) \).

### 2.3.8. The spectral theorem for normal operators.

**Theorem 2.3.28 (Spectral theorem for normal operators).** Let \( T \in L(\mathcal{H}) \) be a normal operator. Then there exists a unique spectral measure \( E \) on \( \mathcal{B}(\sigma(T)) \) such that
\[ T = \int_{\sigma(T)} \lambda \, dE(\lambda). \]

**Proof.** Define
\[ A := \frac{T + T^*}{2}, \quad B := \frac{T - T^*}{2i}. \]
Then \( T = A + iB \) and \( A \) and \( B \) are both selfadjoint. Moreover, due to \( T^*T = TT^* \), the operators \( A \) and \( B \) commute. Now first take the joint spectral measure. Probably one shows that its support is precisely \( \sigma(T) \). \( \square \)

### 2.3.9. Unbounded Selfadjoint Operators.

- dense domains
- explain difference between symmetric and selfadjoint operators
- state the spectral theorem for unbounded operators
Since this Hamiltonian is unbounded, we need to specify a suitable domain of definition \( \mathcal{D}(H) \). For simplicity, we here take the smooth and square integrable functions, \( \mathcal{D}(H) = L^2(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3) \). Then \( \mathcal{H} \) satisfies the requirement \( (1.1.6) \) in the sense that
\[ \langle H\phi|\psi \rangle_\mathcal{H} = \langle \phi|H\psi \rangle_\mathcal{H} \quad \text{for all } \phi, \psi \in \mathcal{D}(H). \]
In mathematical terms, the Hamiltonian is a **symmetric operator** (selfadjointness is a stronger statement; we postpone the details to Section 2.3 below).

### 2.4. Manifolds of Operators

In this book, it will sometimes be useful to endow certain sets of operators with the structure of a manifold. We recall the basic definition; for an introduction to manifolds and to the basics of differential geometry we refer to standard textbooks.
DEFINITION 2.4.1. **(smooth manifold)** An topological manifold \( M \) of dimension \( n \) is a second countable Hausdorff space which is locally homeomorphic to \( \mathbb{R}^n \), i.e. for each point \( p \in M \) there is an open neighborhood \( U \) together with a mapping \( \phi : U \to \mathbb{R}^n \) with the properties that \( \phi(U) \) is an open subset of \( \mathbb{R}^n \) and \( \phi : U \to \phi(U) \) is a homeomorphism. A mapping \( (\phi,U) \) is referred to as a chart. An atlas is a family of charts with the property that every point of \( M \) is in the domain of a chart of this family. A smooth manifold is a topological manifold together whose transitions maps are all smooth, i.e. for all charts \( (\phi,U) \) and \( (\tilde{\phi},\tilde{U}) \), the mapping

\[
\tilde{\phi} \circ \phi|_{\phi(U \cap \tilde{U})} : \phi(U \cap \tilde{U}) \subset \mathbb{R}^n \to \tilde{\phi}(U \cap \tilde{U}) \subset \mathbb{R}^n
\]

is smooth.

We will encounter manifolds as certain sets of operators on a Hilbert space. We begin with a simple example which illustrates the basic concept.

**Example 2.4.2. (Grassmann manifold)** Given \( f \in \mathbb{N} \), we consider \( \mathbb{C}^f \) with the canonical scalar product \( \langle \cdot, \cdot \rangle_{\mathbb{C}^f} \). Let \( \mathcal{G} \) be the set of all orthogonal projection operators to one-dimensional subspaces of \( \mathbb{C}^f \),

\[
\mathcal{G} := \{ \pi_V \text{ orthogonal projection to a one-dimensional subspace } V \subset \mathbb{C}^f \}.
\]

Let us verify that \( \mathcal{G} \) is a smooth manifold of dimension

\[
\dim \mathcal{G} = 2f - 2.
\]

To this end, let \( \pi_V \) be such a projection operator. We choose a unit vector \( v \) such that \( V \) is spanned by \( v \). Next we let \( V^\perp \) be the orthogonal complement of \( V \) and \( W = B_1(0) \cap V^\perp \) its unit ball. We consider the mapping

\[
F : W \to \mathcal{G}, \quad u \mapsto \pi_{\text{span}(v+u)}.
\]

It is verified by direct computation that \( F \) is injective and that its image is an open neighborhood of \( \pi_V \). Also, one easily verifies that it is a local homeomorphism and thus defines a chart

\[
\phi = F^{-1}_{|F(W)} : F(W) \to W \subset V^\perp \simeq \mathbb{C}^{n-1} \simeq \mathbb{R}^{2n-2}.
\]

Moreover, one verifies directly that all the charts obtained in this way form a smooth atlas. This manifold is called Grassmann manifold.

This concept can be generalized to so-called flag manifolds (see for example [73], page 142):

**Definition 2.4.3. (flag manifold)** Given \( r \in \mathbb{N} \), we choose \( r \) integers

\[
0 < d_1 < \cdots < d_r < f.
\]

Consider a sequence \( (L_1,\ldots,L_r) \) of nested subspaces

\[
L_1 \subset \cdots \subset L_r \subset \mathbb{C}^f \quad \text{with} \quad \dim L_i = d_i
\]

for \( i = 1,\ldots,r \). Then the set of such sequences \( (L_1,\ldots,L_r) \) is referred to as the flag manifold \( \mathfrak{F}^{d_1,\ldots,d_r} \).

Here we do not need to verify that a flag manifold is indeed a manifold. Instead, it suffices to consider a specific set of operators which is related to a flag manifold. Namely, choosing again \( \mathcal{H} = \mathbb{C}^f \) as well as integers \( p,q \) with \( p+q < f \), we let \( \mathfrak{F}^{p,q} \) be the set

\[
\mathfrak{F}^{p,q} = \{ A \in \mathfrak{L}(\mathcal{H}) \mid A \text{ is symmetric and has } p \text{ positive and } q \text{ negative eigenvalues} \},
\]

where we count the eigenvalues with multiplicities.
Proposition 2.4.4. Assuming that $f \geq 2n$, the set $\mathcal{F}^{p,q}$ is a smooth manifold of dimension
\[ \dim \mathcal{F}^{p,q} = 2f(p + q) - (p + q)^2. \]

Proof. Let $X \in \mathcal{F}^{p,q}$. We denote its image by $I$ and set $J = I^\perp$. Thus, using a block matrix representation in $C^f = I \oplus J$, the matrix $X$ has the representation
\[ X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}. \]

We now let $A$ be symmetric linear operator on $I$. By choosing its norm sufficiently small, we can arrange that the matrix $(x + A)$ has again $p$ positive and $q$ negative eigenvalues. In particular, this operator is invertible. Next, we choose a linear operator $B : J \to I$.

We now form the matrix
\[ M := \begin{pmatrix} 1 & 0 \\ B(x + A)^{-1} & 1 \end{pmatrix} \begin{pmatrix} x + A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & (x + A)^{-1}B \\ 0 & 1 \end{pmatrix} \]
\[ = \begin{pmatrix} x + A & B \\ B^* & B^*(x + A)^{-1}B \end{pmatrix}. \]

This matrix has again $p$ positive and $q$ negative eigenvalues (for details see Exercise 2.100). Thus for sufficiently small $\varepsilon$ we obtain the mapping
\[ \Lambda : (\text{Symm}(I) \oplus L(I, J)) \cap B_\varepsilon(0) \to \mathcal{F}^{p,q}, \quad (A, B) \mapsto M \]
(where Symm($I$) denotes the symmetric linear operators). Let us verify that (again for sufficiently small $\varepsilon$) this mapping is a homeomorphism to an open neighborhood of $X \in \mathcal{F}^{p,q}$. It is obvious from (2.4.2) that $\Lambda$ is injective. In order to verify that it maps to an open neighborhood of $X$, we let $Y \in \mathcal{F}^{p,q}$ with $\|X - Y\| < \delta$ (with $\delta > 0$ to be specified below). Diagonalizing $Y$ with a unitary matrix $U$, we obtain the block matrix representation
\[ Y = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x + C & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{pmatrix}, \]
where $C$ is a symmetric linear operator on $I$. In the limit $Y \to X$, the image of $Y$ converges to the image of $X$, implying that the matrix $U_{11}$ becomes unitary. Therefore, for sufficiently small $\delta > 0$, the matrix $U_{11}$ is invertible, giving rise to the representation
\[ Y = \begin{pmatrix} 1 & 0 \\ U_{21} U_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} U_{11} (x + C) U_{11}^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (U_{11}^*)^{-1} U_{21}^* \end{pmatrix}. \]

This is indeed of the form (2.4.1), and one can even read off $A$ and $B$,
\[ A = U_{11} (x + C) U_{11}^* - x \]
\[ B = (U_{11} (x + C) U_{11}^*) (U_{11}^*)^{-1} U_{21}^*. \]

We conclude that $\Lambda$ is a bijection to an open neighborhood of $X \in \mathcal{F}^{p,q}$. The continuity of $\Lambda$ and of its inverse are obvious. We have thus constructed a chart around $X$.

Performing the above construction around every point of $X$ gives an atlas. By direct computation one verifies that the transition maps are smooth. We conclude that, with the above atlas, $\mathcal{F}^{p,q}$ is indeed a smooth manifold.

We finally determine its dimension. The linear operator $B$ is represented by a $(p + q) \times (f - p - q)$-matrix, giving rise to $2(p + q)(f - p - q)$ real degrees of freedom. The symmetric
linear operator $A$, on the other hand, is represented by a Hermitian $(p + q) \times (p + q)$-matrix, described by $(p + q)^2$ real parameters. Adding these dimensions concludes the proof. \hfill \Box

## Exercises

**Exercise 2.52** (Borel algebra). (a) Show that the power set of $\mathcal{F}$ (i.e. the set of all subsets) forms a $\sigma$-algebra.

(b) Show that the intersection of $\sigma$-algebras is again a $\sigma$-algebra.

(c) Combine (a) and (b) to conclude that there is a smallest $\sigma$-algebra which contains all open subsets of $\mathcal{F}$.

**Exercise 2.53** (Existence of smooth functions with compact support). Show that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{-\frac{1}{|x|^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

is smooth and has compact support.

Hint: First deal with the case $n = 1$. Show smoothness in this case by induction on the order of differentiation.

**Exercise 2.54.** Let $f \in \mathcal{E}(\mathbb{R}^n)$. For any $n \in \mathbb{N}$ define $f_n \in \mathcal{E}(\mathbb{R}^n)$ by $f_n(x) := f(\frac{x}{n})$. Show that $(f_n)_n \mathcal{E}$-converges to a constant function. If additionally $f \in \mathcal{D}(\mathbb{R}^n)$ and $f \neq 0$ hold, show that $f_n \in \mathcal{D}(\mathbb{R}^n)$ holds as well but this sequence does not converge in $\mathcal{D}(\mathbb{R}^n)$.

**Exercise 2.55.** Show that the following are equivalent ways of saying that $u \in C^\infty(\mathbb{R}^n)$ is a Schwartz function:

1. $\forall \alpha, \beta \in \mathbb{N}^n : \sup_{x \in \mathbb{R}^n} |\langle x \rangle^\alpha \partial^\beta u(x)| < \infty$.
2. $\forall \alpha, \beta \in \mathbb{N}^n : \sup_{x \in \mathbb{R}^n} |\partial^\alpha(\langle x \rangle^\beta u(x))| < \infty$.
3. $\forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \forall x \in \mathbb{R}^n : |\langle x \rangle^\alpha \partial^\beta u(x)| < C$.
4. $\forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \forall x \in \mathbb{R}^n : |\partial^\alpha(\langle x \rangle^\beta u(x))| < C$.
5. $\forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \forall x \in \mathbb{R}^n : |\partial^\alpha u(x)| < \frac{C}{\langle x \rangle^\beta}$.

Here $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ for any $x \in \mathbb{R}^n$.

**Exercise 2.56.** Carry out the details left out in Example 2.2.3.

**Exercise 2.57.** Prove Lemma 2.2.5 for the case $p = \infty$.

**Exercise 2.58** (Moderately growing functions). A function $f \in C^\infty(\mathbb{R}^n)$ is said to be moderately growing if the following holds:

$$\forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} |\langle x \rangle^{-N} \partial^\alpha f(x)| < \infty . \tag{2.4.3}$$

We denote by $\mathcal{O}_M(\mathbb{R}^n)$ the set of moderately growing functions. Show that $\mathcal{O}_M(\mathbb{R}^n)$ is a vector space which contains all polynomials, all Schwartz functions, and all smooth functions with compact support. Is a moderately growing function also a Schwartz function? Give an example of a smooth function which does not belong to $\mathcal{O}_M(\mathbb{R}^n)$.

**Exercise 2.59.** Show that $T : \mathcal{E}(\mathbb{R}) \to \mathbb{C}$, $T(u) := u(0) + u'(1)$ is a distribution.
EXERCISE 2.60. Show that $I : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$, $I(u) = \int_{\mathbb{R}} u(x) \, dx$ defines a tempered distribution. Why does $I$ not define a distribution in $\mathcal{E}'(\mathbb{R})$? Show that for any $[a, b] \subset \mathbb{R}$, the assignment $u \mapsto \int_a^b u(x) \, dx$ defines an element of $\mathcal{E}'(\mathbb{R})$.

EXERCISE 2.61 (Fundamental theorem in calculus of variations in continuous case). Show that any continuous function on $\Omega$ is locally integrable. Suppose now that $T_f = T_g$ for two continuous functions $f, g \in C(\Omega)$. Show that $f = g$.

(Remark: The general case follows by approximation.)

EXERCISE 2.62. Show also that $T_f \in \mathcal{E}'(\Omega)$ if $\text{supp } f$ is compact, and $T_f \in \mathcal{S}'(\Omega)$ if $f \in \mathcal{O}_M(\mathbb{R}^n)$ is of moderate growth.

EXERCISE 2.63 (Integrability of inverse powers). Let $\alpha > 0$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = |x|^{-\alpha}$ for $x \neq 0$ and $f(0) = 0$. Find out for which $\alpha > 0$ one has $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

EXERCISE 2.64 (Logarithmic poles are integrable). Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \log |x|$ for $x \neq 0$ and $f(0) = 0$. Show that $f \in L^1_{\text{loc}}(\mathbb{R})$.

EXERCISE 2.65 (An identity for the Cauchy principal value of $\frac{1}{x}$). Show that for any $u \in \mathcal{D}(\mathbb{R})$ and any even function $\varphi \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\varphi(0) = 1$ one has

$$\langle \text{v.p. } \left( \frac{1}{x} \right), u \rangle = \int_{\mathbb{R}} \frac{u(x) - u(0)\varphi(x)}{x} \, dx,$$

where the right-hand side is to be understood as an honest integral (i.e., without any further manipulations). First explain why this integral exists in the first place.

EXERCISE 2.66. Fill in the details in example 2.2.11. Show furthermore that the $\delta$-distribution and also any regular distribution is a distribution of the type described in this example.

EXERCISE 2.67. For a smooth function $u \in C^\infty(\mathbb{R})$, denote by $u^{(k)} \in C^\infty(\mathbb{R})$ its $k$-th derivative. Show that

$$T(u) := \sum_{k=0}^{\infty} u^{(k)}(k) \quad \forall u \in \mathcal{D}(\mathbb{R})$$

defines a distribution $T \in \mathcal{D}'(\mathbb{R})$. Is $T$ also in $\mathcal{S}'(\mathbb{R})$ or $\mathcal{E}'(\mathbb{R})$?

EXERCISE 2.68. Show that a linear operator $A : \mathcal{E}(\Omega) \to \mathbb{C}$ is sequentially continuous if and only if it is sequentially continuous at $0$ if and only if there exist a compact subset $K \subset \Omega$ and a constant $C > 0$, $m \in \mathbb{N}$ such that

$$\forall u \in \mathcal{E}(\Omega) : |Au| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^\infty(K)}.$$  \hspace{1cm} (2.4.5)

EXERCISE 2.69. Show that a linear operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is sequentially continuous if and only if it is sequentially continuous at $0$ if and only if there exist $m \in \mathbb{N}$ and a constant $C > 0$ such that

$$\forall u \in \mathcal{S}(\mathbb{R}^n) : |Au| \leq C \sum_{|\alpha|, |\beta| \leq m} \|x^\alpha \partial^\beta u\|_{L^\infty(\mathbb{R}^n)}.$$ \hspace{1cm} (2.4.6)

EXERCISE 2.70 (Model $\delta$-sequences). Let $\psi \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x) \, dx = 1$. For any $k \in \mathbb{N}$, define $\psi_k \in C^\infty(\mathbb{R}^n)$ by

$$\psi_k(x) := k^n \psi(kx).$$ \hspace{1cm} (2.4.7)
Show that \((\psi_k)_k\) is a \(\delta\)-sequence. Moreover, show that if \(\text{supp} \psi\) is compact, then \((\psi_k)_k\) is a strict \(\delta\)-sequence.

Remark: \(\delta\)-sequences of this type are called model \(\delta\)-sequences or mollifiers. If additionally \(\psi \geq 0\), one speaks of a positive mollifier and if \(\psi(x) = \varphi(|x|)\) for some \(\varphi \in C_0^\infty(\mathbb{R}^+)\), one speaks of a symmetric mollifier.

**Exercise 2.71.** Show that the following sequences are all delta sequences on \(\mathbb{R}\).

1. The Gaussians \(\psi_k(x) = ke^{-\pi k^2 x^2}\).
2. The Lorentz curves \(\psi_k(x) = \frac{1}{k} \frac{1}{1 + k^2 x^2}\).
3. The Fejer kernel \(\psi_k(x) = \frac{k}{\pi} \frac{\sin(kx)}{kx}\).

You may skip showing the normalization (this is where the constants come from).

**Exercise 2.72.** Show sequential continuity also in those cases of Theorem 2.2.19 which were not explicitly treated in the proof.

**Exercise 2.73 (Oscillatory integrals).** Let \(a, \phi \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^N)\) satisfy the following assumption:

- (A) \(\phi\) is a phase function, i.e. \(\phi\) is real-valued and satisfies \(\phi(x, \lambda \xi) = \lambda \phi(x, \xi)\) for all \(\lambda > 0\) and all \((x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^N_\xi\) (positive homogeneity of degree one in \(\xi\)).
- (B) \(a\) is a symbol of degree \(m\) for some \(m \in \mathbb{R}\), i.e. for any \(\alpha \in \mathbb{N}^n\) and any \(\beta \in \mathbb{N}^N\) there exists a constant \(C_{\alpha \beta} > 0\) such that
  \[ |\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha \beta} (1 + |\xi|)^{m - |\alpha|} \quad \forall (x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^N_\xi. \]  
  \[ (2.4.8) \]
- (C) \(\phi\) has no critical points on the support of \(a\).

For such functions, consider the so-called oscillatory integrals

\[ u(x) = \int e^{i\phi(x, \xi)} a(x, \xi) d\xi, \]

which in general is only well-defined if \(m < -n + 1\). (Why?) Show that \(u\) can however always be interpreted as a tempered distribution in the following way:

1. Show that for every \(\epsilon > 0\) the integral
   \[ u_\epsilon(x) = \int e^{i\phi(x, \xi)} a(x, \xi) e^{-\epsilon |\xi|^2} d\xi \]  
   (2.4.10)
   is well-defined and defines a bounded, smooth function \(u_\epsilon \in C^\infty(\mathbb{R}^n_x)\).
2. Show that \(u_\epsilon\) interpreted as (regular) tempered distribution converges in \(\mathcal{S}'\) as \(\epsilon \to 0\).

Hint: Notice that \(e^{i\phi(x, \xi)} = (i\partial_{x_i} \phi(x, \xi))^{-1} \partial_{x_i} e^{i\phi(x, \xi)}\). Use this and partial integration to show that for \(\varphi \in \mathcal{S}(\mathbb{R}^n_x)\) one can write

\[ \int u_\epsilon(x) \varphi(x) dx = \int_{\mathbb{R}^N_x} e^{-\epsilon |\xi|^2} \left( \int_{\mathbb{R}^n_x} e^{i\phi(x, \xi)} L(a(x, \xi) \varphi(x)) dx \right) d\xi, \]

where \(L\) is some differential operator. Notice that one can choose operators \(L\) of arbitrary high order. Use this and the symbol property of \(a\) to show that in such a way one can obtain a nicely decaying integrand such that the limit \(\epsilon \to 0\) exists.

**Exercise 2.74.** Show that (2.2.14) and (2.2.15) do in fact define distributions again. Cover the different cases \(\mathcal{D}', \mathcal{E}'\) and \(\mathcal{S}'\).
EXERCISE 2.75. Show that (2.2.15) is also compatible with viewing smooth functions as distributions.

EXERCISE 2.76. Compute the distributional derivative of the absolute value function \( |·| \in L^1_{\text{loc}}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R}) \). Show that it is again a regular distribution.

EXERCISE 2.77. For an interval \([a,b] \subset \mathbb{R}\), let \( 1_{[a,b]} \in L^1_{\text{loc}}(\mathbb{R}) \) be the corresponding characteristic function. Compute its derivative as a distribution.

EXERCISE 2.78. By Exercise [2.64] log \( |x| \) is locally integrable on \( \mathbb{R} \). Show that its distributional derivative is the Cauchy principal value \( \text{v.p.}(\frac{1}{x}) \), cf. Exercise ??.

EXERCISE 2.79. Show that \( x \cdot \text{v.p.}(\frac{1}{x}) = 1 \), where \( \text{v.p.}(\frac{1}{x}) \) denotes the Cauchy principal value distribution (cf. exercise ??).

EXERCISE 2.80 (Iterative application of derivatives and multiplication). Let \( T \in \mathcal{E}'(\Omega) \), \( \alpha, \beta \in \mathbb{N}^n \) and \( f, g \in C^\infty(\Omega) \). Show that \( \partial^\alpha (\partial^\beta T) = \partial^\alpha+\beta T \) and \( f(Tg) = (fg)T \).

EXERCISE 2.81 (Leibniz rule for distributions). Either let \( f \in C^\infty(\Omega) \) and \( T \in \mathcal{D}'(\Omega) \) or \( T \in \mathcal{E}'(\Omega) \), or let \( f \in \mathcal{O}_M(\mathbb{R}^n) \) and \( T \in \mathcal{S}'(\mathbb{R}^n) \). Show that the Leibniz rule

\[
\partial^\alpha (fT) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)(\partial^\beta T)
\]

holds for any \( \alpha \in \mathbb{N}^n \).

EXERCISE 2.82. Let \( \alpha \in \mathbb{N}^n \), \( f \in C^\infty(\Omega) \) and \( A = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha \). Show that the maps

\[
u \mapsto \partial^\alpha u, \quad u \mapsto f u, \quad u \mapsto Au
\]

are linear and sequentially continuous on all introduced test function and distribution spaces. In the tempered case, take \( \Omega = \mathbb{R}^n \) and assume that \( f \) and the coefficients of \( A \) are all of moderate growth.

Using the sequential continuity on the test function spaces, give another (very quick) argument why (2.2.14) and (2.2.15) do indeed define distributions again.

EXERCISE 2.83. For \( n \in \mathbb{N} \) let \( G_n : \mathbb{R}^n \to \mathbb{R} \) be defined by\( G_1(x) := x \Theta(x) \) with \( \Theta \) being the Heaviside function, \( G_2(x) := \frac{1}{2} \ln(|x|) \) and \( G_n(x) := -\frac{1}{(n-2)c_n}|x|^2-n \) for \( n \geq 3 \), where \( c_n = |S^{n-1}| = (2\pi)^{\frac{n}{2}}\Gamma(\frac{n}{2}) \) denotes the volume of the \((n-1)\)-dimensional unit sphere. To be complete, set \( G_n(0) := 0 \). Show that \( G_n \) is locally integrable on \( \mathbb{R}^n \) for every \( n \in \mathbb{N} \) and is a Green’s function for the Laplace operator at \( x_0 = 0 \). Use \( G_n \) to also find Green’s functions for the Laplace operator at an arbitrary point \( x_0 \in \mathbb{R}^n \).

EXERCISE 2.84. For \( k > 0 \), define \( G_H : \mathbb{R}^3 \to \mathbb{C} \) by \( G_H^k(x) = -\frac{\exp(ik|x|)}{4\pi|x|} \) and \( G_H^0(0) = 0 \). Show that \( G_H \) is locally integrable and a Green’s function for the so-called Helmholtz operator \( H_k := \Delta + k^2 \). Further show that \lim_{k \to 0} G_H^k = G_3 \in \mathcal{D}'(\mathbb{R}^3) \), where \( G_3 \) is the Green’s function of the Laplace operator defined in exercise 2.83.

EXERCISE 2.85. Define \( G_W : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by \( G_W(x,t) := (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \Theta(t) \), where \( \Theta \) is the Heaviside function \( (G_W(x,0) := 0 \) \). Show that \( G_W \) is locally integrable on \( \mathbb{R}^n \times \mathbb{R} \) and a Green’s function of the heat operator \( W = \partial_t - \Delta \).

EXERCISE 2.86. Define \( G_S : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by \( G_S(x,t) := c_n t^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4ct}\right) \Theta(t) \), where \( \Theta \) is the Heaviside function and \( c_n \) a constant \( (G_S(x,0) := 0 \) \). Show that \( G_S \) is locally
integrable on \( \mathbb{R}^n \times \mathbb{R} \) if and only if \( n = 1 \). Show that
\[
\langle \hat{G}_S, u \rangle := \lim_{\epsilon \to 0} \int_0^\infty \int_{\mathbb{R}^n} G_S(x, t) u(x, t) \, dx \, dt
\]
defines a distribution \( \hat{G}_S \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \). Show further that this distribution is a Green's function of the Schrödinger operator \( S := i\partial_t + \Delta \) if \( c_n \) is chosen appropriately (how?).

**Exercise 2.87.** Define \( G_0^j : \mathbb{R}^2 \to \mathbb{C} \) by \( G_0^j(x, y) := \frac{1}{\pi(x+iy)} \) and \( G_0^j(0) = 0 \). Show that \( G_0^j \) is locally integrable on \( \mathbb{R}^2 \) and a Green's function of the operator \( \partial := \frac{1}{2}(\partial_x + i\partial_y) \).

**Hint:** By Green's theorem one has \( \int_U \partial u(x, y) \, dx \, dy = \frac{1}{2} \int_{\partial U} u(x, y) (dx + idy) \).

**Exercise 2.88.** Define distributions \( G_{\Box}^\pm \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^3) \) by
\[
\langle G_{\Box}^\pm, u \rangle := \int_{\mathbb{R}^3} \frac{1}{4\pi|x|} u(\pm|x|, x) \, d^3x.
\]
Show that this indeed defines distributions and show that \( G_{\Box}^\pm \) are Green's function of the wave operator \( \Box := \partial_t^2 - \Delta \).

**Exercise 2.89 (Fourier transform of a Gaussian).** Compute the Fourier transform of the Gaussian \( f(x) = \exp(-\frac{1}{2}|x|^2) \) by first reducing to \( n = 1 \) and then deriving a well-known ordinary differential equation satisfied by \( \hat{f}(\xi) \).

**Exercise 2.90.** Prove the properties of the Fourier transform on \( \mathcal{S}(\mathbb{R}^n) \) as stated in Theorem 2.2.20. Use the rapid decay of Schwartz functions to estimate integrals. Concerning the Fourier inversion, plug in the functions \( g_\epsilon (\xi) = e^{ix\xi} e^{-|\xi|^2} \) for \( \epsilon > 0 \) into (2.2.21) and examine what happens to both sides in the limit \( \epsilon \to 0 \). The Fourier transform of \( g_\epsilon \) can be computed using exercise 2.89 by completing squares.

**Exercise 2.91.** Prove Theorem 2.2.20 about the Fourier transform of tempered distributions.

**Exercise 2.92.** Compute the Fourier transform of the \( \delta \)-distribution at a general point \( x_0 \in \mathbb{R}^n \) and show that it is a regular distribution.

**Exercise 2.93.** Compute the distributional Fourier transform of the (bounded but not integrable) functions \( e^{ix} \), \( \sin(x) \) and \( \cos(x) \).

**Exercise 2.94.** For \( \sigma \in \mathbb{R} \) given, compute the distributional Fourier transform of the (bounded but not integrable) "complex Gaussian function" \( u(x) = \exp(-i\sigma|x|^2) \).

**Exercise 2.95.** Compute the distributional Fourier transform of the Heaviside function \( \theta \), cf. exercise 2.2.21, using the following trick: First compute the Fourier transform of \( f_\epsilon(x) := \theta(x)e^{-\epsilon x} \) for \( \epsilon > 0 \) and then examine the limit \( \epsilon \to 0 \).

**Exercise 2.96.** Compute the Fourier transform of the Cauchy principal value distribution, cf. example 2.2.13.

**Exercise 2.97.** Compute the distributional Fourier transform of the logarithm (which is locally integrable and of moderate growth at infinity).

**Exercise 2.98.** Let \( T \in \mathcal{S}'(\mathbb{R}^n) \).

1) Suppose that \( \text{supp} \tilde{T} = \{\xi_0\} \). Show that \( T = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha e^{ix\xi_0} \) for certain \( c_\alpha \in \mathbb{C} \), all but finitely many of them being zero.

**Hint:** What do you remember about distributions supported in a point?
(2) Suppose that $T$ is a harmonic distribution, i.e. $\Delta T = 0$, where $\Delta$ is the Laplacian. Show that $T$ must be a polynomial.

(3) Prove Liouville’s theorem: Every bounded harmonic function on $\mathbb{R}^n$ is constant.

**Exercise 2.99.** Let $u \in C^\infty(\Omega_1 \times \Omega_2)$ and $T \in \mathcal{E}'(\Omega_1)$. Using a suitable cutoff function, show that the function $y \mapsto \langle T, u(\cdot, y) \rangle$ is smooth and (2.2.46) holds.

**Exercise 2.100.** Let $X$ be a Hermitian $f \times f$-matrix of rank $p + q$ which (counting multiplicities) has $p$ positive and $q$ negative eigenvalues. Let $A$ be another $f \times f$-matrix (not necessarily Hermitian). Prove the following statements:

(a) The matrix $A^*XA$ has at most $p$ positive and $q$ negative eigenvalues. *Hint:* Consider the maximal positive and negative definite subspaces of the bilinear forms $\langle ., X . \rangle_C^f$ and $\langle ., A^*XA . \rangle_C^f$. Use that

$$\langle u, A^*XA u \rangle_C^f = \langle (Au), X (Au) \rangle_C^f .$$

(b) If $A$ is invertible, then the matrix $A^*XA$ has again $p$ positive and $q$ negative eigenvalues.
CHAPTER 3

Spinors in Curved Space-Time

We give a brief introduction to spinors in curved space-time. We note that the content of this chapter will be needed only in Chapter 9. Therefore, it can be skipped by readers not interested in the geometric constructions of Chapter 9. We follow the approach in [25]; see also [28 §1.1 and §1.5].

3.1. Curved Space-Time and Lorentzian Manifolds

The starting point for general relativity is the observation that a physical process involving gravity can be understood in different ways. Consider for example an observer at rest on earth looking at a freely falling person (e.g. a diver who just jumped from a diving board). The observer at rest argues that the earth’s gravitational force, which he can feel himself, also acts on the freely falling person and accelerates him. The person in free fall, on the other hand, does not feel gravity. He can take the point of view that he himself is at rest, whereas the earth is accelerated towards him. He concludes that there are no gravitational fields, and that the observer on earth merely feels the force of inertia corresponding to his acceleration. Einstein postulated that these two points of view should be equivalent descriptions of the physical process. More generally, it depends on the observer whether one has a gravitational force or an inertial force. In other words, 

*equivalence principle:* no physical experiment can distinguish between gravitational and inertial forces.

In mathematical language, observers correspond to coordinate systems, and so the equivalence principle states that the physical equations should be formulated in general (i.e. “curvilinear”) coordinate systems, and should in all these coordinate systems have the same mathematical structure. This means that the physical equations should be invariant under diffeomorphisms, and thus space-time is to be modeled by a *Lorentzian manifold* \((\mathcal{M}, g)\).

A Lorentzian manifold is “locally Minkowski space” in the sense that at every space-time point \(p \in \mathcal{M}\), the corresponding *tangent space* \(T_p \mathcal{M}\) is a vector space endowed with a scalar product \(\langle \cdot, \cdot \rangle_p\) of signature \((+ - - -)\). Therefore, we can distinguish between spacelike, timelike and null tangent vectors. Defining a non-spacelike curve \(q(\tau)\) by the condition that its tangent vector \(u(\tau) \in T_{q(\tau)} \mathcal{M}\) be everywhere non-spacelike, our above definition of light cones and the notion of causality immediately carry over to a Lorentzian manifold. In a coordinate chart, the scalar product \(\langle \cdot, \cdot \rangle_p\) can be represented in the form (1.2.1) where \(g_{jk}\) is the so-called *metric tensor*. In contrast to Minkowski space, the metric tensor is not a constant matrix but depends on the space-time point, \(g_{jk} = g_{jk}(p)\). Its ten components can be regarded as the relativistic analogue of Newton’s gravitational potential. For every \(p \in \mathcal{M}\) there are coordinate systems in which the metric tensor coincides with the Minkowski metric up to second order,

\[
g_{jk}(p) = \text{diag}(1, -1, -1, -1), \quad \partial_\tau g_{\alpha\beta}(p) = 0. \quad (3.1.1)
\]
Such Gaussian normal coordinates correspond to the reference frame of a “freely falling observer” who feels no gravitational forces. However, it is in general impossible to arrange that also $\partial_{jk}g_{lm}(p) = 0$. This means that by going into a suitable reference frame, the gravitational field can be transformed away locally (= in one point), but not globally. With this in mind, a reference frame corresponding to Gaussian normal coordinates is also called a local inertial frame.

The physical equations can be carried over to a Lorentzian manifold by the prescription that they should in a local inertial frame have the same form as in Minkowski space; this is referred to as the strong equivalence principle. It amounts to replacing all partial derivatives by the corresponding covariant derivatives $\nabla$ of the Levi-Civita connection; we write symbolically

$$\partial \longrightarrow \nabla.$$  

The gravitational field is described via the curvature of space-time. More precisely, the Riemannian curvature tensor is defined by the relations

$$R_{ijkl}^i u^j = \nabla_j \nabla_k u^i - \nabla_k \nabla_j u^i.$$  

(3.1.2)

Contracting indices, one obtains the Ricci tensor $R_{jk} = R_{ik}^i$ and scalar curvature $R = R_j^j$. The relativistic generalization of Newton’s gravitational law are the Einstein equations

$$R_{jk} - \frac{1}{2} R g_{jk} = 8 \pi \kappa T_{jk},$$  

where $\kappa$ is the gravitational constant. Here the energy-momentum tensor $T_{jk}$ gives the distribution of matter and energy in space-time.

3.2. The Dirac Equation in Curved Space-Time

Dirac spinors are often formulated on a manifold using frame bundles, either an orthonormal frame [6, 65] or a Newman-Penrose null frame [83, 16]. We here outline an equivalent formulation of spinors in curved space-time in the framework of a U(2, 2) gauge theory (for details see [25]). We begin with constructions in local coordinates, whereas local issues like topological obstructions to the existence of spin structures will be discussed in Section 3.4 below. We let $\mathcal{M}$ be a 4-dimensional manifold (without Lorentz metric) and define the spinor bundle $S\mathcal{M}$ as a vector bundle over $\mathcal{M}$ with fibre $\mathbb{C}^4$. The fibres are endowed with a scalar product $\langle ., . \rangle$ of signature $(2, 2)$, referred to as the spin scalar product. Sections in the spinor bundle are called spinors or wave functions. In local coordinates, a spinor is represented by a 4-component complex function on space-time, usually denoted by $\psi(x)$. Choosing at every space-time point a pseudo-orthonormal basis $(e_\alpha)_{\alpha=1,...,4}$ in the fibres,

$$\langle e_\alpha | e_\beta \rangle = s_\alpha \delta_{\alpha \beta}, \quad s_1 = s_2 = 1, \ s_3 = s_4 = -1$$  

(3.2.1)

and representing the spinors in this basis, $\psi = \psi^\alpha e_\alpha$, the spin scalar product takes again the form (1.3.6). Clearly, the basis $(e_\alpha)$ is not unique, but at every space-point can be transformed according to

$$e_\alpha \longrightarrow (U^{-1})^\beta_\alpha e_\beta,$$  

where $U$ is an isometry of the spin scalar product, $U \in U(2, 2)$. Under this basis transformation the spinors behave as follows,

$$\psi^\alpha(x) \longrightarrow U^\beta_\alpha(x) \psi^\beta(x).$$  

(3.2.2)
In view of the analogy to gauge theories, we interpret this transformation of the wave functions as a local gauge transformation with gauge group \( G = U(2,2) \). We refer to a choice of the spinor basis \( (e_\alpha) \) as a gauge.

Our goal is to formulate classical Dirac theory in such a way that the above \( U(2,2) \) gauge transformations correspond to a physical symmetry, the \( U(2,2) \) gauge symmetry. To this end, we shall introduce a Dirac-type operator as the basic object on \( M \), from which we will then deduce the Lorentz metric and the gauge potentials. We define a differential operator \( D \) of first order on the wave functions by the requirement that in a chart and gauge it should be of the form

\[
D = iG^j(x) \frac{\partial}{\partial x^j} + B(x) \tag{3.2.3}
\]

with suitable \((4 \times 4)\)-matrices \( G^j \) and \( B \). This definition does not depend on coordinates and gauge, although the form of the matrices \( G^j \) and \( B \) clearly does. More precisely, under a change of coordinates \( x^i \to \tilde{x}^i \) the operator \( (3.2.3) \) transforms into

\[
i \left( G^k(x) \frac{\partial}{\partial \tilde{x}^k} \right) \frac{\partial}{\partial \tilde{x}^j} + B(\tilde{x}) \tag{3.2.4}
\]

whereas a gauge transformation \( \psi \to U\psi \) yields the operator

\[
UDU^{-1} = i(UG^jU^{-1}) \frac{\partial}{\partial x^j} + (UBU^{-1} + iUG^j(\partial_jU^{-1})) \tag{3.2.5}
\]

We define an operator of Dirac type by the requirement that by choosing suitable coordinates and gauge, one can arrange that the coefficient matrices \( G^j \) of the partial derivatives “coincide locally” with the Dirac matrices of Minkowski space.

**Definition 3.2.1.** A differential operator \( D \) of first order is called operator of Dirac type (or Dirac-type operator) if for every \( p \in M \) there is a chart \( (x^i, U) \) around \( p \) and a gauge \( (e_\alpha)_{\alpha = 1, \ldots, 4} \) such that \( D \) is of the form \( (3.2.3) \) with

\[
G^j(p) = \gamma^j \tag{3.2.6}
\]

where the \( \gamma^j \) are the Dirac matrices of Minkowski space in the Dirac representation \( (1.3.3) \).

It may seem unconventional that we defined Dirac-type operators without having a spinor bundle nor a connection. We shall now construct from the Dirac-type operator a gauge-covariant derivative \( D \), also referred to as the spin derivative. To this end, we first write the transformation law \( (3.2.2) \) in the shorter form

\[
\psi(x) \to U(x) \psi(x) \tag{3.2.7}
\]

with \( U \in U(2,2) \). Clearly, partial derivatives of \( \psi \) do not behave well under gauge transformations because we pick up derivatives of \( U \). This problem disappears if instead of partial derivatives we consider gauge-covariant derivatives

\[
D_j = \partial_j - iA_j \tag{3.2.8}
\]

provided that the gauge potentials transform according to

\[
A_j \to UA_jU^{-1} + iU(\partial_jU^{-1}) \tag{3.2.9}
\]

Namely, a short calculation shows that the gauge-covariant derivative behaves under gauge transformations according to

\[
D_j \to U D_j U^{-1} \tag{3.2.10}
\]
and thus the gauge-covariant derivatives of $\psi$ obey the simple transformation rule

$$D_j \psi \rightarrow U D_j \psi.$$  

Our task is to find matrices $A_j$ which transform under local gauge transformations according to (3.2.1). This construction will also reveal the structure of the matrix $B$, and this will finally lead us to the definition of the Dirac operator, which involves precisely the gravitational and electromagnetic fields.

In the chart and gauge where (3.2.6) holds, it is obvious from (1.3.1) that the anti-commutator of the matrices $G^j(p)$ gives the Minkowski metric. Using the transformation rules (3.2.4) and (3.2.5), one sees that in a general coordinate system and gauge, their anti-commutator defines a Lorentz metric,

$$g^{jk}(x) \equiv \frac{1}{2} \{G^j(x), G^k(x)\}.$$  

Moreover, using that the Dirac matrices in Minkowski space are symmetric w.r.t. the spin scalar product (see (1.3.7)), one sees that the same is true for the matrices $G^j$, i.e.

$$\langle G^j \psi | \phi \rangle = \langle \psi | G^j \phi \rangle \quad \text{for all } \psi, \phi.$$  

Via (3.2.11), the Dirac-type operator induces on the manifold a Lorentzian structure. We refer to the matrices $G^j$ as the Dirac matrices in curved space-time. Since we can arrange that these matrices coincide locally with the Dirac matrices of Minkowski space, all manipulations of Dirac matrices can be performed at any given space-time point in an obvious way. In particular, the pseudoscalar operator (1.3.19) now takes the more general form

$$\Gamma(x) = \frac{1}{4!} \varepsilon_{jklm} G^j G^k G^l G^m$$  

where the anti-symmetric tensor $\varepsilon_{jklm} = \sqrt{\det g} \varepsilon_{jklm}$ differs from the anti-symmetric symbol $\epsilon_{jklm}$ by the volume density. The pseudoscalar operator gives us again the notion of even and odd matrices and of chirality (1.3.20). Furthermore, we introduce the bilinear matrices $\sigma^{jk}$ by

$$\sigma^{jk}(x) = \frac{i}{2} [G^j, G^k].$$  

As in Minkowski space, the matrices

$$G^j, \quad \Gamma G^j, \quad \mathbb{1}, \quad i\Gamma, \quad \sigma^{jk}$$  

form a basis of the 16-dimensional (real) vector space of symmetric matrices (symmetric with respect to the spin scalar product $\langle \cdot, \cdot \rangle$). The matrices $G^j$ and $\Gamma G^j$ are odd, whereas $\mathbb{1}$, $i\Gamma$ and $\sigma^{jk}$ are even.

For the construction of the spin connection we must clearly consider derivatives. The Lorentzian metric (3.2.11) induces the Levi-Civita connection $\nabla$, which defines the covariant derivative of tensor fields. Taking covariant derivatives of the Dirac matrices,

$$\nabla_k G^j = \partial_k G^j + \Gamma^j_{kl} G^l,$$  

we obtain an expression which behaves under coordinate transformations like a tensor. However, it is not gauge covariant, because a gauge transformation (3.2.7) yields contributions involving first derivatives of $U$. More precisely, according to (3.2.5),

$$\nabla_k G^j \rightarrow \nabla_k (UG^j U^{-1}) = U(\nabla_k G^j) U^{-1} + (\partial_k U) G^j U^{-1} + U G^j (\partial_k U^{-1})$$

$$= U(\nabla_k G^j) U^{-1} - [U(\partial_k U^{-1}), U G^j U^{-1}].$$  

We can use the second summand in (3.2.15) to partially fix the gauge.
Lemma 3.2.2. For every space-time point \( p \in \mathcal{M} \) there is a gauge such that
\[
\nabla_k G^j(p) = 0 \quad \text{for all indices } j, k. \tag{3.2.16}
\]

Proof. We start with an arbitrary gauge and construct the desired gauge with two subsequent gauge transformations:

1. The matrix \( \partial_j \Gamma \) is odd, because
\[
0 = \partial_j 1 = \partial_j (\Gamma \Gamma) = (\partial_j \Gamma) \Gamma + \Gamma (\partial_j \Gamma).
\]

As a consequence, the matrix \( i \Gamma (\partial_j \Gamma) \) is selfadjoint. We can thus perform a gauge transformation \( U \) with \( U(p) = 1, \partial_j U(p) = \frac{1}{2} (\partial_j \Gamma) \Gamma \). In the new gauge the matrix \( \partial_j \Gamma(p) \) vanishes,
\[
\partial_j \Gamma|_p \longrightarrow \partial_j (U \Gamma U^{-1})|_p = \partial_j \Gamma|_p + \frac{1}{2} [\Gamma (\partial_j \Gamma), \Gamma]|_p = \partial_j \Gamma|_p - \Gamma^2 (\partial_j \Gamma)|_p = 0.
\]

Differentiating the relation \( \{ \Gamma, G^j \} = 0 \), one sees that the matrix \( \nabla_k G^j|_p \) is odd. We thus can represent it in the form
\[
\nabla_k G^j|_p = \Lambda^j_{km} G^m|_p + \Theta^j_{km} \Gamma G^m
\]
with suitable coefficients \( \Lambda^j_{km} \) and \( \Theta^j_{km} \).

This representation can be further simplified: According to Ricci’s Lemma, \( \nabla_n g^{jk} = 0 \). Expressing the metric via the anti-commutation relations and differentiating through with the Leibniz rule, we obtain
\[
0 = \{ \nabla_n G^j, G^k \} + \{ G^j, \nabla_n G^k \} = 2 \Lambda^j_{nm} g^{mk} - \Theta^j_{nm} 2i \Gamma \sigma^{mk} + 2 \Lambda^k_{nm} g^{mj} - \Theta^k_{nm} 2i \Gamma \sigma^{mj}
\]
and thus
\[
\Lambda^j_{nm} g^{mk} = -\Lambda^k_{nm} g^{mj}. \tag{3.2.19}
\]

In the case \( j = k \neq m \), (3.2.18) yields that \( \Theta^j_{nm} = 0 \). For \( j \neq k \), we obtain \( \Theta^j_{nj} \sigma^{ik} + \Theta^k_{nk} \sigma^{kj} = 0 \) and thus \( \Theta^j_{nj} = \Theta^k_{nk} (j \text{ and } k \text{ denote fixed indices, no summation is performed}) \). We conclude that there are coefficients \( \Theta_k \) with
\[
\Theta^j_{km} = \Theta_k \delta^j_m. \tag{3.2.20}
\]

2. We perform a gauge transformation \( U \) with \( U(p) = 1 \) and
\[
\partial_k U = -\frac{1}{2} \Theta_k \Gamma - i \frac{1}{4} \Lambda^m_{kn} g^{nl} \sigma_{ml}.
\]

Using the representation (3.2.17) together with (3.2.19) and (3.2.20), the matrix \( \nabla_k G^j \) transforms into
\[
\nabla_k G^j \longrightarrow \nabla_k G^j + [\partial_k U, G^j]
\]
\[
= \Lambda^j_{km} G^m + \Theta_k \Gamma G^j - \Theta_k \Gamma G^j - i \frac{1}{4} \Lambda^m_{kn} g^{nl} [\sigma_{ml}, G^j]
\]
\[
= \Lambda^j_{km} G^m - i \frac{1}{4} \Lambda^m_{kn} g^{nl} 2i (G_m \delta^j_l - G_l \delta^j_m)
\]
\[
= \Lambda^j_{km} G^m + \frac{1}{2} \Lambda^m_{kn} g^{nj} G_m - \frac{1}{2} \Lambda^j_{km} G^m = 0.
\]
We call a gauge satisfying condition (3.2.16) a normal gauge around $p$. In order to analyze the remaining gauge freedom, we let $U$ be a transformation between two normal gauges. Then according to (3.2.15) and (3.2.16), the commutator $\left[U(\partial_k U^{-1}) , U G^j U^{-1}\right]$ vanishes at $p$ or, equivalently,

$$\left[i(\partial_k U^{-1}) U , G^j\right]|_p = 0 .$$

As is easily verified in the basis (3.2.14) using the anti-commutation relations, a matrix which commutes with all Dirac matrices is a multiple of the identity matrix. Moreover, the matrix

$$i(\partial_j U^{-1}) U$$

is selfadjoint because

$$\left(i(\partial_j U^{-1}) U\right)^* = iU^{-1}(\partial_j U) = -i\partial_j(U^{-1}U) + i(\partial_j U^{-1}) U = i(\partial_j U^{-1}) U .$$

We conclude that the matrix $iU (\partial_j U^{-1})$ is a real multiple of the identity matrix. Transforming it unitarily with $U$, we see that it also coincides with the matrix $iU (\partial_j U^{-1})$. Restricting attention to normal gauges, it is easy to find expressions with the required behavior (3.2.9) under gauge transformations. Namely, setting

$$a_j = \frac{1}{4} \operatorname{Re} \operatorname{Tr} \left(G_j B \right) \mathbb{1} , \quad (3.2.21)$$

where “Tr” denotes the trace of a $4 \times 4$-matrix, one sees from (3.2.5) that

$$a_j \rightarrow a_j + \frac{1}{4} \operatorname{Re} \operatorname{Tr} \left(G_j G^k i(\partial_k U^{-1}) U\right) \mathbb{1} = a_j + iU(\partial_j U^{-1}) .$$

We can identify the $a_j$ with the gauge potentials $A_j$ and use (3.2.8) as the definition of the spin connection.

**Definition 3.2.3.** The spin derivative $D$ is defined by the condition that it behaves under gauge transformations (3.2.7) according to (3.2.10), and that in normal gauges around $p$ it has the form

$$D_j(p) = \frac{\partial}{\partial x^j} - ia_j \quad (3.2.22)$$

with the potentials $a_j$ according to (3.2.21).

In general gauges, the spin derivative can be written as

$$D_j = \frac{\partial}{\partial x^j} - iE_j - ia_j \quad (3.2.23)$$

with additional matrices $E_j(x)$, which involve the Dirac matrices and their first derivatives. The components of $E_j$ are sometimes referred to as spin coefficients. A short calculation shows that the trace of the matrix $E_j$ does not change under gauge transformations, and since it vanishes in normal gauges, we conclude that the matrices $E_j$ are trace-free. It is straightforward to verify that they are given explicitly by (for details see Appendix A)

$$E_j = \frac{i}{2} \Gamma (\partial_j \Gamma) - \frac{i}{16} \operatorname{Tr}(G^m \nabla_j G^n) G_m G_n + \frac{i}{8} \operatorname{Tr}(\Gamma G_j \nabla_m G^m) \Gamma . \quad (3.2.24)$$

In the next two theorems we collect the basic properties of the spin connection.

**Theorem 3.2.4.** The spin derivative satisfies for all wave functions $\psi, \phi$ the equations

$$\left[D_k , G^l\right] + \Gamma_{kl}^j G^j = 0 \quad (3.2.25)$$

$$\partial_j \langle \psi | \phi \rangle = \langle D_j \psi | \phi \rangle + \langle \psi | D_j \phi \rangle . \quad (3.2.26)$$
Proof. The left side of (3.2.25) behaves under gauge transformations according to the adjoint representation \( \rightarrow U \cdot U^{-1} \) of the gauge group. Therefore, it suffices to check (3.2.25) in a normal gauge, where

\[
[D_k, G^j] + \Gamma^l_{kl} G^l = \nabla_k G^j - \frac{i}{4} \text{Re} \, \text{Tr} \left( G_j B \right) [1, G^j] = 0.
\]

Since both sides of (3.2.26) are gauge invariant, it again suffices to consider a normal gauge. The statement is then an immediate consequence of the Leibniz rule for partial derivatives and the fact that the spin derivative differs from the partial derivative by an imaginary multiple of the identity matrix (3.2.22).

The identity (3.2.25) means that the coordinate and gauge invariant derivative of the Dirac matrices vanishes. The relation (3.2.26) shows that the spin connection is compatible with the spin scalar product.

We define torsion \( T \) and curvature \( R \) of the spin connection as the following 2-forms,

\[
T_{jk} = \frac{i}{2} \left( [D_j, G_k] - [D_k, G_j] \right), \quad R_{jk} = \frac{i}{2} [D_j, D_k].
\]

**Theorem 3.2.5.** The spin connection is torsion-free. Curvature has the form

\[
R_{jk} = \frac{1}{8} R_{mnjk} \sigma^{mn} + \frac{1}{2} \left( \partial_j a_k - \partial_k a_j \right),
\]

where \( R_{mnjk} \) is the the Riemannian curvature tensor and the \( a_j \) are given by (3.2.21).

**Proof.** The identity (3.2.25) yields that

\[
[D_j, G_k] = [D_j, g_{kl} G^l] = (\partial_j g_{kl}) G^l - g_{kl} \Gamma^l_{jm} G^m = \Gamma^m_{jk} G_m.
\]

Thus, using that the Levi-Civita connection is torsion-free, we obtain

\[
T_{jk} = \frac{i}{2} \left( \Gamma^m_{jk} - \Gamma^m_{kj} \right) G_m = 0.
\]

Next, again using (3.2.25), we can rewrite the covariant derivative as a spin derivative,

\[
G_l \nabla_k u^l = [D_k, G_l u^l].
\]

Iterating this relation, we can express the Riemann tensor (3.1.2) by

\[
G_i R^i_{jk} u^l = [D_j, [D_k, G_l u^l]] - [D_k, [D_j, G_l u^l]] = [[D_j, D_k], G_l u^l] = -2i \left[ R_{jk}, G_l u^l \right].
\]

This equation determines curvature up to a multiple of the identity matrix,

\[
R_{jk}(x) = \frac{1}{8} R_{mnjk} \sigma^{mn} + \lambda_{jk} 1.
\]

Thus it remains to compute the trace of curvature,

\[
\frac{1}{4} \text{Tr}(R_{jk}) 1 = \frac{1}{8} \text{Tr} \left( \partial_j a_k - \partial_k a_j \right) 1 = \frac{1}{2} \left( \partial_j a_k - \partial_k a_j \right),
\]

where we used (3.2.23) and the fact that the matrices \( E_j \) are trace-free.

We come to the physical interpretation of the above construction. According to Lemma (3.2.2) we can choose a gauge around \( p \) such that the covariant derivatives of the Dirac matrices vanish at \( p \). Moreover, choosing normal coordinates and making a global
(=constant) gauge transformation, we can arrange that \( G(p) = \gamma^j \) and \( \partial_j g_{kl}(p) = 0 \). Then the covariant derivatives at \( p \) reduce to partial derivatives, and we conclude that
\[
G^j(p) = \gamma^j, \quad \partial_k G^j(p) = 0.
\] These equations resemble the conditions for normal coordinates (3.1.1), except that the role of the metric is now played by the Dirac matrices. Indeed, by differentiating (3.2.11) one sees that (3.2.27) implies (3.1.1). Therefore, (3.2.27) is a stronger condition which not only poses a condition for the coordinates, but also for the gauge. We call a coordinate system and gauge where (3.2.27) is satisfied a normal reference frame around \( p \).

In a normal reference frame around \( p \), the Dirac matrices, and via (3.2.11) also the metric, are the same as in Minkowski space up to the order \( \sim (x - p)^2 \). Combining the equivalence principle with the usual minimal coupling procedure in physics, it seems a sensible physical assumption that the Dirac equation at \( p \) should coincide with that in Minkowski space. This implies that there should be a normal gauge such that all gauge potentials vanish at \( p \), and thus the Dirac operator at \( p \) should coincide with the vacuum Dirac operator \( i\partial \). This physical argument makes it possible to specify the zero order term in (3.2.3).

**Definition 3.2.6.** A Dirac-type operator \( D \) is called **Dirac operator** if for any \( p \in M \) there is a normal reference frame around \( p \) such that \( B(p) = 0 \).

Equivalently, the Dirac operator could be defined as a differential operator of first order (3.2.3) with the additional structure that for any \( p \in M \) there is a coordinate chart and gauge such that the following three conditions are satisfied,
\[
G^j(p) = \gamma^j, \quad \partial_k G^j(p) = 0, \quad B(p) = 0.
\] This alternative definition has the disadvantage that it is a-priori not clear whether the second condition \( \partial_k G^j(p) = 0 \) can be satisfied for a general metric. This is the reason why we preferred to begin with only the first condition (Definition 3.2.1), then showed that the second condition can be arranged by choosing suitable coordinates and gauge, and satisfied the third condition at the end (Definition 3.2.6).

In general coordinates and gauge, the Dirac operator can be written as
\[
D = iG^j D_j = iG^j \left( \partial_j - iE_j - ia_j \right),
\] where \( D \) is the spin connection of Definition 3.2.3. The matrices \( E_j \) take into account the gravitational field and are called **spin coefficients**, whereas the \( a_j \) can be identified with the **electromagnetic potential** (compare (1.3.5)). We point out that the gravitational field cannot be introduced into the Dirac equation by the simple replacement rule \( \partial \to D \), because gravity has an effect on both the Dirac matrices and the spin coefficients. But factorizing the gauge group as \( \text{U}(2, 2) = \text{U}(1) \times \text{SU}(2, 2) \), the \( \text{SU}(2, 2) \)-gauge transformations are linked to the gravitational field because they influence \( G^j \) and \( E_j \), whereas the \( \text{U}(1) \) can be identified with the gauge group of electrodynamics. In this sense, we obtain a unified description of electrodynamics and general relativity as a \( \text{U}(2, 2) \) gauge theory. The Dirac equation
\[
(D - m) \psi = 0
\] describes a Dirac particle in the presence of a gravitational and electromagnetic field. According to Theorem 3.2.5 the curvature of the spin connection involves both the Riemann tensor and the electromagnetic field tensor. One can express the classical action in terms of these tensor fields, so that the corresponding Euler-Lagrange equations give rise to the classical Einstein-Dirac-Maxwell equations.
3.3. Computation of the Dirac Operator

For the probabilistic interpretation of the Dirac equation in curved space-time, we choose a space-like hypersurface \( \mathcal{N} \) (corresponding to “space” for an observer) and consider in generalization of \( (1.3.11) \) on solutions of the Dirac equation the scalar product

\[
(\psi|\phi) = \int_{\mathcal{N}} \langle \psi | G^j \nu_j \phi \rangle \, d\mu_{\mathcal{N}} ,
\]

where \( \nu \) is the future-directed normal on \( \mathcal{N} \) and \( d\mu_{\mathcal{N}} \) is the invariant measure on the Riemannian manifold \( \mathcal{N} \). Then \( (\psi|\psi) \) is the normalization integral, which we again normalize to one. Its integrand has the interpretation as the probability density. In analogy to \( (1.3.8) \) the Dirac current is introduced by \( J^k = \langle \psi | G^k \psi \rangle \). Using Theorem \( 3.2.4 \) one sees similar as in Minkowski space that the Dirac current is divergence-free, \( \nabla_k J^k = 0 \).

From the Gauss divergence theorem one obtains that the scalar product \( (3.2.30) \) does not depend on the choice of the hypersurface \( \mathcal{N} \).

In analogy to \( (1.3.15) \), we can introduce the inner product

\[
\langle \psi|\phi \rangle := \int_{\mathcal{M}} \langle \psi|\phi \rangle_x \, d\mu_{\mathcal{M}} ,
\]

in which the wave functions (which need not satisfy the Dirac equation but must have a suitable decay at infinity) are integrated over all of space-time. We finally remark that, using Theorem \( 3.2.4 \) together with Gauss’ divergence theorem, one easily verifies that the Dirac operator is symmetric with respect to this inner product.

### 3.3. Computation of the Dirac Operator

We now explain how the Dirac operator can be computed in an efficient way in a given space-time. Thus suppose that the Lorentzian metric \( g_{ij} \) is given in a chosen chart. The general procedure is to first choose matrices \( G^j(x) \) which are symmetric w.r.t. to the spin scalar product \( (3.2.12) \) (where in our formulation, the spin scalar product is always given by \( (3.2.1) \)) and which satisfy the anti-commutation relations \( (3.2.11) \). Then the spin coefficients as given by \( (3.2.24) \) are obtained by a straightforward computation. Then the spin derivative is given by \( (3.2.23) \) (where \( a_j \) are the components of the electromagnetic potential; they are set to zero if no electromagnetic field is present). The Dirac operator is given by \( (3.3.1) \), i.e.

\[
\mathcal{D} = iG^j D_j = iG^j \partial_j + G^j E_j + G^j a_j .
\]

In this construction, one has a lot of freedom to choose the Dirac matrices \( G^j(x) \) (as described systematically by the \( U(2,2) \)-gauge transformations \( (3.2.7) \) and \( (3.2.10) \)). It is a promising strategy to use this gauge freedom such as to choose Dirac matrices for which the formulas for the spin coefficients \( (3.2.24) \) become as simple as possible. Moreover, one should keep in mind that for the computation of the Dirac operator, one does not need to know all the matrices \( E_j \), but it suffices to compute the combination \( G^j E_j \) in \( (3.3.1) \). Indeed, in many space-times of physical interest, making use of the gauge freedom, the combination \( G^j E_j \) can be computed easily (for details see the computations in black hole geometries in \( [60, 48] \) or various examples in \( [47 \text{ Section 9}] \)). We here illustrate this method by the example of a diagonal metric, in which case it is even unnecessary to compute the Christoffel symbols:
Proposition 3.3.1. Assume that there is a local chart \((x^i)\) in which the metric is diagonal, i.e.

\[
ds^2 = \sum_{i=0}^{3} g_{ii}(x) \, dx_i^2.
\] (3.3.2)

Then there is a gauge in which the Dirac operator (without electromagnetic field) takes the form

\[
D = i G^j \frac{\partial}{\partial x^j} + B,
\] (3.3.3)

where

\[
G^j(x) = g_{jj}(x)^{-\frac{1}{2}} \gamma^j
\] (3.3.4)

\[
B(x) = \frac{i}{2\sqrt{|\det g|}} \partial_j \left( \sqrt{|\det g|} G^j \right).
\] (3.3.5)

(Here \(\gamma^j\) are again the Dirac matrices in Minkowski space).

Proof. With (3.3.4) we have satisfied the anti-commutation relations

\[
\{G^j, G^k\} = 2 g^{jk} 1.
\]

Moreover, the choice (3.3.4) ensures that the pseudoscalar operator is constant, and that all derivatives of the \(G^j\) are in the span of \(\gamma^0, \ldots, \gamma^k\). Therefore, the formula for the zero order term in the Dirac operator (3.3.1) simplifies to

\[
B = -\frac{i}{16} \text{Tr} \left( G_m \left( \nabla_j G_n \right) \right) G^j G^m G^n,
\] (3.3.6)

where \(\nabla_j G_n \equiv \partial_j G_n - \Gamma^k_{jn} G_k\) is the covariant derivative acting on the components of the spinorial matrix. Using the algebra of the Dirac matrices, one finds that (3.3.6) has a vectorial component (obtained by using the anti-commutation relations), and an axial component which is totally antisymmetric in the indices \(j, m, n\). This totally antisymmetric term vanishes for the following reasons: First, since the Levi-Civita connection is torsion-free, we may replace the covariant derivative by a partial derivative. Second, it follows from (3.3.4) that the matrix \(\partial_j G_n\) is a multiple of \(G_n\), implying that the trace \(\text{Tr}(G_m(\partial_j G_n))\) is symmetric in the indices \(m\) and \(n\).

It remains to compute the vectorial component of (3.3.6). A short computation yields

\[
B = \frac{i}{2} \nabla_j G^j,
\]

and the usual formula for the covariant divergence of a vector field gives the result. \(\square\)

3.4. Formulation with Vector Bundles, the Spinor Bundle

So far, the Dirac operator was introduced in a local chart. We intentionally left a large local gauge freedom, having the advantage that this freedom can be used to simplify the form of the Dirac operator. The remaining question is whether our constructions in local charts can be made global to obtain a Dirac operator \(\mathcal{D}\) acting on the sections of the so-called spinor bundle \(\mathcal{S} M\). To this end, we shall consider the Dirac operator in different charts and path the Dirac operators in the overlapping regions.

In preparation, we recall the structures introduced so far, using a more notation which clarifies the dependence on gauge and coordinates. In our local construction at the beginning of Section 3.2, the spinor space at a point \(x \in M\) is simply \(\mathbb{C}^4\) with the
inner product \((3.2.1)\). Using the same notation as in Section 1.4 in Minkowski space, we now denote the spinor space by \((S_x\mathcal{M},\langle\cdot,\cdot\rangle_x)\). Moreover, we denote the linear operators on \(S_x\mathcal{M}\) which are symmetric with respect to the spin scalar product by \(\text{Symm}(S_x\mathcal{M})\). It is a 16-dimensional real vector space spanned by the operators in \((3.2.14)\). Given a Dirac-type operator \(D\), the Dirac matrices \(G^j(x)\) span a four-dimensional subspace \(K_x\) of \(\text{Symm}(S_x\mathcal{M})\),

\[
K_x := \text{span}\{G^0(x),\ldots,G^3(x)\} \subset \text{Symm}(S_x\mathcal{M}),
\]

referred to as a Clifford subspace at \(x\). Contracting a tangent vector \(u\) with the Dirac matrices gives rise to a mapping

\[
\gamma : T_x\mathcal{M} \to K_x, \quad u \mapsto u^j G_j.
\]

Multiplying a spinor by \(\gamma(u)\) is referred to as Clifford multiplication. The anti-commutation relations \((3.2.11)\) can be written as

\[
\frac{1}{2} \{\gamma(u),\gamma(v)\} = g_x(u,v) \mathbf{1}_{S_x\mathcal{M}},
\]

showing that Clifford multiplication encodes the Lorentzian metric.

In view of the transformation law \((3.2.4)\), the Clifford subspace does not depend on the choice of coordinates. But it clearly depends on the gauge. Indeed, in view of \((3.2.5)\), it transforms according to

\[
K_x \to U K_x U^{-1} \quad \text{with} \quad U \in \text{U}(S_x) \quad (3.4.2)
\]

In order to simplify our problem, it is a good idea to arrange by a gauge transformation that the Clifford subspace agrees at every space-time point with the standard Clifford subspace:

**Lemma 3.4.1.** By a gauge transformation \((3.4.2)\) we can arrange that

\[
K_x = \text{span}\{\gamma^0,\ldots,\gamma^3\}
\]

(where \(\gamma^j\) are again the Dirac matrices in the Dirac representation).

**Proof.** We consider the pseudoscalar operator \(\Gamma(x)\) as defined by \((3.2.13)\). By direct computation, one verifies that it satisfies the same relations as in Minkowski space

\[
\Gamma(x)^* = -\Gamma(x) \quad \text{and} \quad \Gamma(x)^2 = \mathbf{1}.
\]

The first relation implies that \(\Gamma(x)\) maps positive definite spinors to negative definite spinors and vice versa. Therefore, there is a pseudo-orthonormal basis of the spinor space in which \(\Gamma(x)\) takes the same form as in Minkowski space,

\[
\Gamma(x) = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad (3.4.3)
\]

Rewriting the change of basis as a gauge transformation, we have arranged by a transformation of the form \((3.4.2)\) that the pseudoscalar operator has the same form as in Minkowski space.

It follows from \((3.2.13)\) and the anti-commutation relations that every vector in \(K\) anti-commutes with \(\Gamma\). Therefore,

\[
K \subset \text{span}\{\gamma^0,\ldots,\gamma^3,\Gamma \gamma^0,\ldots,\Gamma \gamma^3\}
\]
We next show that the vector space $K \cap \text{span}\{\gamma^0, \Gamma \gamma^0\}$ is one-dimensional. To this end, let $u, v \in T_x M$ with $\gamma(u) = (a+b \Gamma) \gamma^0$ and $\gamma(v) = (c+d \Gamma) \gamma^0$ with real coefficients $a, b, c, d$. Then their anti-commutator is computed by

$$\{\gamma(u), \gamma(v)\} = 2(ac - bd) \mathbf{1} + 2(bc - ad) \Gamma,$$

implying that $bc - ad = 0$. This implies that $\gamma(u)$ and $\gamma(v)$ are linearly dependent, giving the claim.

Repeating the last argument for $\gamma^1, \ldots, \gamma^3$, we conclude that there is a (not necessarily pseudo-orthonormal) basis $u_0, \ldots, u_3$ of $T_x M$ such that

$$\gamma(u_j) = (a_j + b_j \Gamma) \gamma_j.$$

Then for any $j \neq k$,

$$\{\gamma(u_j), \gamma(u_k)\} = (b_j a_k - a_j b_k) \Gamma \begin{bmatrix} \gamma_j, \gamma_k \end{bmatrix},$$

implying that the four vectors $(a_j, b_j) \in \mathbb{R}^2$ with $j = 0, \ldots, 3$ are all linearly independent. Therefore, by rescaling the basis vectors $u_j$ we can arrange that

$$\gamma(u_j) = (a + b \Gamma) \gamma_j$$

for real parameters $a$ and $b$.

The signature $(1, 3)$ of the Lorentzian metric implies that $|a| > |b|$. Moreover, by flipping the sign of the vectors $u_j$ if necessary we can arrange that $a > 0$. Therefore, we may represent $K$ as

$$K = \text{span}\{e^{a \Gamma} \gamma^0, \ldots, e^{a \Gamma} \gamma^3\}$$

for some $a \in \mathbb{R}$ (note that $e^{a \Gamma} = \cosh a + \Gamma \sinh a$). Performing the gauge transformation (3.4.2) with $U$ according to

$$U = \exp\left(-\frac{a}{2} \Gamma\right)$$

gives the result. \qed

After these preparations, we are ready to enter the patching construction. Thus let $(x, U)$ and $(\tilde{x}, \tilde{U})$ be two local charts on $(M, g)$ with non-empty overlap $U \cap \tilde{U}$. For technical simplicity, we only consider the case that the space-time is time-oriented. We choose the charts such that $x^0$ and $\tilde{x}^0$ are time functions which increase to the future. Then we can write the Dirac operator in each chart according to (3.2.28), where for clarity we denote the objects in the chart $\tilde{x}$ with an additional tilde. We first consider the case without electromagnetic field where the potentials $a_j$ vanish. According to Lemma 3.4.1, there is no loss of generality to restrict attention to gauges where the Dirac matrices are linear combinations of the Dirac matrices in Minkowski space, i.e.

$$G^j(x) = h^j_k(x) \gamma^k \quad \text{and} \quad \tilde{G}^j(\tilde{x}) = \tilde{h}^j_k(\tilde{x}) \gamma^k.$$

Since $x^0$ is a time coordinate, the bilinear form $<.|G^0(x).>_{x}$ is definite at very space-time point $x$, and similar for the tilde coordinates. We choose the signs of the Dirac matrices such that the bilinear forms $<.|G^0(x).>_{x}$ and $<.|\tilde{G}^0(\tilde{x}).>_{\tilde{y}}$ are all positive definite. Moreover, as explained in Lemma 3.4.1 we choose the gauge such that the pseudoscalar operator (3.2.13) has the same form as in Minkowski space (3.4.3). This implies that the spatial orientations of the two charts are compatible.
The transformation from the chart \((x, U)\) to \((\tilde{x}, \tilde{U})\) involves the coordinate transformation as described by (3.2.4). After this transformation, the Dirac matrices
\[
\tilde{G}^j(\tilde{x}) \quad \text{and} \quad G^k(x) \frac{\partial \tilde{x}^j}{\partial x^k}
\]
will in general not coincide. But since the matrices are all formed as linear combinations of the Dirac matrices in Minkowski space, satisfy the same anti-commutation relations, and have the same time and spatial orientations, they can be obtained from each other by an orthochronous and proper Lorentz transformation, i.e.
\[
\tilde{G}^j(\tilde{x}) = \Lambda^j_l G^k(x) \frac{\partial \tilde{x}^l}{\partial x^k}.
\]
Now we can proceed just as in the proof of Lorentz invariance of the Dirac equation in Minkowski space (see Lemma 1.3.1) to conclude that there is a unitary transformation \(U(x) \in U(S_x)\) of the form
\[
U := \exp \left( \frac{1}{4} \lambda_{lk} \gamma^l \gamma^k \right) \tag{3.4.4}
\]
(with an anti-symmetric tensor \(\lambda_{lk}\)) such that the Dirac matrices agree after the gauge transformation, i.e.
\[
U(\tilde{x}) \tilde{G}^j(\tilde{x}) U(\tilde{x})^{-1} = G^j(\tilde{x}) \frac{\partial \tilde{x}^l}{\partial x^k}.
\]
Since the spin coefficients \(E^j_l\) in (3.2.23) are given explicitly in terms of the Dirac matrices and their derivatives (see (3.2.24)), the lower order terms in the resulting Dirac operators (3.2.28) also agree. Moreover, using that the only matrices which commute with all Dirac matrices are multiples of the identity, one sees that the gauge transformation \(U(\tilde{x})\) of the form (3.4.4) is uniquely determined up to a sign. In this way, to every coordinate transformation, we have found a gauge transformation, unique up to a sign, such that the Dirac operators agree.

With the above construction, we have found a procedure for matching the Dirac operators in two overlapping charts. The involved gauge transformations of the form (3.4.4) are unique up to signs. Therefore, once we have decided on the signs, there is a unique way of identifying the Dirac wave functions in the overlapping region of two charts, such as to obtain Dirac wave functions in a larger patch of the manifold \(\mathcal{M}\). Proceeding inductively, one can hope to obtain Dirac wave functions on all of \(\mathcal{M}\). The subtle point is whether the signs of the transformations can be chosen in a compatible way for all charts. In more mathematical terms, one must satisfy the so-called cocycle conditions, and it turns out that these conditions can be fulfilled if and only if \(\mathcal{M}\) satisfies a topological condition, the so-called spin condition (for details see for example [80], § II.1 and § II.2]). If the spin condition is satisfied, one can identify the spinor spaces via the mappings which patch the charts together. In this way, one obtains a vector bundle over \(\mathcal{M}\), referred to as the spinor bundle \(S\mathcal{M}\). The fibres of the spinor bundle are the spinor spaces \(S_x\mathcal{M}\), which are four-dimensional complex vector spaces endowed with an inner product \(<\cdot,\cdot>_x\) of signature (2, 2). The transformations of the form (3.4.4) generate a group, the so-called spin group denoted by
\[
\text{Spin}^+_x \subset U(S_x) \tag{3.4.5}
\]
(the reason why we write “generated by” is that the operators of the form (3.4.4) do not form a group; see Exercise 3.2). Elements of the spin group act on vectors of the Clifford...
subspace by the adjoint representation,
\[ \gamma(v) \to U \gamma(v) U^{-1}, \]
we obtain another vector of the Clifford subspace, i.e.
\[ U \gamma(v) U^{-1} = \gamma(\mathcal{O}_U(u)) \]  
(3.4.6)
Since the anti-commutation relations remain unchanged, the resulting transformation of the tangent space is an isometry. Indeed, by Lemma 1.3.1 it is a proper orthochronal Lorentz transformation,
\[ \mathcal{O}_U \in SO^\uparrow(T_x \mathcal{M}) \]  
(3.4.7)
The indices \( \uparrow \) in (3.4.5) and (3.4.7) indicate that we restrict attention to orthochronous transformations. We thus obtain the usual commutative diagram
\[ \mathbb{Z}_2 \longrightarrow \text{Spin}_x^\uparrow \longrightarrow SO^\uparrow(T_x \mathcal{M}) \longrightarrow 0. \]

The connection to the usual spin group is obtained as follows. We say that a tangent vector \( u \in T_x \mathcal{M} \) is a unit vector if \( \langle u, u \rangle = \pm 1 \). The spin group is defined by (see for example [6], [80], the concise summary in [4], Section 2 or similarly [65] in the Riemannian setting)
\[ \text{Spin}_x := \{ \text{group generated by } \gamma(u) \gamma(v) \text{ with unit vectors } u, v \in T_x \mathcal{M} \} \]  
(3.4.8)
By expanding the exponential in (3.4.4), one sees that this matrix is generated by even products of Dirac matrices, showing that the group \( \text{Spin}_x^\uparrow \) in (3.4.5) is a subgroup of \( \text{Spin}_x \).

The group \( \text{Spin}_x \) also includes operators which are not unitary but satisfy instead the relation \( U^* U = -1 \). These transformations describe reversals of the time orientation. Working with the general spin group (3.4.8) is of advantage in general dimension or signature. In four-dimensional time-oriented and orientable space-times, however, we can just as well restrict attention to orthochronous proper Lorentz transformations and the gauge transformations in (3.4.5).

We finally mention how to treat an electromagnetic field. Then the starting point is a time-oriented Lorentzian spin manifold \( (\mathcal{M}, g) \) together with an anti-symmetric two-tensor \( F \) (the field tensor). In this situation, after the above coordinate and gauge transformations, the electromagnetic potentials \( a_j \) and \( \tilde{a}_j \) in the two charts will in general not coincide. But, since the field tensor is prescribed, they coincide after a local \( U(1) \)-gauge transformation. Identifying the spinor spaces after this gauge transformation defines the Dirac operator as acting on the spinor bundle \( S \mathcal{M} \). The resulting effective gauge group is \( U(1) \times \text{Spin}_x^\uparrow \). We point out that this effective gauge group is obtained under the condition that the Clifford subspace is fixed at each space-time point according to Lemma 3.4.1. Dropping this condition gives rise to the larger local gauge group \( U(2, 2) \).

### 3.5. The Dirac Solution Space in Globally Hyperbolic Space-Times

We now turn attention to solutions of the Dirac equation. In Minkowski space, a convenient method for constructing solutions is the Fourier transformation (see Section 2.2). However, this method can be used only for PDEs with constant coefficients, and therefore it does not apply to the Dirac equation in curved space-time. Instead, a general method is to solve the Cauchy problem for given initial data, making it possible to identify the solution space of the Dirac equation with the space of suitable initial data. Clearly, for this method to be applicable, one needs to decompose space-time into “space” and “time,” because otherwise it would not be clear how to prescribe initial data at some initial time.
In order to describe the Dirac solutions in all of space-time by initial data, this splitting of space-time into space and time must be performed globally. This splitting will not be unique, as is already clear in Minkowski space because different reference frames gives rise to different splittings. In curved space-time, such a splitting does not even need not exist. For example, in space-times containing closed timelike curves, we cannot expect that the Cauchy problem is well-posed.

The necessary assumptions on space-time needed for a sensible formulation of the Cauchy problem are subsumed in the mathematical notion of global hyperbolicity. We first give the formal definition and then explain its consequences. Let \((\mathcal{M}, g)\) be a Lorentzian manifold. We assume that \(\mathcal{M}\) is time-oriented. Then a parametrized piecewise \(C^1\)-curve \(\gamma(\tau)\) in \(M\) is said to be causal if its tangent vector \(\dot{\gamma}(\tau)\) is causal (i.e. timelike or lightlike) for all \(\tau\) where \(\gamma\) is differentiable. Moreover, it is future-directed and past-directed its tangent vectors are future- and past-directed, respectively. The curve \(\gamma(\tau)\) is said to be inextendible if it cannot be extended as a piecewise \(C^1\)-curve. The manifold \(\mathcal{M}\) is said to satisfy the strong causality condition if there are no almost closed causal curves in the sense that for all \(x \in \mathcal{M}\) and for each open neighborhood \(U\) of \(x\) there is an open neighborhood \(V \subset U\) of \(x\) such that every causal curve in \(\mathcal{M}\) which is starting and ending in \(V\) is entirely contained in \(U\). Moreover, in straightforward generalization of the corresponding notions in Minkowski space as introduced after (1.2.2), we let \(J^+_x\) (and \(J^+_x\)) be the set of all points \(y \in \mathcal{M}\) which can be joined from \(x\) by a future-directed (respectively past-directed) causal curve. The manifold \(\mathcal{M}\) is said to be globally hyperbolic if the strong causality condition holds and if the set \(J^+_x \cap J^+_y\) is compact for all \(x, y \in \mathcal{M}\). For more details on the abstract definitions we refer to [72, Section 6.6], [5, Section 1.3], [8, Section 3.2] or [82, Chapter 14]).

A globally hyperbolic Lorentzian manifold \((\mathcal{M}, g)\) has remarkable properties, as we now explain. First, global hyperbolicity implies that \((\mathcal{M}, g)\) is diffeomorphic to a product,

\[
\mathcal{M} \simeq \mathbb{R} \times N, \tag{3.5.1}
\]

where \(\simeq\) means that there is a smooth diffeomorphism from \(\mathcal{M}\) to \(\mathbb{R} \times N\). Thus every point \(p \in \mathcal{M}\) can be written as \(p = (t, x)\) with \(t \in \mathbb{R}\) and \(x \in N\). One also refers to the above property that \(\mathcal{M}\) admits a smooth foliation \(\mathcal{M} = (\mathcal{N}_t)_{t \in \mathbb{R}}\), where \(\mathcal{N}_t := \{t\} \times N\). Moreover, the foliation can be chosen such as to have the following properties:

(i) Every surface \(\mathcal{N}_t\) is spacelike (meaning that the metric induced by \(g\) on \(\mathcal{N}_t\) is negative definite).

(ii) Every surface \(\mathcal{N}_t\) is a Cauchy surface, meaning that every inextendible timelike curve in \(\mathcal{M}\) intersects \(\mathcal{N}_t\) exactly once.

The function \(t\) is also referred to as a global time function. These above properties of globally hyperbolic manifolds were proven in [9] (for more details and more references see again [5, Section 1.3]).

The property of \(\mathcal{N}_t\) of being a Cauchy surface implies that the Cauchy problem for the Dirac equation is well-posed, as we now explain. To this end, let \((\mathcal{M}, g)\) be a four-dimensional globally hyperbolic space-time. Then the topological splitting (3.5.1) implies that the spin condition mentioned before (3.4.5) is satisfied. Therefore, there is a spinor bundle \((S\mathcal{M}, \langle \cdot, \cdot \rangle)\), being a vector bundle with fibres \(S_x\mathcal{M} \simeq \mathbb{C}^4\) (there may be different spin structures, but we shall not go into this here). Moreover, the Dirac operator \(D\) is well-defined; in local coordinates and local spinor bases it takes the form (3.2.28). In the Cauchy problem, one seeks for solutions of the Dirac equation of mass \(m\) for prescribed
initial data at time $t_0$, i.e.
$$ (\mathcal{D} - m)\psi = 0 \quad \text{with} \quad \psi|_{N_{t_0}} = \psi_0. \quad (3.5.2) $$

The following result holds:

**Theorem 3.5.1.** For smooth initial data $\psi_0 \in C^\infty(N_{t_0}, SM)$, the Cauchy problem (3.5.2) has a unique global solution $\psi \in C^\infty(M, SM)$.

The proof of this theorem uses methods of hyperbolic partial differential equations and will be given in Section 11.4 later in this book.

Having a Cauchy surface is also very useful because we can then a scalar product on the solution space as the spatial integral (3.2.30), where $\mathcal{N}$ is chosen as a Cauchy surface. However, for the integral in (3.2.30) to be well-defined, working with smooth solutions is not suitable. Instead, similar as explained in Minkowski space in Section 1.4, we better assume that the solution has compact support on the Cauchy surface. Due to finite propagation speed for solutions of hyperbolic partial differential equations (as will be made precise in Section 11.2), the following result holds

**Theorem 3.5.2.** For initial data $\psi_0 \in C^\infty(N_{t_0}, SM)$ with compact support, the solution $\psi$ of the Cauchy problem (3.5.2) also has compact support on any other Cauchy surface $\mathcal{N}_t$.

The proof of this theorem will again be given in Section 11.4 below.

Using the same notion as in Section 1.4, we refer to smooth solutions as in the above theorem as having **spatially compact support**. Smooth and spatially compact sections of the spinor bundle are again denoted by $C^\infty_{sc}(M, SM)$. For Dirac solutions in this class, the scalar product (3.2.30) is well-defined. Moreover, due to current conservation, this scalar product does not depend on the choice of the Cauchy surface (as explained after (3.2.30)). Exactly as explained in Section 1.4, taking the completion gives the Hilbert space $(H_m, (\cdot, \cdot))$ of weak solutions of the Dirac equation with the property that their restriction to any Cauchy surface is square integrable (where “square integrable” is defined via (3.2.30)).

### 3.6. Hamiltonian Formulation in Stationary Space-Times

Given a foliation $(\mathcal{N}_t)_{t \in \mathbb{R}}$ by Cauchy surfaces of the globally hyperbolic space-time $(\mathcal{M}, g)$, one can rewrite the Dirac equation in the Hamiltonian form,

$$ i\partial_t \psi = H\psi \quad (3.6.1) $$

with a Hamiltonian $H$. In order to compute $H$ in a local chart, one chooses a coordinate system $(x^i)$ such that $x^0 = t$ coincides with the time function. Then, writing the Dirac operator in (3.2.29) in the form (3.2.28) and solving for the time derivatives, one obtains in generalization of (1.3.14)

$$ H = -(G^0)^{-1} \left( \sum_{\alpha=1}^3 iG^\alpha \left( \partial_\alpha - iE_\alpha - i\alpha_a \right) - m \right) - E_0 - a_0. $$

When analyzing the Dirac equation in the Hamiltonian form, one must be be careful because the Hamiltonian in general is **not symmetric** with respect to the Hilbert space scalar product (3.2.30). This can be seen as follows. For the Dirac equation in Minkowski space, the symmetry of the Hamiltonian is obtained just as for the Schrödinger equation by using that the scalar product is conserved in time (1.1.6). In curved space-time, the scalar product is still conserved (due to current conservation). But when taking the time
derivative, one must take into account that the scalar product itself is time dependent. More precisely, assuming for notational simplicity that the Cauchy surfaces admit global charts,

\[ 0 = \partial_t (\phi \langle \psi) = \frac{\partial}{\partial t} \int_{\mathcal{N}_t} \langle \psi \mid G^j \nu_j \rangle \phi \rangle \, d\mu_{\mathcal{N}_t} \]

\[ = (\partial_t \phi \mid \psi) + (\phi \mid \partial_t \psi) + \int_{\mathbb{R}^3} \langle \psi \mid \left( \partial_t (G^j \nu_j \sqrt{\det g_{\mathcal{N}_t}}) \right) \phi \rangle \, d^3x \]

\[ = -i \left( (H \phi \mid \psi) - (\phi \mid H \psi) \right) + \int_{\mathbb{R}^3} \langle \psi \mid \left( \partial_t (G^j \nu_j \sqrt{\det g_{\mathcal{N}_t}}) \right) \phi \rangle \, d^3x \] (3.6.2)

(where \( g_{\mathcal{N}_t} \) denotes the induced Riemannian metric on the Cauchy surface \( \mathcal{N}_t \)). The integral in the last line is in general non-zero. In this case, the Hamiltonian is obviously not symmetric. At first sight, this might seem surprising because it seems to contradict the axioms of quantum mechanics (for a detailed account on this issue see [2, 3]). However, one should keep in mind that the non-symmetric contributions to the Hamiltonian are needed in order to compensate for the fact that the scalar product itself is time-dependent.

Our interpretation of the above problem is that the Hamiltonian formulation of the Dirac equation is useful only in situations when the integral in (3.6.2) vanishes. This can be arranged if all the coefficients of the metric are time-independent. In other words, space-time should be stationary with corresponding Killing field given by \( \partial_t \). Under these assumptions, the Hamiltonian \( H \) is also time-independent. Moreover, the computation (3.6.2) shows that the operator is symmetric. Using that the time evolution maps smooth and compactly supported initial data on the Cauchy surface at time \( t_0 \) to a smooth and compactly supported solution at an arbitrary time \( t \), one can use abstract methods to construct a selfadjoint extension of \( H \) (see for example [58] for a general situation involving additional boundary conditions). Then the Cauchy problem can be solved immediately using the spectral theorem for selfadjoint operators,

\[ \psi(t, x) = (e^{-itH} \psi_0)(x). \]

This formulation is particularly useful for analyzing the long-time behavior of the solutions (see for example the analysis in the Kerr geometry in [50, 49]).

**Exercises**

**Exercise 3.1.** Verify by elementary integration by parts in a chart that for a diagonal metric (3.3.2), the Dirac operator (3.3.3) is symmetric with respect to the inner product (3.2.31).

**Exercise 3.2.** The goal of this exercise is to show that the unitary operators of the form (3.4.4) do not form a group (in more mathematical language, the spin group is not exponential; for details see [22] and the references therein). We proceed in several steps:

(a) Let \( \lambda_{jk} \) be an anti-symmetric tensor. Show using the anti-commutation relations that

\[ \left( \frac{1}{4} \lambda_{lk} \gamma^l \gamma^k \right)^2 = -\frac{1}{16} \lambda_{lk} \lambda^{lk} 1 + \frac{i}{16 \cdot 4!} \Gamma e^{ijkl} \lambda_{ij} \lambda_{kl}. \] (3.6.3)

(b) Deduce from (a) that the corresponding unitary transformation (3.4.4) is a linear combination of the matrices

\[ 1 , \quad \Gamma , \quad \lambda_{lk} \gamma^l \gamma^k \quad \text{and} \quad \lambda_{lk} \Gamma \gamma^l \gamma^k. \]
(c) Show under the additional assumption $\varepsilon^{ijkl} \lambda_{ij} \lambda_{kl} = 0$ that

$$
\exp \left( \frac{1}{4} \lambda_{lk} \gamma^l \gamma^k \right) = \begin{cases} 
1 & \cos \alpha + \frac{1}{4} \lambda_{lk} \gamma^l \gamma^k \frac{\sin \alpha}{\alpha} \\
1 & \cosh \alpha + \frac{1}{4} \lambda_{lk} \gamma^l \gamma^k \frac{\sinh \alpha}{\alpha}
\end{cases}
$$

if $\lambda_{lk} \lambda^l k > 0$, \quad \text{if } \lambda_{lk} \lambda^l k < 0,

(3.6.4)

where $\alpha := \sqrt{|\lambda_{lk} \lambda^l k|}/4$. \textit{Hint:} Use (3.6.3) in the power series of the exponential.

(c) Choose a specific tensor $\lambda_{lk}$ for which the matrix in (3.6.4) is equal to minus the identity.

(d) We now restrict attention to tensors $\lambda$ for which the corresponding unitary transformation (3.4.4) is a linear combination of the matrices $1$ and $\gamma^0 \gamma^1$. Infer from (b) that in this case, there are real numbers $\alpha, \beta$ such that

$$
\frac{1}{4} \lambda_{lk} \gamma^l \gamma^k = (\alpha + i \Gamma \beta) \gamma^0 \gamma^1.
$$

Deduce that

$$
\exp \left( \frac{1}{4} \lambda_{lk} \gamma^l \gamma^k \right) = \cosh (\alpha + i \Gamma \beta) + \gamma^0 \gamma^1 \sinh (\alpha + i \Gamma \beta).
$$

(e) Show that the last expression involves no contribution $\sim \Gamma$ only if either $\alpha = 0$ or $\beta = 0$ (\textit{Hint:} It might be convenient to work in an eigenvector basis of $i \Gamma$). Infer that in the case $\alpha = 0$, this expression is a linear combination of the matrices $1$ and $\gamma^0 \gamma^1$ only if $\beta \in \pi \mathbb{Z}$. Conclude that

$$
\exp \left( \frac{1}{4} \lambda_{lk} \gamma^l \gamma^k \right) = a 1 + b \gamma^0 \gamma^1 \text{ with } a > 0.
$$

(f) Deduce from (c) and (e) that the matrices of the form (3.4.4) do not form a group.
Part 2

Causal Fermion Systems: Basic Definitions and Fundamental Structures
CHAPTER 4

A Brief Introduction to Causal Fermion Systems

4.1. Motivating Example: A Two-Dimensional Lattice System

In order to motivate causal fermion systems, we begin with the familiar example of a space-time lattice. For simplicity, we consider a two-dimensional lattice (one space and one time dimension), but higher-dimensional lattices can be described similarly. Thus let $\mathcal{M} \subset \mathbb{R}^{1,1}$ be a lattice in two-dimensional Minkowski space. We denote the spacing in time direction by $\Delta t$ and in spatial direction by $\Delta x$ (see Figure 4.1). The usual procedure for setting up equations on a lattice is to replace derivatives by difference quotients, giving rise to an evolution equation which can be solved time step by time step according to deterministic rules.

As a concrete example, let us consider a discretization of the two-dimensional wave equation on the lattice,

$$0 = \Box \phi(t, x) := \frac{1}{(\Delta t)^2} \left( \phi(t + \Delta t, x) - 2\phi(t, x) + \phi(t - \Delta t, x) \right) - \frac{1}{(\Delta x)^2} \left( \phi(t, x + \Delta x) - 2\phi(t, x) + \phi(t, x - \Delta x) \right). \quad (4.1.1)$$

Solving this equation for $\phi(t + \Delta t, x)$ gives a rule for computing $\phi(t + \Delta t, x)$ from the values of $\phi$ at earlier times $t$ and $t - \Delta t$ (see again Figure 4.1).

While this method is very simple and gives well-defined evolution equations, it also has several drawbacks:

- The above method of discretizing the continuum equations is very ad hoc. Why do we choose a regular lattice, why do we work with difference quotients? There are many other ways of discretizing the wave equation.
- The method is not background-free. In order to speak of the “lattice spacing,” the lattice must be thought of as being embedded in a two-dimensional ambient space-time.

![Figure 4.1. Time evolution of a lattice system $\mathcal{M} \subset \mathbb{R}^{1,1}$.](image)
The concept of a space-time lattice is not invariant under general coordinate transformations. In other words, the assumption of a space-time lattice is not compatible with the equivalence principle.

In view of these shortcomings, the following basic question arises:

Can one formulate equations without referring to the nearest neighbor relation and the lattice spacing?

The answer to this question is yes, and we will now see how this can be done in the example of our two-dimensional lattice system. Although our example is somewhat oversimplified, this consideration will lead us quite naturally to the setting of causal fermion systems.

We consider complex-valued wave functions \( \psi_1, \ldots, \psi_f \) on the lattice \( \mathcal{M} \) (for simplicity a finite number, i.e. \( f < \infty \)). These wave functions do not need to satisfy the wave equation. On the complex vector space spanned by these wave functions we introduce a scalar product \( \langle \cdot, \cdot \rangle_\mathcal{H} \) and assume that the wave functions are orthonormal, i.e.

\[
\langle \psi_k | \psi_l \rangle_\mathcal{H} = \delta_{kl} . \tag{4.1.2}
\]

We thus obtain an \( f \)-dimensional Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}) \). Note that the scalar product is given abstractly (meaning that it has no representation in terms of the wave functions as a sum over lattice points). Next, for any lattice point \((t,x) \in \mathcal{M}\) we introduce the so-called local correlation operator \( F(t,x) : \mathcal{H} \to \mathcal{H} \) as a matrix with elements given by

\[
(F(t,x))_{jk}^l = \psi_j(t,x)^* \psi_k(t,x) . \tag{4.1.3}
\]

The diagonal elements of this matrix are the absolute squares \(|\psi_k(t,x)|^2\) of the corresponding wave functions. The off-diagonal elements, on the other hand, tell us about the correlation of the \(j\)th and \(k\)th wave functions at the lattice point \((t,x)\). This is the reason for the name “local correlation operator.” This operator can also be characterized in a basis-invariant way by the relations

\[
\langle \psi, F(t,x) \phi \rangle_\mathcal{H} = \overline{\psi(t,x)} \phi(t,x) ,
\]

to be satisfied for all \(\psi, \phi \in \mathcal{H}\). Taking the complex conjugate, one sees immediately that the matrix defined by \(4.1.3\) is Hermitian. Stated equivalently independent of bases, the local correlation operator is a selfadjoint linear operator on \(\mathcal{H}\). Moreover, a local correlation operator has rank at most one and is positive semi-definite. This can be seen by writing it as

\[
F(t,x) = e^* e \quad \text{with} \quad e : \mathcal{H} \to \mathbb{C}, \quad \psi \mapsto \psi(t,x) . \tag{4.1.4}
\]

It is useful to denote the set of all operators with the above properties by \(\mathcal{F}\),

\[
\mathcal{F} := \{ F \in \text{L}(\mathcal{H}) \mid F \text{ is selfadjoint, positive semi-definite and has rank at most one} \} .
\]

Varying the lattice point, we obtain a mapping (see Figure 4.2)

\[
F : \mathcal{M} \to \mathcal{F} , \quad (t,x) \mapsto F(t,x) .
\]

For clarity, we note that the set \(\mathcal{F}\) is not a vector space, because the linear combination of operators in \(\mathcal{F}\) will in general have rank bigger than one. But it is a conical set in the sense that a positive multiple of any operator in \(\mathcal{F}\) is again in \(\mathcal{F}\) (this is why in Figure 4.2 the set \(\mathcal{F}\) is depicted as a cone).

We point out that the local correlation operators do not involve the lattice spacing or the nearest neighbor relation; instead they contain information only on the local correlations of the wave functions at each lattice point. With this in mind, our strategy for formulating equations which do not involve the structures of the lattice is to work...
exclusively with the local correlation operators, i.e. with the subset $F(\mathcal{M}) \subset \mathcal{F}$. In other words, in Figure 4.2 we want to disregard the lattice on the left and work only with the objects on the right.

How can one set up equations purely in terms of the local correlation operators? In order to explain the general procedure, we consider a finite number of operators $F_1, \ldots, F_L$ of $\mathcal{F}$. Each of these operators can be thought of as giving information on the local correlations of the wave functions at a space-time point. However, this “space-time point” is no longer a lattice point, but at the moment it is merely a point without additional structure. In order to obtain a “space-time” in the usual sense, one needs additional structures and relations between the space-time points. Such relations can be obtained by multiplying the operators. Indeed, the operator product $F_i F_j$ tells us about correlations of the wave functions at different space-time points. Taking the trace of this operator product gives a real number. Our method for formulating physical equations is to set up a variational principle. This variational formulation has the advantage that symmetries give rise to conservation laws by Noether’s theorem (as will be explained in Section 7). Therefore, we want to minimize an action $S$. A simple example is to

$$\text{minimize } S(F_1, \ldots, F_n) := \sum_{i,j=1}^{L} \text{Tr}(F_i F_j)^2$$

under variations of the points $F_1, \ldots, F_L \in \mathcal{F}$. In order to obtain a mathematically sensible variational principle, one needs to impose certain constraints. Here we do not enter the details, because the present example is a bit too simple. Instead, we merely use it as a motivation for the general setting of causal fermion systems, which we now introduce.

4.2. To the General Definition of Causal Fermion Systems

In order to get from our example to the general setting of causal fermion systems, we must extend the above constructions in several steps:

(a) The previous example works similarly in higher dimensions, in particular for a lattice $\mathcal{M} \subset \mathbb{R}^{1,3}$ in four-dimensional Minkowski space. This has no effect on the resulting structure of a finite number of distinguished operators $F_1, \ldots, F_L \in \mathcal{F}$.

(b) Suppose that on the lattice we consider multi-component wave functions $\psi(t, x) \in \mathbb{C}^N$. Then the pointwise product on the right side of (4.1.3) must be replaced by a complex inner product, which we denote by $\langle . | . \rangle$ (in mathematical terms, this inner product is a non-degenerate sesquilinear form; we always use the convention that the wave function on the left is complex conjugated). Thus the definition of the
local correlation operator \( (4.1.3) \) is replaced by

\[
(F(t,x))_k^j = -\langle \psi_j(t,x) | \psi_k(t,x) \rangle
\]

(the minus sign compared to \( (4.1.3) \) merely is a useful convention). The resulting local correlation operator is no longer an operator of rank at most one, but it has rank at most \( N \) (as can be seen for example by writing it similar to \( (4.1.4) \) in the form \( F(t,x) = -e^* e \) with \( e : \mathcal{H} \to \mathbb{C}^N \)). If the inner product \( \langle . | . \rangle \) on \( \mathbb{C}^N \) is positive definite, then the operator \( F(t,x) \) is negative semi-definite. However, in the physical applications, this inner product will not be positive definite. Indeed, a typical example in mind is that of four-component Dirac spinors. The Lorentz invariant inner product \( \bar{\psi} \phi \) on Dirac spinors in Minkowski space (with the usual adjoint spinor \( \bar{\psi} := \psi^\dagger \gamma^0 \)) is indefinite of signature (2, 2). In order to describe systems involving leptons and quarks, one must take direct sums of Dirac spinors, giving the signature \( (n,n) \) with \( n \in 2\mathbb{N} \). With this in mind, we assume more generally that

\[
\langle . | . \rangle \quad \text{has signature} \quad (n,n) \quad \text{with} \quad n \in \mathbb{N}.
\]

Then the resulting local correlation operators are self-adjoint operators of rank at most \( 2n \), which (counting multiplicities) have at most \( n \) positive and at most \( n \) negative eigenvalues.

(c) Finally, it is useful to generalize the setting such as to allow for continuous space-times and for space-times which have both continuous and discrete components. In preparation, we note that the sums over the operators \( F_1, \ldots, F_L \) in \( (4.1.5) \) can be written as integrals,

\[
S(\rho) = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \text{Tr}(F_i F_j)^2,
\]

if \( \rho \) is a measure on \( \mathcal{F} \) chosen as the sum of Dirac measures supported at these operators,

\[
\rho = \sum_{i=1}^L \delta_{F_i}.
\]

In this formulation, the measure plays a double role: First, it distinguishes the points \( F_1, \ldots, F_L \) as those points where the measure is non-zero, as is made mathematically precise by the notion of the support of the measure, i.e.

\[
\text{supp} \rho = \{ F_1, \ldots, F_L \}.
\]

Second, a measure makes it possible to integrate over its support, an operation which for the measure \( (4.2.2) \) reduces to the sum over \( F_1, \ldots, F_L \).

Now one can extend the setting simply by considering \( (4.2.1) \) for more general measures on \( \mathcal{F} \) (like for example regular Borel measures). The main advantage of working with measures is that we get into a mathematical setting in which variational principles like \( (4.1.5) \) can be studied with powerful analytic methods.

This lead us to the following general definition:

**Definition 4.2.1.** \((\text{causal fermion system})\) Given a separable complex Hilbert space \( \mathcal{H} \) with scalar product \( \langle . | . \rangle_{\mathcal{H}} \) and a parameter \( n \in \mathbb{N} \) (the “spin dimension”), we let \( \mathcal{F} \subset \mathcal{L}(\mathcal{H}) \) be the set of all selfadjoint operators on \( \mathcal{H} \) of finite rank, which (counting multiplicities) have at most \( n \) positive and at most \( n \) negative eigenvalues. On \( \mathcal{F} \) we are given a positive
measure $\rho$ (defined on a $\sigma$-algebra of subsets of $\mathcal{F}$), the so-called universal measure. We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a causal fermion system.

This definition is illustrated in Figure 4.3. Now the set $\mathcal{F}$ is invariant in addition under the transformation where an operator is multiplied by a negative number, as is indicated in the figure by the double cones. The support of the measure, denoted by

$$M := \text{supp } \rho ,$$

is referred to as space-time. In generalization of the example of the lattice system, where space-time consisted of discrete points (4.2.3), now the measure $\rho$ can also have continuous components.

The Hilbert space $\mathcal{H}$ can be understood as being spanned by all wave functions of the system, similar as explained for our lattice system after (1.1.2). Indeed, starting from the abstract definition, to every vector $u \in \mathcal{H}$ one can associate a corresponding wave function in space-time, the so-called physical wave function (for details see [36, §1.1.4]). In the applications, the physical wave functions describe fermions. This is the reason for the name “causal fermion system.” The dimension of $\mathcal{H}$ can be regarded as the number of fermionic particles in the system (where in the physical applications we also count the particles of the so-called Dirac sea; see [33] for further explanations of this point).

The above notions evolved over several years. Based on preparations in [28], the present formulation was first given in [44].

### 4.3. Example: Dirac Wave Functions in Minkowski Space

As a further example, we now explain how to construct a causal fermion system in Minkowski space. Recall that in Section 1.4 (and similarly in curved space-time in Section 3.5) we introduced the Hilbert space $(\mathcal{H}_m, (\cdot, \cdot))$ of all solutions of the Dirac equation. We now choose a closed subspace $\mathcal{H}$ of this Hilbert space and denote the scalar product $(\cdot, \cdot)$ restricted to this subspace by $(\cdot, \cdot)_{\mathcal{H}}$. We then obtain the Hilbert space $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$.

By construction, the vectors in this Hilbert space are solutions of the Dirac equation. They can be thought of as the “occupied states” of the system. We prefer the notion of physical wave functions, where “physical” means intuitively that these wave functions are realized in our physical system (whatever this means; we do not enter philosophical issues here). At this point, we do not need to specify $\mathcal{H}$. For more explicit formulas and computations we refer to [36, Section 1.2].

We point out that the functions in $\mathcal{H}$ do not need to be continuous (instead, their restriction to any Cauchy surface merely is an $L^2$-function). Therefore, we cannot evaluate them pointwise at a space-time point $x \in \mathcal{M}$. However, for the following constructions
it is crucial to do so. The way out is to introduce so-called regularization operators \( \mathcal{R}_\varepsilon \) with \( 0 < \varepsilon < \varepsilon_{\text{max}} \) which map \( \mathcal{H} \) to the continuous wave functions,

\[
\mathcal{R}_\varepsilon : \mathcal{H} \rightarrow \mathcal{C}^0(\mathcal{M}, S\mathcal{M}) .
\]

In the limit \( \varepsilon \downarrow 0 \), these operators should go over to the identity,

\[
\mathcal{R}_\varepsilon u \rightarrow u \quad \text{weakly for all } u \in \mathcal{H} .
\]

Here we do not need to specify what precisely we mean by “weak convergence” (for details see [36, Definition 1.2.3]). A simple example of a regularization operator is obtained by mollifying with a test function. Thus we let \( h \in \mathcal{C}_0^\infty(\mathcal{M}, \mathbb{R}) \) be a non-negative test function with

\[
\hat{M} h(x) \, d^4x = 1 .
\]

We define the operators \( \mathcal{R}_\varepsilon \) for \( \varepsilon > 0 \) as the convolution operators

\[
(\mathcal{R}_\varepsilon u)(x) := \frac{1}{\varepsilon^4} \int_\mathcal{M} h(\frac{x-y}{\varepsilon}) \, u(y) \, d^4y .
\]

The physical picture is that on a small length scale (which can be thought of as the Planck length scale \( \varepsilon \approx 10^{-35} \text{m} \), the structure of space-time must be modified. The regularization operators specify what this microscopic structure looks like. We also notice that the construction by regularizing Dirac solutions is quite special. It must be thought of as merely giving examples of causal fermion systems, which serve as a motivation for the much more general abstract setting of causal fermion systems.

For any \( x \in \mathcal{M} \), we consider the bilinear form

\[
b_x : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} , \quad b_x(u, v) = -\langle \mathcal{R}_\varepsilon u(x) \mid \mathcal{R}_\varepsilon v(x) \rangle .
\]  

This bilinear form is well-defined and bounded because \( \mathcal{R}_\varepsilon \) is defined pointwise and because evaluation at \( x \) gives a linear operator of finite rank. Thus for any \( v \in \mathcal{H} \), the anti-linear form \( b_x(., v) : \mathcal{H} \rightarrow \mathbb{C} \) is continuous. By the Fréchet-Riesz theorem (see for example [81, Section 6.3]), there is a unique vector \( w \in \mathcal{H} \) such that \( b_x(u, v) = \langle u \mid v \rangle_{\mathcal{H}} \) for all \( u \in \mathcal{H} \). The mapping \( v \mapsto w \) is linear and bounded. We thus obtain a bounded linear operator \( F^\varepsilon(x) \) on \( \mathcal{H} \) such that

\[
b_x(u, v) = \langle u \mid F^\varepsilon(x) \| v \rangle_{\mathcal{H}} \quad \text{for all } u, v \in \mathcal{H} ,
\]

referred to as the local correlation operator. Taking into account that the inner product on the Dirac spinors at \( x \) has signature \((2, 2)\), the local correlation operator \( F^\varepsilon(x) \) is a symmetric operator on \( \mathcal{H} \) of rank at most four, which has at most two positive and at most two negative eigenvalues.

Thus for any \( \varepsilon \) we obtain a mapping

\[
F^\varepsilon : \mathcal{M} \rightarrow \mathcal{F} ,
\]

where \( \mathcal{F} \subset L(\mathcal{H}) \) is the set of all self-adjoint operators on \( \mathcal{H} \) of finite rank, which (counting multiplicities) have at most two positive and at most two negative eigenvalues. The last step is to drop all structures which we do not consider to be fundamental. Our concept is to work exclusively with the local correlation operators corresponding to the physical wave functions. Thus the basic concept is that all space-time structures (particles, fields, causal structure, geometry, ...) are encoded in the local correlation operators. In order to drop all the other structures, we introduce the universal measure \( \rho^\varepsilon = F^\varepsilon_\ast \mu \) as the push-forward of the volume measure on \( \mathcal{M} \) (for details see Section 2.1 or Exercise 4.2(b)). We thus obtain a causal fermion system (see Def. 12.1).
4.4. The Causal Action Principle

Having explained the general definition of a causal fermion system (see Definition 4.2.1), we can now introduce the variational principle used to describe the dynamics of a causal fermion system, the so-called causal action principle. The mathematical structure of the causal action is similar to the action (4.1.5) given in our example of the lattice system. Its detailed form, however, is far from obvious and is the result of many computations and long considerations. The causal action was first proposed in [28, Section 3.5].

For any \( x, y \in \mathcal{F} \), the product \( xy \) is an operator of rank at most \( 2n \). However, in general it is no longer a selfadjoint operator because \( (xy)^* = yx \), and this is different from \( xy \) unless \( x \) and \( y \) commute. As a consequence, the eigenvalues of the operator \( xy \) are in general complex. We denote these eigenvalues counting algebraic multiplicities by \( \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C} \) (more specifically, denoting the rank of \( xy \) by \( k \leq 2n \), we choose \( \lambda_1^{xy}, \ldots, \lambda_k^{xy} \) as all the non-zero eigenvalues and set \( \lambda_{k+1}^{xy}, \ldots, \lambda_{2n}^{xy} = 0 \)).

We introduce the Lagrangian and the causal action by

\[
\begin{align*}
\text{Lagrangian:} & \quad \mathcal{L}(x, y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left( |\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \\
\text{causal action:} & \quad S(\rho) = \int_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x, y) \, d\rho(x) \, d\rho(y).
\end{align*}
\]

(4.4.1)

(4.4.2)

The causal action principle is to minimize \( S \) by varying the measure \( \rho \) under the following constraints:

- **volume constraint:** \( \rho(\mathcal{F}) = \text{const} \) (4.4.3)
- **trace constraint:** \( \int_{\mathcal{F}} \text{tr}(x) \, d\rho(x) = \text{const} \) (4.4.4)
- **boundedness constraint:** \( \int_{\mathcal{F} \times \mathcal{F}} \left( \sum_{i=1}^{2n} |\lambda_i^{xy}| \right)^2 \, d\rho(x) \, d\rho(y) \leq C \), (4.4.5)

where \( C \) is a given parameter (and \( \text{tr} \) denotes the trace of a linear operator on \( \mathcal{H} \)).

In order to make the causal action principle mathematically well-defined, one needs to specify the class of measures in which to vary \( \rho \). To this end, on \( \mathcal{F} \) we consider the topology induced by the operator norm

\[
\|A\| := \sup \left\{ \|Au\|_\mathcal{H} \mid \|u\|_\mathcal{H} = 1 \right\}.
\]

(4.4.6)

In this topology, the Lagrangian as well as the integrands in (4.4.1) and (4.4.5) are continuous. The \( \sigma \)-algebra generated by the open sets of \( \mathcal{F} \) consists of the so-called Borel sets. A regular Borel measure is a measure on the Borel sets with the property that it is continuous under approximations by compact sets from inside and by open sets from outside (for basics see for example [71, §52]). The right prescription is to vary \( \rho \) within the class of regular Borel measures on \( \mathcal{F} \). There are two settings one can consider:

(a) The **finite-dimensional setting:** \( \dim \mathcal{H} < \infty \) and \( \rho(\mathcal{F}) < \infty \).

In this case, we will prove the existence of minimizers in Chapter 10. This will also clarify the significance of the constraints.

(b) The **infinite-dimensional setting:** \( \dim \mathcal{H} = \infty \) and \( \rho(\mathcal{F}) = \infty \).

This case also seems to make mathematical and physical sense. However, the ex-

F: Sage, dass so die physikalischen Gleichungen formuliert werden. Es werden also diejenigen Maße ausgezeichnet, die physikalisch zulässig sind.
existence theory has not yet been developed. At least, it is known that the Euler-Lagrange equations corresponding to the causal action principle still have a mathematical meaning (for details see [36]). One way of getting along without an existence theory in the infinite-dimensional setting is to take the point of view that on a fundamental physical level, the total volume is finite and the Hilbert space $\mathcal{H}$ is finite-dimensional, whereas the infinite-dimensional setting merely is a mathematical idealization needed in order to describe systems in infinite volume involving an infinite number of quantum particles.

We now explain what space-time and the underlying causal structure is. Given a minimizing measure $\rho$, we again define space-time $M$ as the support of the $\rho$ (4.2.4) (this is illustrated in Exercise 4.2). Thus the space-time points are symmetric linear operators on $\mathcal{H}$. On $M$ we consider the topology induced by $F$ (generated by the sup-norm (4.4.6) on $L(\mathcal{H})$). Moreover, the universal measure $\rho|_M$ restricted to $M$ can be regarded as a volume measure on space-time. This makes space-time to a topological measure space. Furthermore, one has the following notion of causality:

**Definition 4.4.1. (causal structure)** For any $x, y \in F$, the product $xy$ is an operator of rank at most $2n$. We denote its non-trivial eigenvalues (counting algebraic multiplicities) by $\lambda_{1}^{xy}, \ldots, \lambda_{2n}^{xy}$. The points $x$ and $y$ are called spacelike separated if all the $\lambda_{i}^{xy}$ have the same absolute value. They are said to be timelike separated if the $\lambda_{j}^{xy}$ are all real and do not all have the same absolute value. In all other cases (i.e. if the $\lambda_{j}^{xy}$ are not all real and do not all have the same absolute value), the points $x$ and $y$ are said to be lightlike separated.

Restricting the causal structure of $F$ to $M$, we get causal relations in space-time.

The Lagrangian (4.4.1) is compatible with the above notion of causality in the following sense. Suppose that two points $x, y \in F$ are spacelike separated. Then the eigenvalues $\lambda_{i}^{xy}$ all have the same absolute value. Rewriting (4.4.1) as

$$\mathcal{L} = \sum_{i=1}^{2n} |\lambda_{i}^{xy}|^2 - \frac{1}{2n} \sum_{i,j=1}^{2n} |\lambda_{i}^{xy}| |\lambda_{j}^{xy}| = \frac{1}{4n} \sum_{i,j=1}^{2n} \left( |\lambda_{i}^{xy}| - |\lambda_{j}^{xy}| \right)^2,$$

one concludes that the Lagrangian vanishes. Thus pairs of points with spacelike separation do not enter the action. This can be seen in analogy to the usual notion of causality where points with spacelike separation cannot influence each other. This analogy is the reason for the notion “causal” in “causal fermion system” and “causal action principle.”

### 4.5. The Local Trace is Constant

We now derive a first result on minimizing measures of the causal action principle. We do this at such an early stage because the result can be used to simplify the setup of causal fermion systems (see Section 4.6.4 below). The following result was first obtained in [10] (albeit with a different method); see also [36] Proposition 1.4.1]. For simplicity, we here only consider the finite-dimensional setting.

**Proposition 4.5.1.** Consider the causal variational principle in the finite-dimensional setting $\dim \mathcal{H} < \infty$. Let $\rho$ be a minimizer of finite total volume, $\rho(F) < \infty$. Then there is a real constant $c$ such that

$$\text{tr}(x) = c \quad \text{for all } x \in M.$$  

(4.5.1)

We often refer to $\text{tr}(x)$ as the local trace at the point $x$. 
4.5. THE LOCAL TRACE IS CONSTANT

Proof of Proposition 4.5.1. The method is to construct a suitable variation of the universal measure

$$(\rho_\tau)_{\tau \in (-\delta, \delta)} \quad \text{with} \quad \rho_0 = \rho$$

(for some $\delta > 0$). The variation must satisfy the constraints. Then we can make use of the fact that $\rho$ is a minimizer.

For the construction we combine two different methods. One method is to multiply $\rho$ by a positive function $f_\tau : M \to \mathbb{R}^+$,

$$\rho_\tau = f_\tau \rho.$$  (4.5.2)

Clearly, this variation does not change the support of the measure. In order to change the support, one can consider a function $F_\tau : M \to \mathcal{F}$ and take the push-forward measure,

$$\rho_\tau = (F_\tau)_* \rho.$$  (4.5.3)

Combining these methods, we are led to considering the family of measures

$$\rho_\tau = (F_\tau)_* (f_\tau \rho).$$  (4.5.4)

The condition $\rho_0 = \rho$ gives rise to the conditions

$$f_0 \equiv 1 \quad \text{and} \quad F_0 \equiv 1.$$  (4.5.5)

Finally, we assume that the functions $f_\tau$ and $F_\tau$ are defined and smooth in $\tau$.

The ansatz (4.5.4) is particularly convenient for computations. Namely, by definition of the push-forward measure,

$$\int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho_\tau(y) = \int_M \mathcal{L}(x, F_\tau(y)) \, f_\tau(y) \, d\rho(y),$$

and similarly for all other integrals. Next we choose $F_\tau$ in such a way that this integral does not change. To this end we choose

$$F_\tau(x) = \frac{x}{\sqrt{f_\tau(x)}}.$$  (4.5.6)

Using that $\mathcal{L}(x, y)$ is homogeneous in $y$ of degree two, it follows that

$$\int_M \mathcal{L}(x, F_\tau(y)) \, f_\tau(y) \, d\rho(y) = \int_M \mathcal{L} \left( x, \frac{y}{\sqrt{f_\tau(y)}} \right) \, f_\tau(y) \, d\rho(y)$$

$$= \int_M \mathcal{L}(x, y) \, \frac{1}{f_\tau(y)} \, f_\tau(y) \, d\rho(y) = \int_M \mathcal{L}(x, y) \, d\rho(y).$$

Arguing similarly in the variable $x$, one sees that our variation does not change the action. Since the expression $|xy|^2$ is also homogeneous of degree two in $x$ and $y$, the above argument also applies to the functional $\mathcal{T}$, showing that the boundedness constraint is also respected by our variation.

Let us analyze the constraints. In order to satisfy the volume constraint, we make the ansatz

$$f_\tau = 1 + \tau g,$$  (4.5.7)

where $g$ is a bounded function with zero mean,

$$\int_M g(x) \, d\rho(x) = 0 \quad \text{for all} \; \tau \in (-\delta, \delta).$$  (4.5.8)
We finally consider the variation of the trace constraint:
\[
\int_{\mathcal{F}} \text{tr}(x) \, d(\rho - \rho)(x) = \int_M \text{tr} \left( F_\tau(x) \right) f_\tau(x) \, d\rho(x) - \int_M \text{tr}(x) \, d\rho(x)
\]
\[
= \int_M \text{tr} \left( \frac{x}{\sqrt{f_\tau(x)}} \right) f_\tau(x) \, d\rho(x) - \int_M \text{tr}(x) \, d\rho(x)
\]
\[
= \int_M \text{tr}(x) \left( \sqrt{f_\tau(x)} - 1 \right) \, d\rho(x).
\]

Employing again the ansatz (4.5.7) and differentiating with respect to \( \tau \), we obtain for the first variation
\[
\frac{d}{d\tau} \int_{\mathcal{F}} \text{tr}(x) \, d(\rho - \rho)(x) \bigg|_{\tau=0} = \frac{1}{2} \int_M \text{tr}(x) \, g(x) \, d\rho(x).
\]

If the local trace is constant, then the last integral vanishes in view of (4.5.8). This shows that for the considered first variation, all the constraints are satisfied and the action is stationary. Conversely, if the local trace is not constant, then by choosing \( g \) appropriately, one can arrange that the first variation of the local trace is strictly positive,
\[
\frac{d}{d\tau} \int_{\mathcal{F}} \text{tr}(x) \, d(\rho - \rho)(x) \bigg|_{\tau=0} =: d > 0,
\]
whereas the other constraints are satisfied and the action is again stationary. Transforming the measures according to
\[
\rho \rightarrow (G_\tau)_*(\rho_\tau)
\]
with
\[
G_\tau(x) = x \left( \int_K \text{tr}(x) \, d\rho \right) \left( \int_K \text{tr}(x) \, d\rho + \tau \, d \right),
\]
the trace constraint is satisfied to first order. Moreover, as the scaling factor in (4.5.9) is smaller than one, the first variation of the action and of the boundedness constraint are strictly negative, in contradiction to the fact that \( \rho \) is a minimizer. This concludes the proof.

\[\square\]

### 4.6. Simplifications and Generalizations

The causal action principle for causal fermion systems is too complicated and too difficult for analyzing it in one step in full generality. Instead, it is preferable to approach the problem step by step by analyzing simplifications, modifications or generalizations of the setting which capture particular aspects of the full problem. Proceeding in this way also gives a better understanding of the physical meaning of the different structures of a causal fermion system and of the interaction as described by the causal action principle. We now give an outline of the different settings considered so far. In the later parts of this book, we will always consider the setting which is most suitable for the application in mind. Moreover, for pedagogical reasons, in this book we shall sometimes oversimplify the setting for example by assuming that the Lagrangian is smooth (see Section 4.6.5).
4.6. SIMPLIFICATIONS AND GENERALIZATIONS

4.6.1. Prescribing the Eigenvalues. Clearly, the trace constraint (4.4.4) and the boundedness constraint (4.4.5) complicate the analysis. Therefore, it might be a good idea to consider a simplified setting where these constraints are not needed. This can be accomplished most easily by prescribing the eigenvalues of the operators in \( \mathcal{F} \). This method was first proposed in [32, Section 2] in a slightly different formulation. We now explain the method in a way which fits best to our setting. Given \( n \in \mathbb{N} \), we choose real numbers \( \nu_1, \ldots, \nu_{2n} \) with

\[
\nu_1 \leq \cdots \leq \nu_n \leq 0 \leq \nu_{n+1} \leq \cdots \leq \nu_{2n} .
\]

We let \( \mathcal{F} \) be the set of all symmetric operators on \( \mathcal{H} \) of rank \( 2n \) whose eigenvalues (counted with multiplicities) coincide with \( \nu_1, \ldots, \nu_{2n} \). If \( \mathcal{H} \) is finite-dimensional, the set \( \mathcal{F} \) is compact. This is the reason why it is sensible to minimize the causal action (4.4.2) keeping only the volume constraint (4.4.3), which for simplicity we implement by restricting attention to normalized measures,

\[
\rho(\mathcal{F}) = 1 .
\]

The simplest interesting case is obtained by choosing the spin dimension \( n = 1 \) and the Hilbert space \( \mathcal{H} = \mathbb{C}^2 \). We denote the eigenvalues \( \nu_1 \) and \( \nu_2 \) in (4.6.1) by

\[
\nu_{1/2} = 1 \pm \tau \quad \text{with} \quad \tau > 1 .
\]

Then \( \mathcal{F} \) consists of all Hermitian \( 2 \times 2 \)-matrices \( F \) which have eigenvalues \( \nu_1 \) and \( \nu_2 \). Such a matrix can be represented with the help of Pauli matrices as

\[
\mathcal{F} = \left\{ F = \tau \vec{x} \vec{\sigma} + 1 \quad \text{with} \quad \vec{x} \in S^2 \subset \mathbb{R}^3 \right\}.
\]

Thus the set \( \mathcal{F} \) can be identified with the unit sphere \( S^2 \). In this setting, a straightforward computation yields for the Lagrangian (4.4.1)

\[
\mathcal{L}(x, y) = \max \left( 0, D(x, y) \right) \quad \text{with} \quad D(x, y) = 2\tau^2 \left( 1 + \langle x, y \rangle \right) \left( 2 - \tau^2 \left( 1 - \langle x, y \rangle \right) \right) ,
\]

and \( \langle x, y \rangle \) denotes the Euclidean scalar product of two unit vectors \( x, y \in S^2 \subset \mathbb{R}^3 \) (thus \( \langle x, y \rangle = \cos \vartheta \), where \( \vartheta \) is the angle between \( x \) and \( y \)). The resulting so-called causal variational principle on the sphere was introduced in [32, Chapter 1] and analyzed in [59, Sections 2 and 5] and more recently in [7].

4.6.2. Causal Variational Principles in the Compact Setting. Generalizing the causal variational principle on the sphere, one can replace \( \mathcal{F} \) by a smooth compact manifold of dimension \( m \geq 1 \). The Lagrangian \( \mathcal{L} \) can be defined in analogy to (4.6.3) as the positive part of a smooth function \( D \), i.e.

\[
\mathcal{L}(x, y) = \max \left( 0, D(x, y) \right) \quad \text{with} \quad D \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}) .
\]

Even more generally, one can choose \( \mathcal{L} \) as a non-negative continuous function,

\[
\mathcal{L}(x, y) \in C^0(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+0) .
\]

We usually assume that

(i) \( \mathcal{L} \) is symmetric: \( \mathcal{L}(x, y) = \mathcal{L}(y, x) \) for all \( x, y \in \mathcal{F} \).

(ii) \( \mathcal{L} \) is strictly positive on the diagonal: \( \mathcal{L}(x, x) > 0 \) for all \( x \in \mathcal{F} \).
The causal variational principle is to minimize the causal action
\[ S = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y) \]
under variations of the universal measure in the class of all regular Borel measures on \( \mathcal{F} \) which are normalized, i.e.
\[ \rho(\mathcal{F}) = 1. \]
This so-called compact setting was introduced in [59, Section 1.2]. It is the preferable choice for studying phenomena for which the detailed form of the Lagrangian as well as the constraints of the causal action principle are irrelevant.

Giving a minimizing measure \( \rho \), the Lagrangian induces on space-time \( M := \text{supp} \rho \) a causal structure. Namely, two space-time points \( x, y \in M \) are said to be timelike and space-like separated if \( \mathcal{L}(x, y) > 0 \) and \( \mathcal{L}(x, y) = 0 \), respectively. But of course, spinorial wave functions are missing in this setting.

### 4.6.3. The Non-Compact Setting.
In the compact setting, space-time \( M \) clearly is a compact subset of \( \mathcal{F} \). This is not suitable for describing situations when space-time has an asymptotic future or past or when space-time has singularities (like at the big bang or inside a black hole). For studying such situations, it is preferable to work in the so-called non-compact setting introduced in [53, Section 2.1]. Thus we now let \( \mathcal{F} \) be a non-compact smooth manifold of dimension \( m \geq 1 \). We choose the Lagrangian again in the form (4.6.4) or (4.6.5) again with the properties (i) and (ii) on page 93. In the non-compact setting, it is not sensible to work with normalized measures. Instead, the total volume \( \rho(\mathcal{F}) \) is infinite. In order to introduce a well-defined variational principle, we let \( \tilde{\rho} \) be another regular Borel measure on \( \mathcal{F} \) which satisfies the conditions
\[ |\tilde{\rho} - \rho|(\mathcal{F}) < \infty \quad \text{and} \quad (\tilde{\rho} - \rho)(\mathcal{F}) = 0 \]
(4.6.6)
(where \( |.| \) denotes the total variation of a signed measure; see Section 2.1). We then consider the difference of the actions defined by
\[ (S(\tilde{\rho}) - S(\rho)) = \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y) \]
\[ + \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y) + \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y). \]
(4.6.7)
The measure \( \rho \) is said to be a minimizer of the causal action if this difference is non-negative for all \( \tilde{\rho} \) satisfying (4.6.6),
\[ (S(\tilde{\rho}) - S(\rho)) \geq 0. \]

### 4.6.4. How Causal Fermion Systems Fit into the Non-Compact Setting.
Under mild technical assumptions on the minimizing measure, the causal action principle for causal fermion systems is a special case of the causal variational principle in the non-compact setting, as we now explain.

Since for minimizers of the causal action principle, all operators in \( M \) have the same trace (see Proposition 4.5.1), we can simplify the setting by restricting attention to linear operators in \( \mathcal{F} \) which have constant trace. Then the trace constraint can be disregarded. We now implement this simplification by modifying our setting. At the same time, we
implement the boundedness constraint by a Lagrange multiplier term. Here we apply this method naively by modifying the Lagrangian to

\[
L_\kappa(x, y) := \frac{1}{4n} \sum_{i,j=1}^{2n} \left( |\lambda^{xy}_i| - |\lambda^{xy}_j| \right)^2 + \kappa \left( \sum_{i=1}^{2n} |\lambda^{xy}_i| \right)^2 ,
\]

where \( \kappa > 0 \) is the Lagrange multiplier. The justification for this procedure as given in [10] is a bit subtle, and for brevity we shall not give these constructions here. It is important to note that, in contrast to the usual Lagrange multiplier, where a minimizer under constraints merely gives rise to a critical point of the Lagrangian including the Lagrange multipliers, here we obtain again a minimizer of the effective action (for details see [10, Theorem 3.13]).

Finally, we make a mild technical simplification. A space-time point \( x \in M \) is said to be regular if \( x \) has the maximal possible rank \( 2n \). Otherwise, the space-time point is singular. In physical applications, all space-time points are regular, except maybe at singularities like the center of black holes. With this in mind, we here assume that the causal fermion system is regular in the sense that all space-time points are regular. This assumption has the advantage the set of all regular points of \( \mathcal{F} \) is a smooth manifold (see Proposition 2.4.4 in Section 2.4). We remark that in the case that \( \mathcal{H} \) is infinite-dimensional, the set of regular points of \( \mathcal{F} \) could be endowed with the structure of a Banach manifold.

These considerations lead us to the following setting: Let \( (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}) \) be a complex Hilbert space. Moreover, we are given parameters \( n \in \mathbb{N} \) (the spin dimension), \( c > 0 \) (the constraint for the local trace) and \( \kappa > 0 \) (the Lagrange multiplier of the boundedness constraint). We let \( \mathcal{F}^{\text{reg}} \subset L(\mathcal{H}) \) be the set of all self-adjoint operators \( F \) on \( \mathcal{H} \) with the following properties:

(a) \( F \) has finite rank and (counting multiplicities) has \( n \) positive and \( n \) negative eigenvalues.

(b) The local trace is constant, i.e.

\[
\text{tr}(F) = c .
\]

On \( \mathcal{F}^{\text{reg}} \) we again consider the topology induced by the sup-norm on \( L(\mathcal{H}) \). We assume that \( \rho \) is a minimizer of the causal action principle, meaning that the action

\[
S_\kappa(\rho) = \iint_{\mathcal{F} \times \mathcal{F}} L_\kappa(x, y) \, d\rho(x) \, d\rho(y)
\]

is minimal under variations which preserve the total volume.

In order to avoid misunderstandings, we point out that the above description of causal fermion systems in \( \mathcal{F}^{\text{reg}} \) is not a suitable setting for the existence theory. Indeed, \( \mathcal{F}^{\text{reg}} \) is not locally compact, because the boundary points in \( \mathcal{F} \) are missing. More specifically, considering a minimizing sequence \( (\rho_n)_{n \in \mathbb{N}} \) of measures in \( \mathcal{F}^{\text{reg}} \), the limiting measure might well be supported also on \( \mathcal{F} \setminus \mathcal{F}^{\text{reg}} \). For this reason, there is no existence theory on \( \mathcal{F}^{\text{reg}} \). But if a minimizing measure is given, it seems sensible to assume that the resulting causal fermion system is regular. Under this assumption, the analysis of the causal fermion system can be carried out exclusively in \( \mathcal{F}^{\text{reg}} \), whereas \( \mathcal{F} \) is no longer needed. For a convenient and compact notation, in such situations we shall even omit the superscript “reg”, so that \( \mathcal{F} \) denotes the set of all selfadjoint operators on \( \mathcal{H} \) with the above properties (a) and (b). Moreover, we shall omit the subscript \( \kappa \). Thus, with a
slight abuse of notation, we shall denote the Lagrangian including the Lagrange multiplier term (4.6.8) by \( \mathcal{L} \).

4.6.5. The Smooth Setting. In order to bypass differentiability issues, it is sometimes a fair simplification to assume that the Lagrangian is smooth,

\[
\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+) .
\]

(4.6.9)

Clearly, the Lagrangian of the causal action (4.4.1) is not smooth, and this fact is indeed responsible for interesting effects like the results on the singular support in \([59, 7]\). In view of these results, the smoothness assumption (4.6.9) is unphysical. But nevertheless, it is useful in certain situations (see for example \([39]\)).

4.6.6. The Lower Semi-Continuous Setting. As a further generalization, one may relax the continuity of the Lagrangian in (4.6.5) by the condition that \( \mathcal{L} \) merely be lower semi-continuous, i.e. for all sequences \( x_n \to x \) and \( y_n' \to y \),

\[
\mathcal{L}(x, y) \leq \liminf_{n,n' \to \infty} \mathcal{L}(x_n, y_{n'}) .
\]

Since the Lagrangian of the causal action (4.4.1) is continuous, lower semi-continuity is an unphysical generalization. Nevertheless, this setting is useful for two reasons: First, because from the point of view of calculus of variations, it is a natural generalization to which most methods still apply. And second, because lower semi-continuous Lagrangians are convenient for formulating explicit examples (like the lattice model in \([53, \text{Section 5}]\)).

4.6.7. Riemannian Fermion Systems. Clearly, causal fermion systems were developed for describing space-times. Also, the resulting structures are closely tied to structures in space-time. Nevertheless, by slightly generalizing the definition, it also applies to the Euclidean and Riemannian setting:

**Definition 4.6.1.** Given a complex Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}) \) and parameters \( p, q \in \mathbb{N}_0 \) with \( p \leq q \), we let \( \mathcal{F} \subset \text{L}(\mathcal{H}) \) be the set of all self-adjoint operators on \( \mathcal{H} \) of finite rank, which (counting with multiplicities) have at most \( p \) positive and at most \( q \) negative eigenvalues. On \( \mathcal{F} \) we are given a positive measure \( \rho \) (defined on a \( \sigma \)-algebra of subsets of \( \mathcal{F} \)), the so-called universal measure. We refer to \( (\mathcal{H}, \mathcal{F}, \rho) \) as a topological fermion system of spin signature \( (p, q) \).

In the case \( p = q \), we call \( (\mathcal{H}, \mathcal{F}, \rho) \) a causal fermion system of spin dimension \( n := p \). If \( p = 0 \), we call \( (\mathcal{H}, \mathcal{F}, \rho) \) a Riemannian fermion system of spin dimension \( n := q \).

This definition was first given in \([47]\), where also many examples and applications are worked out.

For a Riemannian fermion system, the causal structure of Definition 4.4.1 becomes trivial. Namely, in this case the operator \( -y \) is positive semi-definite, so that its square root \( \sqrt{-y} \) is well-defined as a positive semi-definite operator. Using that the spectrum is invariant under cyclic permutations, it follows that

\[
xy = -x \sqrt{-y} \sqrt{-y} \quad \text{is isospectral to} \quad \sqrt{-y} (-x) \sqrt{-y}, \tag{4.6.10}
\]

and the last operator product is obviously positive semi-definite. As a consequence, the eigenvalues of \( xy \) are all real and non-negative. As a consequence, the points \( x \) and \( y \) are necessarily timelike separated, except in the trivial case that all eigenvalues of \( xy \) coincide. In view of this shortcoming, it is not clear whether the causal action principle is sensible for Riemannian fermion systems. It is conceivable that other variational principles are of mathematical interest, but this has not yet been explored.


Exercises

Exercise 4.1. (a causal fermion system on $\ell_2$) Let $(\mathcal{H} = \ell_2, \langle \cdot, \cdot \rangle)$ be the Hilbert space of square-summable complex-valued sequences. Thus, writing the vectors of $\mathcal{H}$ as $u = (u_i)_{i \in \mathbb{N}}$, the scalar product is defined by

$$\langle u | v \rangle = \sum_{i=1}^{\infty} u_i v_i .$$

For any $k \in \mathbb{N}$, we let $x_k$ be an operator on $\mathcal{H}$ defined by

$$(x_k u)_k = u_{k+1}, \quad (x_k u)_{k+1} = u_k$$

and $(x_k u)_i = 0$ for all $i \not\in \{k, k+1\}$. In other words,

$$x_k u = \left( 0, \ldots, 0, u_{k+1}, u_k, 0, \ldots \right) . \quad (4.6.11)$$

Finally, we let $\mu$ be the counting measure on $\mathbb{N}$ (i.e. $\mu(X) = \# X$ equals the number of elements of $X \subset \mathbb{N}$).

(a) Show that every operator $x_k$ has rank two, is symmetric, and has one positive and one negative eigenvalue. Make yourself familiar with the concept that every operator is a point in $F$ (as introduced in Definition 4.2.1) for spin dimension $n = 1$.

(b) Let $F : \mathbb{N} \to F$ be the mapping which to every $k$ associates the corresponding operator $x_k$. Show that the push-forward measure $\rho = F_* \mu$ defined by $\rho(\Omega) = \mu(F^{-1}(\Omega))$ defines a measure on $F$. Show that this measure can also be characterized by

$$\rho(\Omega) = \# \{ k \in \mathbb{N} | x_k \in \Omega \} .$$

(c) Show that $(\mathcal{H}, F, \rho)$ is a causal fermion system of spin dimension one.

Exercise 4.2. (support of a measure)

(a) We return to the example of Exercise 4.1. Show that the support of $\rho$ consists precisely of all the operators $x_k$.

(b) In order to illustrate how to encode geometric information in the support of a measure, let $\mathcal{M} \subset \mathbb{R}^3$ be a smooth surface described in a parametrization $\Phi$. Thus given an open subset $\Omega \subset \mathbb{R}^2$, we consider a smooth injective map

$$\Phi : \Omega \to \mathbb{R}^3$$

with the property that $D\Phi|_p : \mathbb{R}^2 \to \mathbb{R}^3$ has rank two for all $p \in \Omega$. Then the surface $\mathcal{M}$ is defined as the image $\Phi(\Omega) \subset \mathbb{R}^3$. We now introduce the measure $\rho$ as the push-forward measure of the Lebesgue measure on $\mathbb{R}^2$: Let $\mu$ be the Lebesgue measure on $\mathbb{R}^2$. We define a set $U \subset \mathbb{R}^3$ to be $\rho$-measurable if and only if its preimage $\Phi^{-1}(U) \subset \mathbb{R}^2$ is $\mu$-measurable. On the $\rho$-measurable sets we define the measure $\rho$ by

$$\rho(U) = \mu(\Phi^{-1}(U)) .$$

Verify that the $\rho$-measurable sets form a $\sigma$-algebra, and that $\rho$ is a measure. What are the sets of $\rho$-measure zero? What is the support of the measure $\rho$?

Suppose that $\Phi$ is no longer assumed to be injective. Is $\rho$ still a well-defined measure? Is $\rho$ well-defined if $\Phi$ is only assumed to be continuous? What are the minimal regularity assumptions on $\Phi$ needed for the push-forward measure to be well-defined? What is the support of $\rho$ in this general setting?
Exercise 4.3. This exercise shows that the trace constraint ensures that the action is non-zero. Let \((\mathcal{H}, \mathcal{F}, \rho)\) be a causal fermion system of spin dimension \(n\).

(a) Assume that \(\text{tr}(x) \neq 0\). Show that \(\mathcal{L}(x, x) > 0\). (For a quantitative statement of this fact in the setting of discrete space-times see [30] Proposition 4.3.)

(b) Assume that \(\int_{\mathcal{F}} \text{tr}(x) \, d\rho \neq 0\). Show that \(S(\rho) > 0\).

Exercise 4.4. (The two-dimensional Dirac sphere) We let \(M = S^2 \subset \mathbb{R}^3\) with \(d\mu\) the surface area measure, normalized such that \(\mu(S^2) = 1\). Furthermore, we choose \(n = 1\) and \(f = 2\). For a given parameter \(\tau > 1\) we introduce the mapping \(F : M \to \mathcal{F}\) by

\[
F(\vec{x}) = \tau \vec{x} \vec{\sigma} + 1.
\]

Then \(\sigma(F(x)) = \{1 + \tau, 1 - \tau\}\), and thus \(F(x)\) has one positive and one negative eigenvalue. Furthermore, a symmetry argument shows that the identity constraint (C2) is satisfied. Using identities between Pauli matrices

\[
\sigma^i \sigma^j = \delta^i_j + i \epsilon^{ijk} \sigma^k,
\]

one obtains

\[
F(\vec{x}) F(\vec{y}) = (1 + \tau^2 \vec{x} \vec{y}) 1 + \tau (\vec{x} + \vec{y}) \vec{\sigma} + i \tau^2 (\vec{x} \wedge \vec{y}) \vec{\sigma}.
\]

A straightforward calculation yields for the eigenvalues of this matrix

\[
\lambda_{1/2} = 1 + \tau^2 \cos \vartheta \pm \tau \sqrt{1 + \cos \vartheta} \sqrt{2 - \tau^2 (1 - \cos \vartheta)},
\]

where \(\vartheta\) denotes the angle \(\vartheta\) between \(\vec{x}\) and \(\vec{y}\). If \(\vartheta\) is sufficiently small, the term \((1 - \cos \vartheta)\) is close to zero, and thus the arguments of the square roots are all positive. However, if \(\vartheta\) becomes so large that

\[
\vartheta \geq \vartheta_{\text{max}} := \arccos \left(1 - \frac{2}{\tau^2}\right),
\]

the argument of the last square root in (4.6.14) becomes negative, so that the \(\lambda_{1/2}\) form a complex conjugate pair. The calculation

\[
\lambda_1 \lambda_2 = \det(F(\vec{x}) F(\vec{y})) = \det(F(\vec{x})) \det(F(\vec{y})) = (1 + \tau)^2 (1 - \tau)^2 > 0
\]

shows that if the \(\lambda_{1/2}\) are both real, then they have the same sign. Hence the Lagrangian simplifies to

\[
\mathcal{L}[A_{xy}] = \mathcal{L}(\cos \vartheta) = \frac{(\lambda_1 - \lambda_2)^2}{2} \Theta(\vartheta_{\text{max}} - \vartheta) = 2 \tau^2 (1 + \cos \vartheta) (2 - \tau^2 (1 - \cos \vartheta)) \Theta(\vartheta_{\text{max}} - \vartheta).
\]

Using this formula in (4.4.2), we can carry out the integrals to obtain

\[
S[F] = \frac{1}{2} \int_0^{\vartheta_{\text{max}}} \mathcal{L}(\cos \vartheta) \sin \vartheta \, d\vartheta = 4 - \frac{4}{3\tau^2}.
\]

Similarly, the functional \(\mathcal{T}\) can be computed to be

\[
\mathcal{T}[F] = 4 \tau^2 (\tau^2 - 2) + 12 - \frac{8}{3\tau^2}.
\]

Hence the action (4.6.15) is bounded uniformly in \(\tau\), although the function \(F\) (4.6.12), as well as the functional \(\mathcal{T}\) (4.6.16), diverge as \(\tau \to \infty\).

Exercise 4.5. This exercise explains why the causal action principle is ill-posed in the case \(\dim \mathcal{H} = \infty\) and \(\rho(\mathcal{F}) < \infty\). The underlying estimates were first given in the setting of discrete space-times in [30] Lemma 5.1.
(a) Let $\mathcal{H}_0$ be a finite-dimensional Hilbert space of dimension $n$ and $(\mathcal{H}, \rho_0, \mathcal{F}_0)$ be a causal fermion system of finite total volume $\rho_0(\mathcal{F}_0)$. Let $\iota : \mathcal{H}_0 \rightarrow \mathcal{H}$ be an isometric embedding. Construct a causal fermion system $(\mathcal{H}, \rho, \mathcal{F})$ which has the same action, the same total volume and the same values for the trace and boundedness constraints as the causal fermion system $(\mathcal{H}_0, \rho_0, \mathcal{F}_0)$.

(b) Let $\mathcal{H}_1 = \mathcal{H}_0 \oplus \mathcal{H}_0$. Construct a causal fermion system $(\mathcal{H}_1, \rho_1, \mathcal{F}_1)$ which has the same total volume and the same value of the trace constraint as $(\mathcal{H}, \rho_0, \mathcal{F}_0)$ but a smaller action and a smaller value of the boundedness constraint. *Hint:* Let $F_{1/2} : L(\mathcal{H}_0) \rightarrow L(\mathcal{H}_1)$ be the linear mappings

$$
(F_1(A))(u \oplus v) = (Au) \oplus 0, \quad (F_2(A))(u \oplus v) = 0 \oplus (Av).
$$

Show that $F_{1/2}$ maps $\mathcal{F}_0$ to $\mathcal{F}_1$. Define $\rho_1$ by

$$
\rho_1 = \frac{1}{2}((F_1)_* \rho + (F_2)_* \rho).
$$

(c) Iterate the construction in (b) and apply (a) to obtain a series of universal measures on $\mathcal{F}$ of fixed total volume and with fixed value of the trace constraint, for which the action and the values of the boundedness constraint tend to zero. Do these universal measures converge? If yes, what is the limit?

**EXERCISE 4.6. (embedding of $S_x\mathcal{M}$ into $S_{F(x)}$)** The goal of this exercise is to explain how the fibres of the spinor bundle $S\mathcal{M}$ are related to the spin spaces $S_x$ of the corresponding causal fermion system. In order to keep the setting as simple as possible, we let $(\mathcal{M}, g)$ be Minkowski space and $\mathcal{H}$ a finite-dimensional subspace of the Dirac solution space $\mathcal{H}_m$, consisting of smooth wave functions of spatially compact support, i.e.

$$
\mathcal{H} \subset C^\infty_S(\mathcal{M}, S\mathcal{M}) \cap \mathcal{H}_m \quad \text{finite-dimensional}.
$$

We again let $F(x)$ be the local correlation operators, i.e.

$$
\langle \psi|F(x)\psi \rangle = -\langle \psi(x)|\phi(x)\rangle \quad \text{for all } \psi, \phi \in \mathcal{H}
$$

(since $\mathcal{H}$ consists of smooth functions, we may leave out the regularization operators). Defining the universal measure again by $d\rho = F_* (d^4 x)$, we again obtain a causal fermion system of spin dimension $n = 2$. We next introduce the *evaluation map* $e_x$ by

$$
e_x : \mathcal{H} \rightarrow S_x \mathcal{M}, \quad e_x(\psi) = \psi(x).
$$

Restricting the evaluation mapping to the spin space $S_{F(x)}$ at the space-time point $F(x)$ (defined as in the lecture as the image of the operator $F(x)$), we obtain a mapping

$$
e_{x|S_{F(x)}} : S_{F(x)} \rightarrow S_x \mathcal{M}.
$$

(a) Show that $e_{x|S_{F(x)}}$ is an isometric embedding.

(b) Show that for all $u \in \mathcal{H}$ and $x \in \mathcal{M}$,

$$
e_{x|S_{F(x)}}(\psi u(F(x))) = u(x).
$$

**EXERCISE 4.7. (identification of $S\mathcal{M}$ with $S\mathcal{M}$)** In the setting of the previous exercise, we now make two additional assumptions:

(i) The mapping $F : \mathcal{M} \rightarrow \mathcal{F}$ is injective and its image is closed in $\mathcal{F}$.

(ii) The resulting causal fermion system is regular in the sense that for all $x \in \mathcal{M}$, the operator $F(x)$ has rank $2n$.

Using the results of the previous exercise, explain how the following objects can be identified:
(a) $x$ with $F(x)$
(b) $M$ with $M$
(c) The spinor space $S_x M$ with the corresponding spin space $S_F(x)$
(d) $u \in \mathcal{H}$ with its corresponding physical wave function $\psi^u$

**Exercise 4.8. (the space $C^0(M, SM)$)** In the lecture we introduced a wave function $\psi$ as a mapping from $M$ to $H$ such that $\psi(x) \in S_x$ for all $x \in M$. It is most convenient to define continuity of a wave function by the requirement that for all $x \in M$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\left\| \sqrt{|y|} \psi(y) - \sqrt{|x|} \psi(x) \right\|_{\mathcal{H}} < \varepsilon \quad \text{for all } y \in M \text{ with } \|y - x\| \leq \delta.$$ 

Show that, using this definition, every physical wave function is continuous. Thus, denoting the space of continuous wave functions by $C^0(M, SM)$, we obtain an embedding

$$\mathcal{H} \hookrightarrow C^0(M, SM).$$

**Hint:** You may use the inequality

$$\left\| \sqrt{|y|} - \sqrt{|x|} \right\| \leq \|y - x\| \uparrow \|y + x\| \uparrow.$$ 

**Exercise 4.9.** This exercise is devoted to analyzing general properties of the spectrum of the closed chain.

(a) As in Definition 4.4.1, we let $x$ and $y$ be symmetric operators of finite rank on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. Show that there is a finite-dimensional subspace $I \subset \mathcal{H}$ on which both $x$ and $y$ are invariant. By choosing an orthonormal basis of $I$ and restricting the operators to $I$, we may represent both $x$ and $y$ by Hermitian matrices. Therefore, the remainder of this exercise is formulated for simplicity in terms of Hermitian matrices.

(b) Show that for any matrix $Z$, the characteristic polynomials of $Z$ and of its adjoint $Z^*$ (being the transposed complex conjugate matrix) are related by complex conjugation, i.e. $\det(Z^* - \lambda \mathbb{1}) = \overline{\det(Z - \lambda \mathbb{1})}$.

(c) Let $X$ and $Y$ be symmetric matrices. Show that the characteristic polynomials of the matrices $XY$ and $YX$ coincide.

(d) Combine (b) and (c) to conclude that the characteristic polynomial of $XY$ has real coefficients, i.e. $\det(XY - \lambda \mathbb{1}) = \overline{\det(XY - \lambda \mathbb{1})}$. Infer that the spectrum of the matrix product $XY$ is symmetric about the real axis, i.e.

$$\det(XY - \lambda \mathbb{1}) = 0 \quad \implies \quad \det(XY - \bar{\lambda} \mathbb{1}) = 0.$$  \hspace{1cm} (4.6.17)

(e) For the closed chain (5.0.2), the mathematical setting is somewhat different, because $A_{xy}$ is a symmetric operator on the indefinite inner product space $(S_x, \langle \cdot, \cdot \rangle_x)$. On the other hand, we saw after (5.0.2) that $A_{xy}$ is isospectral to $xy$. Indeed, the symmetry result (4.6.17) can be used to prove a corresponding statement for $A_{xy}$,

$$\det(A_{xy} - \lambda \mathbb{1}) = 0 \quad \implies \quad \det(A_{xy} - \bar{\lambda} \mathbb{1}) = 0.$$  \hspace{1cm} (4.6.18)

This result is well-known in the theory of indefinite inner product spaces (see for example the textbooks [13, 67] or [30 Section 3]). In order to derive it from (4.6.17), one can proceed as follows: First, represent the indefinite inner product in the form $\langle \cdot, \cdot \rangle = \langle \cdot, S \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is a scalar product and $S$ is an invertible operator which is symmetric (with respect to this scalar product). Next, show that the operator $B := A_{xy} S$ is symmetric (again with respect to this scalar product). Finally, write the closed chain as $A_{xy} = BS^{-1}$ and apply (4.6.17).
 CHAPTER 5

Inherent Structures

The concept is that a causal fermion system describes space-time as well as all the
structures therein (like the causal and metric structures, particles, fields, etc.). In order
to recover all these structures, we simply give structures of the causal fermion system
suitable names. We point out that we do not introduce additional structures, but we
merely work with structures which are already encoded in the causal fermion system. We
call these structures as being inherent in the causal fermion system. We now introduce
and explain some of these inherent structures (for a more complete account see 36,
Chapter 1).

The causal action principle depends crucially on the eigenvalues of the operator pro-
duct $xy$ with $x, y \in \mathcal{F}$. For computing these eigenvalues, it is convenient not to consider
this operator product on the (possibly infinite-dimensional) Hilbert space $\mathcal{H}$, but instead
to restrict attention to a finite-dimensional subspace of $\mathcal{H}$, chosen such that the oper-
tor product vanishes on the orthogonal complement of this subspace. This construction
leads us to the spin spaces and to the kernel of the fermionic projector, which we now
introduce. For every $x \in \mathcal{F}$ we define the spin space $S_x$ by $S_x = x(\mathcal{H})$; it is a subspace
of $\mathcal{H}$ of dimension at most $2n$ (see Figure 5.1). Moreover, we let

$$\pi_x : \mathcal{H} \rightarrow S_x$$

be the orthogonal projection in $\mathcal{H}$ on the subspace $S_x \subset \mathcal{H}$. For any $x, y \in M$ we define
the kernel of the fermionic projector $P(x, y)$ by (see Figure 5.2).

![Figure 5.1. The spin spaces](image1)

![Figure 5.2. The kernel of the fermionic projector](image2)
\[ P(x, y) = \pi_x y |_{S_y} : S_y \to S_x \]  
(5.0.1)

(Where \( \pi_x \) is again the orthogonal projection on the subspace \( x(\mathcal{H}) \subset \mathcal{H} \). Taking the trace of \ref{5.0.1} in the case \( x = y \), one finds that \( \text{tr}(x) = \text{Tr}_{S_x}(P(x, x)) \), making it possible to express the integrand of the trace constraint \ref{4.4.4} in terms of the kernel of the fermionic projector. In order to also express the eigenvalues of the operator \( xy \) in terms of the kernel of the fermionic projector, we introduce the \textit{closed chain} \( A_{xy} \) as the product

\[ A_{xy} = P(x, y) P(y, x) : S_x \to S_x . \]  
(5.0.2)

Computing powers of the closed chain, one obtains

\[ A_{xy} = (\pi_x y)(\pi_y x)|_{S_x} = \pi_x y x|_{S_x} \quad \text{and} \quad (A_{xy})^p = \pi_x (yx)^p|_{S_x} \ . \]

Taking the trace, one sees in particular that

\[ \text{Tr}_{S_x}(A_{xy}^p) = \text{tr}[(yx)^p] = \text{tr}[(xy)^p] \]  
(5.0.3)

(Where the last identity simply is the invariance of the trace under cyclic permutations).

Since all our operators have finite rank, there is a finite-dimensional subspace \( I \) of \( \mathcal{H} \) such that \( xy \) maps \( I \) to itself and vanishes on the orthogonal complement of \( I \). Then the non-trivial eigenvalues of the operator product \( xy \) are given as the zeros of the characteristic polynomial of the restriction \( xy|_I : I \to I \). The coefficients of this characteristic polynomial (like the trace, the determinant, etc.) are symmetric polynomials in the eigenvalues and can therefore be expressed in terms of traces of powers of \( A_{xy} \). We conclude that the eigenvalues of the closed chain coincide with the non-trivial eigenvalues \( \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \) of the operator \( xy \) in Definition \ref{4.4.1}. This argument also shows that the operator product \( xy \) and \( yx \) are isospectral. In particular, one sees that kernel of the fermionic projector encodes the causal structure of \( M \). Moreover, we see that the causal structure is symmetric in \( x \) and \( y \). The main advantage of working with the kernel of the fermionic projector is that the closed chain \ref{5.0.2} is a linear operator on a vector space of dimension at most \( 2n \), making it possible to compute the \( \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \) as the eigenvalues of a finite matrix.

Next, it is very convenient to arrange that the kernel of the fermionic projector is symmetric in the sense that

\[ P(x, y)^* = P(y, x) . \]  
(5.0.4)

To this end, one chooses on the spin space \( S_x \) the \textit{spin scalar product} \( \langle . | . \rangle_x \) by

\[ \langle u | v \rangle_x = -\langle u | xv \rangle_{\mathcal{H}} \quad \text{for all} \quad u, v \in S_x . \]  
(5.0.5)

Due to the factor \( x \) on the right, this definition really makes the kernel of the fermionic projector symmetric, as is verified by the computation

\[ \langle u | P(x, y) v \rangle_x = -\langle u | x P(x, y) v \rangle_{\mathcal{H}} = -\langle u | xy v \rangle_{\mathcal{H}} \]
\[ = -\langle \pi_y x u | y v \rangle_{\mathcal{H}} = \langle P(y, x) u | v \rangle_y \]

(Where \( u \in S_x \) and \( v \in S_y \). The spin space \( (S_x, \langle . | . \rangle_x) \) is an \textit{indefinite} inner product space of signature \( (p, q) \) with \( p, q \leq n \) (for textbooks on indefinite inner product spaces see \cite{13, 67}). In this way, indefinite inner product spaces arise naturally when analyzing the mathematical structure of the causal action principle.

The kernel of the fermionic projector plays a central role in the analysis for several reasons:

- The Lagrangian can be expressed in terms of \( P(x, y) \) (via the closed chain \ref{5.0.2} and its eigenvalues).
Figure 5.3. The physical wave function

- Being a mapping from one spin space to another, \( P(x, y) \) gives relations between different space-time points. In this way, it carries geometric information. This aspect was explained for non-mathematicians in [62]. For a mathematical formulation see [42] and Chapter 9 later in this lecture.

- The kernel of the fermionic projector also encodes all the wave functions of the system. In order to see the connection, for a vector \( u \in \mathcal{H} \) one introduces the corresponding physical wave function \( \psi^u \) as (see Figure 5.3)

\[
\psi^u : M \to \mathcal{H}, \quad \psi^u(x) = \pi_x u \in S_x M.
\]

Then, choosing an orthonormal basis \( \{e_i\} \) of \( \mathcal{H} \) and using the completeness relation as well as (5.0.5), one obtains for any \( \phi \in S_y \)

\[
P(x, y) \phi = \pi_x y |S_y \phi = \sum_i \pi_x e_i \langle e_i | y \phi \rangle_{\mathcal{H}} = -\sum_i \psi^{e_i}(x) \langle \psi^{e_i}(y) \rangle \phi^y,
\]

showing that \( P(x, y) \) is indeed composed of all the physical wave functions, i.e. in a bra/ket notation

\[
P(x, y) = -\sum_i \langle \psi^{e_i}(x) \rangle \langle \psi^{e_i}(y) \rangle \quad (5.0.6)
\]

As is explained in [51] and worked out in detail in [36, Section 1.2], starting from Dirac spinors in Minowski space, the physical wave functions can indeed be identified with the regularized wave functions used in the construction of the local correlation operators (4.3.1), i.e.

\[
\psi^u(x) \in S_x M \quad \text{can be identified with} \quad (\mathcal{R}_u(x)) \in S_x \mathcal{M}.
\]

Using this identification, the relation (5.0.6) can be computed in Minkowski space and gives a direct method for analyzing the causal fermion system. We will come back to this approach in Chapter 13 later in this lecture.

The last point is also the reason why the kernel of the fermionic projector was the starting point when developing the theory [28].
CHAPTER 6

The Euler-Lagrange Equations

6.1. The Euler-Lagrange Equations

Let $\rho$ be a minimizer of the causal variational principle in the non-compact setting (see Section 4.6.3). We now derive the Euler-Lagrange (EL) equations, following the method in the compact setting \[59, Lemma 3.4\]. We again define space-time as the support of $\rho$, $M := \text{supp} \rho \subset \mathcal{F}$.

In order to make sense of the variations, we make the following assumptions:

(i) The measure $\rho$ is \textit{locally finite} (meaning that any $x \in \mathcal{F}$ has an open neighborhood $U$ with $\rho(U) < \infty$).

(ii) The function $L(x, \cdot)$ is $\rho$-integrable for all $x \in \mathcal{F}$, giving a bounded continuous function on $\mathcal{F}$.

We introduce the function

$$\ell(x) = \int_\mathcal{F} L(x, y) \, d\rho(y) - s : \mathcal{F} \to \mathbb{R} \text{ bounded and lower semi-continuous} ,$$

(6.1.1)

where the parameter $s \in \mathbb{R}$ will be specified below.

**Lemma 6.1.1. (The Euler-Lagrange equations)** Let $\rho$ be a minimizer of the causal action. Then

$$\ell|_{\text{supp} \rho} \equiv \inf_\mathcal{F} \ell .$$

(6.1.2)

**Proof.** Given $x_0 \in \text{supp} \rho$, we choose an open neighborhood $U$ with $0 < \rho(U) < \infty$.

For any $y \in \mathcal{F}$ we consider the family of measures $(\tilde{\rho}_\tau)_{\tau \in [0,1)}$ given by

$$\tilde{\rho}_\tau = \chi_{M \setminus U} \rho + (1 - \tau) \chi_U \rho + \tau \rho(U) \delta_y$$

(6.1.3)

(where $\delta_y$ is the Dirac measure supported at $y$). Then

$$\tilde{\rho}_\tau - \rho = -\tau \chi_U \rho + \tau \rho(U) \delta_y = \tau (\rho(U) \delta_y - \chi_U \rho) ,$$

(6.1.4)

implying that $\tilde{\rho}_\tau$ satisfies the volume constraint. Hence

$$0 \leq \{S(\tilde{\rho}) - S(\rho)\} = 2\tau \left( \rho(U) \left( \ell(y) + s \right) - \int_U \left( \ell(x) + s \right) \, d\rho(x) \right) + O(\tau^2) .$$

As a consequence, the linear term must be non-negative,

$$\ell(y) \geq \frac{1}{\rho(U)} \int_U \ell(x) \, d\rho(x) .$$

(6.1.5)

Assume that (6.1.2) is false. Then there is $x_0 \in \text{supp} \rho$ and $y \in \mathcal{F}$ such that $\ell(x_0) > \ell(y)$. Lower semi-continuity of $\ell$ implies that there is an open neighborhood $U$ of $x_0$ such that $\ell(x) > \ell(y)$ for all $x \in U$, in contradiction to (6.1.5). This gives the result. \[\square\]
We always choose the parameter $s$ such that the infimum of $\ell$ in (6.1.2) is zero. Then the EL equations read

$$\ell|_{\text{supp } \rho} \equiv \inf_{\mathcal{F}} \ell = 0 .$$

(6.1.6)

6.2. The Weak Euler-Lagrange Equations in the Smooth Setting

The EL equations (6.1.6) make a statement on the function $\ell$ even at points $x \in \mathcal{F}$ which are far away from space-time $M$ (see the left of Figure 6.1). In this way, the EL equations contain much more information than conventional physical equations formulated in space-time. At present, it is unclear how this additional information can be used or interpreted. One way of understanding this situation is to take the point of view that all information on the physical system must be obtained by performing observations or measurements in space-time, which would mean that the information contained in $\ell$ away from $M$ is inaccessible for principal reasons. Here we shall not take sides or discuss whether or to which extent this point of view is correct. Instead, we simply note that it seems preferable and physically sensible to restrict attention to the function $\ell$ in an arbitrarily small neighborhood $U$ of $M$ in $\mathcal{F}$ (see the right of Figure 6.1). In practice, this means that we shall evaluate $\ell$ as well as its derivatives only on $M$. In this way, the causal action principle gives rise to an interaction described by equations in space-time.

This concept leads us to the so-called weak EL equations, which we now introduce. For technical simplicity, we again restrict attention to the smooth setting introduced in Section 4.6.5 (for a more general derivation see [53], Section 4). In this case, the minimality of $\ell$ implies that the derivative of $\ell$ vanishes on $M$, i.e.

$$\ell|_M \equiv 0 \quad \text{and} \quad D\ell|_M \equiv 0$$

(6.2.1)

(where $D\ell(p) : T_p\mathcal{F} \to \mathbb{R}$ is the derivative). In order to combine these two equations in a compact form, it is convenient to consider a pair $u := (a, u)$ consisting of a real-valued function $a$ on $M$ and a vector field $u$ on $T\mathcal{F}$ along $M$, and to denote the combination of multiplication and directional derivative by

$$\nabla_u \ell(x) := a(x) \ell(x) + (Du) \ell(x).$$

(6.2.2)

Then the equations (6.2.1) imply that $\nabla_u \ell(x)$ vanishes for all $x \in M$,

$$\nabla_u \ell|_M = 0 \quad \text{for all } u \in J_0^\infty.$$

(6.2.3)

These are the so-called weak EL equations. For brevity, a solution of the weak EL equations is also referred to as a critical measure. The pair $u = (a, u)$ is referred to as a jet. This jet is a vector in a corresponding jet space $J_\infty$ defined by

$$u = (a, u) \in J_\infty := C_\infty(M, \mathbb{R}) \oplus C_\infty(M, T\mathcal{F}).$$
6.3. The Linearized Field Equations in the Smooth Setting

Linearized fields often appear in physics, for example when describing linearized oscillations or linear waves. In order to derive the linearized equations, one typically considers a family $G_\tau$ of solutions. The parameter $\tau$ can be thought of as the “amplitude” of the perturbation, and $G_\tau|_{\tau=0}$ describes the unperturbed system (for example an anharmonic oscillator at rest or the vacuum). Then the derivative
\[ \frac{d}{d\tau} G_\tau \big|_{\tau=0} \]
is the linearized field; it satisfies a linear equation obtained by differentiating the nonlinear equation (like Hamilton’s equation for the anharmonic oscillator or the nonlinear wave equation) with respect to $\tau$.

The concept of linearization is also fruitful in the context of causal fermion systems. Since the system is described by the universal measure, the above family of solutions now corresponds to a family of measures $(\tilde{\rho}_\tau)_{\tau \in (0,\delta)}$ which are all critical points of the causal action principle. The basic question is how to vary the measure. Indeed, there are many ways of varying. We here begin with a simple method, which we will generalize and discuss afterwards. In order to keep the presentation as simple as possible, we again restrict attention to the smooth case. We choose a family $f_\tau$ of positive weight functions and a family $F_\tau$ of mapping from $M$ to $\mathcal{F}$. These functions should all be smooth, also in the parameter $\tau$, i.e.
\[ f \in C^\infty([0,\delta) \times M, \mathbb{R}^+) \quad \text{and} \quad F \in C^\infty([0,\delta) \times M, \mathcal{F}) . \]
We multiply $\rho$ by $f_\tau$ and then take the push-forward under $F_\tau$,
\[ \tilde{\rho}_\tau := (F_\tau)_*(f_\tau \rho) . \quad (6.3.1) \]
We assume that for $\tau = 0$ the variation is trivial,
\[ f_0 \equiv 1 \quad \text{and} \quad F_0 \equiv 1 . \quad (6.3.2) \]
Since multiplying by a positive function leaves the support unchanged, the support of the measure is transformed only by $F_\tau$; more precisely,
\[ \text{supp } \tilde{\rho}_\tau = F_\tau(\text{supp } \rho) \quad (6.3.3) \]
(see Exercise 6.1).

The assumption that all the measures $\tilde{\rho}_\tau$ are critical means that they all satisfy the weak EL equations (6.2.3). Taking into account that the support of the measures changes according to (6.3.3), we know that for all $u \in \mathcal{Y}_0^\infty$ and all $x \in M$,
\[ 0 = \nabla_u \left( \int_{\mathcal{F}} \mathcal{L}(F_\tau(x), y) \ d\tilde{\rho}_\tau(y) - s \right) \]
\[ = \nabla_u \left( \int_{\mathcal{F}} \mathcal{L}(F_\tau(x), F_\tau(y)) \ f_\tau(y) \ d\rho(y) - s \right) , \]
where in the last line we used the definition of the push-forward measure. It is convenient to multiply this equation by $f_\tau(x)$. We can write this factor inside the brackets,
\[ 0 = \nabla_u \left( \int_{\mathcal{F}} f_\tau(x) \mathcal{L}(F_\tau(x), F_\tau(y)) \ f_\tau(y) \ d\rho(y) - f_\tau(x) s \right) , \]
because the terms obtained when the derivative $\nabla_u$ acts on $f_\tau(x)$ vanish in view of the weak EL equations (6.2.3). Since this equation holds for all $\tau \in [0, \delta)$, we can differentiate at $\tau = 0$ to obtain the equation
\[
\nabla_u \left( \int_\mathcal{F} (\nabla_{1,v} + \nabla_{2,v}) \mathcal{L}(x, y) \, d\rho(y) - \nabla_v s \right) = 0,
\]
where $v$ is the jet generated by the functions $f_\tau$ and $F_\tau$,
\[
v := \frac{d}{d\tau} \left( f_\tau, F_\tau \right) \Big|_{\tau=0} \in J_\infty.
\]
Here $\nabla_{1,v}$ and $\nabla_{2,v}$ denote derivatives acting on the first and second argument of the Lagrangian, respectively. Here and throughout this book, we use the following conventions for partial derivatives and jet derivatives:

(i) Partial and jet derivatives with an index $i \in \{1, 2\}$ only act on the respective variable of the function $\mathcal{L}$. This implies, for example, that the derivatives commute,
\[
\nabla_{1,v} \nabla_{1,u} \mathcal{L}(x, y) = \nabla_{1,u} \nabla_{1,v} \mathcal{L}(x, y).
\]
(ii) The partial or jet derivatives which do not carry an index act as partial derivatives on the corresponding argument of the Lagrangian. This implies, for example, that
\[
\nabla_u \int_\mathcal{F} \nabla_{1,v} \mathcal{L}(x, y) \, d\rho(y) = \int_\mathcal{F} \nabla_{1,u} \nabla_{1,v} \mathcal{L}(x, y) \, d\rho(y).
\]

We point out that jets are never differentiated. This is a very convenient convention. In (6.3.4) this convention can be used because the terms obtained if the derivative $\nabla_u$ acted on the jet $v$ again vanish as a consequence of the weak EL equations. We remark that, from a differential geometric perspective, defining higher derivatives on $\mathcal{F}$ would make it necessary to introduce a connection on $\mathcal{F}$. While this could be done, we here use the simpler method that higher derivatives on $\mathcal{F}$ are defined as partial derivatives carried out in a distinguished charts. More precisely, around each point $x \in \mathcal{F}$ we distinguish a chart and carry out derivatives as partial derivatives acting on each tensor component.

**Definition 6.3.1.** A jet $v \in J_\infty$ is referred to as a solution of the linearized field equations if
\[
\langle u, \Delta v \rangle(x) := \nabla_u \left( \int_\mathcal{F} (\nabla_{1,v} + \nabla_{2,v}) \mathcal{L}(x, y) \, d\rho(y) - \nabla_v s \right) = 0 \quad (6.3.6)
\]
for all $u \in J_\infty$ and all $x \in M$. The vector space of all linearized solutions is denoted by $J_{\text{lin}} \subset J_\infty$.

We conclude this section with a discussion of our ansatz (6.3.1). Intuitively speaking, this ansatz means that the support of the measure is changed smoothly as a whole. In particular, if $M$ is a smooth four-dimensional submanifold of $\mathcal{F}$, then the varied measure $M_\tau$ will again have this property. In physical terms, measures where $M$ has such a manifold structure describe classical space-times. Consequently, the ansatz (6.3.1) and the corresponding linearization (6.3.5) correspond to classical fields in a classical space-time. In contrast, if the support $M := \text{supp} \, \rho$ of the universal measure does not have the structure of a four-dimensional manifold, then we refer to $M$ as a quantum space-time. The notion “quantum space-time” appears in the literature in different contexts with rather different meanings. We here take the above notion as the mathematical definition of what we mean by a quantum space-time. Likewise, we refer to a variation of $\rho$ which does not preserve the manifold structure of $M$ as a quantum field. Here the notion
“quantum field” does have a well-defined meaning at least in perturbative quantum field theory, and our definition must be justified by working out the corresponding limiting case. Here we cannot enter the details of these constructions as given in [46, 45] but merely explain the qualitative picture.

For simplicity of the presentation, we assume that the unperturbed measure $\rho$ describes a classical space-time $M$ (for example Minkowski space $M \cong \mathbb{R}^4$). As just explained, the ansatz (6.3.1) changes the support of the measure smoothly as a whole (see Figure 6.2 (a)). More generally, one can consider the situation where the measure $\rho$ “disintegrates” into several “components” which are perturbed differently (see Figure 6.2 (b)).

To this end, we choose a parameter $L \in \mathbb{N}$ (the “number of subsystems”) and consider mappings $f_a \in C^\infty([0, \delta) \times M, \mathbb{R}^+)$, $F_a \in C^\infty([0, \delta) \times M, \mathcal{F})$ with $a = 1, \ldots, L$.

For the so-called universal measure with fragmentation in generalization of (6.3.1) we make the ansatz

$$\tilde{\rho}_\tau = \frac{1}{L} \sum_{a=1}^L (F_a)\ast(f_a, \tau \rho).$$

(6.3.7)

The larger $L$ is chosen, the more freedom we have in perturbing the measure. We point out that we may choose $L$ arbitrarily large. In the limit $L \to \infty$, one can even describe situations where the support of the measure $\rho$ is “enlarged” by the perturbation as shown in Figure 6.2 (c).

Assuming again that the family $({\tilde{\rho}}_\tau)_{\tau \in [0, \delta)}$ satisfies the weak EL equations for all $\tau$, we can again linearize in $\tau$ to obtain the corresponding linearized field equations. They again have the form (8.5.2), but now with $v$ being the “averaged jet”

$$v = \frac{1}{L} \sum_{a=1}^L v_a \quad \text{with} \quad v_a = \frac{d}{d\tau}(f_a, \tau F_a)\big|_{\tau=0}.$$ 

Therefore, for linearized fields the fragmentation does not give anything essentially new. But on the nonlinear level, fragmentation yields interesting additional effects. We refer the interested reader for more details to [40, Section 5] and [39, Section 5] as well as to the applications to quantum field theory in [45].

In view of this consideration, the only restriction in describing linear perturbations of a measure $\rho$ by a jet $v$ of the form (6.3.3) is that the support of the measure $\rho$ is changed continuously in $\tau$, in the sense the support $\text{supp} \tilde{\rho}_\tau$ lies in a small neighborhood of $M$ (for details see Exercise 6.2). In particular, we do not cover variations of the form (6.1.3) where part of the measure is “transported” to a a point $y \in \mathcal{F}$ which may be far away from $M$. The reason for disregarding such variations is that, similar as explained before introducing the weak EL equations in Section 6.2 (see Figure 6.1), analyzing the
EL equations outside a small neighborhood of $M$ does not seem to be relevant for the physical applications.

**Exercises**

**Exercise 6.1.** Let $F : \mathcal{F} \to \mathcal{F}$ be continuous and $\rho$ a measure on $\mathcal{F}$. Show that

$$\text{supp } F_{\ast} \rho = F(\text{supp } \rho).$$

*Hint:* Recall the definition of the support of a measure (2.1.1) and use that the preimage of an open set under a continuous mapping is open.

**Exercise 6.2.** (a) Assume that $\mathcal{F}$ is locally compact. Moreover, assume that $F \in C^0([0, \delta] \times M, \mathcal{F})$ is continuous and that its preimage of any compact set is compact. Then for any $y \notin M$ there is $\tau_0 \in (0, \delta)$ such that

$$y \notin \text{supp } \hat{\rho}_\tau \quad \text{for all } \tau \in [0, \tau_0]$$

(where $\hat{\rho}_\tau$ are again the measures (6.3.1)) *Hint:* Use the result of Exercise 6.1

(b) Show that this result remains valid for the variation (6.3.7) with a finite number of subsystems.

(c) What happens for an infinite number of subsystems? Also, is the assumption necessary that the preimage of a compact set under $F$ is compact?
CHAPTER 7

Surface Layer Integrals and Conservation Laws

7.1. The Concept of Surface Layer Integrals

In daily life we experience space and objects therein. These objects are usually described by densities, and integrating these densities over space gives particle numbers, charges, the total energy, etc. In mathematical terms, the densities are typically described as the normal components of vector fields on a Cauchy surface, and conservation laws express that the values of these integrals do not depend on the choice of the Cauchy surface, i.e.

\[ \int_{\mathcal{N}_1} J^k \nu_k \, d\mu_{\mathcal{N}_1}(x) = \int_{\mathcal{N}_2} J^k \nu_k \, d\mu_{\mathcal{N}_2}(x), \]  

(7.1.1)

where \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are two Cauchy surfaces, \( \nu \) is the future-directed normal, and \( d\mu_{\mathcal{N}_1/2} \) is the induced volume measure.

In the abstract setting of causal fermion systems, surface integrals like (7.1.1) are undefined. Instead, one considers so-called surface layer integrals, as we now explain. In general terms, a surface layer integral is a double integral of the form

\[ \int_{\Omega} \left( \int_{M \setminus \Omega} \cdots \mathcal{L}(x,y) \, d\rho(y) \right) \, d\rho(x), \]  

(7.1.2)

where one variable is integrated over a subset \( \Omega \subset M \), and the other variable is integrated over the complement of \( \Omega \). In order to explain the basic idea, we make the assumption that the Lagrangian is of short range in the following sense. We let \( d \in C^0(M \times M, \mathbb{R}_+^0) \) be a distance function on \( M \) (since \( M \) is compact, any two such distance functions are equivalent). The assumption of short range means that \( \mathcal{L} \) vanishes on distances larger than \( \delta \), i.e.

\[ d(x,y) > \delta \implies \mathcal{L}(x,y) = 0 \]  

(7.1.3)

Then the surface layer integral (7.1.2) only involves pairs \((x,y)\) of distance at most \( \delta \), where \( x \) is in \( \Omega \) and \( y \) is in the complement \( M \setminus \Omega \). Thus the integral only involves points in a layer around the boundary of \( \Omega \) of width \( \delta \), i.e.

\[ x, y \in B_\delta(\partial \Omega). \]

Therefore, a double integral of the form (7.1.2) can be regarded as an approximation of a surface integral on the length scale \( \delta \), as shown in Figure 7.1. Therefore, in the setting of causal variational principles, they take the role of surface integrals in Lorentzian geometry.

We remark that in applications in Minkowski space or on a Lorentzian manifold, the Lagrangian typically decays on the Compton scale \( 1/m \) (where \( m \) denotes the rest mass of the Dirac particles).

7.2. Noether-Like Theorems

We closely follow the presentation in [52]. We now want to introduce a symmetry condition for the Lagrangian \( \mathcal{L}(x,y) \) of a causal variational principle. The most obvious
method would be to consider a one-parameter group of diffeomorphisms $\Phi_\tau$,
\[
\Phi : \mathbb{R} \times \mathcal{F} \to \mathcal{F} \quad \text{with} \quad \Phi_\tau \Phi_\tau' = \Phi_{\tau+\tau'} \tag{7.2.1}
\]
and to impose that $\mathcal{L}$ be invariant under these diffeomorphisms in the sense that
\[
\mathcal{L}(x,y) = \mathcal{L}(\Phi_\tau(x),\Phi_\tau(y)) \quad \text{for all } \tau \in \mathbb{R} \text{ and } x,y \in \mathcal{F}. \tag{7.2.2}
\]
However, this condition is unnecessarily strong for two reasons. First, it suffices to consider families which are defined locally for $\tau \in (-\tau_{\text{max}},\tau_{\text{max}})$. Second, the mapping $\Phi$ does not need to be defined on all of $\mathcal{F}$. Instead, it is more appropriate to impose the symmetry condition only on space-time $M \subset \mathcal{F}$. This leads us to consider instead of (7.2.1) a mapping
\[
\Phi : (-\tau_{\text{max}},\tau_{\text{max}}) \times M \to \mathcal{F} \quad \text{with} \quad \Phi(0,.) = 1. \tag{7.2.3}
\]
We also write $\Phi_\tau(x) \equiv \Phi(\tau,x)$ and refer to $\Phi_\tau$ as a \textit{variation} of $M$ in $\mathcal{F}$. Next, we need to specify what we mean by “smoothness” of this variation. This is a subtle point because in view of the results in [59], the universal measure does not need to be smooth (in the sense that it cannot in general be written as a smooth function times the Lebesgue measure), and therefore the function $\ell$ will in general only be Lipschitz continuous. Our Noether-like theorems require only that the function $\ell$ be differentiable in the direction of the variations:

**Definition 7.2.1.** A variation $\Phi_\tau$ of the form (7.2.3) is \textbf{continuously differentiable} if the composition
\[
\ell \circ \Phi : (-\tau_{\text{max}},\tau_{\text{max}}) \times M \to \mathbb{R}
\]
is continuous and if its partial derivative $\partial_\tau (\ell \circ \Phi)$ exists and is continuous.

The next question is how to adapt the symmetry condition (7.2.2) to the mapping $\Phi$ defined only on $(-\tau_{\text{max}},\tau_{\text{max}}) \times M$. This is not obvious because setting $\tilde{x} = \Phi_\tau(x)$ and using the group property, the condition (7.2.2) can be written equivalently as
\[
\mathcal{L}(\Phi_{-\tau}(\tilde{x}),y) = \mathcal{L}(\tilde{x},\Phi_\tau(y)) \quad \text{for all } \tau \in \mathbb{R} \text{ and } \tilde{x},y \in \mathcal{F}. \tag{7.2.4}
\]
But if we restrict attention to pairs $x,y \in M$, the equations in (7.2.2) and (7.2.4) are different. It turns out that the correct procedure is to work with the expression in (7.2.4).

**Definition 7.2.2.** A variation $\Phi_\tau$ of the form (7.2.3) is a \textbf{symmetry} of the Lagrangian if
\[
\mathcal{L}(x,\Phi_\tau(y)) = \mathcal{L}(\Phi_{-\tau}(x),y) \quad \text{for all } \tau \in (-\tau_{\text{max}},\tau_{\text{max}}) \text{ and } x,y \in M. \tag{7.2.5}
\]
We now state our Noether-like theorem.
Theorem 7.2.3. Let \( \rho \) be a critical measure and \( \Phi_\tau \) a continuously differentiable symmetry of the Lagrangian. Then for any compact subset \( \Omega \subset M \), we have

\[
\frac{d}{d\tau} \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \left( \mathcal{L}(\Phi_\tau(x),y) - \mathcal{L}(\Phi_{-\tau}(x),y) \right) \bigg|_{\tau=0} = 0. \tag{7.2.6}
\]

Before coming to the proof, we explain the connection to surface layer integrals. To this end, let us assume that \( \Phi_\tau \) and the Lagrangian are differentiable in the sense that the derivatives

\[
\frac{d}{d\tau} \Phi_\tau(x) \bigg|_{\tau=0} =: u(x) \quad \text{and} \quad \frac{d}{d\tau} \mathcal{L}(\Phi_\tau(x),y) \bigg|_{\tau=0}
\]

exist for all \( x, y \in M \) and are continuous on \( M \) respectively \( M \times M \). Then one may exchange differentiation and integration in (7.2.6) and apply the chain rule to obtain

\[
\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) D_u(x) \mathcal{L}(x,y) = 0,
\]

where \( D_u(x) \) is the derivative in the direction of the vector field \( u(x) \). This expression is a surface layer integral as in (7.1.2). In general, the derivatives in (7.2.7) need not exist, because we merely imposed the weaker differentiability assumption of Definition 7.2.1. In this case, the statement of the theorem implies that the derivative of the integral in (7.2.6) exists and vanishes.

Proof of Theorem 7.2.3. Integrating (7.2.5) over \( \Omega \times \Omega \) gives

\[
0 = \int_{\Omega \times \Omega} \left( \mathcal{L}(x,\Phi_\tau(y)) - \mathcal{L}(\Phi_{-\tau}(x),y) \right) d\rho(x) d\rho(y)
\]

\[
= \int_\Omega d\rho(x) \int_M d\rho(y) \chi_\Omega(y) \left( \mathcal{L}(\Phi_\tau(x),y) - \mathcal{L}(\Phi_{-\tau}(x),y) \right),
\]

where in the last step we used the symmetry of the Lagrangian (7.2.5) as well as the symmetry of the integrand in \( x \) and \( y \). We rewrite the characteristic function \( \chi_\Omega(y) \) as \( 1 - (1 - \chi_\Omega(y)) \), multiply out and use the definition of \( \ell \), (6.1.1). We thus obtain

\[
0 = \int_\Omega \left( \ell(\Phi_\tau(x)) - \ell(\Phi_{-\tau}(x)) \right) d\rho(x)
\]

\[
- \int_\Omega d\rho(x) \int_M d\rho(y) \chi_{M\setminus\Omega}(y) \left( \mathcal{L}(\Phi_\tau(x),y) - \mathcal{L}(\Phi_{-\tau}(x),y) \right).
\]

We thus obtain the identity

\[
\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \left( \mathcal{L}(\Phi_\tau(x),y) - \mathcal{L}(\Phi_{-\tau}(x),y) \right)
\]

\[
= \int_\Omega \left( \ell(\Phi_\tau(x)) - \ell(\Phi_{-\tau}(x)) \right) d\rho(x). \tag{7.2.8}
\]

Using that \( \ell(\Phi_\tau(x)) \) is continuously differentiable (see Definition 7.2.1) and that \( \Omega \) is compact, we conclude that the right side of this equation is differentiable at \( \tau = 0 \). Moreover, we are allowed to exchange the \( \tau \)-differentiation with integration. The EL equations (6.1.6) imply that

\[
\frac{d}{d\tau} \ell(\Phi_\tau(x)) \bigg|_{\tau=0} = \frac{d}{d\tau} \ell(\Phi_{-\tau}(x)) \bigg|_{\tau=0}. \tag{7.2.9}
\]

Hence the right side of (7.2.8) is differentiable at \( \tau = 0 \), and the derivative vanishes. This gives the result. \( \square \)
This theorem has the following connection to conservation laws. Let us assume that $M$ admits a sensible notion of “spatial infinity” and that the vector field $\partial_\tau \Phi$ has suitable decay properties at spatial infinity. Then one can choose a sequence $\Omega_n \subset M$ of compact sets which form an exhaustion of a set $\Omega$ which extends up to spatial infinity (see Figure 7.2 (a) and (b)). Considering the surface layer integrals for $\Omega_n$ and passing to limit, one concludes that also the surface layer integral corresponding to $\Omega$ vanishes. Let us assume that the boundary $\partial \Omega$ has two components $N_1$ and $N_2$ (as in Figure 7.2 (b)). Then the above theorem implies that the surface layer integrals over $N_1$ and $N_2$ coincide (where the surface layer integral over $N$ is defined as the surface layer integral corresponding to a set $\Omega_N$ with $\partial \Omega_N = N$ as shown in Figure 7.2 (c)). In other words, the quantity

$\left. \frac{d}{d\tau} \int_{\Omega_N} d\rho(x) \int_{M \setminus \Omega_N} d\rho(y) \left( \mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right) \right|_{\tau=0}$

is well-defined and independent of the choice of $N$. In this setting, the surfaces $N$ can be interpreted as Cauchy surfaces, and the conservation law of Theorem 7.2.3 means that the surface layer integral is preserved under the time evolution.

As a concrete example of a conservation law, we now consider current conservation. To this end, we consider the setting of causal fermion systems. Let $\mathcal{A}$ be a bounded symmetric operator on $\mathcal{H}$ and

$\mathcal{U}_\tau := \exp(i\tau \mathcal{A}) \quad (7.2.10)$

be the corresponding one-parameter family of unitary transformations. We introduce the mapping

$\Phi_\tau : \mathbb{R} \times \mathcal{F} \to \mathcal{F}, \quad \Phi(\tau, x) = \mathcal{U}_\tau x \mathcal{U}_\tau^{-1} \quad (7.2.11)$

(since $\mathcal{U}_\tau$ is unitary, it clearly preserves the eigenvalues of the operator $x$ as well as the local trace). Restricting this mapping to $(-\tau_{\max}, \tau_{\max}) \times M$, we obtain a variation

$\Phi : (-\tau_{\max}, \tau_{\max}) \times M \to \mathcal{F} \quad \text{with} \quad \Phi(0, \cdot) = 1 \quad (7.2.12)$

**Lemma 7.2.4.** The variation $\Phi_\tau$ given by (7.2.11) is a symmetry of the Lagrangian.

**Proof.** We first recall that the Lagrangian $\mathcal{L}(x, y)$ in (7.3.6) is defined in terms of the spectrum of the operator product $xy$. The calculation

$x \Phi_\tau(y) = x \mathcal{U}_\tau y \mathcal{U}_\tau^{-1} = \mathcal{U} (\mathcal{U}_\tau^{-1} x \mathcal{U}_\tau y) \mathcal{U}_\tau^{-1} = \mathcal{U} (\Phi_{-\tau}(x) y) \mathcal{U}_\tau^{-1}$

shows that the operators $x \Phi_\tau(y)$ and $\Phi_{-\tau}(x) y$ are unitarily equivalent and therefore isospectral. This concludes the proof. \qed

We cannot in general expect that the variation $\Phi_\tau$ is continuously differentiable in the sense of Definition 7.2.1. Therefore, we must include the differentiability of $\ell \circ \Phi$ as an assumption in the following theorem.
Theorem 7.2.5. Given a bounded symmetric operator \( A \) on \( \mathcal{H} \), we let \( \Phi_\tau \) be the variation (7.2.11). Assume that the mapping \( \ell \circ \Phi : (-\tau_{\text{max}}, \tau_{\text{max}}) \times \mathcal{M} \to \mathbb{R} \) is continuously differentiable in the sense that it is continuous and that \( \partial_\tau (\ell \circ \Phi) \) exists and is also continuous on \((-\tau_{\text{max}}, \tau_{\text{max}}) \times \mathcal{M} \). Then for any compact subset \( \Omega \subset \mathcal{M} \), the conservation law (7.2.6) holds.

In [52, Section 5.2] it is shown that for Dirac systems in Minkowski space, this conservation law gives back the conservation of the Dirac current. We now see that this conservation law corresponds to a much more general conservation law for causal fermion systems. We finally remark that the conservation laws for energy-momentum can also be obtained from Theorem 7.2.3, assuming that the causal fermion system has symmetries as described by generalized Killing symmetries. We refer the interested reader to [52, Section 6].

7.3. A Class of Conservation Laws in the Smooth Setting

In the previous section we saw that surface layer integrals can be used to formulate Noether-like theorems which relate symmetries to conservation laws. We shall now see that there are conservation laws even in the absence of symmetries. Instead, these conservation laws are closely tied to the structure of the linearized field equations as derived in Section 6.3. In order to focus on the essence of the construction, we again restrict attention to the smooth setting introduced in Section 4.6.5.

The basic idea of the construction is explained in the following proposition:

Proposition 7.3.1. Let \( \Omega \subset \mathcal{M} \) be a compact. Then for any solution \( v \in \mathcal{J}^{\text{lin}} \) of the linearized field equations (8.5.2),

\[
I_1^\Omega := \int_\Omega \delta \rho(x) \int_{M \setminus \Omega} \delta \rho(y) \left( \nabla_{1,v} - \nabla_{2,v} \right) \mathcal{L}(x,y) = \int_\Omega \nabla_{\rho} \mathfrak{s} \, d\rho. \quad (7.3.1)
\]

Proof. In view of the anti-symmetry of the integrand,

\[
\int_\Omega \delta \rho(x) \int_\Omega \delta \rho(y) \left( \nabla_{1,v} - \nabla_{2,v} \right) \mathcal{L}(x,y) = 0.
\]

Adding this equation to the left side of (7.3.1), we obtain

\[
\int_\Omega \delta \rho(x) \int_{M \setminus \Omega} \delta \rho(y) \left( \nabla_{1,v} - \nabla_{2,v} \right) \mathcal{L}(x,y)
= \int_\Omega \delta \rho(x) \int_M \delta \rho(y) \left( \nabla_{1,v} - \nabla_{2,v} \right) \mathcal{L}(x,y)
= \int_\Omega \delta \rho(x) \left( 2 \nabla_{\rho} \left( \ell(x) + \mathfrak{s} \right) - (\Delta v)(x) - \nabla_{\rho} \mathfrak{s} \right),
\]

where in the last line we used the definitions of \( \ell \) and \( \Delta \) (see 6.1.1 and 8.5.2). Applying the weak EL equations (6.2.3) and the linearized field equations (8.5.2) gives the result. \( \square \)

We remark that the identity (7.3.1) has a similar structure as the conservation law in the Noether-like theorem (7.2.0). In order to make the connection precise, one describes the symmetry \( \Phi_\tau \) infinitesimally by a jet \( v \) with vanishing scalar component,

\[
v(x) := \left. \frac{d}{d\tau} \left( 0, \Phi_\tau(x) \right) \right|_{\tau=0}.
\]
Using the symmetry property \(7.2.5\), one immediately verifies that this jet satisfies the linearized field equations \(8.5.2\). Therefore, Proposition \(8.3.1\) applies, and the right side vanishes because \(v\) because \(v\) has no scalar component. We thus recover the identity obtained by carrying out the \(\tau\)-derivative in \(7.2.6\).

We conclude that Proposition \(8.3.1\) is a generalization of Theorem \(7.2.3\). Instead of imposing symmetries, the identity \(8.3.1\) is a consequence of the linearized field equations. Again choosing \(\Omega\) as the region between two Cauchy surfaces (see Figure 7.2), one obtains a relation between the surface layer integrals around \(N_1\) and \(N_2\). If the scalar component of \(v\) vanishes, we obtain a conservation law. Otherwise, the right side of \(8.3.1\) tells us how the surface layer integral changes in time.

We now generalize Proposition \(8.3.1\). The idea is to integrate anti-symmetric expressions in \(x\) and \(y\) which involve higher derivatives of the Lagrangian. We again restrict attention to the smooth setting (for the general proof see [54]). Let \(\tilde{\rho}_{s,t}\) with \(s,t \in (-\delta, \delta)\) be a two-parameter family of universal measures which are solutions of the weak EL equations. We assume that these measures are of the form

\[
\tilde{\rho}_{s,t} = (F_{s,t})_\ast (f_{s,t} \, \rho) , \tag{7.3.2}
\]

where \(f_{s,t}\) and \(F_{s,t}\) are smooth,

\[
f \in C^\infty((-\delta, \delta)^2 \times \mathcal{I}, \mathbb{R}^+) \quad \text{and} \quad F \in C^\infty((-\delta, \delta)^2 \times \mathcal{I}, \mathcal{F}) , \tag{7.3.3}
\]

and are trivial in the case \(s = t = 0\),

\[
f_{0,0} \equiv 1 , \quad F_{0,0} = 1 . \tag{7.3.4}
\]

Moreover, we make the following regularity assumptions:

\((r1)\) For all \(x \in M, p, q \geq 0\) and \(r \in \{0, 1\}\), the following partial derivatives exist and can be interchanged with integration:

\[
\int_M \partial_x^p \partial_x^q \partial_t^k \mathcal{L}(F_{s+s',t}(x), F_{s,t}(y)) \bigg|_{s'=s=t=0} \, d\rho(y) = \partial_x^p \partial_x^q \partial_t^k \int_M \mathcal{L}(F_{s+s',t}(x), F_{s,t}(y)) \, d\rho(y) \bigg|_{s'=s=t=0} .
\]

**Theorem 7.3.2.** Let \(f\) and \(F\) be as in \((7.3.3)\) and \((7.3.4)\) which satisfy the above regularity assumption \((r1)\). Moreover, assume that the measures \(\tilde{\rho}_{s,t}\) given by \((7.3.2)\) satisfy the weak EL equations for all \(s\) and \(t\). Then for every compact \(\Omega \subset M\) and every \(k \in \mathbb{N}\),

\[
(I_{k+1})^\Omega := \int \int_{\Omega} \, d\rho(x) \, d\rho(y) \times (\partial_{1,s} - \partial_{2,s}) \left( \partial_{1,t} + \partial_{2,t} \right)^k f_{s,t}(x) \, \mathcal{L}(F_{s,t}(x), F_{s,t}(y)) \, f_{s,t}(y) \bigg|_{s=t=0} = s \int \partial_x \partial_t \partial_x^k f_{s,t}(x) \bigg|_{s=t=0} \, d\rho(x) . \tag{7.3.5}
\]

**Proof.** Introducing the short notation

\[
L(x_{s,t}, y_{s,t}) = f_{s,t}(x) \, \mathcal{V}(F_{s,t}(x), F_{s,t}(y)) \, f_{s,t}(y) , \tag{7.3.6}
\]

the weak EL equations \((8.5.2)\) read

\[
\nabla u \left( \int_M L(x_{s,t}, y_{s,t}) \, d\rho(y) - s \, f_{s,t}(x) \right) = 0 \quad \text{for all} \ u \in \mathcal{J}_0^\infty.
\]
In particular for any \( k \geq 0 \) and any vector \( v = v^s \partial_s + v^t \partial_t \), we obtain

\[
\int_M \partial_1 \cdot (\partial_1 + \partial_2)^k L(x_{s,t}, y_{s,t}) \, d\rho(y) \bigg|_{s=t=0} = 5 \partial_s \partial_t^k f_{s,t}(x) \quad (7.3.7)
\]

\[
\int_M (\partial_1 + \partial_2)^{k+1} L(x_{s,t}, y_{s,t}) \, d\rho(y) \bigg|_{s=t=0} = 5 \partial_s \partial_t^{k+1} f_{s,t}(x) \quad (7.3.8)
\]

(the derivatives exist and can be exchanged with the integration according to the above regularity assumption (r1)). Differentiating the last equation with respect to \( v^s \) and dividing by \( k + 1 \), we obtain

\[
\int_M (\partial_1 + \partial_2) (\partial_1 + \partial_2)^k L(x_{s,t}, y_{s,t}) \, d\rho(y) = 5 \partial_s \partial_t^k f_{s,t}(x) .
\]

Subtracting twice the identity (7.3.7), we obtain for any \( k \geq 0 \) the equation

\[
\int_M (\partial_1 + \partial_2) (\partial_1 + \partial_2)^k L(x_{s,t}, y_{s,t}) \, d\rho(y) = 5 \partial_s \partial_t^k f_{s,t}(x) .
\]

Integrating the last equation over \( \Omega \) gives

\[
\int \Omega \, d\rho(x) \int_M d\rho(y) (\partial_1 - \partial_2) (\partial_1 + \partial_2)^k L(x_{s,t}, y_{s,t}) = 5 \int \Omega \, d\rho(x) \partial_s \partial_t^k f_{s,t}(x) . \quad (7.3.9)
\]

On the other hand, since the integrand is anti-symmetric in its arguments \( x \) and \( y \), we also know that

\[
\int \Omega \, d\rho(x) \int \Omega \, d\rho(y) (\partial_1 - \partial_2) (\partial_1 + \partial_2)^k L(x_{s,t}, y_{s,t}) = 0 . \quad (7.3.10)
\]

Subtracting this equation from (7.3.9) and evaluating at \( s = t = 0 \) gives the result. \( \square \)

7.4. The Symplectic Form and the Surface Layer Inner Product

For the applications, the most important surface layer integrals are \( I^1_1 \) (see Proposition 7.3.1) and \( I^1_2 \) (see Theorem 7.3.2 in the case \( k = 1 \)). We now have a closer look at the surface layer integral \( I^1_2 \). It is defined by

\[
I^1_2 = \int \Omega \, d\rho(x) \int_M \Omega \, d\rho(y) \int \Omega \, d\rho(x) \times (\partial_1 - \partial_2)(\partial_1 + \partial_2) f_{s,t}(x) L(F_{s,t}(x), F_{s,t}(y)) f_{s,t}(y) \bigg|_{s=t=0}
\]

and satisfies for any compact subset \( \Omega \subset M \) the identity

\[
I^1_2 = \frac{1}{5} \int \Omega \, d\rho(x) \partial_s \partial_t f_{s,t}(x) \bigg|_{s=t=0} \, d\rho(x) . \quad (7.4.2)
\]

These formulas simplify considerably if we anti-symmetrize in the parameters \( s \) and \( t \). Namely, in the formula for \( I^1_2 \) reduces to the surface layer integral

\[
\int \Omega \, d\rho(x) \int_M \Omega \, d\rho(y) (\partial_1 - \partial_2)(\partial_1 - \partial_2) f_{s,t}(x) L(F_{s,t}(x), F_{s,t}(y)) f_{s,t}(y) \bigg|_{s=t=0} .
\]
Since this expression involves only first partial derivatives, it can be rewritten with jet derivatives as

$$
\sigma^\Omega_\rho (u, v) := \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) (\nabla_u \nabla_v - \nabla_v \nabla_u) \mathcal{L}(x, y),
$$

(7.4.3)

where the jets $u$ and $v$ are the linearized solutions

$$
u = \partial_t (f_{s,t}, F_{s,t}) \big|_{s=t=0} \quad \text{and} \quad v = \partial_t (f_{s,t}, F_{s,t}) \big|_{s=t=0}.
$$

(7.4.4)

Moreover, the right side of (7.4.2) vanishes when antisymmetrizing in $s$ and $t$. We conclude that

$$
\sigma^\Omega_\rho (u, v) = 0 \quad \text{for every compact } \Omega \subset M.
$$

Choosing $\Omega$ again as explained in Figure 7.2 we obtain a conservation law for a surface layer integral over a neighborhood of a hypersurface $N$ which extends to spatial infinity. We refer to $\sigma^\Omega_\rho$ as the symplectic form (the connection to symplectic geometry will be explained after (7.4.6) below).

Symmetrizing $I_2^\sigma$ in the parameters $s$ and $t$ gives the surface layer integral

$$
\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) (\partial_1 \partial_2 - \partial_2 \partial_1) f_{s,t}(x) \mathcal{L}(F_{s,t}(x), F_{s,t}(y)) f_{s,t}(y) \big|_{s=t=0}.
$$

(7.4.5)

This expression is has a more difficult structure because it involves second partial derivatives. Such second partial derivatives cannot be expressed directly in terms of second jet derivatives, because the derivatives of the jets also need to be taken into account. In a differential geometric language, defining second derivatives would make it necessary to introduce a connection on $\mathcal{F}$. As explained after (ii) on page 108 we here use the simpler method of taking second partial derivatives in distinguished charts. Then it is useful to introduce the surface layer inner product $\langle \cdot, \cdot \rangle_\rho^\Omega$ as the contribution to (7.4.5) involving second derivatives of the Lagrangian, i.e.

$$
\langle u, v \rangle_\rho^\Omega := \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) (\nabla_{1,u} \nabla_{1,v} - \nabla_{2,u} \nabla_{2,v}) \mathcal{L}(x, y),
$$

where the jets $u$ and $v$ are again the linearized solutions (7.4.4). We point out that, in contrast to the symplectic form, the surface layer inner product does not correspond to a conservation law. This has two reasons: First because the right side of (7.4.2) gives rise to a volume term, and second because the derivatives of the jets $u$ and $v$ gives additional correction terms. For the details and the interpretation of these correction terms we refer to 54. Here we only remark that the significance of the surface layer inner product is that it is an approximate conservation law. In particular, it can be used for estimating solutions of the linearized field equations and for proving existence. We will come back to these applications in Section 12.

We finally comment on the name symplectic form. Clearly, this name is taken from symplectic geometry, where it refers to a closed and non-degenerate two-form $\sigma$ on a manifold which we denote by $\mathcal{B}$. The connection to the surface layer integral (16.3.2) is obtained if we assume that the set of all critical measures of the form (6.3.1) forms a smooth manifold $\mathcal{B}$ (which may be a infinite-dimensional Banach manifold). In this case, a jet $\mathfrak{v}$ describing first variations of a measure (6.3.3) is a tangent vector in $T_p \mathcal{B}$. Consequently, the jet space $\mathfrak{J}$ can be identified with the tangent space $T_p \mathcal{B}$. The surface layer integral (16.3.2) can be regarded as a mapping

$$
\sigma^\Omega_\rho : T_p \mathcal{B} \times T_p \mathcal{B} \to \mathbb{R}.
$$

(7.4.6)
7.5. Commutator Jets

In the previous sections we saw that symmetries of causal fermion systems give rise to conservation laws which can be expressed in terms of surface layer integrals (Sections 7.1 and 7.2). A convenient method for getting this connection is to first use the symmetries to construct solutions of the linearized field equations and then to consider the corresponding conservation laws of Section 7.4 (see the explanation after Proposition 7.3.1). In this and the next section, we shall apply this method to the inherent symmetries of a causal fermion system: the invariance under unitary transformations of $\mathcal{H}$ (this section) and the invariance of the universal measure under diffeomorphisms of $M$ combined with a multiplication of $\rho$ by a smooth weight function (Section 7.6).

The causal action principle is unitarily invariant in the following sense. Let $U \in U(\mathcal{H})$ be a unitary transformation. Given a measure $\rho$ on $\mathcal{F}$, we can unitarily transform the measure by setting

$$\rho_U(\Omega) = \rho(U^{-1}\Omega U)$$

for $\Omega \subset \mathcal{F}$. Thus in bra/ket notation, $A = |\psi\rangle\langle\psi|$. By exponentiating we obtain a family of unitary operators $(U_\tau)_{\tau \in \mathbb{R}}$,

$$U_\tau := \exp(i\tau A)$$

Due to the unitary invariance of the causal action, the corresponding measures $U_\tau \rho$ (defined by (7.5.1)) all satisfy the EL equations. Consequently, the jet $v$ generating this transformation,

$$v := (0, v) \in \mathfrak{J}^{\text{lin}} \quad \text{with} \quad v(x) = \frac{d}{d\tau} \left. (U_\tau x U_\tau^{-1}) \right|_{\tau=0} = i[A, x] ,$$

is a solution of the linearized field equations. We thus obtain a mapping

$$j : \mathcal{H} \to \mathfrak{J}^{\text{lin}} , \quad \psi \mapsto v .$$
In view of (7.5.2) and (7.5.4), this mapping is positive homogeneous of rank two in the sense that
\[ j(\lambda v) = |\lambda|^2 j(v) \quad \text{for all } \lambda \in \mathbb{C}. \]
Due to the commutator in (7.5.4), we refer to the image of \( j \) as the commutator jets \( J^c \subset J^{\text{lin}} \).

We now evaluate the surface layer integral \( I_\Omega^0 \) for commutator jets. In order to clarify the dependence on \( \rho \), in what follows we also denote the surface layer integral \( I_\Omega^0 \) by \( I_\Omega^\rho \).

Since commutator jets have no scalar components, the corresponding surface layer integral
\[ C_\rho^\Omega(u) := I_\rho^\Omega(j(u)) \quad \text{with } u \in \mathcal{H} \] (7.5.5)
does not depend on the choice of the set \( \Omega \) (as explained in Figure 7.2). This surface layer defines a functional on \( \mathcal{H} \) which is again positive homogeneous of degree two, i.e.
\[ C_\rho^\Omega(\lambda u) = |\lambda|^2 C_\rho^\Omega(u) \quad \text{for all } u \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}. \]

Therefore, we can use the polarization formula to define a sesquilinear form,
\[ C_\rho^\Omega(u, v) := \frac{1}{4} \left( C_\rho^\Omega(u + v) - C_\rho^\Omega(u - v) \right) - \frac{i}{4} \left( C_\rho^\Omega(u + iv) - C_\rho^\Omega(u - iv) \right). \]

In [52] Section 5 it was shown that, taking the continuum limit in Minkowski space, this sesquilinear form coincides, up to a constant, with the scalar product \( \langle u | v \rangle_{\mathcal{H}} \). We give this property the following name:

**Definition 7.5.1.** Given a critical measure \( \rho \) and a subset \( \Omega \subset M \), the surface layer integral \( I_\rho^\Omega \) is said to represent the scalar product if there is a non-zero real constant \( c \) such that for every symmetric operator \( A \) of finite rank, the corresponding commutator jet as defined by (7.5.4) has the property that
\[ I_\rho^\Omega(u) = c \text{ tr } A. \] (7.5.6)

In view of the conservation law of Proposition 7.3.1 this property remains valid if \( \Omega \) is changed by a compact subset of \( M \). The typical situation in mind is that \( \Omega \) is the past of a Cauchy surface (as shown in Figure 7.2 (c)).

In this book, we shall not assume that \( I_\rho^\Omega \) represents the scalar product. Instead, it suffices to make the following weaker assumption. We assume that the sesquilinear form \( C_\rho^\Omega \) is equivalent to the scalar product in the sense that
\[ C_\rho^\Omega(u, v) = \langle u | C_\rho v \rangle_{\mathcal{H}}, \]
where \( C_\rho \) is a bounded linear operator on \( \mathcal{H} \) with bounded inverse. Under this assumption, the Hilbert space scalar product can be expressed by
\[ \langle u | v \rangle_{\mathcal{H}} = C_\rho^\Omega(u, C_\rho^{-1} v). \]

In this way, the Hilbert space scalar product can be represented by a surface layer integral involving the physical wave functions in space-time.

**7.6. Inner Solutions**

We now introduce an additional assumption on causal fermion systems and explain why it is useful in some applications.

**Definition 7.6.1.** Space-time \( M := \text{supp } \rho \) has a smooth manifold structure if the following conditions hold:

(i) \( M \) is diffeomorphic to a smooth oriented manifold \( M^k \) of dimension \( k \).
(ii) In a chart \((x,U)\) of \(\mathcal{M}^k\), the universal measure is absolutely continuous with respect to the Lebesgue measure with a smooth, strictly positive weight function,

\[
d\rho = h(x)\, d^k x \quad \text{with} \quad h \in C^\infty(\mathcal{M}^k, \mathbb{R}^+) .
\]

Even though there is no reason why physical space-time should have a smooth manifold structure on the Planck scale, this assumption is clearly justified on the macroscopic scale of atomic and gravitational physics. With this in mind, the assumption of a smooth manifold structure seems admissible in all applications in which the microscopic structure of space-time should be irrelevant. Before going on, we point out that one should carefully the assumption of a smooth manifold structure from the smooth setting introduced in Section 4.6.5. Indeed, the smoothness of \(\mathcal{L}\) does not imply that \(M\) is a smooth manifold, nor vice versa.

The fact that \(\rho\) is defined independent of charts implies that the function \(h\) in (7.6.1) transform like a tensor density. Thus on the overlap of two charts \((x,U)\) and \((\tilde{x},\tilde{U})\), we know that

\[
h(x)\, d^k x = \tilde{h}(\tilde{x})\, d^k \tilde{x}.
\]

and thus

\[
h(x) = \det \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right) \tilde{h}(\tilde{x}) .
\]

This transformation law makes it possible to define the covariant divergence of a vector field \(v\) on \(M \simeq \mathcal{M}^k\) in a local chart by

\[
div v = \frac{1}{h} \partial_j (h v^j) \quad (7.6.2)
\]

(where following the Einstein summation convention we sum over \(j = 0, \ldots, k\)). Alternatively, the divergence of a vector field \(v \in \Gamma(M, TM)\) can be defined independent of charts by the relation

\[
\int_M \text{div} v \, \eta(x) \, d\rho = - \int_M D_v \eta(x) \, d\rho(x) ,
\]

to be satisfied for all test functions \(\eta \in C^\infty_0(M, \mathbb{R})\).

**Definition 7.6.2.** An *inner solution* is a jet \(v \in J^1\) of the form

\[
v = (\text{div} v, v) \quad \text{with} \quad v \in \Gamma(M, TM) .
\]

The vector space of all inner solution is denoted by \(J^{\text{in}} \subset J^1\).

The name “inner solution” is justified by the following lemma:

**Lemma 7.6.3.** Every inner solution \(v \in J^{\text{in}}\) of compact support is a solution of the linearized field equations, i.e.

\[
\langle u, \Delta v \rangle_M = 0 \quad \text{for all} \ u \in J^{\text{test}} .
\]

**Proof.** Applying the Gauss divergence theorem, one finds that for every \(f \in C^1_0(M, \mathbb{R})\),

\[
\int_M \nabla v f \, d\rho = \int_M (\text{div} v \, f + D_v f) \, d\rho = \int_M \text{div} (fv) \, d\rho = 0 .
\]
Likewise, in the linearized field equations we may integrate by parts in $y$,

\[
\langle u, v \rangle_M = \nabla_u \left( \int_M \left( \nabla_{1,v} + \nabla_{2,v} \right) \mathcal{L}_\kappa(x, y) - \nabla_v \mathcal{S} \right)
\]

where the last term vanishes in view of the EL equations. Moreover, we used that the function $\nabla_u \mathcal{L}_\kappa$ vanishes identically on $M$ in view of the weak EL equations. Therefore, this function is differentiable in the direction of every vector field on $M$, and this directional derivative is zero. □

This result also holds for inner solutions $v$ of non-compact support, provided that the vector field $v$ has suitable decay properties at infinity. For details we refer to [46, Section 3].

We now turn to the question which scalar components can be realized by an inner solution. This question can be answered in a more generality by applying Moser’s theorem (see for example [79, Section XVIII, §2] or straightforward generalizations to non-compact manifolds). For simplicity, we here make additional assumptions which make it possible to use hyperbolic methods.

**Theorem 7.6.4.** Assume that $M \simeq \mathcal{M}^k$ has a smooth manifold structure. Moreover, assume that $\mathcal{M}$ is topologically of the form $\mathcal{M} = \mathbb{R} \times \mathcal{N}$ with a manifold $\mathcal{N}$ which admits a complete Riemannian metric $g_{\mathcal{N}}$. Let $a \in C^\infty_{sc}(\mathcal{M}, \mathbb{R})$ be a smooth function with spatially compact support. Then there is a vector field $v \in C^\infty_{sc}(\mathcal{M}, T\mathcal{M})$ again with spatially compact support such that the jet $\mathbf{v} := (a, v)$ is an inner solution.

**Proof.** Our task is to solve the equation $\text{div } v = a$, which can be written equivalently as

\[
\partial_j (h v^j) = h a .
\]  
(7.6.3)

We first consider the case that $a$ has compact support. In order to solve the partial differential equation (7.6.3), it is useful to choose a Lorentzian metric with

\[
ds^2 = dt^2 - g_{\mathcal{N}}.
\]

Here the choice of the Riemannian metric $g_{\mathcal{N}}$ is irrelevant, and the arbitrariness in choosing this metric corresponds to the fact that (7.6.3) is an underdetermined equation which admits many different solutions. Let $\square$ be the corresponding wave operator. Using for example retarded Green’s operators, there is a solution $\phi \in C^\infty_{sc}(\mathcal{M}, \mathbb{R})$ with $\square \phi = h a$ (for details see Section 11 below). Then the vector field

\[
v^j := \frac{1}{h} g^{jk} \partial_k \phi
\]  
(7.6.4)

satisfies (7.6.3) (note that, in view of (7.6.1), we may divide by $h$ to again obtain a smooth vector field with spatially compact support).

In the case that $a$ merely has spatially compact support, we decompose $a$ as

\[
a = a_+ + a_-,
\]

where $a_+$ is supported in the set $\{t > 0\}$ and $a_-$ is supported in the set $\{t < 1\}$. Denoting the advanced and retarded Green’s operators of the scalar wave equation in Minkowski
space by $S^\vee$ and $S^\wedge$, respectively, the function
\[ \phi := S^\wedge(ha_+) + S^\vee(ha_-) \]
is a well-defined solution of the equation $\Box \phi = ha$ which is smooth and has spatially compact support. Therefore, we can again define the vector field $v$ by (7.6.4). This gives the result.

Inner solutions can be regarded as infinitesimal generators of transformations of $M$ which leave the measure $\rho$ unchanged. Therefore, inner solutions do not change the causal fermion system, but merely describe symmetry transformations of the universal measure. The result of the previous proposition can be used to change the scalar component of solutions arbitrarily. More precisely, considering a system of finite spatial volume for a scattering process taking place in finite time, all the jets describing the interaction will have spatially compact support. Therefore, we can compensate the scalar components by corresponding inner solutions. With this in mind, if space-time has a smooth manifold structure (see Definition 7.6.1), then one may always restrict attention to jets with vanishing scalar components. This is a major simplification. In particular, Proposition 7.3.1 gives rise to the conservation law for the surface layer integral
\[ I^\Omega_\rho(v) = \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( D_{1,v} - D_{2,v} \right) \mathcal{L}(x,y), \tag{7.6.5} \]
where $\Omega$ can again be chosen as in Figure 7.2 (c)). This conservation law plays a central role when working out the connection to quantum field theory in [46, 45].

Exercises
CHAPTER 8

Positive Functionals Arising from Second Variations

8.1. Overview of the Positivity Results

Given a minimizer of a variational principle, second variations are always non-negative. This basic observation goes back to Legendre and Jacobi, who used it to analyze the question whether classical trajectories minimize the action and geodesics minimize arc length \[68\]. In classical field theory (like electrodynamics or general relativity), second variations are less useful because in these theories the action is unbounded from below, so that instead of minimizing one merely seeks for critical points of the action. However, in the theory of causal fermion systems, the action is indeed bounded from below, and physical space-time should be described by a minimizer. Therefore, second variations should give rise to positive functionals in space-time. In this chapter, we show that this is indeed the case, and we also compute these positive functionals.

For technical simplicity, we again restrict attention to causal variational principles in the noncompact smooth setting (see Sections 4.6.3 and 4.6.5). We closely follow the constructions in \[40\]. For the variations we again consider families of measures \((\tilde{\rho}_\tau)_{\tau \in [0,\delta]}\) of the form \(6.3.1\), but now the measures in this family do not need to be critical. But we assume that \(\tilde{\rho} = \tilde{\rho}_0\) should be a minimizer of the causal action principle. Moreover, the variation should satisfy the volume constraint \(4.6.6\). For technical simplicity, we restrict attention to variations which are compactly supported (i.e. trivial outside a compact set; for details see Section 8.3). Then the variation of the measure \(6.3.1\) is described infinitesimally by a pair of a real-valued function and a vector field on \(M\) (for details see Section 8.2), both with compact support,

\[
(a,u) := \frac{d}{d\tau}(f_\tau, F_\tau) \bigg|_{\tau=0} \in \mathcal{J}_0^\infty := C_0^\infty(M,\mathbb{R}) \oplus C_0^\infty(M,T\mathcal{F}).
\]

We now state the main result of this chapter:

**Theorem 8.1.1.** Let \(\rho\) be a minimizer of a causal variational principle in the noncompact smooth setting (for details see Sections 4.6.3 and 4.6.5). Then the following two quadratic functionals on \(\mathcal{J}_0^\infty\) are positive:

\[
\int_M \nabla^2 \ell|_x(u,u) \, d\rho(x) \geq 0 \quad \text{(8.1.1)}
\]
\[
\int_M d\rho(x) \int_M d\rho(y) \nabla_{1,u} \nabla_{2,u} \mathcal{L}(x,y) + \int_M \nabla^2 \ell|_x(u,u) \, d\rho(x) \geq 0. \quad \text{(8.1.2)}
\]

We remark that the following converse of this theorem also holds: Assume that \(\rho\) is a Radon measure which satisfies \(6.2.3\) as well as the inequalities \(8.1.1\) and \(8.1.2\). If the inequality \(8.1.2\) is strict for every non-zero \(u \in \mathcal{J}_0^\infty\), then \(\rho\) is an isolated local minimum within the class of compactly supported variations with local fragmentation \(6.3.7\). For the proof we refer to \[40\] Section 5].
The above theorem has various applications. First, the positive bilinear forms can be used to endow jet space with a Hilbert space structure where in the scalar product the jets are integrated over space-time (see Section 8.4). Second, it gives rise to a positive surface layer integral, which in turn induces a spatial scalar product on the linearized solutions (see Section 8.5). Finally, as will be explained in Chapter 12 below, the above functionals are very useful for getting energy estimates.

8.2. Positivity of the Hessian of $\ell$

Let $\rho$ be a minimizer of the causal action. According to the EL equations (6.1.2), the function $\ell$ is minimal on $M$. This clearly implies that its Hessian (as computed in any chart) is positive semi-definite, i.e.

$$D^2\ell(x) \geq 0 \quad \text{for all } x \in M := \text{supp } \rho.$$  \hfill (8.2.1)

This is the first non-negative quantity obtained from the fact that $\rho$ is a minimizer. In view of the weak EL equations (6.2.3), the zero and first order derivatives of $\ell$ vanish for all $x \in M$. Adding such lower derivative terms, we can write (8.2.1) with jet derivatives as

$$\nabla^2\ell|_x(u,u) \geq 0 \quad \text{for all } x \in M ,$$

where, following our conventions (i) and (ii) on page 108,

$$\nabla^2\ell|_x(u,u) := a(x)^2 \ell(x) + 2 a(x) D_u \ell(x) + D^2\ell|x(u,u).$$

Integrating over $M$ gives the following result:

**Proposition 8.2.1.** Let $\rho$ be a minimizer of the causal action. Then

$$\int_M \nabla^2\ell|_x(u,u) \, d\rho(x) \geq 0 \quad \text{for all } u \in J^\infty_0 .$$

8.3. Positivity of Second Variations Generated by Jets

In this section we analyze second variations for a special class of variations of the measure $\rho$ to obtain another positive functional on jets. Similar as in [53, Section 3] we consider measures of the form

$$\tilde{\rho}_\tau = (F_\tau)_* (f_\tau \rho) \quad \text{for } \tau \in (-\tau_{\text{max}}, \tau_{\text{max}})$$  \hfill (8.3.1)

with smooth mappings

$$f \in C^\infty((-\tau_{\text{max}}, \tau_{\text{max}}) \times M, \mathbb{R}^+) \quad \text{and} \quad F \in C^\infty((-\tau_{\text{max}}, \tau_{\text{max}}) \times M, \mathcal{F}) ,$$

where the star denotes the push-forward measure defined by $((F_\tau)_* \mu)(\Omega) = \mu(F_\tau^{-1}(\Omega))$ (where $\Omega \subset \mathcal{F}$; for basics see for example [12, Section 3.6]). We assume that for $\tau = 0$ the variation is trivial,

$$f_0 \equiv 1 \quad \text{and} \quad F_0 \equiv \mathbb{1} .$$  \hfill (8.3.2)

Moreover, for technical simplicity we assume that $F_\tau$ and $f_\tau$ are compactly supported, meaning that they are trivial outside a compact set $K \subset M$, i.e.

$$f_{\tau}|_{M \setminus K} \equiv 1 \quad \text{and} \quad F_{\tau}|_{M \setminus K} \equiv \mathbb{1} .$$  \hfill (8.3.3)

Finally, in order to satisfy the volume constraint on the right side of (4.6.6), we assume that

$$\int_K f_\tau(x) \, d\rho(x) = \rho(K) \quad \text{for all } \tau \in (-\tau_{\text{max}}, \tau_{\text{max}}) .$$  \hfill (8.3.4)
Then the transformation (8.3.1) is described infinitesimally by a smooth and compactly supported jet,
\[ u = (a, \hat{u}) := (\tilde{f}_0, \tilde{F}_0) \in \mathcal{J}^\infty_0. \]
Moreover, differentiating the volume constraint (8.3.4) (and \( \nabla \)) acting on the first argument of the Lagrangian). Moreover, the second variation is
\[ y \]
In the last line we can carry out the equations (6.2.3) with (8.3.5) (and a similar formula for \( \ddot{\hat{f}} \) scalar variation \( f \)).

Then the transformation (8.3.1) is described infinitesimally by a smooth and compactly supported jet,
\[ u = (a, \hat{u}) := (\tilde{f}_0, \tilde{F}_0) \in \mathcal{J}^\infty_0. \]
Moreover, differentiating the volume constraint (8.3.4) gives
\[ \frac{\partial}{\partial x} \text{subject to the condition that the jet } u \text{ must satisfy the volume constraint (8.3.5).} \]
In the next proposition we remove this condition with a limiting procedure:

**PROPOSITION 8.3.1.** Let \( \rho \) be a minimizer of the causal action. Then
\[ \int_M d\rho(x) \int_M d\rho(y) \nabla_{1,u} \nabla_{2,u} \mathcal{L}(x, y) + \int_M \nabla^2 \ell|_x(u, u) d\rho(x) \geq 0, \]
subject to the condition that the jet \( u \) must satisfy the volume constraint (8.3.5). In the next proposition we remove this condition with a limiting procedure:

**PROOF.** Let \( u = (a, u) \in \mathcal{J}^\infty_0 \) be a jet which violates the volume constraint (8.3.5). Then, choosing a compact set \( \Omega \subset M \) with \( \rho(\Omega) > 0 \), the jet \( \tilde{u} := (\tilde{a}, u) \) with
\[ \hat{a}(x) = a(x) - c(\Omega) \chi_\Omega(x) \quad \text{and} \quad c(\Omega) := \frac{1}{\rho(\Omega)} \int_\Omega a(x) d\rho(x) \]
(where \( \chi_\Omega \) is the characteristic function) satisfies the volume constraint. Choosing the scalar variation \( f_\tau = (1 - \tau) + \tau \hat{a} \) and a family of diffeomorphisms \( F_\tau \) with \( \tilde{F}_0 = u \), we
obtain a variation which satisfies the volume constraint (note that \( \tilde{f} = 0 \)). Clearly, due to the characteristic function, the jet \( \tilde{u} \) is no longer smooth, but it has again compact support, and an approximation argument using Lebesgue’s dominated convergence theorem shows that the inequality \([8.3.7]\) also holds for \( \tilde{u} \). Expanding in powers of \( c \), we thus obtain the inequality

\[
0 \leq \int_K d\rho(x) \int_K d\rho(y) \nabla_{1,u} \nabla_{2,u} \mathcal{L}(x, y) + \int_K \nabla^2 \ell|_x(x, u) \, d\rho(x) \\
- 2c \int_M d\rho(x) \int_K d\rho(y) \chi_\Omega(x) \nabla_{2,u} \mathcal{L}(x, y) \\
+ c^2 \int_M d\rho(x) \int_M d\rho(y) \chi_\Omega(y) \mathcal{L}(x, y) \\
+ \int_M \left( -2c \chi_\Omega(x) \nabla_u \ell(x) + c^2 \chi_\Omega(x)^2 \ell(x) \right) d\rho(x)
\]

(the integrand in the last line arises from the contributions to \( \nabla^2 \ell|_x(x, u) \) involving the scalar components of the jets). The last line vanishes due to the weak EL equations \([6.2.3]\). Hence

\[
\int_K d\rho(x) \int_K d\rho(y) \nabla_{1,u} \nabla_{2,u} \mathcal{L}(x, y) + \int_K \nabla^2 \ell|_x(x, u) \, d\rho(x) \\
\geq 2c \int_K d\rho(x) \int_K d\rho(y) \nabla_{1,u} \mathcal{L}(x, y) - c^2 \int_K d\rho(x) \int_K d\rho(y) \mathcal{L}(x, y) =: A(\Omega).
\]

We now let \( (\Omega_n)_{n \in \mathbb{N}} \) be an exhaustion of \( M \) by compact sets. We distinguish the two cases when \( \rho(M) \) is finite and infinite and treat these cases separately. If the total volume \( \rho(M) \) is finite, one can take the limit \( n \to \infty \) with Lebesgue’s dominated convergence theorem to obtain

\[
\lim_{n \to \infty} \int_K d\rho(x) \int_{\Omega_n} d\rho(y) \nabla_{1,u} \mathcal{L}(x, y) = \int_K d\rho(x) \int_K d\rho(y) \nabla_{1,u} \mathcal{L}(x, y) \\
\overset{[??]}{=} \int_K \nabla_u \left( \ell(x) + s \right) \, d\rho(x) = s \int_K a(x) \, d\rho(x)
\]

\[
\lim_{n \to \infty} A(\Omega_n) = 2c(M) s \int_K a(x) \, d\rho(x) - c(M)^2 \rho(M) s \\
= \frac{\nu}{2 \rho(M)} \left( \int_K a(x) \, d\rho(x) \right)^2 \geq 0,
\]

where in the last line we substituted the value of \( c(M) \) in \([8.3.8]\).

In the remaining case that the volume \( \rho(M) \) is infinite, we estimate the terms as follows,

\[
c(\Omega_n)^2 \int_K d\rho(x) \int_K d\rho(y) \mathcal{L}(x, y) \\
\leq c(\Omega_n)^2 \int_K d\rho(x) \int_K d\rho(y) \mathcal{L}(x, y) = c(\Omega_n)^2 s \rho(K) \to 0
\]

\[
\int_K d\rho(x) \int_{\Omega_n} d\rho(y) \nabla_{1,u} \mathcal{L}(x, y) \to \int_K d\rho(x) \int_K d\rho(y) \nabla_{1,u} \mathcal{L}(x, y) \\
\overset{[??]}{=} \int_K \nabla_u \left( \ell(x) + s \right) \, d\rho(x) = s \int_K a(x) \, d\rho(x).
\]
8.5. APPLICATION: A POSITIVE SURFACE LAYER INTEGRAL

As a consequence, \( A(\Omega_n) \) converges to zero as \( n \to \infty \). This concludes the proof. □

**Proof of Theorem 8.1.1.** The inequalities (8.1.1) and (8.1.2) were derived in Propositions 8.2.1 and 8.3.1. □

8.4. Application: Hilbert Spaces of Jets

As an application, we now explain how our positive functionals can be used to endow the space of jets in space-time with Hilbert space structures. These Hilbert space structures should be very useful because they make functional analytic tools applicable to the analysis of the jet spaces and the causal action principle. We now introduce the following bilinear forms on \( J_\infty \):

\[
\langle u, v \rangle := \int_M d\rho(x) \int_M d\rho(y) \nabla_{1,u} \nabla_{2,v} L(x, y) + \int_M \nabla^2 \ell|_{x}(u, v) \, d\rho(x) \quad (8.4.1)
\]

\[
\langle \langle u, v \rangle \rangle := \langle u, v \rangle + \int_M \nabla^2 \ell|_{x}(u, v) \, d\rho(x) . \quad (8.4.2)
\]

By Theorem 8.1.1 both bilinear forms are positive semi-definite. Thus dividing out the null space and forming the completion gives real Hilbert spaces of jets denoted by \( \mathcal{H}_\langle \langle \cdot, \cdot \rangle \rangle \) and \( \mathcal{H}_\langle \cdot, \cdot \rangle \), respectively. Obviously,

\[
\langle u, u \rangle \leq \langle \langle u, u \rangle \rangle ,
\]

giving rise to a norm-decreasing mapping \( \mathcal{H}_\langle \langle \cdot, \cdot \rangle \rangle \to \mathcal{H}_\langle \cdot, \cdot \rangle \).

For the scalar components of the jets, the two scalar products (8.4.1) and (8.4.2) obviously agree. But they are quite different for the vector components. In order to understand this difference, it is instructive to consider a jet \( u = (0, u) \) which describes a symmetry of the Lagrangian, i.e. (for details see [52, Section 3.1])

\[
(D_{1,u} + D_{2,u}) L(x, y) = 0 \quad \text{for all } x, y \in M .
\]

For this jet, a direct computation shows that

\[
\langle u, u \rangle = 0 .
\]

Hence symmetry transformations lie in the kernel of the bilinear form \( \langle \cdot, \cdot \rangle \) and thus correspond to the zero vector in \( \mathcal{H}_\langle \cdot, \cdot \rangle \). For example, in the setting of causal fermion systems, jets describing global phase transformations (see [52, Section 5.1]) are not contained in \( \mathcal{H}_\langle \cdot, \cdot \rangle \). Generally speaking, the scalar product \( \langle \cdot, \cdot \rangle \) makes it possible to disregard symmetry transformations of the causal fermion system. However, jets describing symmetry transformations do in general correspond to non-zero vectors of the Hilbert space \( \mathcal{H}_\langle \langle \cdot, \cdot \rangle \rangle \).

8.5. Application: A Positive Surface Layer Integral

In the setting of causal variational principles, the usual integrals over hypersurfaces in space-time are undefined. Instead, one considers so-called surface layer integrals. In general terms, a surface layer integral is a double integral of the form

\[
\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \cdots L(x, y) , \quad (8.5.1)
\]

where \( \Omega \) is a subset of \( M \) and \( \cdots \) stands for a differential operator acting on the Lagrangian. The structure of such surface layer integrals can be understood most easily in the special situation that the Lagrangian is of short range in the sense that \( L(x, y) \)
vanishes unless $x$ and $y$ are close together. In this situation, we only get a contribution to the double integral if both $x$ and $y$ are close to the boundary $\partial \Omega$. With this in mind, surface layer integrals can be understood as an adaptation of surface integrals to the setting of causal variational principles (for a more detailed explanation see Section 2.3)

In [52], it is shown that there are conservation laws expressed in terms of surface layer integrals which generalize the well-known charge and current conservation. In [53], a conserved surface layer integral was found which generalizes the symplectic form to the setting of causal variational principles. We now derive a surface layer integral which is not necessarily conserved, but which has a definite sign. A jet $v \in J$ (not necessarily with compact support) is referred to as a solution of the linearized field equations if it satisfies the equation (for details and the motivation see [53])

$$\nabla_u \left( \int_M (\nabla_{1,v} + \nabla_{2,v}) \mathcal{L}(x,y) \, d\rho(y) - \nabla_y s \right) = 0 \quad (8.5.2)$$

for all $u \in J_\infty$ and all $x \in M$.

The following proposition shows that there is a positive surface layer integral. Similar as explained at the beginning of Section 8.4, this can be used to endow the jet space with a Hilbert structure. But in contrast to the scalar products in Section 8.4 where the jets were integrated over space-time, here the scalar product is given as a surface layer integral. This should be very useful for analyzing the dynamics of jets in space-time.

**Proposition 8.5.1.** Assume that $v$ is a solution of the linearized field equations (8.5.2). Then for any compact $\Omega \subset M$, the following surface layer integral is positive,

$$- \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \nabla_{1,v} \nabla_{2,v} \mathcal{L}(x,y) \geq 0.$$

**Proof.** Denoting the components of $v$ by $v = (b,v)$, we evaluate (8.5.2) for $u = v$ and integrate over $\Omega$. The resulting integrals can be rewritten as follows,

$$0 = \int_\Omega d\rho(x) \int_M d\rho(y) \nabla_{1,v}(\nabla_{1,v} + \nabla_{2,v}) \mathcal{L}(x,y) - s \int_\Omega b(x)^2 \, d\rho(x)$$

$$= \int_\Omega \nabla^2 \ell(x,v) \, d\rho(x) + \int_\Omega d\rho(x) \int_M d\rho(y) \nabla_{1,v} \nabla_{2,v} \mathcal{L}(x,y)$$

$$= \int_\Omega \nabla^2 \ell(x,v) \, d\rho(x) + \int_\Omega d\rho(x) \int_M d\rho(y) \nabla_{1,v} \nabla_{2,v} \mathcal{L}(x,y) \quad (8.5.3)$$

$$+ \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \nabla_{1,v} \nabla_{2,v} \mathcal{L}(x,y). \quad (8.5.4)$$

Using characteristic functions, the expression (8.5.3) can be written as

$$\int_M \nabla^2 \ell(x,\chi_\Omega v, \chi_\Omega v) \, d\rho(x) + \int_\Omega d\rho(x) \int_M d\rho(y) \nabla_{1,\chi_\Omega v} \nabla_{2,\chi_\Omega v} \mathcal{L}(x,y).$$

Approximating the jet $\chi_\Omega v$ by smooth jets with compact support, one finds that the integrals in (8.5.3) are non-negative by Proposition 8.3.1. This gives the result.

We finally remark that in [37] Section 6] the surface layer integral in the last proposition is computed in Minkowski space.
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9.1. Constructing a Causal Fermion System from the Fermionic Projector

We now explain how one can construct a causal fermion system from a given fermionic projector \( P(x, y) \) as constructed in Section ???. We closely follow the presentation in [56, Section 4].

Our task is to introduce an ultraviolet regularization. This is done most conveniently with so-called regularization operators.

**Definition 9.1.1.** A family \( \{R_\varepsilon\}_{\varepsilon > 0} \) of bounded linear operators on \( \mathcal{H}_m \) are called **regularization operators** if they have the following properties:

(i) Solutions of the Dirac equation are mapped to continuous solutions,

\[ R_\varepsilon : \mathcal{H}_m \rightarrow C^0(M, SM) \cap \mathcal{H}_m \]

(ii) For every \( \varepsilon > 0 \) and \( x \in M \), there is a constant \( c > 0 \) such that

\[ \| (R_\varepsilon \psi_m)(x) \| \leq c \| \psi_m \| \quad \forall \psi_m \in \mathcal{H}_m. \]

(9.1.1)

(iii) In the limit \( \varepsilon \searrow 0 \), the regularization operators go over to the identity with strong convergence of \( R_\varepsilon \) and \( R_\varepsilon^* \), i.e.

\[ R_\varepsilon \psi_m, R_\varepsilon^* \psi_m \searrow \psi_m \text{ in } \mathcal{H}_m \quad \forall \psi_m \in \mathcal{H}_m. \]

There are many possibilities to choose regularization operators. As a typical example, one can choose finite-dimensional subspaces \( \mathcal{H}^{(n)} \subset C^{\infty}_{sc}(M, SM) \cap \mathcal{H}_m \) which are an exhaustion of \( \mathcal{H}_m \) in the sense that \( \mathcal{H}^{(0)} \subset \mathcal{H}^{(1)} \subset \cdots \) and \( \mathcal{H}_m = \cup_n \mathcal{H}^{(n)} \). Setting \( n(\varepsilon) = \max\{0, 1/\varepsilon\} \cap \mathbb{N} \), we can introduce the operators \( R_\varepsilon \) as the orthogonal projection operators to \( \mathcal{H}^{(n(\varepsilon))} \). An alternative method is to choose a Cauchy hypersurface \( N \), to mollify the restriction \( \psi_m|_N \) to the Cauchy surface on the length scale \( \varepsilon \), and to define \( R_\varepsilon \psi_m \) as the solution of the Cauchy problem for the mollified initial data.

Given regularization operators \( R_\varepsilon \), for any \( \varepsilon > 0 \) we introduce the **particle space** \( (\mathcal{H}_{\text{particle}}, \langle . | . \rangle_{\mathcal{H}_{\text{particle}}}) \) as the Hilbert space

\[ \mathcal{H}_{\text{particle}} = \ker \left( R_\varepsilon \chi_{(-\infty,0)}(S) \right)^\perp, \quad \langle . | . \rangle_{\mathcal{H}_{\text{particle}}} = \langle . | . \rangle_{\mathcal{H}_{\text{particle}} \times \mathcal{H}_{\text{particle}}}. \]

Next, for any \( x \in M \) we consider the bilinear form

\[ b : \mathcal{H}_{\text{particle}} \times \mathcal{H}_{\text{particle}} \rightarrow \mathbb{C}, \quad b(\psi_m, \phi_m) = -\langle (R_\varepsilon \psi_m)(x) \mid (R_\varepsilon \phi_m)(x) \rangle_x. \]

This bilinear form is bounded in view of (9.1.1). The **local correlation operator** \( F^\varepsilon(x) \) is defined as the signature operator of this bilinear form, i.e.

\[ b(\psi_m, \phi_m) = \langle \psi_m \mid F^\varepsilon(x) \phi_m \rangle_{\mathcal{H}_{\text{particle}}} \quad \text{for all } \psi_m, \phi_m \in \mathcal{H}_{\text{particle}}. \]

Taking into account that the spin scalar operator has signature \((2,2)\), the local correlation operator is a symmetric operator in \( \mathcal{L}(\mathcal{H}_{\text{particle}}) \) of rank at most four, which has
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at most two positive and at most two negative eigenvalues. Finally, we introduce the
universal measure \( \rho = F^* \mu_M \) as the push-forward of the volume measure on \( M \) under
the mapping \( F^* \) (thus \( \rho(\Omega) := \mu_M((F^*)^{-1}(\Omega)) \)). Omitting the subscript “particle”, we
thus obtain a causal fermion system of spin dimension (see Definition 4.2.1).

9.2. Geometric Structures of a Causal Fermion System

We now outline constructions from [42] which give general notions of a connection
and curvature (see Theorem 9.2.9, Definition 9.2.11 and Definition 9.2.12). We closely
follow the presentation in [44].

9.2.1. Construction of the Spin Connection. Let \( (M, F, \rho) \) be a
causal fermion system of spin dimension \( n \) in the particle representation
\( \hat{\mathcal{H}} \). We now construct additional
objects, leading us to the more familiar space-time representation. First, on
\( F \) we consider
the topology induced by the operator norm \( \| A \| := \sup\{\|Au\|_{\mathcal{H}} \text{ with } \|u\|_{\mathcal{H}} = 1\} \). For
every \( x \in F \) we define the
spin space \( S_x \) by
\[
S_x = x(\hat{\mathcal{H}}); \text{ it is a subspace of } \mathcal{H} \text{ of dimension at most } 2n.
\]
On \( S_x \) we introduce the
spin scalar product \( \langle . | . \rangle_x \) by
\[
\langle u | v \rangle_x = -\langle xu | v \rangle_{\mathcal{H}} \quad (\text{for all } u, v \in S_x);
\]
(9.2.1)

it is an indefinite inner product of signature \((p, q)\) with \( p, q \leq n \). We define space-time
\( \mathcal{M} = \text{supp } \rho \). It is a closed subset of \( F \), and
by restricting the causal structure of \( F \) to \( \mathcal{M} \), we get causal relations in space-time. A
wave function \( \psi \) is defined as a function which to every \( x \in \mathcal{M} \) associates a vector of the
corresponding spin space,
\[
\psi : \mathcal{M} \to \hat{\mathcal{H}} \text{ with } \psi(x) \in S_x \text{ for all } x \in \mathcal{M}.
\]
(9.2.2)

On the wave functions we introduce the indefinite inner product
\[
\langle \psi | \phi \rangle = \int_\mathcal{M} \langle \psi(x) | \phi(x) \rangle_x \, d\rho(x).
\]
(9.2.3)

In order to ensure that the last integral converges, we also introduce the norm \( \| . \| \) by
\[
\| \psi \|^2 = \int_\mathcal{M} \langle \psi(x) | x \psi(x) \rangle_{\mathcal{H}} \, d\rho(x)
\]
(where \( |x| \) is the absolute value of the operator \( x \) on \( \hat{\mathcal{H}} \)). The one-particle space \( \mathcal{K} \)

is defined as the space of wave functions for which the norm \( \| . \| \) is finite, with the topology
induced by this norm, and endowed with the inner product \( \langle . | . \rangle \). Then \((\mathcal{K}, \langle . | . \rangle)\) is a Krein space (see [13]). Next, for any \( x, y \in \mathcal{M} \) we define the kernel of the fermionic operator \( P(x, y) \) by
\[
P(x, y) = \pi_x y : S_y \to S_x,
\]
(9.2.4)

where \( \pi_x \) is the orthogonal projection onto the subspace \( S_x \subset \mathcal{H} \). The closed chain is
defined as the product
\[
A_{xy} = P(x, y) P(y, x) : S_x \to S_x.
\]

As it is an endomorphism of \( S_x \), we can compute its eigenvalues. The calculation
\( A_{xy} = (\pi_x y)(\pi_y x) = \pi_x y x \) shows that these eigenvalues coincide precisely with the non-trivial
eigenvalues \( \lambda_{xy}^1, \ldots, \lambda_{xy}^{2n} \) of the operator \( xy \) as considered in Definition 4.4.1. In this
way, the kernel of the fermionic operator encodes the causal structure of \( M \). Choosing a
suitable dense domain of definition \( D(P) \), we can regard \( P(x, y) \) as the integral kernel of a corresponding operator \( P \),

\[
P : D(P) \subset K \rightarrow K, \quad (P\psi)(x) = \int_M P(x, y) \psi(y) \, d\rho(y), \quad (9.2.5)
\]

referred to as the *fermionic operator*. We collect two properties of the fermionic operator:

(A) \( P \) is symmetric in the sense that \(<P\psi|\phi> = <\psi|P\phi> \) for all \( \psi, \phi \in D(P) \):

According to the definitions \((9.2.4)\) and \((9.2.1)\),

\[
<P(x, y) \psi(y) | \psi(x)>_x = -<(x \pi_x y \psi(y)) | x \phi(x)>_\pi_x = -<\psi(y) | P(y, x) \psi(x)>_y.
\]

We now integrate over \( x \) and \( y \) and apply \((9.2.5)\) and \((9.2.3)\).

(B) \((-P)\) is positive in the sense that \(<\psi|(-P)\psi> \geq 0 \) for all \( \psi \in D(P) \):

This follows immediately from the calculation

\[
<\psi|(-P)\psi> = -\int_{M \times M} <\psi(x) | P(x, y) \psi(y)>_x \, d\rho(x) \, d\rho(y)
\]

\[
= \int_{M \times M} \langle \psi(x) | x \pi_x y \psi(y)\rangle_{\pi_x} \, d\rho(x) \, d\rho(y) = \langle \phi | \phi \rangle_{\pi_x} \geq 0,
\]

where we again used \((9.2.3)\) and \((9.2.4)\) and set

\[
\phi = \int_M x \psi(x) \, d\rho(x).
\]

The space-time representation of the causal fermion system consists of the Krein space \( (K, <.|.>) \), whose vectors are represented as functions on \( M \) (see \((9.2.2)\), \((9.2.3)\)), together with the fermionic operator \( P \) in the integral representation \((9.2.5)\) with the above properties (A) and (B).

Having Dirac spinors in a four-dimensional space-time in mind, from now on we assume that the spin dimension \( n = 2 \). Moreover, we only consider space-time points \( x \in M \) which are regular in the sense that the corresponding spin spaces \( S_x \) have the maximal dimension four.

An important structure from spin geometry missing so far is Clifford multiplication. To this end, we need a Clifford algebra represented by symmetric operators on \( S_x \). For convenience, we first consider Clifford algebras with the maximal number of five generators; later we reduce to four space-time dimensions (see Definition \(9.2.14\) below). We denote the set of symmetric linear endomorphisms of \( S_x \) by \( \text{Symm}(S_x) \); it is a 16-dimensional real vector space.

**Definition 9.2.1.** A five-dimensional subspace \( K \subset \text{Symm}(S_x) \) is called a *Clifford subspace* if the following conditions hold:

(i) For any \( u, v \in K \), the anti-commutator \( \{u, v\} \equiv uv + vu \) is a multiple of the identity on \( S_x \).
(ii) The bilinear form \( \langle ., . \rangle \) on \( K \) defined by

\[
\frac{1}{2} \{ u, v \} = \langle u, v \rangle \mathbb{1} \quad \text{for all } u, v \in K
\]

is non-degenerate and has signature \((1, 4)\).

In view of the situation in spin geometry, we would like to distinguish a specific Clifford subspace. In order to partially fix the freedom in choosing Clifford subspaces, it is useful to impose that \( K \) should contain a given so-called sign operator.

**Definition 9.2.2.** An operator \( v \in \text{Symm}(S_x) \) is called a sign operator if \( v^2 = \mathbb{1} \) and if the inner product \( \langle . | . \rangle : S_x \times S_x \to \mathbb{C} \) is positive definite.

**Definition 9.2.3.** For a given sign operator \( v \), the set of Clifford extensions \( \mathcal{T}^v \) is defined as the set of all Clifford subspaces containing \( v \),

\[
\mathcal{T}^v = \{ K \text{ Clifford subspace with } v \in K \}.
\]

Considering \( x \) as an operator on \( S_x \), this operator has by definition of the spin dimension two positive and two negative eigenvalues. Moreover, the calculation

\[
\langle u | (-x) u \rangle_x = \langle u | x^2 u \rangle_x > 0 \quad \text{for all } u \in S_x \setminus \{0\}
\]

shows that the operator \((-x)\) is positive definite on \( S_x \). Thus we can introduce a unique sign operator \( s_x \) by demanding that the eigenspaces of \( s_x \) corresponding to the eigenvalues \( \pm 1 \) are precisely the positive and negative spectral subspaces of the operator \((-x)\). This sign operator is referred to as the Euclidean sign operator.

A straightforward calculation shows that for two Clifford extensions \( K, \tilde{K} \in \mathcal{T}^v \), there is a unitary transformation \( U \in e^{iR_{s_x}} \) such that \( \tilde{K} = UKU^{-1} \) (for details see [42, Section 3]). By dividing out this group action, we obtain a five-dimensional vector space, endowed with the inner product \( \langle ., . \rangle \). Taking for \( v \) the Euclidean signature operator, we regard this vector space as a generalization of the usual tangent space.

**Definition 9.2.4.** The tangent space \( T_x \) is defined by

\[
T_x = \mathcal{T}^x_s / \exp(iR_{s_x}).
\]

It is endowed with an inner product \( \langle ., . \rangle \) of signature \((1, 4)\).

We next consider two space-time points, for which we need to make the following assumption.

**Definition 9.2.5.** Two points \( x, y \in M \) are said to be properly time-like separated if the closed chain \( A_{xy} \) has a strictly positive spectrum and if the corresponding eigenspaces are definite subspaces of \( S_x \).

This definition clearly implies that \( x \) and \( y \) are time-like separated (see Definition 4.4.1). Moreover, the eigenspaces of \( A_{xy} \) are definite if and only if those of \( A_{yx} \) are, showing that Definition 9.2.5 is again symmetric in \( x \) and \( y \). As a consequence, the spin space can be decomposed uniquely into an orthogonal direct sum \( S_x = I^+ \oplus I^- \) of a positive definite subspace \( I^+ \) and a negative definite subspace \( I^- \) of \( A_{xy} \). This allows us to introduce a unique sign operator \( v_{xy} \) by demanding that its eigenspaces corresponding to the eigenvalues \( \pm 1 \) are the subspaces \( I^\pm \). This sign operator is referred to as the directional sign operator of \( A_{xy} \). Having two sign operators \( s_x \) and \( v_{xy} \) at our disposal, we can distinguish unique corresponding Clifford extensions, provided that the two sign operators satisfy the following generic condition.
9.2. GEOMETRIC STRUCTURES OF A CAUSAL FERMION SYSTEM

**Definition 9.2.6.** Two sign operators \( v, \tilde{v} \) are said to be *generically separated* if their commutator \([v, \tilde{v}]\) has rank four.

**Lemma 9.2.7.** Assume that the sign operators \( s_x \) and \( v_{xy} \) are generically separated. Then there are unique Clifford extensions \( K_{x}^{(y)} \in T^{s_x} \) and \( K_{xy} \in T^{v_{xy}} \) and a unique operator \( \rho \in K_{x}^{(y)} \cap K_{xy} \) with the following properties:

(i) The relations \( \{s_x, \rho\} = 0 = \{v_{xy}, \rho\} \) hold.

(ii) The operator \( U_{xy} := e^{i\rho} \) transforms one Clifford extension to the other,

\[
K_{xy} = U_{xy} K_{x}^{(y)} U_{xy}^{-1}.
\]

(iii) If \( \{s_x, v_{xy}\} \) is a multiple of the identity, then \( \rho = 0 \).

The operator \( \rho \) depends continuously on \( s_x \) and \( v_{xy} \).

We refer to \( U_{xy} \) as the *synchronization map*. Exchanging the roles of \( x \) and \( y \), we also have two sign operators \( s_y \) and \( v_{yx} \) at the point \( y \). Assuming that these sign operators are again generically separated, we also obtain a unique Clifford extension \( K_{yx} \in T^{v_{yx}} \).

After these preparations, we can now explain the construction of the spin connection \( D \) (for details see [42, Section 3]). For two space-time points \( x, y \in M \) with the above properties, we want to introduce an operator

\[
D_{x,y} : S_y \rightarrow S_x
\]

(generally speaking, by the subscript \( xy \) we always denote an object at the point \( x \), whereas the additional comma \( x,y \) denotes an operator which maps an object at \( y \) to an object at \( x \)). It is natural to demand that \( D_{x,y} \) is unitary, that \( D_{y,x} \) is its inverse, and that these operators map the directional sign operators at \( x \) and \( y \) to each other,

\[
D_{x,y} = (D_{y,x})^* = (D_{y,x})^{-1}
\]

(9.2.6)

\[
v_{xy} = D_{x,y} v_{yx} D_{y,x}.
\]

(9.2.7)

The obvious idea for constructing an operator with these properties is to take a polar decomposition of \( P(x, y) \); this amounts to setting

\[
D_{x,y} = A_{xy}^{-\frac{1}{2}} P(x, y).
\]

(9.2.8)

This definition has the shortcoming that it is not compatible with the chosen Clifford extensions. In particular, it does not give rise to a connection on the corresponding tangent spaces. In order to resolve this problem, we modify \( 9.2.8 \) by the ansatz

\[
D_{x,y} = e^{i\varphi_{xy}} v_{xy} A_{xy}^{-\frac{1}{2}} P(x, y)
\]

(9.2.9)

with a free real parameter \( \varphi_{xy} \). In order to comply with \( 9.2.6 \), we need to demand that

\[
\varphi_{xy} = -\varphi_{yx} \mod 2\pi;
\]

(9.2.10)

then \( 9.2.7 \) is again satisfied. We can now use the freedom in choosing \( \varphi_{xy} \) to arrange that the distinguished Clifford subspaces \( K_{xy} \) and \( K_{yx} \) are mapped onto each other,

\[
K_{xy} = D_{x,y} K_{yx} D_{y,x}.
\]

(9.2.11)

It turns out that this condition determines \( \varphi_{xy} \) up to multiples of \( \frac{\pi}{2} \). In order to fix \( \varphi_{xy} \) uniquely in agreement with \( 9.2.10 \), we need to assume that \( \varphi_{xy} \) is not a multiple of \( \frac{\pi}{2} \). This leads us to the following definition.

**Definition 9.2.8.** Two points \( x, y \in M \) are called *spin connectable* if the following conditions hold:
(a) The points \( x \) and \( y \) are properly timelike separated (note that this already implies that \( x \) and \( y \) are regular as defined in Section 9.2.1).
(b) The Euclidean sign operators \( s_x \) and \( s_y \) are generically separated from the directional sign operators \( v_{xy} \) and \( v_{yx} \), respectively.
(c) Employing the ansatz (9.2.9), the phases \( \varphi_{xy} \) which satisfy condition (9.2.11) are not multiples of \( \frac{\pi}{4} \).

We denote the set of points which are spin connectable to \( x \) by \( I(x) \). It is straightforward to verify that \( I(x) \) is an open subset of \( M \).

Under these assumptions, we can fix \( \varphi_{xy} \) uniquely by imposing that

\[
\varphi_{xy} \in \left( -\frac{\pi}{2}, -\frac{\pi}{4} \right) \cup \left( \frac{\pi}{4}, \frac{\pi}{2} \right),
\]

(9.2.12)
giving the following result (for the proofs see [42, Section 3.3]).

**Theorem 9.2.9.** Assume that two points \( x, y \in M \) are spin connectable. Then there is a unique spin connection \( D_{x,y} : S_y \to S_x \) of the form (9.2.9) having the properties (9.2.6), (9.2.7), (9.2.11) and (9.2.12).

### 9.2.2. A Time Direction, the Metric Connection and Curvature

We now outline a few further constructions from [42, Section 3]. First, for spin connectable points we can distinguish a direction of time.

**Definition 9.2.10.** Assume that the points \( x, y \in M \) are spin connectable. We say that \( y \) lies in the future of \( x \) if the phase \( \varphi_{xy} \) as defined by (9.2.9) and (9.2.12) is positive. Otherwise, \( y \) is said to lie in the past of \( x \).

According to (9.2.10), \( y \) lies in the future of \( x \) if and only if \( x \) lies in the past of \( y \). By distinguishing a direction of time, we get a structure similar to a causal set (see for example [14]). However, in contrast to a causal set, our notion of “lies in the future of” is not necessarily transitive.

The spin connection induces a connection on the corresponding tangent spaces, as we now explain. Suppose that \( u_y \in T_y \). Then, according to Definition 9.2.4 and Lemma 9.2.7, we can consider \( u_y \) as a vector of the representative \( K_y^{(x)} \in T^{s_y} \). By applying the synchronization map, we obtain a vector in \( K_{yx} \),

\[
u_{yx} := U_{yx} \ u_y \ U_{yx}^{-1} \in K_{yx}.
\]

According to (9.2.11), we can now “parallel transport” the vector to the Clifford subspace \( K_{xy} \),

\[
u_{xy} := D_{x,y} \ u_{yx} \ D_{y,x} \in K_{xy}.
\]

Finally, we apply the inverse of the synchronization map to obtain the vector

\[
u_x := U_{xy}^{-1} \ u_{xy} \ U_{xy} \in K_x^{(y)}.
\]

As \( K_x^{(y)} \) is a representative of the tangent space \( T_x \) and all transformations were unitary, we obtain an isometry from \( T_y \) to \( T_x \).

**Definition 9.2.11.** The isometry between the tangent spaces defined by

\[
\nabla_{x,y} : T_y \to T_x : u_y \mapsto u_x
\]

is referred to as the metric connection corresponding to the spin connection \( D \).

We next introduce a notion of curvature.
9.3. Correspondence to Lorentzian Spin Geometry

Definition 9.2.12. Suppose that three points \( x, y, z \in M \) are pairwise spin connectable. Then the associated metric curvature \( R \) is defined by

\[
R(x, y, z) = \nabla_{x,y} \nabla_{y,z} \nabla_{z,x} : T_x \to T_x.
\] (9.2.13)

The metric curvature \( R(x, y, z) \) can be thought of as a discrete analog of the holonomy of the Levi-Civita connection on a manifold, where a tangent vector is parallel transported along a loop starting and ending at \( x \). On a manifold, the curvature at \( x \) is immediately obtained from the holonomy by considering the loops in a small neighborhood of \( x \). With this in mind, Definition 9.2.12 indeed generalizes the usual notion of curvature to causal fermion systems.

The following construction relates directional sign operators to vectors of the tangent space. Suppose that \( y \) is spin connectable to \( x \). By synchronizing the directional sign operator \( v_{xy} \), we obtain the vector

\[
\hat{y}_x := U_{xy}^{-1} v_{xy} U_{xy} \in K_x(y).
\] (9.2.14)

As \( K_x(y) \in T^s_x \) is a representative of the tangent space, we can regard \( \hat{y}_x \) as a tangent vector. We thus obtain a mapping

\[
I(x) \to T_x : y \mapsto \hat{y}_x.
\]

We refer to \( \hat{y}_x \) as the directional tangent vector of \( y \) in \( T_x \). As \( v_{xy} \) is a sign operator and the transformations in (9.2.14) are unitary, the directional tangent vector is a timelike unit vector with the additional property that the inner product \( \langle \hat{y}_x, \cdot \rangle_x \) is positive definite.

We finally explain how to reduce the dimension of the tangent space to four, with the desired Lorentzian signature \((1, 3)\).

Definition 9.2.13. The fermion system is called chirally symmetric if to every \( x \in M \) we can associate a spacelike vector \( u(x) \in T_x \) which is orthogonal to all directional tangent vectors,

\[
\langle u(x), \hat{y}_x \rangle = 0 \quad \text{for all } y \in I(x),
\]

and is parallel with respect to the metric connection, i.e.

\[
u(x) = \nabla_{x,y} u(y) \nabla_{y,x} \quad \text{for all } y \in I(x).
\]

Definition 9.2.14. For a chirally symmetric fermion system, we introduce the reduced tangent space \( T_{x}^{\text{red}} \) by

\[
T_{x}^{\text{red}} = \langle u_x \rangle^\perp \subset T_x.
\]

Clearly, the reduced tangent space has dimension four and signature \((1, 3)\). Moreover, the operator \( \nabla_{x,y} \) maps the reduced tangent spaces isometrically to each other. The local operator \( \Gamma := -iu/\sqrt{-u^2} \) takes the role of the pseudoscalar matrix.

9.3. Correspondence to Lorentzian Spin Geometry

We also explain how these notions correspond to the usual objects of differential geometry in Minkowski space (Theorem 9.3.1) and on a globally hyperbolic Lorentzian manifold (Theorem 9.3.2). We closely follow the presentation in [44, Section 3.3].

We let \((M, g)\) be a time-oriented Lorentzian spin manifold with spinor bundle \( SM \) (thus \( S_x M \) is a 4-dimensional complex vector space endowed with an inner product \( \langle \cdot, \cdot \rangle_x \) of signature \((2, 2)\)). Assume that \( \gamma(t) \) is a smooth, future-directed and timelike curve, for simplicity parametrized by the arc length, defined on the interval \([0, T]\) with \( \gamma(0) = y \).
and $\gamma(T) = x$. Then the parallel transport of tangent vectors along $\gamma$ with respect to the Levi-Civita connection $\nabla^{\text{LC}}$ gives rise to the isometry $\nabla^{\text{LC}}_{x,y} : T_y \to T_x$.

In order to compare with the metric connection $\nabla$ of Definition 9.2.11, we subdivide $\gamma$ (for simplicity with equal spacing, although a non-uniform spacing would work just as well). Thus for any given $N$, we define the points $x_0, \ldots, x_N$ by

$$x_n = \gamma(t_n) \quad \text{with} \quad t_n = \frac{nT}{N}.$$  

We define the parallel transport $\nabla^N_{x,y}$ by successively composing the parallel transport between neighboring points,

$$\nabla^N_{x,y} := \nabla_{x_N,x_{N-1}} \nabla_{x_{N-1},x_{N-2}} \cdots \nabla_{x_1,x_0} : T_y \to T_x.$$  

Our first theorem gives a connection to the Minkowski vacuum. For any $\varepsilon > 0$ we regularize on the scale $\varepsilon > 0$ by inserting a convergence generating factor into the integrand of the Dirac sea in Minkowski space (9.3.1)

$$P^\varepsilon(x,y) = \int \frac{d^4k}{(2\pi)^4} \left(\delta(k^2 - m^2) \Theta(-k^0) e^{iky} e^{-ik(x-y)}\right).$$

This function can indeed be realized as the kernel of the fermionic operator $\rho^\varepsilon$ corresponding to a causal fermion system $(H, F, \rho^\varepsilon)$. Here the measure $\rho^\varepsilon$ is the push-forward of the volume measure in Minkowski space by an operator $F^\varepsilon$, being an ultraviolet regularization of the operator $F$, similar as constructed in Section 9.1. Moreover, for technical convenience we assume that the manifold coincides with Minkowski space to the past of any Cauchy hypersurface (for details see [42], Section 4).

**Theorem 9.3.1.** For given $\gamma$, we consider the family of regularized fermionic projectors of the vacuum $(P^\varepsilon)_{\varepsilon>0}$ as given by (9.3.1). Then for a generic curve $\gamma$ and for every $N \in \mathbb{N}$, there is $\varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $n = 1, \ldots, N$, the points $x_n$ and $x_{n-1}$ are spin connectable, and $x_{n+1}$ lies in the future of $x_n$ (according to Definition 9.2.10). Moreover,

$$\nabla^{\text{LC}}_{x,y} = \lim_{N \to \infty} \lim_{\varepsilon \to 0} \nabla^N_{x,y}.$$  

By a generic curve we mean that the admissible curves are dense in the $C^\infty$-topology (i.e., for any smooth $\gamma$ and every $K \in \mathbb{N}$, there is a sequence $\gamma_\ell$ of admissible curves such that $D^k \gamma_\ell \to D^k \gamma$ uniformly for all $k = 0, \ldots, K$). The restriction to generic curves is needed in order to ensure that the Euclidean and directional sign operators are generically separated (see Definition 9.2.8 (b)). The proof of the above theorem is given in [42], Section 4).

Clearly, in this theorem the connection $\nabla^{\text{LC}}_{x,y}$ is trivial. In order to show that our connection also coincides with the Levi-Civita connection in the case with curvature, in [42], Section 5] a globally hyperbolic Lorentzian manifold is considered. For technical simplicity, we assume that the manifold is flat Minkowski space in the past of a given Cauchy hypersurface.

**Theorem 9.3.2.** Let $(M, g)$ be a globally hyperbolic manifold which is isometric to Minkowski space in the past of a given Cauchy-hypersurface $N$. For given $\gamma$, we consider the family of regularized fermionic projectors $(P^\varepsilon)_{\varepsilon>0}$ such that $P^\varepsilon(x,y)$ coincides with the distribution (9.3.1) if $x$ and $y$ lie in the past of $N$. Then for a generic curve $\gamma$ and
for every sufficiently large \( N \), there is \( \varepsilon_0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and all \( n = 1, \ldots, N \), the points \( x_n \) and \( x_{n-1} \) are spin connectable, and \( x_{n+1} \) lies in the future of \( x_n \) (according to Definition 9.2.11). Moreover,

\[
\lim_{N \to \infty} \lim_{\varepsilon \searrow 0} \nabla^N_{x,y} \nabla^L_{x,y} = \mathcal{O}\left( L(\gamma) \left( \frac{\nabla R}{m^2} \right) \left( 1 + \mathcal{O}\left( \frac{\text{scal}}{m^2} \right) \right) \right),
\]

where \( R \) denotes the Riemann curvature tensor, \( \text{scal} \) is scalar curvature, and \( L(\gamma) \) is the length of the curve \( \gamma \).

Thus the metric connection of Definition 9.2.11 indeed coincides with the Levi-Civita connection, up to higher order curvature corrections. For detailed explanations and the proof we refer to [42, Section 5].

We conclude this section by pointing to a few additional constructions in [42] which cannot be explained consistently in this short survey article. First, there is the subtle point that the unitary transformation \( U \in \exp(i\mathbb{R}s_x) \) which is used to identify two representatives \( K, \tilde{K} \in T_x \) via the relation \( \tilde{K} = UKU^{-1} \) (see Definition 9.2.4) is not unique. More precisely, the operator \( U \) can be transformed according to

\[
U \to -U \quad \text{and} \quad U \to s_x U.
\]

As a consequence, the metric connection (see Definition 9.2.11) is defined only up to the transformation

\[
\nabla_{x,y} u \to s_x(\nabla_{x,y} u) s_x.
\]

Note that this transformation maps representatives of the same tangent vector into each other, so that \( \nabla_{x,y} u \in T_x \) is still a well-defined tangent vector. But we get an ambiguity when composing the metric connection several times (as for example in the expression for the metric curvature in Definition 9.2.12). This ambiguity can be removed by considering parity-preserving systems as introduced in [42, Section 3.4].

At first sight, one might conjecture that Theorem 9.3.2 should also apply to the spin connection in the sense that

\[
D^L_{x,y} = \lim_{N \to \infty} \lim_{\varepsilon \searrow 0} D^N_{x,y},
\]

where \( D^L_{x,y} \) is the spin connection on \( SM \) induced by the Levi-Civita connection and

\[
D^N_{x,y} := D_{x_N,x_{N-1}}D_{x_{N-1},x_{N-2}} \cdots D_{x_1,x_0} : S_y \to S_x \tag{9.3.2}
\]

(and \( D \) is the spin connection of Theorem 9.3.9). It turns out that this conjecture is false. But the conjecture becomes true if we replace (9.3.2) by the operator product

\[
D^N_{(x,y)} := D_{x_N,x_{N-1}}U_{x_{N-1}}^{(x_N|x_{N-2})}D_{x_{N-1},x_{N-2}}U_{x_{N-2}}^{(x_{N-1}|x_{N-3})} \cdots U_{x_1}^{(x_2|x_0)}D_{x_1,x_0}.
\]

Here the intermediate factors \( U^{(\cdot | \cdot)} \) are the so-called splice maps given by

\[
U^{(z|y)} = U_{xz} V U_{xy}^{-1},
\]

where \( U_{xz} \) and \( U_{xy} \) are synchronization maps, and \( V \in \exp(i\mathbb{R}s_x) \) is an operator which identifies the representatives \( K_{xy}, K_{zx} \in T_x \) (for details see [42, Section 3.7 and Section 5]). The splice maps also enter the spin curvature \( \mathcal{R} \), which is defined in analogy to the metric curvature (9.2.13) by

\[
\mathcal{R}(x, y, z) = U^{(z|y)}_{x,y} D_{x,y} U^{(x|z)}_{y,z} D_{y,z} U^{(y|x)}_{z,x} D_{z,x} : S_x \to S_x.
\]

Exercises
Part 3

Mathematical Methods and Rigorous Constructions
CHAPTER 10

Measure-Theoretic Methods

The goal of this section is to prove the existence of minimizers of the causal action principle in the case that \( \mathcal{H} \) is finite dimensional,

\[
\dim \mathcal{H} = f < \infty .
\] (10.0.1)

and that the total volume is finite, \( \rho(\mathcal{F}) < \infty \). By rescaling we can always arrange that the measure is normalized, i.e.

\[
\rho(\mathcal{F}) = 1 .
\] (10.0.2)

Moreover, the spin dimension \( n \in \mathbb{N} \) is kept fixed. The existence proof was first given in [32] in a slightly more general setting.

Our general strategy is to apply the direct method of the calculus of variations, which can be summarized as follows:

(a) Choose a minimizing sequence, i.e. a sequence of measures \( (\rho_k) \) which satisfy the constraint such that

\[
\mathcal{S}[\rho_k] \to \inf_{\rho} \mathcal{S}[\rho] .
\]

Such a minimizing sequence always exists by definition of the infimum (note that the action and therefore also its infimum are non-negative).

(b) Show that a subsequence of the measures converges in a suitable sense,

\[
\rho_k \xrightarrow{??} \rho .
\]

The question marks indicate that we still need to specify in which sense the sequence should converge (convergence in which space, strong or weak convergence, etc.).

(c) Finally, one must show that the action is lower semi-continuous, i.e.

\[
\mathcal{S}[\rho] \leq \liminf_{k \to \infty} \mathcal{S}[\rho_k] .
\]

Also, one must prove that the limit measure \( \rho \) satisfies the constraints.

Once these three steps have been carried out, the measure \( \rho \) is a desired minimizer. We point out that this procedure does not give a unique minimizer, simply because there may be different minimizing sequences, and because the choice of the subsequences may involve an arbitrariness. Indeed, for the causal action principle we do not expect uniqueness. There should be many different minimizers, which describe different physical systems (like the vacuum, a system involving particles and fields, etc.). This intuitive picture is confirmed by the numerical study in [59] which shows that even if the dimension of \( \mathcal{H} \) is small (equal to two) there are indeed many different minimizers.

We first illustrate the constraints by a few examples (Section 10.1). In preparation of the existence proof, we need to learn the measure-theoretic tools needed for the above program (Section 10.2–10.5). Applying these tools is not quite straightforward, but works out in the end (Sections 10.6–10.7).
10. Examples Illustrating the Constraints

We now give a few examples which show why the constraints (4.4.3)–(4.4.5) are needed. These examples also illustrate the difficulties which we will encounter in the existence proof. Generally speaking, the constraints are needed in order to avoid trivial minimizers and in order for the variational principle to be well-posed. For the volume constraint, this is quite obvious: If we dropped the constraint of fixed total volume (4.4.3), the measure $\rho = 0$ would be a trivial minimizer. The trace constraint is already less obvious. Its role is explained in the next two examples.

**Example 10.1.1. (necessity of the trace constraint)** Let $x$ be the operator with the matrix representation

\[
x = \text{diag}(1, \ldots, 1, -1, \ldots, -1, 0, \ldots)
\]

(10.1.1)

Moreover, we choose $\rho$ as a multiple of the Dirac measure supported at $x$. Then $T > 0$ but $S = 0$. ♦

**Example 10.1.2. (nontriviality of the action with trace constraint)** Now suppose that let $\rho$ is a normalized measure which satisfies the trace constraint in a non-trivial way, i.e.

\[
\int T(x) \, d\rho(x) = \text{const} \neq 0.
\]

Let us prove that the action is non-zero. This will show that the trace constraint really avoids trivial minimizers of the causal action principle.

(a) Since the integral over the trace is non-zero, there is a point $x$ in the support of $\rho$ with $\text{tr}(x) \neq 0$. We denote the non-trivial eigenvalues of $x$ by $\nu_1, \ldots, \nu_{2n}$ and order them according to

\[
\nu_1 \leq \cdots \leq \nu_n \leq 0 \leq \nu_{n+1} \leq \cdots \leq \nu_{2n}.
\]

(10.1.2)

The fact that the trace of $x$ is non-zero clearly implies that the $\nu_i$ do not all have the same absolute value. As a consequence, the nontrivial eigenvalues of the operator product $x^2$ given by $\lambda_{j}^{xx} = \nu_j^2$ are all non-negative and not all equal. According to (4.4.7), this implies that $L(x, x) > 0$.

(b) Since the Lagrangian is continuous in both arguments, there is an open neighborhood $U \subset \mathcal{F}$ of $x$ such that $L(y, z) > 0$ for all $y, z \in U$. Since $x$ is in the support of $\rho$, we know that $\rho(U) > 0$. As a consequence,

\[
S \geq \int_U d\rho(x) \int_U d\rho(y) \, L(x, y) > 0
\]

(because if the integrals vanished, then the integrand would have to be zero almost everywhere, a contradiction).

We remark that this argument is quantified in [30, Proposition 4.3]. ♦

We now come to the boundedness constraint. In order to explain how it comes about, we give an explicit example with $(4 \times 4)$-matrices (for a similar example with $(2 \times 2)$-matrices see Exercise 4.4).

**Example 10.1.3. (necessity of the boundedness constraint)** The following example explains why the boundedness constraint (4.4.5) is needed to ensure the existence
10.1. EXAMPLES ILLUSTRATING THE CONSTRAINTS

of minimizers. It was first given in [32, Example 2.9]. Let $\mathcal{H} = \mathbb{C}^4$. We introduce the four $4 \times 4$-matrices acting on $\mathcal{H}$ by

$$
\gamma^\alpha = \begin{pmatrix} \sigma^\alpha & 0 \\ 0 & -\sigma^\alpha \end{pmatrix}, \quad \alpha = 1, 2, 3 \quad \text{and} \quad \gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

(where the $\sigma^\alpha$ are again the Pauli matrices (??)). For a given parameter $\tau > 1$ consider the following mapping from the sphere $S^3 \subset \mathbb{R}^4$ to the linear operators on $\mathcal{H}$,

$$
F : S^3 \to \mathbb{L}(\mathcal{H}), \quad F(x) = \sum_{i=1}^{4} \tau x^i \gamma^i + 1.
$$

(a) *The matrices $F(x)$ have two positive and two negative eigenvalues:*
Since the computation of the eigenvalues of $4 \times 4$-matrices is tedious, it is preferable to proceed as follows. The matrices $\gamma^j$ are the Dirac matrices of Euclidean $\mathbb{R}^4$, satisfying the anti-commutation relations

$$
\{\gamma^i, \gamma^j\} = 2 \delta^{ij} 1 \quad (i, j = 1, \ldots, 4).
$$

As a consequence,

$$
F(x) - 1 = \sum_{i=1}^{4} \tau x^i \gamma^i
$$

$$
(F(x) - 1)^2 = \sum_{i,j=1}^{4} \tau^2 x^i x^j \gamma^i \gamma^j = \tau^2 \sum_{i,j=1}^{4} x^i x^j \{\gamma^i, \gamma^j\}
$$

$$
= \tau^2 \sum_{i,j=1}^{4} x^i x^j 2 \delta_{ij} 1 = \tau^2 1.
$$

Hence the matrix $F(x)$ satisfies the polynomial equation

$$
(F(x) - 1)^2 = \tau^2 1.
$$

We conclude that $F(x)$ has the eigenvalues

$$
\nu_{\pm} = 1 \pm \tau.
$$

Since $F(x) - 1$ is trace-free, each eigenvalue must appear with multiplicity two. Using that $\tau > 1$, we conclude that $F(x)$ really has two positive and two negative eigenvalues.

(b) *Construction of a causal fermion system:*
Let $\mu$ be the normalized Lebesgue measure on $S^3 \subset \mathbb{R}^4$. Setting $\rho = F_\ast \mu$ defines a causal fermion system of spin dimension two and total volume one. Since the matrices $F(x)$ all have trace four, we also know that

$$
\int_{S^3} \text{tr}(x) \, d\rho(x) = \int_{S^3} \text{tr}(F(x)) \, d\mu(x) = 4.
$$

Therefore, the volume constraint (4.4.3) and the trace constraint (4.4.4) are satisfied, both with constants independent of $\tau$. 

(c) **Computation of the eigenvalues of** $F(x) F(y)$:

Again, this can be computed most conveniently using the Clifford relations.

\[
F(x) F(y) = \left( \sum_{i=1}^{4} \tau x^i \gamma^i + 1 \right) \left( \sum_{j=1}^{4} \tau y^j \gamma^j + 1 \right)
\]

\[
= (1 + \tau^2 \langle x, y \rangle) 1 + \tau \sum_{i=1}^{4} (x^i + y^i) \gamma^i + \frac{\tau^2}{2} \sum_{i,j=1}^{4} x^i y^j [\gamma^i, \gamma^j] . \tag{10.1.3}
\]

Using that

\[
\gamma^i [\gamma^i, \gamma^j] = - [\gamma^i, \gamma^j] \gamma^i ,
\]

we conclude that

\[
(F(x) F(y) - (1 + \tau^2 \langle x, y \rangle) 1)^2 = \tau^2 \sum_{i=1}^{4} (x^i + y^i)^2 + \left( \frac{\tau^2}{2} \sum_{i,j=1}^{4} x^i y^j [\gamma^i, \gamma^j] \right)^2 .
\]

This can be simplified with the relations

\[
\sum_{i=1}^{4} (x^i + y^i)^2 = 2 + 2 \langle x, y \rangle
\]

\[
\left( \sum_{i,j=1}^{4} x^i y^j [\gamma^i, \gamma^j] \right)^2 = -4 \sin^2 \vartheta = -4 \left( 1 - \langle x, y \rangle^2 \right) ,
\]

where $\vartheta$ is the angle between the vectors $x, y \in \mathbb{R}^4$. Putting all this together, we find that the product $F(x) F(y)$ satisfies the polynomial equation

\[
\left( F(x) F(y) - (1 + \tau^2 \langle x, y \rangle) 1 \right)^2 = 2 \tau^2 (1 + \langle x, y \rangle) - \tau^4 (1 - \langle x, y \rangle)^2
\]

\[
= \tau^2 \left( 1 + \langle x, y \rangle \right) \left( 2 - \tau^2 (1 - \langle x, y \rangle) \right) .
\]

Taking the square root, the zeros of this polynomial are computed by

\[
\lambda_{1,2} = 1 + \tau^2 \langle x, y \rangle \pm \tau \sqrt{1 + \langle x, y \rangle} \sqrt{2 - \tau^2 (1 - \langle x, y \rangle)} . \tag{10.1.4}
\]

Moreover, taking the trace of \((10.1.3)\), one finds

\[
\text{tr} \left( F(x) F(y) \right) = 4 \left( 1 + \tau^2 \langle x, y \rangle \right) .
\]

This implies that each eigenvalue in \((10.1.3)\) has algebraic multiplicity two.

(d) **Computation of the Lagrangian:**

Again denoting the angle between the vectors $x, y \in \mathbb{R}^4$ by $\vartheta$. Then if $\vartheta$ is sufficiently small, then the term $(1 - \langle x, y \rangle)$ is close to zero, and thus the arguments of the square roots are all positive. However, if $\vartheta$ becomes so large that

\[
\vartheta \geq \vartheta_{\text{max}} := \arccos \left( 1 - \frac{2}{\tau^2} \right) ,
\]

then the argument of the last square root in \((10.1.4)\) becomes negative, so that the $\lambda_{1,2}$ form a complex conjugate pair. Moreover, a short calculation shows that

\[
\lambda_1 \lambda_2 = (1 + \tau)^2 (1 - \tau)^2 > 0 ,
\]
implying that if the \( \lambda_{1/2} \) are both real, then they have the same sign. Using this information in (4.4.7), the Lagrangian simplifies to

\[
\mathcal{L}(F(x), F(y)) = \frac{1}{8} \sum_{i,j=1}^{4} \left( |\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 = \frac{1}{2} \sum_{i,j=1}^{2} \left( |\lambda_i| - |\lambda_j| \right)^2
\]

\[
= \frac{1}{2} \Theta(\vartheta_{\text{max}} - \vartheta) \sum_{i,j=1}^{2} (\lambda_i - \lambda_j)^2 = \Theta(\vartheta_{\text{max}} - \vartheta)(\lambda_1 - \lambda_2)^2
\]

\[
= 4\tau^2 (1 + \cos \vartheta) \left( 2 - \tau^2 (1 - \cos \vartheta) \right) \Theta(\vartheta_{\text{max}} - \vartheta) .
\]

(e) Computation the action:
Inserting this Lagrangian in (4.4.2) and using the definition of the push-forward measure, we obtain

\[
S = \int_{S^3} d\mu(x) \int_{S^3} d\mu(y) \mathcal{L}(F(x), F(y))
\]

\[
= \int_{S^3} d\mu(y) \mathcal{L}(F(x), F(y)) = \frac{2}{\pi} \int_{0}^{\vartheta_{\text{max}}} \mathcal{L}(\cos \vartheta) \sin^2 \vartheta d\vartheta
\]

\[
= \frac{512}{15\pi \tau} + O(\tau^{-2}) .
\]

Thus setting \( F_k = F|_{\tau=k} \), we have constructed a divergent minimizing sequence. However, the integral in the boundedness constraint (4.4.5) also diverges as \( k \to \infty \). This example shows that leaving out the boundedness constraint, there is no minimizer. \( \Diamond \)

We finally remark that this example is not as artificial or academic as it might appear at first sight. Indeed, as found out by Niki Kilbertus in his master thesis [77], when discretizing a Dirac system in \( \mathbb{R} \times S^3 \) (the latter can be thought of as a “spatial compactification” of Minkowski space), then in the simplest case of four occupied Dirac states (referred to as “one shell,” i.e dim \( \mathcal{H} = 4 \), this system reduces precisely to the last example. To my opinion, this is very good news, because it shows that “Clifford structures tend to make the causal action small.”

### 10.2. The Banach-Alaoglu Theorem

For our purposes, it suffices to consider the case that the Banach space is separable, which is due to Banach (Alaoglu proved the generalization to non-separable Banach spaces; this makes use of Tychonoff’s theorem and goes beyond what we need here). Indeed, the idea of proof of the theorem can be traced back to Eduard Helly’s doctoral thesis in 1912, where the closely related “Helly’s selection theorem” is proved (of course without reference to Banach spaces, which were introduced later). We closely follow the presentation in [81] Section 10.3.

Let \((E, \|\cdot\|_E)\) be a separable (real or complex) Banach space and \((E^*, \|\cdot\|_{E^*})\) its dual with the usual sup-norm, i.e.

\[
\|\phi\|_{E^*} = \sup_{u \in E, \|u\|=1} |\phi(u)| .
\]

A sequence \((\phi_n)_{n \in \mathbb{N}}\) in \(E^*\) is said to be weak*-convergent to \(\phi \in E^*\) if

\[
\lim_{n \to \infty} \phi_n(u) = \phi(u) \quad \text{for all } u \in E .
\]

F: Könnte man eventuell in die "mathematische preliminaries" integrieren?
Theorem 10.2.1. (Banach-Alaoglu in the separable case) Let \( E \) be a separable Banach space. Then every bounded sequence in \( E^* \) has a weak*-convergent subsequence.

Proof. Let \( \phi_n \) be a bounded sequence in \( E^* \), i.e. there is a constant \( c > 0 \) with
\[
\|\phi_n\|_{E^*} \leq c \quad \text{for all } n \in \mathbb{N}.
\] (10.2.2)
We let \((u_\ell)_{\ell \in \mathbb{N}}\) be a sequence in \( E \) which is dense in \( E \). Combining (10.2.2) with (10.2.1), the estimate
\[
|\phi_n(u_1)| \leq \|\phi_n\|_{E^*} \|u_1\|_E \leq c \|u_1\|_E
\] (10.2.3)
shows that \((\phi_n(u_1))_{n \in \mathbb{N}}\) is a bounded sequence. Thus we can choose a convergent subsequence. By inductively choosing subsequences and taking the diagonal sequence, we obtain a subsequence \((\phi_{n_j})\) such that the limit \(\lim_{j \to \infty} \phi_{n_j}(u_\ell)\) exists for all \(\ell \in \mathbb{N}\). Hence setting
\[
\phi(u_\ell) := \lim_{j \to \infty} \phi_{n_j}(u_\ell),
\]
we obtain a densely defined functional. Taking the limit in (10.2.3) (and the similar inequalities for \(u_2, u_3, \ldots\)), one sees that this functional is again continuous. Therefore, it has a unique continuous extension to \( E \). By continuity, the resulting functional \(\phi \in E^*\) satisfies the relations
\[
\phi(u) = \lim_{j \to \infty} \phi_{n_j}(u) \quad \text{for all } u \in \mathcal{H}.
\]
This concludes the proof. \( \square \)

10.3. The Riesz Representation Theorem

In this and the next section we provide the necessary background in measure theory. For basics on measure theory we recommend the textbooks [90, 71, 12]. The book [24] is also a good reference. However, it only considers measures in \( \mathbb{R}^n \), but otherwise goes far beyond what we need here.

For our purposes, we can always assume that the base space \( \mathcal{K} \) is a compact topological space. Recall that the \( \sigma \)-algebra generated by all open sets of \( \mathcal{K} \) is referred to as the Borel algebra. A measure on the Borel algebra is called Borel measure. A Borel measure \( \mu \) is called regular if it can be approximated from the outside by open and from the inside by compact sets, i.e. if for every Borel set \( B \),
\[
\sup \{ \mu(K) \mid K \subset B \text{ compact} \} = \mu(B) = \inf \{ \mu(\Omega) \mid \Omega \supset B \text{ open} \}.
\]
In order to avoid confusion, we remark that by a measure we always mean a positive measure (signed measures will not at all be considered in this lecture). A bounded measure, also referred to as a measure of finite total volume, is a measure \( \mu \) with \( \mu(\mathcal{K}) < \infty \). Often, we normalize the measure such that \( \mu(\mathcal{K}) = 1 \). We remark that a Radon measure is a regular Borel measure with the property that \( \mu(C) \) is finite for every compact \( C \). Clearly, if the base space is compact, a Radon measure is the same as a bounded regular Borel measure.

In words, the Riesz representation theorem makes it possible to represent a linear functionals on the Banach space of continuous functions of a topological space by a regular Borel measures on this topological space. We remark that we already came across the Riesz representation theorem in the functional analysis course, where it was used for the construction of spectral measures. However, back then we only proved it in the special case that the topological space was an interval of the real line. We now state the general theorem and outline its proof, mainly following the presentation in [71 §56].
As a simple example, one can choose \( K \) as the closed unit ball in \( \mathbb{R}^n \). Restricting the Lebesgue measure to the Borel subsets of \( K \) gives a Radon measure. The Lebesgue measure itself is a completion of this Radon measure obtained by extending the \( \sigma \)-algebra of measurable sets by all subsets of Borel sets of measure zero. Since this completion is a rather trivial extension, in what follows we prefer to work with Radon measures or normalized regular Borel measures.

**Theorem 10.3.1. (Riesz representation theorem)** Let \( K \) be a compact topological space, and \( E = C^0(K, \mathbb{R}) \) the Banach space of continuous functions on \( K \) with the usual \( \sup \)-norm,

\[
\| f \| = \sup_{x \in K} |f(x)|.
\]

Let \( \Lambda \in E^* \) be a continuous linear function which is positive in the sense that \( \Lambda(f) \geq 0 \) for all nonnegative function \( f \in C^0(K, \mathbb{R}) \).

Then there is a unique regular Borel measure \( \mu \) such that

\[
\Lambda(f) = \int_K f \, d\mu \quad \text{for all } f \in C^0(K, \mathbb{R}).
\]

**Outline of the Proof.** We follow the strategy in [71, §56]. Given a Borel set \( A \subset K \), we set

\[
\lambda(A) = \inf \{ \Lambda(f) \mid f \in C^0(K, \mathbb{R}) \text{ and } f \geq \chi_A \} \in \mathbb{R}_0^+.
\]

Intuitively speaking, \( \lambda \) gives us the desired “volume” of the set \( A \). But there is the technical problem that \( \lambda \) is in general not a regular Borel measure. Instead, it merely is a content, meaning that it has the following properties:

(i) non-negative and finite: \( 0 \leq \lambda(A) < \infty \)

(ii) monotone: \( C, D \) compact and \( C \subset D \implies \lambda(C) \leq \lambda(D) \)

(iii) additivity: \( C, D \) compact and disjoint \( \implies \lambda(C \cup D) = \lambda(C) + \lambda(D) \)

(iv) subadditivity: \( C, D \) compact \( \implies \lambda(C \cup D) \leq \lambda(C) + \lambda(D) \)

At this stage, we are in a similar situation as in the elementary measure theory course after saying that a cube of length \( \ell \) in \( \mathbb{R}^3 \) should have volume \( \ell^3 \). In order to get from this “volume measure” to a measure in the mathematical sense, one has to proceed in several steps invoking the subtle and clever constructions of measure theory (due to Lebesgue, Hahn, Carathéodory, Borel, Riesz and others) in order to get a mapping from a \( \sigma \)-algebra to the non-negative real numbers which is \( \sigma \)-additive. In simple terms, repeating these constructions starting from the above content gives the desired Borel measure \( \mu \). For brevity, we here merely outline the constructions and refer for details to textbooks on measure theory (like for example [71, Chapter X]).

The first step is to approximate (or exhaust) from inside by compact sets. Thus one introduces the inner content \( \lambda_* \) by

\[
\lambda_*(U) = \sup \{ \lambda(C) \mid C \subset U \text{ compact} \}.
\]

This inner content is countably subadditive and countably additive. The second step is to exhaust from outside by open sets. This gives the outer measure \( \mu^* \),

\[
\mu^*(U) = \inf \{ \lambda_*(\Omega) \mid \Omega \supset U \text{ open} \}.
\]

The outer measure is defined for any subset of \( K \). Therefore, it remains to distinguish the measurable sets. This is accomplished by Carathéodory’s criterion, which defines a
set $A \subset \mathcal{K}$ to be *measurable* if
\[ \mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B) \] (10.3.1)
for every subset $B \subset \mathcal{K}$. Then Carathéodory’s lemma (for a concise proof see for example [15, Lemma 2.8]) implies that the measurable sets form a $\sigma$-algebra, and that the restriction of $\mu^*$ to the measurable sets is indeed a measure, denoted by $\mu$.

In order to complete the proof, one still needs to verify that every Borel set is $\mu$-measurable. Moreover, it remains to show that the resulting Borel measure is regular. To this end, one first needs to show that the content $\lambda$ is regular in the following sense:

(iv) regular: For every compact $C$,
\[ \lambda(C) = \inf \{ \lambda(D) \mid D \text{ compact and } C \subset \overset{\circ}{D} \} . \]

As the proofs of these remaining points are quite straightforward and not very instructive, we refer for the details to [71, §54–§56]. □

10.4. The Radon-Nikodym Theorem

As explained at the beginning of Section 10.3, we again restrict attention to the case that the base space $\mathcal{K}$ is a compact topological space. We give the proof of the Radon-Nikodym theorem by von Neumann following the presentation in [90, Chapter 6]. An alternative method of proof is given in [71, 24].

**Definition 10.4.1.** A Radon measure $\lambda$ is **absolutely continuous** with respect to another Radon measure $\nu$, denoted by
\[ \lambda \ll \nu , \]
if the implication
\[ \nu(E) = 0 \implies \lambda(E) = 0 \]
holds for any Borel set $E$. The measure $\lambda$ is **concentrated** on the Borel set $A$ if $\lambda(E) = \lambda(E \cap A)$ for all Borel sets $E$. The measures $\lambda$ and $\mu$ are **mutually singular**, denoted by
\[ \lambda \perp \mu , \]
if there are disjoint Borel sets $A$ and $B$ such that $\lambda$ is concentrated in $A$ and $\nu$ is concentrated in $B$.

In order to avoid confusion, we point out that the supports of two mutually singular measures are not necessarily disjoint, as one sees in the simple example of the Lebesgue measure on $(0,1)$ and the Dirac measure supported at the origin,
\[ \lambda := dx|_{(0,1)} \quad \text{and} \quad \mu = \delta_0 . \]
Since the support is by definition a closed set (see (2.1.1)), the support of $dx|_{(0,1)}$ contains the origin, which is precisely the support of the Dirac measure. But clearly, the two measures are concentrated on the sets $(0,1)$ and $\{0\}$, respectively, and are thus mutually singular.

**Theorem 10.4.2.** (Radon-Nikodym) Let $\mu$ and $\lambda$ be Radon measures on the compact topological space $\mathcal{K}$.

(a) There is a unique pair of Borel measures $\lambda_a$ and $\lambda_s$ such that
\[ \lambda = \lambda_a + \lambda_s \quad \text{and} \quad \lambda_a \ll \mu , \quad \lambda_s \perp \mu . \]
(b) There is a unique function \( h \in L^1(K, d\mu) \) such that
\[
\lambda_a(E) = \int_E h \, d\mu \quad \text{for every Borel set } E. \tag{10.4.1}
\]
The pair \((\lambda_a, \lambda_s)\) is also referred to as the **Lebesgue decomposition** of \( \lambda \) with respect to \( \mu \).

**Proof of Theorem 10.4.2.** The uniqueness of the decomposition is easily seen as follows: Suppose that \((\lambda_a', \lambda_s')\) is another Lebesgue decomposition. Then
\[
\lambda_a' - \lambda_a = \lambda_s' - \lambda_s. \tag{10.4.2}
\]
Since \( \lambda_s \perp \mu \) and \( \lambda_s' \perp \mu \), the measures \( \lambda_s \) and \( \lambda_s' \) are concentrated in a Borel set \( A \) with \( \mu(A) = 0 \). Evaluating (10.4.2) on Borel subsets of \( A \), the left side vanishes, because \( \lambda_a \) and \( \lambda_a' \) are both absolutely continuous with respect to \( \mu \). Hence \( \lambda_s' - \lambda_s = 0 \). Using this relation in (10.4.2), we also conclude that
\[
\lambda_a' - \lambda_a = 0. \tag{10.4.3}
\]
This proves uniqueness.

For the existence proof, we let \( \rho \) be the measure \( \rho = \lambda + \mu \). Then
\[
\hat{K} f \, d\rho = \hat{K} f \, d\lambda + \hat{K} f \, d\mu \tag{10.4.4}
\]
for any non-negative Borel function \( f \). If \( f \in L^2(K, d\rho) \), the Schwarz inequality gives
\[
\left| \int_K f \, d\lambda \right| \leq \int_K |f| \, d\rho \leq \sqrt{\rho(K)} \|f\|_{L^2(K,d\rho)}. \tag{10.4.5}
\]
Therefore, the mapping \( f \mapsto \int_K f \, d\lambda \) is a bounded linear functional on \( L^2(K, d\rho) \). By the Fréchet-Riesz theorem, we can represent this linear functional by a function \( g \in L^2(K, d\rho) \), i.e.
\[
\int_K f \, d\lambda = \int_K g f \, d\rho \quad \text{for all } f \in L^2(K, d\rho). \tag{10.4.6}
\]
We next want to show that, by modifying \( g \) on a set of \( \rho \)-measure zero, we can arrange that \( g \) takes values in the interval \([0,1]\). To this end, we let \( E \) be any Borel set with \( \rho(E) > 0 \). Evaluating (10.4.6) for \( f = \chi_E \), we obtain
\[
0 \leq \frac{1}{\rho(E)} \int_E g \, d\rho = \frac{\lambda(E)}{\rho(E)} \leq 1.
\]
Now the claim follows from elementary measure theory (see for example [90, Theorem 1.40]).

Using (10.4.3), we can rewrite (10.4.4) as
\[
\int_K (1 - g) f \, d\lambda = \int_K g f \, d\mu \quad \text{for all non-negative } f \in L^2(K, d\rho). \tag{10.4.7}
\]
We introduce the Borel sets
\[
A = \{ x \in K \mid 0 \leq g(x) < 1 \} \quad \text{and} \quad B = \{ x \in K \mid g(x) = 1 \}.
\]
and define the measures \( \lambda_a \) and \( \lambda_s \) by
\[
d\lambda_a = \chi_A \, d\lambda \quad \text{and} \quad d\lambda_s = \chi_B \, d\lambda.
\]
Choosing \( f = \chi_B \) in (10.4.7), one sees that \( \mu(B) = 0 \), implying that \( \lambda_s \perp \mu \).

Moreover, since \( g \) is bounded, we can evaluate (10.4.7) for
\[
f = (1 + g + \cdots + g^n) \chi_E
\]
for any \( n \in \mathbb{N} \) and any Borel set \( E \). Using the same transformation with “telescopic sums” as in the evaluation of the geometric or Neumann series, we obtain

\[
\int_E (1 - g^{n+1}) \, d\lambda = \int_E g \left(1 + g + \cdots + g^n\right) \, d\mu .
\] (10.4.6)

At every point of \( B \), the factor \((1 - g^{n+1})\) in the integrand on the left vanishes. At every point of \( A \), on the other hand, the factor \((1 - g^{n+1})\) is monotone increasing in \( n \) and converges to one. Hence Lebesgue’s monotone convergence theorem implies that the left side of (10.4.6) converges to

\[
\lim_{n \to \infty} \int_E (1 - g^{n+1}) \, d\lambda = \lambda(E \cap A) .
\]

The integrand on the right side of (10.4.6), on the other hand, is monotone increasing in \( n \), so that the limit

\[
h(x) := \lim_{n \to \infty} g(x) \left(1 + g(x) + \cdots + g^n(x)\right) \quad \text{exists in } \mathbb{R}_0^+ \cup \{\infty\} .
\]

Moreover, the monotone convergence theorem implies that

\[
\lim_{n \to \infty} \int_E g \left(1 + g + \cdots + g^n\right) \, d\mu = \int_E h \, d\mu \in \mathbb{R}_0^+ \cup \{\infty\} .
\]

We conclude that, in the limit \( n \to \infty \), the relation (10.4.6) yields

\[
\lambda_a(E) = \lambda(E \cap A) = \int_E h \, d\mu \quad \text{for any Borel set } E.
\]

Choosing \( E = \mathcal{K} \), one sees that \( h \in L^1(\mathcal{K}, d\mu) \). This concludes the proof of (10.4.1). Finally, the representation (10.4.1) implies that \( \lambda_a \ll \mu \). □

### 10.5. Weak Compactness for Measures on Compact Metric Spaces

We shall apply the Banach-Alaoglu theorem for a specific Banach space, namely the continuous functions on a compact metric space. We first verify that this Banach space is indeed separable. Then we shall prove a compactness result for Radon measures on this space.

**Proposition 10.5.1.** Let \( \mathcal{K} \) be a compact metric space. Then the Banach space \( C^0(\mathcal{K}, \mathbb{R}) \) is separable.

**Proof.** The proposition is a consequence of the Stone-Weierstrass theorem, whose proof can be found for example in [20] 7.3.1. We closely follow the proof given in [20] 7.4.4.

Covering \( \mathcal{K} \) by a finite number of open balls of radii 1, 1/2, 1/3, \ldots, one gets an enumerable basis of the open sets \((U_n)_{n \in \mathbb{N}}\). For any \( n \in \mathbb{N} \), we let \( g_n \) be the continuous function

\[
g_n(x) = d(x, \mathcal{K} \setminus U_n) .
\]

Clearly, the algebra generated by these functions (by taking finite products and finite linear combinations) is again separable. Therefore, it suffices to show that this algebra is dense in \( C^0(\mathcal{K}, \mathbb{R}) \). To this end, we need to verify the assumptions of the Stone-Weierstrass theorem. The only assumption which is not obvious is that the algebra separates the points. This can be seen as follows: Let \( x \) and \( y \) be two distinct points in \( \mathcal{K} \). Since the \((U_n)\) are a basis of the topology, there is \( U_n \) with \( x \in U_n \) and \( y \notin U_n \). As a consequence, \( g_n(x) > 0 \) but \( g_n(y) = 0 \). □
Theorem 10.5.2. Let \( \rho_n \) be a series of regular Borel measures on \( C^0(\mathcal{K}, \mathbb{R}) \) which are bounded in the sense that there is a constant \( c > 0 \) with
\[
\rho_n(\mathcal{K}) \leq c \quad \text{for all } n.
\]
Then there is a subsequence \( (\rho_{n_k}) \) which converges as a measure, i.e.
\[
\lim_{k \to \infty} \int_{\mathcal{K}} f \, d\rho_{n_k} = \int_{\mathcal{K}} f \, d\rho \quad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R}).
\]
Moreover, the total volume converges, i.e.
\[
\rho(\mathcal{K}) = \lim_{k \to \infty} \rho_{n_k}(\mathcal{K}). \tag{10.5.1}
\]

Proof. Via
\[
\phi_n(f) := \int_{\mathcal{K}} f \, d\rho_n,
\]
every measure can be identified with a positive linear functional on \( E := C^0(\mathcal{K}, \mathbb{R}) \). Since \( E \) is separable (Proposition [10.5.1]), we can apply the Banach-Alaoglu theorem in the separable case (Theorem [10.2.1]) to conclude that there is a weak*-convergent subsequence, i.e.
\[
\lim_{k \to \infty} \phi_{n_k}(f) = \phi(f) \quad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R}). \tag{10.5.2}
\]
Clearly, since all \( \phi_{n_k} \) are positive, the same is true for the limit \( \phi \). Therefore, the Riesz representation theorem (Theorem [10.3.1]) makes it possible to represent \( \phi \) by a regular Borel measure \( \rho \), i.e.
\[
\phi(f) = \int_{\mathcal{K}} f \, d\rho \quad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R}). \tag{10.5.3}
\]
Choosing \( f \) as the constant function, one obtains (10.5.1). This concludes the proof. □

10.6. Moment Measures

We again assume that the Hilbert space is finite-dimensional (10.0.1) and the measure \( \rho \) is normalized (10.0.2). We consider \( \mathcal{F} \) with the metric induced by the sup-norm on \( L(\mathcal{F}) \), i.e.
\[
d(p, q) = \|p - q\|
\]
(and \( \| \cdot \| \) as in (10.4.6)). The basic difficulty in applying the abstract theorems is that \( \mathcal{F} \) is not compact (indeed, it is a star-shaped in the sense that \( p \in \mathcal{F} \) implies \( \lambda p \in \mathcal{F} \) for all \( \lambda \in \mathbb{R} \)). If the metric space is non-compact, the abstract results no longer apply, as is obvious in the following simple example.

Example 10.6.1. Consider the Banach space \( C^0_0(\mathbb{R}, \mathbb{R}) \) of compactly supported continuous functions (this is indeed the correct Banach space to consider, because the space \( C^0(\mathbb{R}, \mathbb{R}) \) of all bounded continuous functions is not separable and is “too large” for the applications). We then consider the sequence of Dirac measures \( \rho_n = \delta_n \) supported at \( n \in \mathbb{N} \). Then for any \( f \in C^0_0(\mathbb{R}, \mathbb{R}) \),
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f \, d\rho_n = \lim_{n \to \infty} f(n) = 0.
\]
Hence the sequence \( (\rho_n)_{n \in \mathbb{N}} \) converges as a measure to zero. Thus the limiting measure is no longer normalized. This shows that Theorem 10.5.2 fails to hold if the base space is non-compact. □
The way out is to make use of the fact that the causal action as well as the constraints are formed of functionals which are homogeneous under the scaling \( p \to \lambda p \) of degree zero, one or two. This makes it possible to restrict attention to a compact subset of \( \mathcal{F} \), and to consider three measures on this compact set.

**Definition 10.6.2.** Let \( \mathcal{K} \) be the compact metric space
\[
\mathcal{K} = \{ p \in \mathcal{F} \text{ with } \| p \| = 1 \} \cup \{ 0 \}.
\]
For a given measure \( \rho \) on \( \mathcal{F} \), we define the measurable sets of \( \mathcal{K} \) by the requirement that the sets \( R^+ \Omega = \{ \lambda p \mid \lambda \in \mathbb{R}^+, p \in \Omega \} \) and \( R^- \Omega \) should be \( \rho \)-measurable in \( \mathcal{F} \). We introduce the measures \( m^{(0)}, m^{(1)}_+, m^{(1)}_- \) by
\[
\begin{align*}
m^{(0)}(\Omega) &= \frac{1}{2} \rho(R^+ \Omega \setminus \{0\}) + \frac{1}{2} \rho(R^- \Omega \setminus \{0\}) + \rho(\Omega \cap \{0\}) & (10.6.1) \\
m^{(1)}_+(\Omega) &= \frac{1}{2} \int_{R^+ \Omega} \| p \| \, d\rho(p) & (10.6.2) \\
m^{(1)}_-(\Omega) &= \frac{1}{2} \int_{R^- \Omega} \| p \| \, d\rho(p) & (10.6.3) \\
m^{(2)}(\Omega) &= \frac{1}{2} \int_{R^+ \Omega} \| p \|^2 \, d\rho(p) + \frac{1}{2} \int_{R^- \Omega} \| p \|^2 \, d\rho(p). & (10.6.4)
\end{align*}
\]
The measures \( m^{(l)}_+ \) and \( m^{(l)}_- \) are referred to as the \( l \)th moment measure.

We remark that, as a short notation, it is convenient to abbreviate the difference of the first moment measures by
\[
m^{(1)}(\Omega) := m^{(1)}_+(\Omega) - m^{(1)}_-(\Omega). \quad (10.6.5)
\]
In mathematical terms, \( m^{(1)} \) can be regarded as a signed measure (see for example [71, §28] or [90, Chapter 6]). For simplicity, we here avoid the concept of signed measures by working instead with the measures \( m^{(1)}_\pm \). Nevertheless, we introduce an \( m^{(1)} \)-integral as a short notation for the difference of the integrals with respect to \( m^{(1)}_+ \) and \( m^{(1)}_- \), i.e.
\[
\int_{\mathcal{K}} h \, d m^{(1)} := \int_{\mathcal{K}} h \, d m^{(1)}_+ - \int_{\mathcal{K}} h \, d m^{(1)}_-,
\]
where we always assume for simplicity that \( h \) is continuous.

The \( \rho \)-integrals of homogeneous functions can be rewritten as integrals over \( \mathcal{K} \) using the moment measures, as we now make precise.

**Definition 10.6.3.** A function \( h \in C^0(\mathcal{F}) \) is called homogeneous of degree \( \ell \) with \( \ell \in \{0, 1, 2\} \) if
\[
h(\nu x) = \nu^\ell h(x) \quad \text{for all } \nu \in \mathbb{R} \text{ and } x \in \mathcal{F}. \quad (10.6.6)
\]

**Lemma 10.6.4.** Let \( h \in C^0(\mathcal{F}) \) be a function which is homogeneous of degree \( \ell \) with \( \ell \in \{0, 1, 2\} \). Then
\[
\int_{\mathcal{F}} h \, d\rho = \int_{\mathcal{K}} h \, d m^{(l)}.
\]

**Proof.** We first note that, using the homogeneity \([10.6.6]\), the function \( h \) is uniquely determined by its restriction to \( \mathcal{K} \). Moreover, using an approximation theorem with Lebesgue’s dominated convergence theorem, it suffices to consider a function \( h \) which is
homogeneous of degree $\ell$ and simple in the sense that their restriction to $\mathcal{K}$ takes a finite number of values, i.e.

$$h|_{\mathcal{K}} = \sum_{i=1}^{N} c_i \chi_{\Omega_i}$$

with Borel sets $\Omega_1, \ldots, \Omega_N \subset \mathcal{K}$ and real coefficients $c_1, \ldots, c_N$. For such functions,

$$\int_{\mathcal{F}} h \, d\rho = \sum_{i=1}^{N} c_i \int_{\mathcal{F} + \Omega_i} ||p||^\ell \, d\rho(p) = \sum_{i=1}^{N} c_i \, m^\ell(\Omega_i) = \int_{\mathcal{K}} h \, dm^{(\ell)}.$$ 

This concludes the proof. □

Applying this lemma, the normalization $\rho(\mathcal{F}) = 1$ can be expressed in terms of the moment measures as

$$m^{(0)}(\mathcal{K}) = 1,$$  \hspace{1cm} (10.6.7)

whereas the action (4.4.2) as well as the functionals in the constraints (4.4.5) and (4.4.4) can be written as

$$S(\rho) = \int_{\mathcal{K} \times \mathcal{K}} \mathcal{L}(p, q) \, dm^{(2)}(p) \, dm^{(2)}(q)$$  \hspace{1cm} (10.6.8)

$$T(\rho) = \int_{\mathcal{K} \times \mathcal{K}} |pq|^2 \, dm^{(2)}(p) \, dm^{(2)}(q)$$  \hspace{1cm} (10.6.9)

$$\int_{\mathcal{F}} \text{tr}(x) \, d\rho(x) = \int_{\mathcal{K}} \text{tr}(p) \, dm^{(1)}_+(p) - \int_{\mathcal{K}} \text{tr}(p) \, dm^{(1)}_-(p).$$  \hspace{1cm} (10.6.10)

Working with the moment measures has the advantage that they are measures on the compact space $\mathcal{K}$. We also learn that two measures $\rho$ and $\tilde{\rho}$ whose moment measures coincide yield the same values for the functionals $S$ and $T$ as well as for the integral (10.6.10) entering the trace constraint. It is most convenient to work exclusively with the moment measures. At the very end, we shall then construct a suitable representative $\rho$ of the limiting moment measures. A key step for making this method work is the following a-priori estimate.

**Lemma 10.6.5.** There is a constant $\varepsilon = \varepsilon(f, n) > 0$ such that for every measure $\rho$ on $\mathcal{F}$ the corresponding moment measures (see Definition 10.6.2) satisfy for all measurable $\Omega \subset \mathcal{K}$ the following inequalities:

$$\left( m^{(1)}_+(\Omega) + m^{(1)}_-(\Omega) \right)^2 \leq m^{(0)}(\Omega) \, m^{(2)}(\Omega)$$  \hspace{1cm} (10.6.11)

$$m^{(2)}(\mathcal{K}) \leq \frac{\sqrt{T(\rho)}}{\varepsilon}. \hspace{1cm} (10.6.12)$$

**Proof.** The inequality (10.6.11) follows immediately from Hölder’s inequality,

$$\left| 2(m^{(1)}_+(\Omega) + m^{(1)}_-(\Omega)) \right|^2 \leq \left( \int_{\mathcal{F} \Omega} ||p||^2 \, d\rho(p) \right)^2 \leq \rho(\mathcal{R}\Omega) \int_{\mathcal{F} \Omega} ||p||^2 \, d\rho(p) \leq 4m^{(0)}(\Omega) \, m^{(2)}(\Omega).$$

In order to prove (10.6.12), we introduce the mapping

$$\phi : \mathcal{K} \times \mathcal{K} \to \mathbb{R} : (p, q) \mapsto |pq|.$$ 

Clearly, $\phi$ is continuous and
\( \phi(p, p) = |p^2| = \text{Tr}(p^2) = \|p\|^2 = 1 \).

Thus every point \( r \in K \) has a neighborhood \( U(r) \subset K \) with
\[
\phi(p, q) \geq \frac{1}{2} \quad \text{for all } p, q \in U(r).
\] (10.6.13)

Since \( K \) is compact, there is a finite number of points \( r_1, \ldots, r_N \) such that the corresponding sets \( U_i := U(r_i) \) cover \( K \). Due to the additivity property of measures, there is an index \( i \in \{1, \ldots, N\} \) such that
\[
m^2(U_i) \geq \frac{m^2(K)^2}{N^2}.
\] (10.6.14)

We write \( T \) in the form (10.6.9) and apply (10.6.13) as well as (10.6.14) to obtain
\[
T(\rho) \geq \iint_{U_i \times U_i} |pq|^2 \, dm^2(p) \, dm^2(q) \geq \frac{1}{2} m^2(U_i)^2 \geq \frac{m^2(K)^2}{2N^2}.
\]

Setting \( \varepsilon = \frac{1}{\sqrt{2N}} \), the result follows. \( \square \)

10.7. Existence of Minimizers of the Causal Action Principle

We now return to the strategy of the direct method of the calculus of variations described at the beginning of Chapter 10. Thus we let \( \rho_k \) be a minimizing sequence satisfying our constraints (10.0.2) as well as (4.4.4) and (4.4.5) (for fixed constants). Then, in view of Lemma 10.6.5 we know that the moment measures are uniformly bounded measures on the compact metric space \( K \). Applying the compactness result of Theorem 10.5.2 (based on the Banach-Alaoglu theorem and the Riesz representation theorem) we conclude that that for a suitable subsequence (which we again denote by \( (\rho_k) \)), these measures converge in the \( C^0(K)^* \)-topology to regular Borel measures,
\[ m^{(0)}_k \to m^{(0)}, \quad m^{(1)}_{k,\pm} \to m^{(1)}_\pm \quad \text{and} \quad m^{(2)}_k \to m^{(2)}, \]
which again have the properties (10.6.7), (10.6.11) and (10.6.12).

We next form the Radon-Nikodym decompositions of \( m^{(1)}_\pm \) and \( m^{(2)} \) with respect to \( m^{(0)} \). The inequality (10.6.11) shows that every set of \( m^{(0)} \)-measure zero is also a set of measure zero with respect to \( m^{(1)}_\pm \) and \( m^{(1)} \). In other words, the measures \( m^{(1)}_\pm \) are absolutely continuous with respect to \( m^{(0)} \). Hence applying Theorem 10.4.2 we obtain the Radon-Nikodym decompositions
\[
dm^{(1)}_\pm = f_\pm \, dm^{(0)} \quad \text{with} \quad f_\pm \in L^1(K, dm^{(0)}).
\]

As a consequence, the signed measure \( m^{(1)} \) in (10.6.5) has the decomposition
\[
dm^{(1)} = f^{(1)} \, dm^{(0)} \quad \text{with} \quad f^{(1)} := f_+ - f_- \in L^1(K, dm^{(0)}).
\] (10.7.1)

As we do not know if \( m^{(2)} \) is also absolutely continuous with respect to \( m^{(0)} \), Theorem 10.4.2 gives the decomposition
\[
dm^{(2)} = f^{(2)} \, dm^{(0)} + dm^{(2)}_{\text{sing}} \quad \text{with} \quad f^{(2)} \in L^1(K, dm^{(0)}),
\] (10.7.2)

where the measure \( dm^{(2)}_{\text{sing}} \) is singular with respect to \( m^{(0)} \).
Lemma 10.7.1. The functions $f^{(1)}$ and $f^{(2)}$ in the Radon-Nikodym decompositions (10.7.1) and (10.7.2) can be chosen such that

$$|f^{(1)}|^2 \leq f^{(2)}.$$

Proof. Since $m^{(2)}_{\text{sing}} \perp m^{(0)}$, there is a Borel set $V$ such that

$$\chi_V dm^{(0)} = dm^{(0)} \quad \text{and} \quad \chi_V dm^{(2)}_{\text{sing}} = 0.$$  

Then, using the Radon-Nikodym decompositions (10.7.1) and (10.7.2) in (10.6.11), we obtain for any Borel set $U \subset V$ the inequality

$$\left| \int_U f^{(1)} dm^{(0)} \right|^2 \leq m^{(0)}(U) \int_U f^{(2)} dm^{(0)}.$$  

If the function $f^{(1)}$ does not change signs on $U$, we conclude that

$$\inf_U |f^{(1)}|^2 \leq \sup_U f^{(2)}.$$  

By decomposing $U$ into the two sets where $f^{(1)}$ is positive and negative, respectively, one readily sees that this inequality even holds for any Borel set $U \subset V$ the inequality

$$\left| \int_U f^{(1)} dm^{(0)} \right|^2 \leq m^{(0)}(U) \int_U f^{(2)} dm^{(0)}.$$  

In particular, we conclude that $f^{(1)}$ even lies in $L^2(K, dm^{(0)})$. Setting $f = f^{(1)}$ and $dn = (f^{(2)} - |f|^2) dm^{(0)} + dm^{(2)}_{\text{sing}}$, we obtain the decomposition

$$dm^{(1)} = f dm^{(0)}, \quad dm^{(2)} = |f|^2 dm^{(0)} + dn,$$  

where $f \in L^2(K, dm^{(0)})$, and $n$ is a positive measure which need not be absolutely continuous with respect to $m^{(0)}$. From the definition (10.6.5) it is clear that $f$ is odd, i.e.

$$f(-p) = -f(p) \quad \text{for all } p \in K.$$  

The remaining task is to represent the limiting moment measures $m^{(l)}$ in (10.7.3) by a measure $\rho$. Unfortunately, there is the basic problem that such a measure can exist only if $m^{(2)}$ is absolutely continuous with respect to $m^{(0)}$, as the following consideration shows: Assume conversely that $m^{(2)}$ is not absolutely continuous with respect to $m^{(0)}$. Then there is a measurable set $\Omega \subset K$ with $m^{(0)}(\Omega) = 0$ and $m^{(2)}(\Omega) \neq 0$. Assume furthermore that there is a measure $\rho$ on $F$ which represents the limiting moment measures in the sense that (10.6.1)–(10.6.4) hold. From (10.6.1) we conclude that the set $\mathbb{R}\Omega \subset F$ has $\rho$-measure zero. But then the integral (10.6.4) also vanishes, a contradiction.

This problem can also be understood in terms of the limiting sequence $\rho_k$. We cannot exclude that there is a star-shaped region $\mathbb{R}\Omega \subset F$ such that the measures $\rho_k(\mathbb{R}\Omega)$ tend to zero, but the corresponding moment integrals (10.6.4) have a non-zero limit. Using a notion from the calculus of variations for curvature functionals, we refer to this phenomenon as the possibility of bubbling. This bubbling effect is illustrated by the following example.

Example 10.7.2. (Bubbling) We choose $f = 2$ and $n = 1$. Furthermore, we let $\mathcal{M} = [0, 1]$ with $\mu$ the Lebesgue measure. For any parameters $\kappa \geq 0$ and $\varepsilon \in (0, \frac{1}{2})$, we introduce
the mapping \( F_\varepsilon : \mathcal{M} \rightarrow \mathcal{F} \) by
\[
F_\varepsilon(x) = \frac{1}{1 - 2\varepsilon} \times \begin{cases} 
-\kappa \varepsilon^{-1/2} \sigma^3 & \text{if } x \leq \varepsilon \\
1 + \sigma^1 \cos(\nu x) + \sigma^2 \sin(\nu x) & \text{if } \varepsilon < x \leq 1 - \varepsilon \\
\kappa \varepsilon^{-1/2} \sigma^3 & \text{if } x > 1 - \varepsilon ,
\end{cases}
\]
where we set \( \nu = 2\pi/(1 - 2\varepsilon) \) (and \( \sigma^j \) are the Pauli matrices). The corresponding measure \( \rho_\varepsilon \) on \( \mathcal{F} \) has the following properties. On the set
\[
S := \{1 + v^1 \sigma^1 + v^2 \sigma^2 \text{ with } (v^1)^2 + (v^2)^2 = 1\},
\]
which can be identified with a circle \( S^1 \), \( \rho_\varepsilon \) is a multiple of the Lebesgue measure. Moreover, \( \rho_\varepsilon \) is supported at the two points
\[
p_{\pm} := \pm \frac{\kappa \varepsilon^{-1/2} \sigma^3}{1 - 2\varepsilon} \text{ with } \rho_\varepsilon(\{p_+\}) = \rho_\varepsilon(\{p_-\}) = \varepsilon . \tag{10.7.5}
\]
A short calculation shows that the trace constraint is satisfied. Furthermore, the separations of the points \( p_+ \) and \( p_- \) from each other and from \( S \) are either spacelike or just in the boundary case between spacelike and timelike. Thus for computing the action, we only need to take into account the pairs \( (p_+, p_+), (p_-, p_-) \) as well as pairs \( (x, y) \) with \( x, y \in S \). A straightforward computation yields
\[
S(\rho_\varepsilon) = \frac{3}{(1 - 2\varepsilon)^2}, \quad \mathcal{T}(\rho_\varepsilon) = \frac{6}{(1 - 2\varepsilon)^2} + \frac{16\kappa^2}{(1 - 2\varepsilon)^3} + \frac{16\kappa^4}{(1 - 2\varepsilon)^4} . \tag{10.7.6}
\]
Let us consider the limit \( \varepsilon \searrow 0 \). From (10.7.6) we see that the functionals \( S \) and \( \mathcal{T} \) converge,
\[
\lim_{\varepsilon \searrow 0} S = 3, \quad \lim_{\varepsilon \searrow 0} \mathcal{T} = 6 + 16(\kappa^2 + \kappa^4) . \tag{10.7.7}
\]
Moreover, there are clearly no convergence problems on the set \( S \). Thus it remains to consider the situation at the two points \( \hat{p}_{\pm} \), (10.7.5), which move to infinity as \( \varepsilon \) tends to zero. These two points enter the moment measures only at the corresponding normalized points \( \hat{p}_{\pm} = p_{\pm}/||p_{\pm}|| \in \mathcal{K} \). A short calculation shows that the limiting moment measures \( m^{(\ell)} = \lim_{\varepsilon \searrow 0} m_\varepsilon^{(\ell)} \) satisfy the relations
\[
m^{(0)}(\{\hat{p}_{\pm}\}) = m^{(1)}(\{\hat{p}_{\pm}\}) = 0 \quad \text{but} \quad m^{(2)}(\{\hat{p}_{\pm}\}) = \kappa^2 > 0 .
\]
Hence \( m^{(2)} \) is indeed not absolutely continuous with respect to \( m^{(0)} \).

To avoid misunderstandings, we point out that this example does not show that bubbling really occurs for minimizing sequences, because we do not know whether the family \( (\rho_\varepsilon)_{0 < \varepsilon < 1/2} \) is minimizing. But at least, our example shows that bubbling makes it possible to arrange arbitrary large values of \( \mathcal{T} \) without increasing the action \( S \) (see (10.7.7) for large \( \kappa \)).

In order to handle possible bubbling phenomena, it is important to observe that the second moment measure does not enter the trace constraint. Therefore, by taking out the term \( dm \) in (10.7.3) we decrease the functionals \( S \) and \( \mathcal{T} \) (see (10.6.8) and (10.6.9)), without affecting the trace constraint. It remains to show that the resulting moment measure can indeed be realized by a measure \( \rho \). This is shown in the next lemma.

**Lemma 10.7.3.** For any normalized regular Borel measure \( m^{(0)} \) on \( \mathcal{K} \) and any function \( f \in L^2(\mathcal{K}, \mathbb{R}) \), there is a normalized regular Borel measure \( \hat{m} \) whose moment measures \( \hat{m}^{(\ell)} \) are given by
\[
\hat{m}^{(0)} = m^{(0)}, \quad d\hat{m}^{(1)} = f \ dm^{(0)}, \quad d\hat{m}^{(2)} = |f|^2 \ dm^{(0)} . \tag{10.7.8}
\]
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Proof. We introduce the mapping

\[ F : \mathcal{K} \to \mathcal{F}, \quad F(x) = f(x) \, x. \]

Then the measure \( \tilde{\rho} := F_* \mu^{(0)} \) does the trick. \( \square \)

This concludes the existence proof. We finally remark that the fact that we dropped the measure \( d\mu \) in (10.7.3) implies that the value of \( \mathcal{T} \) might decrease in the limit. This is the only reason why the boundedness constraint (14.3.5) is formulated as an inequality, and not as an equality. It is not clear if the causal action principle also admits minimizers if the inequality in (14.3.5) is replaced by an equality. My conjecture is that the answer is yes. But, although I thought about it for a while, I could not come up with a proof. For the physical applications, it makes no difference if (14.4.5) is an equality or an inequality, because one then works with the corresponding Euler-Lagrange equations, into which the constraints enter via Lagrange multiplier terms (for details see [10]). Here we shall not enter the analysis of the Euler-Lagrange equations, but refer instead to the detailed treatment in [36].

Exercises

Exercise 10.1. Let \( \Lambda \) be the functional

\[ \Lambda : C^0([0,1],\mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) = \sup_{x \in [0,1]} f(x). \]

Can this functional be represented by a measure? Analyze how your findings are compatible with the Riesz representation theorem.

Exercise 10.2. Let \( \rho \) be the Borel measure on \([0,\pi]\) given by

\[ \rho(\Omega) = \int_{\Omega} \sin x \, dx + \sum_{n=1}^{\infty} \frac{1}{n^2} \chi_{\Omega} \left( \frac{1}{n} \right). \]

Compute the Lebesgue decomposition of \( \rho \) with respect to the Lebesgue measure.

Exercise 10.3. Let \( \mathcal{F} = \mathbb{R}^2 \) and \( \mathcal{K} = S^1 \cup \{0\} \) be a compact subset of \( \mathcal{F} \). Given a Borel measure \( \rho \) on \( \mathcal{F} \), the moment measures \( \mu^{(0)}, \mu^{(1)} \) and \( \mu^{(2)} \) can be defined just as in the lecture. Compute these moment measures for the following choices of \( \rho \):

(a) \( \rho = F_* \mu_{S^1} \), where \( F : S^1 \hookrightarrow \mathbb{R}^2 \) is the natural injection and \( \mu_{S^1} \) the normalized Lebesgue measure on \( S^1 \).

(b) \( \rho = \delta_{(0,0)} + \delta_{(1,1)} + \delta_{(5,0)} \) (where \( \delta_{(x,y)} \) denotes the Dirac measure supported at \((x,y) \in \mathbb{R}^2\)).

(c) \( \rho = F_* \mu_{\mathbb{R}} \), where

\[ F : \mathbb{R} \to \mathbb{R}^2 \quad F(x) = (x,2) \]

and \( \mu_{\mathbb{R}} \) is the Lebesgue measure.

Exercise 10.4. (a) Let \((\rho_n)_{n \in \mathbb{N}}\) be a sequence of normalized regular Borel measures on \( \mathbb{R} \) with the property that there is a constant \( c > 0 \) such that

\[ \int_{-\infty}^{\infty} x^2 \, d\rho_n(x) \leq c \quad \text{for all } n. \]

Show that a subsequence converges again to a normalized Borel measure on \( \mathbb{R} \).

Hint: Apply the compactness result of the lecture to the measures restricted to the interval \([-L,L]\) and analyze the behavior as \( L \to \infty \).
(b) More generally, assume that for a given non-negative function \( f(x) \),
\[
\int_{-\infty}^{\infty} f(x) \, dp_n(x) \leq c \quad \text{for all } n.
\]
Which condition on \( f \) ensures that the a subsequence of the measures converges to a normalized Borel measure? Justify your result by a counter example.

**Exercise 10.5.** Let \( \mathcal{M} \subset \mathbb{R} \) be a closed embedded submanifold of \( \mathbb{R}^3 \). We choose a compact set \( K \subset \mathbb{R}^3 \) which contains \( \mathcal{M} \). On \( C^0(K, \mathbb{R}) \) we introduce the functional
\[
\Lambda : C^0(K, \mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) = \int_{\mathcal{M}} f(x) \, d\mu_{\mathcal{M}}(x)
\]
(where \( d\mu_{\mathcal{M}} \) is the volume measure corresponding to the induced Riemannian metric on \( \mathcal{M} \)). Show that this functional is linear, bounded and positive. Apply the Riesz representation theorem to represent this functional by a Borel measure on \( K \). What is the support of this measure?
11.1. The Cauchy Problem, Linear Symmetric Hyperbolic Systems

In this section, we want to prove that the Cauchy problem (15.0.1) in the presence of an external potential has a unique global solution. Moreover, we want to show that the finite speed of propagation as postulated by special relativity is indeed respected by the solutions of the Dirac equation. For later purposes, it is preferable to include an inhomogeneity. Thus in generalization of (15.0.1), we consider the Cauchy problem

\[(i\partial + B - m)\psi = \phi \in C^\infty(M,SM), \quad \psi|_{t_0} = \psi_0 \in C^\infty(\mathbb{R}^3,SM)\]

for given \(\phi\) and \(\psi_0\). In order to make the standard methods available, we multiply the equation by \(-i\gamma^0\),

\[\begin{align*}
C_4 \partial_t \psi + \gamma^0 \gamma \nabla \psi &= -i\gamma^0 \phi.
\end{align*}\]

Now the matrices in front of the derivatives are all Hermitian (with respect to the standard scalar product on \(C^4\)). Moreover, the matrix in front of the time derivative is positive definite. Kurt Otto Friedrichs \([66]\) observed that these properties are precisely what is needed in order to get a well-posed Cauchy problem. He combined these properties in the notion of a \textit{symmetric hyperbolic system}. We now give their general definition. More specifically, we consider a system of \(N\) complex-valued equations with spatial coordinates \(\vec{x} \in \mathbb{R}^m\) and time \(t\) in an interval \([0,T]\) with \(T > 0\). The initial data will always be prescribed at time \(t = 0\). For notational clarity, we denote partial derivatives in spatial direction by \(\nabla\).

**Definition 11.1.1.** A linear system of differential equations of the form

\[A^0(t,\vec{x}) \partial_t u(t,\vec{x}) + \sum_{\alpha=1}^m A^\alpha(t,\vec{x}) \nabla_\alpha u(t,\vec{x}) + B(t,\vec{x}) u(t,\vec{x}) = w(t,\vec{x})\]

is called \textbf{symmetric hyperbolic} if

(i) The matrices \(A^0\) and \(A^\alpha\) are Hermitian,

\[(A^0)^\dagger = A^0 \quad \text{and} \quad (A^\alpha)^\dagger = A^\alpha\]

(where \(\dagger\) denotes the adjoint with respect to the canonical scalar product on \(\mathbb{C}^N\)).

(ii) The matrix \(A^0\) is uniformly positive definite, i.e. there is a positive constant \(C\) such that

\[A^0(t,\vec{x}) > C \mathbf{1}_{\mathbb{C}^N} \quad \text{for all} \ (t,\vec{x}) \in ([0,T] \times \mathbb{R}^m).\]

In the case \(w \equiv 0\), the linear system is called \textbf{homogeneous}.
A good reference for linear symmetric hyperbolic systems is the book by Fritz John [75, Section 5.3] (who was Friedrichs’ colleague at the Courant Institute). Our presentation was also influenced by [88]. We remark that the concept of symmetric hyperbolic systems can be extended to nonlinear equations of the form

\[ A^0(t,\vec{x},u) \partial_t u(t,\vec{x}) + \sum_{\alpha=1}^m A^\alpha(t,\vec{x},u) \nabla_\alpha u(t,\vec{x}) + B(t,\vec{x},u) = 0, \]

where the matrices \( A^0 \) and \( A^\alpha \) should again satisfy the above conditions (i) and (ii). For details we refer to [95, Section 16] or [89, Section 7]. Having the Dirac equation in mind, we always restrict attention to linear systems. We also note that an alternative method for proving existence of fundamental solutions is to work with the so-called Riesz distributions (for a good textbook see [5]). Yet another method is to work with estimates in the interaction picture [18].

It is a remarkable fact that all partial differential equations in relativistic physics as well as most wave-type equations can be rewritten as a symmetric hyperbolic system. As an illustration, we now explain this reformulation in the example of a scalar hyperbolic equation.

**Example 11.1.2.** Consider a scalar hyperbolic equation of the form

\[ \partial_{tt} \phi(t,\vec{x}) = \sum_{\alpha,\beta=1}^m a_{\alpha\beta}(t,\vec{x}) \nabla_\alpha \nabla_\beta \phi + \sum_{\alpha=1}^m b_\alpha(t,\vec{x}) \nabla_\alpha \phi + c(t,\vec{x}) \partial_t \phi + d(t,\vec{x}) \phi \quad (11.1.3) \]

with \( (a_{\alpha\beta}) \) a symmetric, uniformly positive matrix (in the case \( a_{\alpha\beta} = \delta_{\alpha\beta} \) and \( b, c, d = 0 \), one gets the scalar wave equation). Now the initial data prescribes \( \phi \) and its first time derivatives,

\[ \phi|_{t=0} = \phi_0 \in C^\infty(M), \quad \partial_t \phi|_{t=0} = \phi_1 \in C^\infty(M). \quad (11.1.4) \]

In order to rewrite the equation as a symmetric hyperbolic system, we introduce the vector \( u \) with \( k := m + 2 \) components by

\[ u_1 = \nabla_1 \phi, \ldots, u_m = \nabla_m \phi, \quad u_{m+1} = \partial_t \phi, \quad u_{m+2} = \phi. \quad (11.1.5) \]

Then the system

\[
\begin{aligned}
- \sum_{\beta=1}^m a_{\alpha\beta} \partial_{t} u_\beta \\
- \sum_{\alpha=1}^m b_\alpha u_\alpha + \partial_t u_{m+1} - c u_{m+1} - d u_{m+2} \\
0 \\
- u_{m+1} + \partial_t u_{m+2} = 0
\end{aligned}
\]

is symmetric hyperbolic (as one verifies by direct inspection). Also, a short calculation shows that if \( \phi \) is a smooth solution of the scalar equation (11.1.3), then the corresponding vector \( u \) is a solution of the system (11.1.6). Conversely, assume that \( u \) is a smooth solution of (11.1.6) which satisfies the initial condition \( u|_{t=0} = u_0 \), where \( u_0 \) is determined by \( \phi_0 \) and \( \phi_1 \) via (11.1.5). Setting \( \phi = u_{m+2} \), the last line in (11.1.6) shows that \( u_{m+1} = \partial_t \phi \). Moreover, the first line in (11.1.6) implies that \( \partial_t u_\beta = \nabla_\beta u_{m+1} = \partial_t \nabla_\beta \phi \). Integrating over \( t \) and using that the relation \( u_\beta = \nabla_\beta \phi \) holds initially, we conclude that this relation holds for all times. Finally, the second line in (11.1.6) yields that \( \phi \) satisfies the scalar hyperbolic equation (11.1.3). In this sense, the Cauchy problem for the system (11.1.6) is equivalent to that for the scalar equation (11.1.3). ◊
11.2. Finite Propagation Speed and Uniqueness of Solutions

For what follows, it is convenient to combine the time and spatial coordinates to a space-time vector \( x = (t, \vec{x}) \in [0, T] \times \mathbb{R}^m \). We denote the space-time dimension by \( n = m + 1 \). Moreover, setting \( \partial_0 \equiv \partial_t \), we use latin space-time indices \( i \in \{0, \ldots, m\} \) and use the Einstein summation convention. Then our linear system (11.1.2) can be written in the compact form

\[
A^j(x) \partial_j u(x) + B(x) u(x) = w(x) .
\]

Next, a direction in space-time can be described by a vector \( \xi = (\xi_i)_{i=0,\ldots,m} \in \mathbb{R}^{m+1} \). Contracting with the matrices \( A^j(x) \), we obtain the Hermitian \( N \times N \)-matrix

\[
A(x, \xi) := A^j(x) \xi_j ,
\]

referred to as the characteristic matrix. Note that in the example of the Dirac equation (11.1.1), the index \( i \) is a vector index in Minkowski space, and \( \xi \) should be regarded as a covector (i.e. a vector in the cotangent bundle). One should keep in mind that, despite the suggestive notation, the equation (11.2.1) should not be considered as being covariant, because it corresponds to the Hamiltonian formulation (11.1.1), where a time manifestly covariant direction is distinguished.

The determinant of the characteristic matrix is referred to as the characteristic polynomial, being a polynomial in the components \( \xi_i \). For our purposes, it is most helpful to consider whether the characteristic matrix is positive or negative definite. If the vector \( \xi = (\tau, \vec{0}) \) points in the time direction, then \( A(x, \xi) = \tau A^0 \), which in view of Definition (11.1.1) is a definite matrix. By continuity, \( A(x, \xi) \) is definite if the spatial components of \( \xi \) are sufficiently small. In the example of the Dirac equation (11.1.1), the fact that

\[
A(x, \xi) = 1 \xi_0 + \gamma^0 \vec{\gamma} \xi \quad \text{has eigenvalues} \quad \xi_0 \pm |\vec{\xi}|
\]

shows that \( A(x, \xi) \) is definite if and only if \( \xi \) is a timelike vector. Moreover, it is positive definite if and only if \( \xi \) is future-directed and timelike. This suggests that the causal properties of the equation are encoded in the positivity of the characteristic matrix. We simply use this connection to define the causal structure for a general symmetric hyperbolic system.

**Definition 11.2.1.** The vector \( \xi \in \mathbb{R}^{m+1} \) is called timelike at the space-time point \( x \) if the characteristic matrix \( A(x, \xi) \) is definite. A timelike vector is called future-directed if \( A(x, \xi) \) is positive definite. If the characteristic polynomial vanishes, then the vector \( \xi \) is called lightlike. A hypersurface \( \mathcal{H} \subset [0, T] \times \mathbb{R}^m \) with normal \( \nu \) is called space-like if the matrix \( A(x, \nu) \) is positive definite for all \( x \in \mathcal{H} \).

The notion “normal” used here requires an explanation. The simplest method is to represent the hypersurface locally as the zero set of a function \( \phi(x) \). Then the normal can be defined as the gradient of \( \phi \). In this way, the gradient is a co-vector, so that the contraction \( A^j \nu_j = A^j \partial_j \phi \) is well-defined without referring to a scalar product. In particular, the last definition is independent of the choice of a scalar product on space-time vectors in \( \mathbb{R}^n \). We always choose the normal to be future-directed and we normalize it with respect to the Euclidean scalar product on \( \mathbb{R}^{m+1} \), but these are merely a conventions.

We shall now explain why and in which sense the solutions of symmetric hyperbolic systems comply with this notion of causality.

**Definition 11.2.2.** Let \( u \) be a smooth solution of the linear symmetric hyperbolic system (11.2.1). A subset \( K \) of the initial value surface \( \{t = 0\} \) determines the solution
at a space-time point $x \in [0,T] \times \mathbb{R}^m$ if every smooth solution of the system which coincides on $K$ with $u$, also coincides with $u$ at $x$. The domain of determination of $K$ is the set of all space-time points at which the solution is determined by the initial data on $K$.

**Definition 11.2.3.** An open subset $L \subset (0,T) \times \mathbb{R}^m$ is called a lens-shaped region if $L$ is relatively compact in $\mathbb{R}^n$ and if its boundary $\partial L$ is contained in the union of two smooth hypersurfaces $S_0$ and $S_1$ whose intersection with $[0,T] \times \mathbb{R}^m$ is space-like. We set $(\partial L)^+ = \partial L \cap S_1$ and $(\partial L)^- = \partial L \cap S_0$, where we adopt the convention that $(\partial L)^+$ lies to the future of $(\partial L)^-$.

Figure 11.1 shows typical examples of lens-shaped regions. Often, one chooses $S_0 = \{ t = 0 \}$ as the initial data surface. Moreover, it is often convenient to write the hypersurface $S_1$ as a graph $S_1 = \{ (t,\vec{x}) \mid t = f(\vec{x}) \}$. In this case, $S_1$ is the zero set of the function $\phi(t,\vec{x}) = t - f(\vec{x})$, and the normal $\nu$ is the gradient of this function, i.e. $(\nu_j)_{j=0,\ldots,m} = (1, \nabla_1 f, \ldots, \nabla_m f)$.

We first consider the homogeneous equation

$$\begin{equation}
(A^j \partial_j + B) u = 0. \tag{11.2.3}
\end{equation}$$

The idea for analyzing the domain of determination is to multiply this equation by a suitable test function and to integrate over a lens-shaped region. More precisely, we consider the equation

$$\begin{equation}
0 = \int_L e^{-Kt} 2 \text{Re} \langle u, (A^j \partial_j + B)u \rangle d^n x, \tag{11.2.4}
\end{equation}$$

where $\langle ., . \rangle$ denotes the canonical scalar product on $\mathbb{C}^N$, and $K > 0$ a positive parameter to be determined later. Since the $A^j$ are Hermitian, we have

$$\begin{equation}
\partial_j \langle u, A^j u \rangle = 2 \text{Re} \langle u, A^j \partial_j u \rangle + \langle u, (\partial_j A^j) u \rangle, \tag{11.2.5}
\end{equation}$$

and using this identity in (11.2.4) gives

$$\begin{equation}
0 = \int_L e^{-Kt} \left( \partial_j \langle u, A^j u \rangle + \langle u, (B + B^*) - (\partial_j A^j) u \rangle \right) d^n x. \tag{11.2.6}
\end{equation}$$

In the first term we integrate by parts with the Gauß divergence theorem,

$$\begin{equation}
\int_L e^{-Kt} \partial_j \langle u, A^j u \rangle d^n x = K \int_L e^{-Kt} \langle u, A^0 u \rangle d^n x
+ \int_{(\partial L)_+} e^{-Kt} \langle u, \nu_j A^j u \rangle d\mu_{\partial L_+} - \int_{(\partial L)_-} e^{-Kt} \langle u, \nu_j A^j u \rangle d\mu_{\partial L_-}. \tag{11.2.7}
\end{equation}$$
11.2. FINITE PROPAGATION SPEED AND UNIQUENESS OF SOLUTIONS

We now use (11.2.7) in (11.2.6) and solve for the surface integral over \((\partial L)_+\),

\[
\int_{(\partial L)_+} e^{-Kt} \langle u, v_j A^j u \rangle \, d\mu_{\partial L_+} = \int_{(\partial L)_-} e^{-Kt} \langle u, v_j A^j u \rangle \, d\mu_{\partial L_-} \\
+ \int_{L} e^{-Kt} \langle u, (-K - B - B^* + (\partial_j A^j)) u \rangle \, d^nx.
\]

This identity is the basis for the following uniqueness results.

**Theorem 11.2.4.** Let \(u_1\) and \(u_2\) be two smooth solutions of the linear symmetric hyperbolic system (11.1.2) which coincide on the past boundary of a lens-shaped region \(L\),

\[u_1|_{(\partial L)_-} = u_2|_{(\partial L)_-}.\]

Then \(u_1\) and \(u_2\) coincide in the whole set \(L\).

**Proof.** The function \(u := u_1 - u_2\) is a solution of the homogeneous system (11.2.3) with \(u|_{(\partial L)_-} = 0\). Hence (11.2.8) simplifies to

\[
\int_{(\partial L)_+} e^{-Kt} \langle u, v_j A^j u \rangle \, d\mu_{\partial L_+} = \int_{L} e^{-Kt} \langle u, (-K - B - B^* + (\partial_j A^j)) u \rangle \, d^nx.
\]

Assume that \(u\) does not vanish identically in \(L\). By choosing \(K\) sufficiently large, we can then arrange that the right side becomes negative. However, since \(\partial L_+\) is a space-like hypersurface, the left side is non-negative. This is a contradiction. □

As an immediate consequence, we obtain the following uniqueness result for solutions of the Cauchy problem.

**Corollary 11.2.5.** Let \(u_1\) and \(u_2\) be two smooth solutions of the linear symmetric hyperbolic system (11.1.2) with the same initial at \(t = 0\). Then \(u_1 \equiv u_2\) in a neighborhood of the initial data surface.

If the matrices \(A^j\) are uniformly bounded and \(A^0\) is uniformly positive, then \(u_1 \equiv u_2\) in \([0, T] \times \mathbb{R}^m\).

**Proof.** The local uniqueness result follows immediately by covering the initial data surface by lens-shaped regions (see the left of Figure 11.2). For the global uniqueness, for any \(x_0 = (t_0, \vec{x}_0) \in [0, T] \times \mathbb{R}^m\) our task is to choose a lens-shaped region which contains \(x_0\) and whose past boundary \(S_0\) is contained in the surface \(\{t = 0\}\). We need to rule out the situation shown on the right of Figure 11.2 that the hypersurface \(S_1\) does not intersect \(S_0\), in which case we would not get a relatively compact lens-shaped region.

To this end, we must use that the matrices \(A^j\) are uniformly bounded by assumption. As a consequence, there is \(\varepsilon > 0\) such that the inequality \(\|\nabla f\| \leq \varepsilon\) implies that the hypersurface \(S_1 = \{(t = f(\vec{x}), \vec{x})\}\) is spacelike. Possibly after decreasing \(\varepsilon\), we may choose

\[
f(\vec{x}) = t_0 + \varepsilon \left(1 - \sqrt{1 + \|\vec{x} - \vec{x}_0\|^2}\right).
\]

This concludes the proof. □
By a suitable choice of lens-shaped region one can get an upper bound for the maximal propagation speed. For the Dirac equation, where the causal structure of Definition 11.2.1 coincides with that of Minkowski space in view of (11.2.2), one can choose for $S_1$ a family of space-like hypersurfaces which converge to the boundary of a light cone (see Figure 11.3). This shows that the maximal propagation speed for Dirac waves is indeed the speed of light.

11.3. Global Existence of Smooth Solutions

We now write the linear system (11.2.1) as

$$Lu = w \quad \text{with} \quad L := A^j \partial_j + B,$$

where we again sum over $j = 0, \ldots, m$. Going back to the formula for the divergence (11.2.5) and using the equation, we obtain

$$\partial_j \langle u, A^j u \rangle + \langle u, Cu \rangle = 2 \text{Re} \langle u, w \rangle,$$

$$C := B + B^* - (\partial_j A^j).$$

In what follows, for any $\lambda \in [0, T]$ we consider the time strip

$$R_\lambda = [0, \lambda] \times \mathbb{R}^m.$$

- Betone, dass man wegen der endlichen Ausbreitungsgeschwindigkeit das Cauchy-Problem immer auf ganz $\mathbb{R}^m$ ausdehnen kann. Später kann man die erhaltenen Lösungen wieder geeignet zusammenkleben.

We assume that the functions $A^j$, $B$ and $w$ are smooth and uniformly bounded in $R_\lambda$. Moreover, we assume that $w$ has spatially compact support (meaning that $w(t,.) \in C_0^\infty(\mathbb{R}^m)$ for all $t \in [0, \lambda]$). We denote the $s$-times continuously differential functions on $R_\lambda$ with spatially compact support by $C^s(R_\lambda)$. The function spaces

$$C^s(R_\lambda) \quad \text{and} \quad \overline{C^s(R_\lambda)}$$

are defined as the functions which in addition vanish at $t = 0$ and $t = \lambda$, respectively.

We want to solve the Cauchy problem

$$Lu = w, \quad u|_{t=0} = u_0 \in C_0^\infty(\mathbb{R}^m)$$

in $C^s(R_T)$. First of all, we may restrict attention to the case $u_0 \equiv 0$,

$$Lu = w, \quad u|_{t=0} \equiv 0.$$

To see this, let $u$ be a solution of the above Cauchy problem. Choosing a function $v \in C^\infty(R_T)$ which at $t = 0$ coincides with $u_0$, then the function $\tilde{u} := (u - v)$ satisfies the equation $L\tilde{u} = \tilde{w}$ with $\tilde{w} = w + A^j \partial_j v + Bv$ and vanishes at $t = 0$. If conversely $\tilde{u}$ is a solution of the corresponding Cauchy problem with zero initial data, then $u := \tilde{u} + v$ is a solution of the original problem (11.3.3).
We first derive so-called energy estimates. To this end, we integrate (11.3.1) over \(R_T\), integrate by parts with the Gauß divergence theorem and use that the initial values at \(t = 0\) vanish. We thus obtain
\[
E(\lambda) := \int_{t=\lambda}^{\infty} \langle u, A^0 u \rangle \, dt = \int_0^\lambda dt \int_{\mathbb{R}^m} \left( 2 \text{Re} \langle u, w \rangle - \langle u, Cu \rangle \right) \, dx.
\]
(11.3.5)
Since the matrix \(C\) is uniformly bounded and \(A_0\) is uniformly positive, there is a constant \(K > 1\) such that
\[
|\langle u, Cu \rangle| \leq K \langle u, A_0 u \rangle.
\]
Moreover, the linear term in \(u\) can be estimated with the Schwarz inequality by
\[
2 \text{Re} \langle u, w \rangle \leq \mu \langle u, u \rangle + \frac{1}{\mu^2} \langle w, w \rangle \leq \langle u, A_0 u \rangle + \frac{1}{\mu^2} \langle w, A_0 w \rangle
\]
with a suitable constant \(\mu > 0\). Applying these estimates in (11.3.5) gives
\[
E(\lambda) \leq (K + 1) \int_0^\lambda E(t) \, dt + \frac{1}{\mu^2} \int_{R_T} \langle w, A_0 w \rangle \, dx.
\]
Writing this inequality as
\[
\frac{d}{d\lambda} e^{-(K+1)\lambda} \int_0^\lambda E(t) \, dt \leq e^{-(K+1)\lambda} \frac{1}{\mu^2} \int_{R_T} \langle w, A_0 w \rangle \, dx,
\]
we can integrate over \(\lambda\) to obtain
\[
\int_0^T E(\lambda) \, d\lambda \leq \frac{e^{(K+1)T} - 1}{K + 1} \frac{1}{\mu^2} \int_{R_T} \langle w, A_0 w \rangle \, dx.
\]
(11.3.6)
Finally, we apply the mean value theorem and use that the exponential function is monotone to conclude that
\[
\int_0^T E(\lambda) \, d\lambda \leq \frac{T}{\mu^2} e^{(K+1)T} \int_{R_T} \langle w, A_0 w \rangle \, dx.
\]
This is the desired energy estimate. Before going on, we point out that the notion of “energy” used for the quantity \(E(\lambda)\) does in general not coincide with the physical energy. In fact, for the Dirac equation (11.1.1), \(E(\lambda)\) has the interpretation as the electric charge. Following Example 11.1.2 for the scalar wave equation \(\square \phi = 0\), we find
\[
E(\lambda) = \int_{\mathbb{R}^m} \left( |\partial_t \phi|^2 + |\nabla \phi|^2 + |\phi|^2 \right) \, dx.
\]
(11.3.7)
This differs from the physical energy by the last summand \(|\phi|^2\) (and an overall factor of two). The name “energy” for \(E(\lambda)\) was motivated by the fact, considering only the highest derivative terms, the expression (11.3.7) is indeed the physical energy. We point out that, in contrast to the physical energy, the quantity \(E(\lambda)\) does in general depend on time. The point is that (11.3.6) gives an \(a\)-priori control of the energy in terms of the inhomogeneity. The exponential factor in (11.3.6) can be understood in analogy to a Grönewall estimate.

For the following construction, it is convenient to introduce on \(C^1(R_T)\) the scalar product
\[
(u, v) = \int_{R_T} \langle u, A^0 v \rangle \, dx.
\]
(11.3.8)
We denote the corresponding norm by \( \| \cdot \| \). Setting furthermore
\[
\Gamma^2 = \frac{T}{\mu^2} e^{(K+1)T},
\]
the energy estimate can be written in the compact form
\[
(u, u) \leq \Gamma^2 (w, w).
\]
This inequality holds for every solution \( u \) of the differential equation \( Lu = w \) which vanishes at \( t = 0 \). Noting that every function \( u \in C^1(R_T) \) is a solution of this differential equation with inhomogeneity \( w := Lu \), we obtain
\[
\|u\| \leq \Gamma \|Lu\| \quad \text{for all } u \in C^1(R_T).
\]
This is the form of the energy estimates suitable for an abstract existence proof. In preparation, we want to introduce the notion of a \textit{weak solution}. As the space of test functions we choose \( C^1(R_T) \); this guarantees that integrating by parts does not yield boundary terms at \( t = T \). For a classical solution \( u \in C^1(R_T) \), we obtain
\[
(v, w) = (\tilde{L}v, u) \quad \text{for all } v \in C^1(R_T),
\]
where \( \tilde{L} \) is the formal adjoint of \( L \) with respect to the scalar product (11.3.8), i.e.
\[
\tilde{L} = -\tilde{A}^i\partial_j + \tilde{B} \quad \text{with} \quad \tilde{A}^i = (A^0)^{-1} A^i A^0, \quad \tilde{B} = (A^0)^{-1} (B^i A^0 - \partial_j (A^j A^0)).
\]
If conversely (11.3.10) holds for all \( u \in C^1(R_T) \), then \( u \) is indeed a solution of the Cauchy problem (11.3.4) (the relation \( u|_{t=0} = 0 \) is verified by considering the boundary terms obtained after integrating by parts). Hence we can use (11.3.10) as the definition of a weak solution of the Cauchy problem. Note that the operator \( -\tilde{L} \) is again symmetric hyperbolic and has the same boundedness and positivity properties as \( L \). Hence, repeating the above arguments, we obtain similar to (11.3.9) the "dual estimate"
\[
\|v\| \leq \tilde{\Gamma} \|\tilde{L}v\| \quad \text{for all } v \in C^1(R_T).
\]
We now want to show the existence of weak solutions with the help of the Fréchet-Riesz theorem (for basics on functional analysis see [86] or [81]). To this end, we first introduce on \( C^1(R_T) \) the scalar product
\[
\langle v, v' \rangle = (\tilde{L}v, \tilde{L}v).
\]
This scalar product is indeed positive definite, because for any \( v \neq 0 \),
\[
\langle v, v \rangle = (\tilde{L}v, \tilde{L}v) \geq \tilde{\Gamma}^{-2} (v, v) \neq 0,
\]
where in the last step we applied (11.2.3). Forming the completion, we obtain the Hilbert space \( (\mathcal{H}, (\cdot, \cdot)) \). We denote the corresponding norm by \( \| \cdot \| \). In view of (11.2.3) and (11.3.12), we know that every vector \( v \in \mathcal{H} \) is a function in \( L^2(R_T, d^n x) \). Moreover, we know from (11.3.12) that \( \tilde{L}v \) is also in \( L^2(R_T, d^n x) \). We remark that in functional analytic language, the space \( \mathcal{H} \) can be identified with the Sobolev space \( W^{1,2}(R_T) \), but we do not need this here.

We now consider for \( w \in C^0(R_T) \) and \( v \in C^1(R_T) \) the linear functional \( (v, w) \). In view of the estimate
\[
|\langle v, w \rangle| \leq \|v\| \|w\| \leq \tilde{\Gamma} \|w\| \|v\|,
\]
this functional is continuous in \( v \in \mathcal{H} \). The Fréchet-Riesz theorem shows that there is \( U \in \mathcal{H} \) with
\[
\langle v, w \rangle_{L^2(R_T)} = \langle v, U \rangle = (\tilde{L}v, \tilde{L}U) \quad \text{for all } v \in \mathcal{H}.
\]
Hence the function $u := \tilde{L}u \in L^2(R_T, d^m x)$ satisfies the equation (11.3.10) and is thus the desired weak solution. Note that all our methods apply for arbitrarily large $T$. We have thus proved the global existence of weak solutions.

We next want to show that the solutions are smooth. Thus our task is to show that our constructed weak solution $u$ is of the class $C^s(R_T)$, where $s \geq 1$ can be chosen arbitrarily large. We first show that a linear symmetric hyperbolic system can be "enlarged" to include the partial derivatives of $\phi$.

**Lemma 11.3.1.** Suppose that the system $A^j \partial_j u + Bu = w$ is symmetric hyperbolic. Then there is a symmetric hyperbolic system of the form

$$\tilde{A}^j \partial_j \Psi + \tilde{B} \Psi = \tilde{w}$$

for the vector $\Psi := (\partial_t u, \nabla_1 u, \ldots, \nabla_m u, u) \in \mathbb{C}^{n+1}$.

**Proof.** Let $i$ be a fixed space-time index. We differentiate the equation $Lu = w$,

$$\partial_i w = \partial_i Lu = L \partial_i u + (\partial_i A^j) \partial_j u + (\partial_i B) u.$$ 

This equation can be written as

$$\tilde{A}^j \partial_j \Psi_i + \tilde{B}_i \Psi_j + (\partial_i B) u = \tilde{w}_i,$$

where we set

$$\tilde{B}_i = B \delta_i + (\partial_i A^j) \quad \text{and} \quad \tilde{w}_i = \partial_i w.$$ 

Combining these equations with the equation $Lu = w$, we obtain a system of the form (11.3.13), where the matrices $\tilde{A}^j$ are block diagonal in the sense that

$$\tilde{A}^j = ((\tilde{A}^j)^\alpha_{\beta})_{\alpha, \beta = 0, \ldots, m+1} \quad \text{with} \quad (\tilde{A}^j)^\alpha_{\beta} = A^j \delta_{\beta}^{\alpha}.$$ 

Obviously, this system is again symmetric hyperbolic. \hfill $\square$

Iterating this lemma, we obtain (at least in principle) a symmetric hyperbolic system for $u$ and all its partial derivatives up to any given order $s$. Since the corresponding weak solution is in $L^2(R_T)$, we conclude that $u$ and all its weak partial derivatives are square integrable. The next lemma, which is a special case of the general Sobolev embedding theorems (see for example [23] Section II.5, or [94] Section 4), gives smoothness of the solution.

**Lemma 11.3.2.** Let $s > \frac{m}{2}$ be an integer. If a function $g$ on $\mathbb{R}^m$ is $s$ times weakly differentiable and

$$\int_{\mathbb{R}^m} |\nabla^\alpha g|^2 d^m x < C$$

for all multi-indices $\alpha$ with $|\alpha| \leq s$, then $g$ is bounded, $g \in L^\infty(\mathbb{R}^m)$. Likewise, if $g$ is $s + l + 1$ times weakly differentiable with $l \geq 1$ and (11.3.14) holds for all $\alpha$ with $|\alpha| \leq s + l + 1$, then $g \in C^l(\mathbb{R}^m)$. 

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Proof. We apply the Schwarz inequality to the Fourier transform,

\[ |g(x)|^2 = \left| \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} \hat{g}(k) e^{-ikx} \right|^2 \]

\[ = \left| \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} (1 + |k|^2)^{-\frac{s}{2}} (1 + |k|^2)^{\frac{s}{2}} \hat{g}(k) e^{-ikx} \right|^2 \]

\[ \leq c_m \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} (1 + |k|^2)^s |\hat{g}(k)|^2 , \]

where the constant \( c_m \) is finite due to our choice of \( s \),

\[ c_m = \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} (1 + |k|^2)^{-s} < \infty . \]

Using the Plancherel formula together with the fact that a factor \( k^2 \) corresponds to a Laplacian in position space, we obtain

\[ \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} (1 + |k|^2)^s |\hat{g}(k)|^2 = \sum_{\ell=0}^{s} \left( \frac{n}{\ell} \right) \| \nabla^\ell g \|_{L^2(\mathbb{R}^m)}^2 < c . \]

Hence \( \sqrt{c_m} \) is an \( L^\infty \)-bound for \( g \).

Next, if \( g \) is \( s + 1 \) times weakly differentiable, then \( \| Dg \|_{L^\infty(\mathbb{R}^m)} < c \). As a consequence, the mean value theorem yields \( |g(x) - g(y)| \leq c|x - y| \), so that \( g \) is Lipschitz continuous. Finally, if \( g \) is \( s + l + 1 \) times weakly differentiable, then all partial derivatives \( \nabla^\alpha g \) of order \( |\alpha| \leq l \) are Lipschitz continuous, so that \( g \in C^l(\mathbb{R}^m) \).

More precisely, in order to apply this lemma, we fix a time \( t \) and consider the solution \( u(\lambda, .) \). The identity (11.3.5) implies that \( E(\lambda) \) is controlled in terms of \( \|w\| \) and \( \|u\| \).

After interatively applying Lemma (11.3.1) we conclude that the weak derivatives of \( u(\lambda, .) \) exist to any order and are in \( L^2(\mathbb{R}^m) \). It follows that \( u(\lambda, .) \) is smooth. Finally, one uses the equation to conclude that \( u \) is also smooth in the time variable.

The results of this section can be summarized as follows.

Theorem 11.3.3. Consider the Cauchy problem

\[ (A^0 \partial_t + \sum_{a=1}^{m} a^0 \nabla_a + B) u = w \in C^\infty([0, T] \times \mathbb{R}^m) , \quad u|_{t=0} = u_0 \in C^\infty(\mathbb{R}^m) . \]

Assume that the matrices \( A^0, A^1 \) and \( B \) as well as the functions \( w \) and \( u_0 \) are smooth. Moreover, assume that all these functions as well as all their partial derivatives are uniformly bounded (where the bound may depend on the order of the derivatives). Then the Cauchy problem has a smooth solution on \([0, T] \times \mathbb{R}^m \).

This theorem also applies in the case \( T = \infty \), giving global existence of a smooth solution.

We finally show that the solutions depend smoothly on parameters.

Corollary 11.3.4. Suppose that the matrices \( A^1, B \) and the functions \( w, u_0 \) depend smoothly on a parameter \( \lambda \). Then the family of solutions \( u(\lambda) \) is also smooth in \( \lambda \).

Proof. First, similar as explained after (11.3.3), we may restrict attention to the case \( u_0 = 0 \). Differentiating the equation \( Lu = w \) with respect to \( \lambda \), we obtain

\[ Lu_\lambda = (\partial_\lambda L) u + \partial_\lambda w =: \tilde{w} , \]

where \( u_\lambda \) stands for the formal derivative \( \partial_\lambda u \). This is a symmetric hyperbolic system for \( u_\lambda \). According to Theorem 11.3.3 we know that \( u \) and therefore \( \tilde{w} \) are smooth.
Hence, again applying Theorem 11.3.3, we conclude there exists a smooth solution \( u_\lambda \). Considering the limit of the difference quotients, one verifies that \( u_\lambda \) really coincides with \( \partial_\lambda u(\lambda) \) for our given family of solutions \( u(\lambda) \). The higher \( \lambda \)-derivatives can be treated inductively.

11.4. The Cauchy Problem in Globally Hyperbolic Space-Times

In this section, we explain how symmetric hyperbolic systems can be used for solving the Cauchy problem for the Dirac equation in globally hyperbolic space-times. The reader not interested in curved space-time may skip this section.

Proof of Theorem 3.5.2

11.5. Existence of Causal Green’s Operators

We now return to the Dirac equation in an external potential \( \mathcal{B} \). Then the previous existence and uniqueness results can be stated in a slightly stronger version.

Theorem 11.5.1. Consider the Cauchy problem for the Dirac equation (15.0.1) for a smooth potential \( \mathcal{B} \). Then there is a unique global solution \( \psi \in \mathcal{C}^\infty(M, SM) \).

Proof. It remains to show that we do not need to assume that \( \mathcal{B} \) and \( \psi_0 \) as well as all their partial derivatives are uniformly bounded. The reason is that, as explained at the end of Section 11.2, the propagation speed is the speed of light. Therefore, in the proof of Corollary 11.2.5, we can choose the lens-shaped region independent of \( \mathcal{B} \), making it unnecessary to impose bounds on \( \mathcal{B} \) and its partial derivatives. For the existence of solutions, in order to construct \( \psi(x) \) it suffices to consider the equation in the compact region \( J^\wedge(x) \cap \{ t \geq t_0 \} \). On this compact region, smoothness immediately gives uniform bounds for \( \mathcal{B} \), \( \psi \) and all their partial derivatives.

Next, we explain how the previous existence and uniqueness results give rise to the existence of causal Green’s functions, being defined as distributions. Our main tool is the Schwartz kernel theorem (see [74, Section 5.2] or [94, Section 4.6]). For clarity, we denote the objects in the external potential with an additional tilde. We begin with the causal fundamental solution and generalize Lemma ??.

Theorem 11.5.2. Assume that the external potential \( \mathcal{B} \) is smooth and that \( \mathcal{B} \) and all its partial derivatives are uniformly bounded in Minkowski space. Then for any \( t, t_0 \) there is a unique distribution \( \tilde{k}_m(t, .; t_0, .) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3) \) such that the solution of the Cauchy problem (15.0.1) has the representation

\[
\psi(t, \vec{x}) = 2\pi \int_N \tilde{k}_m(t, \vec{x}; t_0, \vec{y}) \gamma^0 \psi_0(\vec{y}) \, d^3 y. \tag{11.5.1}
\]

The integral kernel \( k_m \) is also a distribution in space-time, \( k_m \in \mathcal{D}'(M \times M) \) It is a distributional solution of the Dirac equation,

\[
(i\partial_x + \mathcal{B} - m) \tilde{k}_m(x, y) = 0. \tag{11.5.2}
\]

Proof. The energy estimates combined with the Sobolev embedding of Lemma 11.3.2 showed that there is \( k \in \mathbb{N} \) and a constant \( C = C(t, t_0, \vec{x}, \mathcal{B}) \) such that the solution \( \psi(t, .) \) of the Cauchy problem is bounded in terms of the initial data by

\[
|\psi(t, \vec{x})| \leq C |\psi_0|_{C^k}, \tag{11.5.3}
\]
where \(|\psi|^2 := \langle \psi | \gamma^0 \psi \rangle\), and the \(C^k\)-norm is defined by
\[
|\psi|_{C^k} = \max_{|\beta| \leq k} \sup_{\vec{x} \in \mathbb{R}^3} |\nabla^\beta \psi(\vec{x})|.
\]
Moreover, this estimate is locally uniform in \(\vec{x}\), meaning that for any compact set \(K \subset \mathbb{R}^3\), there is a constant \(C\) such that (11.5.3) holds for all \(\vec{x} \in K\). This makes it possible to apply the Schwartz kernel theorem [24, Theorem 5.2.1], showing that \(\tilde{k}_m(t, ; t_0, .) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)\).

Next, we note that the constant \(C\) in (11.5.3) can also be chosen locally uniformly in \(t\) and \(t_0\). Thus, after evaluating weakly in \(t\) and \(t_0\), we may again apply the Schwartz kernel theorem to obtain that \(\tilde{k}_m \in \mathcal{D}'(M \times M)\). Finally, the distributional equation (11.5.2) follows immediately from the fact that (11.5.1) is satisfies the Dirac equation for any choice of \(\psi_0\).

Having defined the causal fundamental solution, we can introduce the causal Green’s functions by decomposing \(\tilde{k}_m\) in time in such a way that the relation (11.4.2) extends to the setting in an external potential. Namely, for any \(t, t_0\) we introduce the distribution \(\tilde{s}^\vee_m(t, ; t_0, .), \tilde{s}^\wedge_m(t, ; t_0, .) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)\) by
\[
\left\{
\begin{array}{ll}
\tilde{s}^\vee_m(t, ; t_0, .) = 2\pi i \tilde{k}_m(t, ; t_0, .) \Theta(t_0 - t) \\
\tilde{s}^\wedge_m(t, ; t_0, .) = -2\pi i \tilde{k}_m(t, ; t_0, .) \Theta(t_0 - t)
\end{array}
\right.
\]
(11.5.4)
(where \(\Theta\) denotes the Heaviside function).

**Theorem 11.5.3.** Assume that the external potential \(B\) is smooth and that \(B\) and all its partial derivatives are uniformly bounded in Minkowski space. Then there are unique distributions
\[
\tilde{s}^\vee_m, \tilde{s}^\wedge_m \in \mathcal{D}'(M \times M)
\]
which satisfy the distributional equations
\[
(i\partial_x + B - m) s_m(x, y) = \delta^4(x - y) \quad (11.5.5)
\]
and are supported in the upper respectively lower light cone,
\[
\text{supp } \tilde{s}^\vee_m(x, .) \subset J^+ \quad \text{supp } \tilde{s}^\wedge_m(x, .) \subset J^- .
\]
(11.5.6)

**Proof.** It is clear by construction and the fact that the constant \(C\) in (11.5.3) can be chosen locally uniformly in \(x\) and \(y\) that the causal Green’s functions are well-defined distributions in \(\mathcal{D}'(M \times M)\). The support property (11.5.6) follows immediately from finite propagation speed as explained at the end of Section 11.2. The uniqueness of the Green’s functions is clear from the uniqueness of solutions of the Cauchy problem. In order to derive the distributional equations (11.5.5), we only consider the retarded Green’s function (the argument for the advanced Green’s function is analogous). Then, according to (11.5.1) and (11.5.4),
\[
\Theta(t_0 - t) \psi(t, \vec{x}) = i \int_N \tilde{s}^\vee_m(t, \vec{x}; t_0, \vec{y}) \gamma^0 \psi_0(\vec{y}) d^3 y ,
\]
where \(\psi\) is the solution of the corresponding Cauchy problem. Applying the Dirac operator in the distributional sense yields
\[
i \gamma^0 \delta(t_0 - t) \psi_0(t, \vec{x}) = i(D_x - m) \int_N \tilde{s}^\vee_m(t, \vec{x}; t_0, \vec{y}) \gamma^0 \psi_0(\vec{y}) d^3 y .
\]
We now choose the initial values as the restriction of a test function in space-time, \(\psi_0 = \phi|_{t=t_0}\) with \(\phi \in C_0^\infty(M, SM)\). Then we can integrate over \(t_0\) to obtain.
11.6. A Polynomial Estimate in Time

We now derive an estimate which shows that the solutions of the Dirac equation increase at most polynomially in time. This result will be needed in Section 15.2.1. For the proof we adapt standard methods of the theory of partial differential equations to the Dirac equation. In generalization of (11.6.1), we denote the spatial Sobolev norms by

\[ \| \phi \|_{W^{a,2}}^2 = \sum_{\alpha \text{ with } |\alpha| \leq a} \int_{\mathbb{R}^3} |\nabla^\alpha \phi(x)|^2 \, dx. \]

**Lemma 11.6.1.** We are given two non-negative integers \( a \) and \( b \) as well as a smooth time-dependent potential \( \mathcal{B} \). In the case \( a > 0 \) and \( b \geq 0 \), we assume furthermore that the spatial derivatives of \( \mathcal{B} \) decay faster than linearly for large times in the sense that

\[ |\nabla \mathcal{B}(t)|_{C^{a-1}} \leq \frac{c}{1 + |t|^{1+\varepsilon}} \quad (11.6.1) \]

for suitable constants \( c, \varepsilon > 0 \). Then there is a constant \( C = C(c, \varepsilon, a, b) \) such that every family of solutions \( \psi \in \mathcal{H}^\infty \) of the Dirac equation (1.3.13) for varying mass parameter can be estimated for all times in terms of the boundary values at \( t = 0 \) by

\[ \| \partial_m^a \psi_m |t| \|_{W^{a,2}} \leq C \left( 1 + |t|^b \right) \sum_{p=0}^{b} \| \partial_m^p \psi_m |t=0\|_{W^{a,2}}. \]

**Proof.** We choose a multi-index \( \alpha \) of length \( a := |\alpha| \) and a non-negative integer \( b \). Differentiating the Dirac equation (1.3.13) with respect to the mass parameter and to the spatial variables gives

\[ (i\partial + \mathcal{B} - m) \nabla^\alpha \partial_m^b \psi_m = b \nabla^\alpha \partial_m^{b-1} \psi_m - \nabla^\alpha (\mathcal{B} \partial_m^b \psi_m) + \mathcal{B} \nabla^\alpha \partial_m^b \psi_m. \]

Introducing the abbreviations

\[ \Xi := \nabla^\alpha \partial_m^b \psi_m \quad \text{and} \quad \phi := b \nabla^\alpha \partial_m^{b-1} \psi_m - \nabla^\alpha (\mathcal{B} \partial_m^b \psi_m) + \mathcal{B} \nabla^\alpha \partial_m^b \psi_m, \]

we rewrite this equation as the inhomogeneous Dirac equation

\[ (\mathcal{D} - m) \Xi = \phi. \]

A calculation similar to current conservation yields

\[ -i\partial_j <\Xi | \gamma^j \Xi> = <(\mathcal{D} - m) \Xi | \Xi> - <\Xi | (\mathcal{D} - m) \Xi> = <\phi | \Xi> - <\Xi | \phi>. \]

Integrating over the equal time hypersurfaces and using the Schwarz inequality, we obtain

\[ |\partial_t (\Xi | \Xi) |_t \leq 2 \| \Xi | \_t \| \| \phi | \_t \|, \]

and thus

\[ |\partial_t \| \Xi | \_t \| \leq \| \phi | \_t \|. \]

Substituting the specific forms of \( \Xi \) and \( \phi \) and using the Schwarz and triangle inequalities, we obtain the estimate

\[ |\partial_t \| \nabla^\alpha \partial_m^b \psi_m |t| \_t \| \leq b \| \nabla^\alpha \partial_m^{b-1} \psi_m |t| \_t \| + c a \| \nabla \mathcal{B}(t) |_{C^{a-1}} \| \partial_m^b \psi_m |t| \|_{W^{a-1,2}}, \quad (11.6.2) \]

where we used the notation (15.2.2).
We now proceed inductively in the maximal total order \( a + b \) of the derivatives. In the case \( a = b = 0 \), the claim follows immediately from the unitarity of the time evolution. In order to prove the induction step, we note that in (11.6.2), the order of differentiation of the wave function on the right hand side is smaller than that on the left hand side at least by one. In the case \( a = 0 \) and \( b \geq 0 \), the induction hypothesis yields the inequality

\[
|\partial_t \| \partial^b_m \psi_m |_t \| | \leq b \| \partial^{b-1}_m \psi_m |_t \| \leq b C \left( 1 + |t|^{b-1} \right) \sum_{p=0}^{b-1} \| \partial^p_m \psi_m |_{t=0} \|
\]

and integrating this inequality from 0 to \( t \) gives the result. In the case \( a > 0 \) and \( b \geq 0 \), we apply (11.6.1) together with the induction hypothesis to obtain

\[
|\partial_t \| \partial^b_m \psi_m |_t \|_{W^{a,2}} | \leq b C \left( 1 + |t|^{b-1} \right) \sum_{p=0}^{b-1} \| \partial^p_m \psi_m |_{t=0} \|_{W^{a,2}}
\]

\[
+ c C \frac{1 + |t|^b}{1 + |t|^{1+\epsilon}} \sum_{p=0}^{b} \| \partial^p_m \psi_m |_{t=0} \|_{W^{a-1,2}}
\]

Again integrating over \( t \) gives the result. \( \square \)

Exercises
CHAPTER 12

Energy Methods for the Linearized Field Equations

• Mein Vorschlag: Wir sollten hier die Konstruktionen aus \[17\] einbauen. Das zeigt gut die Analogie zu den hyperbolischen PDEs aus Kapitel \[11\]

• Die vorbereitenden Konstruktionen aus \[17\] (Zeitrichtung, asymptotischer Abfall von Jets im Unendlichen) sollten wir vielleicht in Teil 2 integrieren.

• Man sollte auch die Energie-Abschätzungen des vorherigen Kapitels so anpassen, dass alles gut zusammenpasst.

Exercises
Functional Analytic Methods in Space-Time

When constructing a causal fermion system in Minkowski space in Section 4.3, we chose \( \mathcal{H} \) as a subspace of the solution space \( \mathcal{H}_m \) of the Dirac equation. In principle, one can choose \( \mathcal{H} \) as one likes, and different choices give rise to different causal fermion systems. If one wants to describe a physical system, one must specify the subspace \( \mathcal{H} \subset \mathcal{H}_m \), and it is important to do it right. It is not obvious what “right” and “wrong” should be. Generally speaking, \( \mathcal{H} \) can be thought of as the “occupied states” of the physical system in mind. If we want to describe the vacuum in Minkowski space (i.e. no particles and no interaction is present), then the natural and only physically reasonable choice is to let \( \mathcal{H} \) be the subspace of all negative-frequency solutions of the Dirac equation. This choice corresponds to the physical concept of the Dirac sea as introduced by Dirac in 1930, which led to the prediction of anti-particles (discovered shortly afterwards in 1932, earning Dirac the Nobel prize in 1933). Following these physical concepts, it is also clear that if particles and anti-particles (but no interaction of the matter) is present, then \( \mathcal{H} \) is obtained from the subspace of all negative-frequency solutions by occupying additional particle states and by creating “holes” in the sea corresponding to the anti-particle states. Once an interaction (for example an electromagnetic field) is present, it is no longer clear how \( \mathcal{H} \) is to be chosen. The reason is that as soon as the fields are time-dependent, the notion of positive and negative frequency solutions breaks down, so that there is no obvious decomposition of the solution space into two subspaces. But for the description of the physical system, a decomposition of the solution space is needed, and taking the “wrong” decomposition makes no physical sense and also gives rise to serious artificial mathematical difficulties.

We now explain a functional analytic method which gives rise to a canonical decomposition of the solution space into two subspaces, even in the time-dependent situation. In the static situation, this splitting reduces to the canonical frequency splitting. This splitting is “right” in the sense that it gives rise to a physically reasonable ground state of the system (a so-called Hadamard state, as we will learn in Chapter 16 later in this lecture). Moreover, when performing our construction perturbatively, one can compute the singularities of \( P(x, y) \) explicitly working exclusively with bounded line integrals. These explicit computations are the backbone of the analysis of the continuum limit in [36]. Here we shall not enter the perturbation expansion and all the explicit computations. Instead, we restrict attention to the functional analytic constructions.

13.1. General Setting and Basic Ideas

We now recall our setting and explain the idea of the construction. We first summarize the structures of Section 1 using a more general notation, which has the advantage that it can be used just as well if Minkowski space is replaced by a globally hyperbolic spacetime. Thus the reader who is familiar with general relativity and Lorentzian geometry, in what follows can consider \((\mathcal{M}, g)\) as a globally hyperbolic Lorentzian manifold with
spinor bundle \((\mathcal{S}M,\langle\cdot,\cdot\rangle)\). The Dirac equation is written as
\[
(D - m)\psi_m = 0
\]
(13.1.1)
(here the subscript \(m\) indicates the mass of the solution; this is of advantage because later on, we shall consider families of solutions with a varying mass parameter). In Minkowski space, one chooses \(D = i\partial + B\) such as to get back to (1.3.13). In a globally hyperbolic space-time, the Dirac operator is a first order differential operator, but the coefficients depend on the metric (for details see Chapter 3). Next, we let \(\mathcal{N}\) be any Cauchy surface. Then the scalar product (1.3.11) on the solutions can be written more generally as
\[
(\psi_m|\phi_m)_m = 2\pi \hat{\mathcal{N}} \langle \psi_m|\phi_m \rangle_x d\mu_N(x),
\]
(13.1.2)
where \(\nu\) its future-directed normal and \(d\mu_N\) is the volume measure given by the induced Riemannian metric on \(\mathcal{N}\) (in Minkowski space and \(\mathcal{N} = \{t = \text{const}\}\), the normal has the components \(\nu^i = (1,0,0,0)\) and \(d\mu_N = d^3x\), giving back (1.3.11)). Similar to the computation (1.3.9), the vector field \(\langle \psi_m|\gamma^j\phi_m \rangle_x\) is again divergence-free, implying that this scalar product is independent of the choice of the Cauchy surface (for details see [56, Section 2]). Forming the completion gives the Hilbert space \((\mathcal{H}_m,\langle\cdot,\cdot\rangle_m)\).

For the following constructions, we make use of another structure which was not used so far. Namely, given two wave functions \(\psi\) and \(\phi\) (not necessarily solutions of the Dirac equation), one can integrate their pointwise inner product \(\langle \psi|\phi \rangle_x\) over space-time.

In order for this integral to be well-defined, one can proceed for example as follows. We denote the smooth sections of the spinor bundle by \(C^\infty(\mathcal{M},\mathcal{S}M)\). Similarly, \(C^\infty_0(\mathcal{M},\mathcal{S}M)\) denotes the smooth sections with compact support. On the wave functions, one has the Lorentz invariant inner product
\[
\langle \cdot,\cdot \rangle : C^\infty(\mathcal{M},\mathcal{S}M) \times C^\infty_0(\mathcal{M},\mathcal{S}M) \to \mathbb{C},
\]
\[
\langle \psi|\phi \rangle = \int_\mathcal{M} \langle \psi|\phi \rangle_x d\mu_M.
\]
(13.1.3)

In order to explain the basic idea of the construction as first given in [56], let us assume for simplicity that the integral in (13.1.3) exists for all solutions \(\psi_m,\phi_m \in \mathcal{H}_m\). This condition is not satisfied in Minkowski space because the time integral in (13.1.3) in general diverges. But it is indeed satisfied in space-times of finite lifetime (for details see [56, Section 3.2]). Then the space-time inner product can be extended by continuity to a bilinear form
\[
\langle \cdot,\cdot \rangle : \mathcal{H}_m \times \mathcal{H}_m \to \mathbb{C},
\]
which is bounded, i.e.
\[
|\langle \phi_m|\psi_m \rangle| \leq c \|\phi_m\|_m \|\psi_m\|_m
\]
(13.1.4)
(where \(\|\cdot\|_m = (\cdot,\cdot)^{\frac{1}{2}}_m\) is the norm on \(\mathcal{H}_m\)). Then, applying the Fréchet-Riesz theorem, we can uniquely represent this inner product with a signature operator \(S\),
\[
S : \mathcal{H}_m \to \mathcal{H}_m \quad \text{with} \quad \langle \phi_m|\psi_m \rangle = (\phi_m|S|\psi_m)_m.
\]
(13.1.5)
We refer to \(S\) as the **fermionic signature operator**. It is obviously a symmetric operator. Moreover, it is bounded according to (13.1.4). We conclude that it is self-adjoint. The spectral theorem gives the spectral decomposition
\[
S = \int_{\sigma(S)} \lambda dE_\lambda,
\]
where \( E_\lambda \) is the spectral measure (see for example \([86]\)). The spectral measure gives rise to the spectral calculus
\[
f(S) = \int_{\sigma(S)} f(\lambda) \, dE_\lambda : \mathcal{H}_m \to \mathcal{H}_m,
\]
where \( f \) is a bounded Borel function on \( \sigma(S) \subset \mathbb{R} \). Choosing \( f \) as a characteristic function, one obtains the operators \( \chi_{(0,\infty)}(S) \) and \( \chi_{(-\infty,0)}(S) \). Their images are referred to as the positive and negative spectral subspace of \( \mathcal{H}_m \), respectively. In this way, one obtains the desired decomposition of the solution space into two subspaces. We remark that the fermionic signature operator also gives a setting for doing spectral geometry and index theory with Lorentzian signature. We will not enter this subject here but refer the interested reader to the paper \([42, 38]\).

The basic shortcoming of the above construction is that in many physically interesting space-times (like Minkowski space) the inequality \((13.1.4)\) fails to be true. The idea to bypass this problem is to make use of the fact that a typical solution \( \psi \in C^\infty(\mathcal{M}, S\mathcal{M}) \cap \mathcal{H}_m \) of the Dirac equation oscillates for large times. If we consider families of solutions with varying mass parameter, then the wave functions for different values of \( m \) typically have different phases. Therefore, integrating over the mass parameter leads to destructive interference, giving rise to decay in time. In order to make this idea mathematically precise, one considers families of solutions \( (\psi_m)_{m \in I} \) of the family of Dirac equations \((13.1.1)\) with the mass parameter \( m \) varying in an open interval \( I \). We need to assume that \( I \) does not contain the origin, because our methods for dealing with infinite lifetime do not apply in the massless case \( m = 0 \) (this seems no physical restriction because all known fermions in nature have a non-zero rest mass). By symmetry, it suffices to consider positive masses. Thus we choose
\[
I := (m_L, m_R) \subset \mathbb{R} \quad \text{with parameters } m_L, m_R > 0. \quad (13.1.6)
\]
We always choose the family of solutions \( (\psi_m)_{m \in I} \) in the class \( C^\infty_{sc,0}(\mathcal{M} \times I, S\mathcal{M}) \) of smooth solutions with spatially compact support in Minkowski space \( \mathcal{M} \) which depend smoothly on \( m \) and vanish identically for \( m \) outside a compact subset of \( I \). Then the “decay due to destructive interference” can be made precise by demanding that there is a constant \( c > 0 \) such that
\[
|\langle \int_I \phi_m \, dm \mid \int_I \psi_m \, dm' \rangle| \leq c \int_I \|\phi_m\|_m \|\psi_m\|_m \, dm \quad (13.1.7)
\]
for all families of solutions \( (\psi_m)_{m \in I}, (\phi_m)_{m \in I} \in C^\infty_{sc,0}(\mathcal{M} \times I, S\mathcal{M}) \). The point is that we integrate over the mass parameter before taking the space-time inner product. Intuitively speaking, integrating over the mass parameter generates a decay of the wave function, making sure that the time integral converges. The inequality \((13.1.7)\) is one variant of the so-called mass oscillation property. If \((13.1.7)\) holds, we shall prove that there is a representation
\[
|\langle \int_I \phi_m \, dm \mid \int_I \psi_m \, dm' \rangle| = \int_I \langle \phi_m \mid \hat{\delta}_m \psi_m \rangle \, dm,
\]
which for every \( m \in I \) uniquely defines the fermionic signature operator \( \hat{\delta}_m \). This operator is bounded and symmetric with respect to the scalar product \((13.1.2)\). Moreover, it does not depend on the choice of the interval \( I \). Now the positive and negative spectral subspaces of the operator \( \hat{\delta}_m \) again yield the desired splitting of the solution space into two subspaces.
13.2. The Mass Oscillation Properties

In a space-time of infinite life time, the space-time inner product $\langle \psi_m | \phi_m \rangle$ of two solutions $\psi_m, \phi_m \in \mathcal{H}_m$ is in general ill-defined, because the time integral in (13.1.3) may diverge. In order to avoid this difficulty, we shall consider families of solutions with a variable mass parameter. The so-called mass oscillation property will make sense of the space-time integral in (13.1.3) after integrating over the mass parameter.

We consider the mass parameter in a bounded open interval $I$ (13.1.6). For a given Cauchy surface $\mathcal{N}$, we consider a function $\psi_N(x, m) \in S_xM$ with $x \in \mathcal{N}$ and $m \in I$. We assume that this wave function is smooth and has compact support in both variables, $\psi_N \in C_0^\infty(\mathcal{N} \times I, SM)$. For every $m \in I$, we let $\psi(., m)$ be the solution of the Cauchy problem for initial data $\psi_N(., m)$,

$$(D - m) \psi(x, m) = 0, \quad \psi(x, m) = \psi_N(x, m) \quad \forall x \in \mathcal{N}. \quad (13.2.1)$$

Since the solution of the Cauchy problem is smooth and depends smoothly on parameters, we know that $\psi \in C^\infty(\mathcal{M} \times I, SM)$. Moreover, due to finite propagation speed, $\psi(., m)$ has spatially compact support. Finally, the solution is clearly compactly supported in the mass parameter $m$. We summarize these properties by writing

$$\psi \in C^\infty_{sc, 0}(\mathcal{M} \times I, SM), \quad (13.2.2)$$

where $C^\infty_{sc, 0}(\mathcal{M} \times I, SM)$ denotes the smooth wave functions with spatially compact support which are also compactly supported in $I$. We often denote the dependence on $m$ by a subscript, $\psi_m(x) := \psi(x, m)$. Then for any fixed $m$, we can take the scalar product (13.1.2). On families of solutions $\psi, \phi \in C^\infty_{sc, 0}(\mathcal{M} \times I, SM)$ of (13.2.1), we introduce a scalar product by integrating over the mass parameter,

$$\langle \psi | \phi \rangle := \int_I (\psi_m | \phi_m)_m \, dm \quad (13.2.3)$$

(where $dm$ is the Lebesgue measure). Forming the completion gives the Hilbert space $(\mathcal{H}, (.|.) )$. It consists of measurable functions $\psi(x, m)$ such that for almost all $m \in I$, the function $\psi(., m)$ is a weak solution of the Dirac equation which is square integrable over any Cauchy surface. Moreover, this spatial integral is integrable over $m \in I$, so that the scalar product (13.2.3) is well-defined. We denote the norm on $\mathcal{H}$ by $|| \cdot ||$.

For the applications, it is useful to introduce a subspace of the solutions of the form (13.2.2):

**Definition 13.2.1.** We let $\mathcal{H}^\infty \subset C^\infty_{sc, 0}(\mathcal{M} \times I, SM) \cap \mathcal{H}$ be a subspace of the smooth solutions with the following properties:

(i) $\mathcal{H}^\infty$ is invariant under multiplication by smooth functions in the mass parameter,

$$\eta(\psi(x, m)) \in \mathcal{H}^\infty \quad \forall \psi \in \mathcal{H}^\infty, \eta \in C^\infty(I).$$

(ii) For every $m \in I$, the set $\mathcal{H}^\infty_m := \{ \psi(., m) \mid \psi \in \mathcal{H}^\infty \}$ is a dense subspace of $\mathcal{H}_m$,

$$\mathcal{H}^\infty_m \subset \mathcal{H}_m \quad \forall m \in I.$$

We refer to $\mathcal{H}^\infty$ as the **domain** for the mass oscillation property.

The simplest choice is to set $\mathcal{H}^\infty = C^\infty_{sc, 0}(\mathcal{M} \times I, SM) \cap \mathcal{H}$, but in some applications it is preferable to choose $\mathcal{H}^\infty$ as a proper subspace of $C^\infty_{sc, 0}(\mathcal{M} \times I, SM) \cap \mathcal{H}$.

Our motivation for considering a variable mass parameter is that integrating over the mass parameter should improve the decay properties of the wave function for large times (similar as explained in the introduction in the vacuum Minkowski space). This decay
for large times should also make it possible to integrate the Dirac operator in the inner product (13.1.3) by parts without boundary terms,

$$<Dp\psi|p\phi> = <p\psi|Dp\phi>,$$

implying that the solutions for different mass parameters should be orthogonal with respect to this inner product. Instead of acting with the Dirac operator, it is technically easier to work with the operator of multiplication by $m$, which we denote by

$$T : \mathcal{H} \to \mathcal{H}, \quad (T\psi)_m = m\psi_m.$$

In view of property (i) in Definition 13.2.1, this operator leaves $\mathcal{H}^\infty$ invariant,

$$T|_{\mathcal{H}^\infty} : \mathcal{H}^\infty \to \mathcal{H}^\infty.$$

Moreover, $T$ is a symmetric operator, and it is bounded because the interval $I$ is,

$$T^* = T \in L(\mathcal{H}).$$

Finally, integrating over $m$ gives the operation

$$p : \mathcal{H}^\infty \to C^\infty_{sc}(M,SM), \quad p\psi = \int_I \psi_m dm.$$

We point out for clarity that $p\psi$ no longer satisfies a Dirac equation. The following notions were introduced in [57], and we refer the reader to this paper for more details.

**Definition 13.2.2.** The Dirac operator $D = i\partial + B$ on Minkowski space $M$ has the weak mass oscillation property in the interval $I = (m_L, m_R)$ with domain $\mathcal{H}^\infty$ if the following conditions hold:

(a) For every $\psi, \phi \in \mathcal{H}^\infty$, the function $<p\phi|p\psi>$ is integrable on $M$. Moreover, there is a constant $c = c(\psi)$ such that

$$|<p\phi|p\psi>| \leq c\|\phi\| \quad \text{for all } \phi \in \mathcal{H}^\infty.$$  

(b) For all $\psi, \phi \in \mathcal{H}^\infty$,

$$<pT\psi|p\phi> = <p\psi|pT\phi>.$$  

**Definition 13.2.3.** The Dirac operator $D = i\partial + B$ on Minkowski space $M$ has the strong mass oscillation property in the interval $I = (m_L, m_R)$ with domain $\mathcal{H}^\infty$ if there is a constant $c > 0$ such that

$$|<p\psi|p\phi>| \leq c\int_I \|\phi_m\|_m \|\psi_m\|_m dm \quad \text{for all } \psi, \phi \in \mathcal{H}^\infty.$$  

**13.3. The Fermionic Signature Operator**

In this section we give abstract constructions based on the mass oscillation property. We first assume that the weak mass oscillation property of Definition 13.2.2 holds. Then, in view of the inequality (13.2.6), every $\psi \in \mathcal{H}^\infty$ gives rise to a bounded linear functional on $\mathcal{H}^\infty$. By continuity, this linear functional can be uniquely extended to $\mathcal{H}$. The Fréchet-Riesz theorem allows us to represent this linear functional by a vector $u \in \mathcal{H}$, i.e.

$$(u|\phi) = <p\psi|p\phi> \quad \forall \phi \in \mathcal{H}.$$  

Varying $\psi$, we obtain the linear mapping

$$S : \mathcal{H}^\infty \to \mathcal{H}, \quad (S\psi|\phi) = <p\psi|p\phi> \quad \forall \phi \in \mathcal{H}.$$  

This operator is symmetric because

$$(S\psi|\phi) = <p\psi|p\phi> = (\psi|S\phi) \quad \forall \phi, \psi \in \mathcal{H}^\infty.$$
Moreover, (13.2.7) implies that the operators $S$ and $T$ commute,
\[ ST = TS : \mathcal{H}^\infty \to \mathcal{H}. \] (13.3.1)
Thus the weak mass oscillation property makes it possible to introduce $S$ as a densely defined symmetric operator on $\mathcal{H}$. It is indeed possible to construct a self-adjoint extension of the operator $S^2$ (using the Friedrich's extension), giving rise to a functional calculus with corresponding spectral measure (for details see [57, Section 3]). In this setting the operator $S$ and the spectral measure are operators on the Hilbert space $\mathcal{H}$ which involves an integration over the mass parameter. In simple terms, this implies that all objects are defined only for almost all values of $m$ (with respect to the Lebesgue measure on $I \subset \mathbb{R}$), and they can be modified arbitrarily on subsets of $I$ of measure zero. But it does not seem possible to “evaluate pointwise in the mass” by constructing operators $S_m$ which act on the Hilbert space $\mathcal{H}_m$ for fixed mass.

In view of this shortcoming, we shall not enter the spectral calculus based on the weak mass oscillation operator. Instead, we move on to the strong mass oscillation property, which makes life much easier because it implies that $S$ is a bounded operator.

**Theorem 13.3.1.** The following statements are equivalent:

(i) The strong mass oscillation property holds.

(ii) There is a constant $c > 0$ such that for all $\psi, \phi \in \mathcal{H}^\infty$, the following two relations hold:
\[
|<p \psi | p \phi> | \leq c \| \psi \| \| \phi \| \] (13.3.2)
\[
<p T \psi | p \phi> = <p \psi | p T \phi>. \] (13.3.3)

(iii) There is a family of linear operators $S_m \in L(\mathcal{H}_m)$ which are uniformly bounded,
\[ \sup_{m \in I} \| S_m \| < \infty, \]
such that
\[
<p \psi | p \phi> = \int_I (\psi_m | S_m \phi_m)_m \, dm \quad \forall \psi, \phi \in \mathcal{H}^\infty. \] (13.3.4)

**Proof.** The implication (iii)⇒(i) follows immediately from the estimate
\[
|<p \psi | p \phi> | \leq \int_I |(\psi_m | S_m \phi_m)_m| \, dm \leq \sup_{m \in I} \| S_m \| \int_I \| \psi_m \| \| \phi_m \| \, dm.
\]
In order to prove the implication (i)⇒(ii), we first apply the Schwarz inequality to (13.2.8) to obtain
\[
|<p \psi | p \phi> | \leq c \int_I \| \phi_m \| \| \psi_m \| \, dm
\]
\[
\leq c \left( \int_I \| \phi_m \|_m^2 \, dm \right)^{\frac{1}{2}} \left( \int_I \| \psi_m \|_m^2 \, dm \right)^{\frac{1}{2}} = c \| \phi \| \| \psi \| ,
\]
proving (13.3.2). Next, for given $N \in \mathbb{N}$ we subdivide the interval $I = (m_L, m_R)$ by choosing the intermediate points
\[
m_\ell = \frac{\ell}{N} (m_R - m_L) + m_L, \quad \ell = 0, \ldots, N.
\]
Moreover, we choose non-negative test functions \( \eta_1, \ldots, \eta_N \in C_0^\infty(\mathbb{R}) \) which form a partition of unity and are supported in small subintervals, meaning that
\[
\sum_{\ell=1}^{N} \eta_{\ell}|_I = 1|_I \quad \text{and} \quad \text{supp} \eta_{\ell} \subset (m_{\ell-1}, m_{\ell+1}) ,
\]
where we set \( m_{-1} = m_L - 1 \) and \( m_{N+1} = m_R + 1 \). For any smooth function \( \eta \in C_0^\infty(\mathbb{R}) \) we define the bounded linear operator \( \eta(T) : \mathcal{H}^\infty \to \mathcal{H}^\infty \) by
\[
(\eta(T)\psi)_m = \eta(m) \psi_m .
\]
Then by linearity,
\[
\langle pT\psi|p\phi \rangle - \langle p\psi|pT\phi \rangle
= \sum_{\ell, \ell' = 1}^{N} \left( \langle pT\eta_\ell(T)\psi | p\eta_{\ell'}(T)\phi \rangle - \langle p\eta_\ell(T)\psi | pT\eta_{\ell'}(T)\phi \rangle \right)
= \sum_{\ell, \ell' = 1}^{N} \left( \langle p(T-m_\ell)\eta_\ell(T)\psi | p\eta_{\ell'}(T)\phi \rangle - \langle p\eta_\ell(T)\psi | p(T-m_\ell)\eta_{\ell'}(T)\phi \rangle \right).
\]
Taking the absolute value and applying (13.3.8), we obtain
\[
|\langle pT\psi|p\phi \rangle - \langle p\psi|pT\phi \rangle| \leq c \sum_{\ell, \ell' = 1}^{N} \int_I |m - m_\ell| \eta_\ell(m) \eta_{\ell'}(m) \| \phi_m \|_m \| \psi_m \|_m \ dm .
\]
In view of the second property in (13.3.5), we only get a contribution if \( |\ell - \ell'| \leq 1 \). Moreover, we know that \( |m - m_\ell| \leq 2 |I|/N \) on the support of \( \eta_\ell \). Thus
\[
|\langle pT\psi|p\phi \rangle - \langle p\psi|pT\phi \rangle| \leq \frac{6c |I|}{N} \sum_{\ell = 1}^{N} \int_I \eta_\ell(m) \| \phi_m \|_m \| \psi_m \|_m \ dm
= \frac{6c |I|}{N} \int_I \| \phi_m \|_m \| \psi_m \|_m \ dm .
\]
Since \( N \) is arbitrary, we obtain (13.3.3).

It remains to prove the implication (ii)\( \Rightarrow \) (iii). Combining (13.3.2) with the Fréchet-Riesz theorem, there is a bounded operator \( S \in L(\mathcal{H}) \) with
\[
\langle p\psi|p\phi \rangle = (\psi|S\phi) \quad \forall \psi, \phi \in \mathcal{H}^\infty .
\]
The relation (13.3.3) implies that the operators \( S \) and \( T \) commute. Moreover, these two operators are obviously symmetric and thus self-adjoint. Hence the spectral theorem for commuting self-adjoint operators implies that there is a spectral measure \( F \) on \( \sigma(S) \times I \) such that
\[
S^{p} T^{q} = \int_{\sigma(S) \times I} \nu^{p} m^{q} dF_{\nu,m} \quad \forall \ p, q \in \mathbb{N} .
\]
For given \( \psi, \phi \in \mathcal{H}^\infty \), we introduce the Borel measure \( \mu_{\psi,\phi} \) on \( I \) by
\[
\mu_{\psi,\phi}(\Omega) = \int_{\sigma(S) \times \Omega} \nu \ d(\psi|F_{\nu,m}\phi) .
\]
Then \( \mu_{\psi,\phi}(I) = (\psi|S\phi) \) and
\[
\mu_{\psi,\phi}(\Omega) = \int_{\sigma(S) \times \Omega} \nu \ d(\chi_{\Omega}(T) \psi | F_{\nu,m} \chi_{\Omega}(T) \phi) = (\chi_{\Omega}(T) \psi | S \chi_{\Omega}(T) \phi) .
\]
Since the operator $S$ is bounded, we conclude that
\[
|\mu_{\psi,\phi}(\Omega)| \leq c \|\chi_{\Omega}(T)\| \|\psi\| \|\phi\| \leq c \left( \int_{\Omega} \|\psi\|_m^2 \, dm \int_{\Omega} \|\phi\|_{m'}^2 \, dm' \right)^{1/2} \leq c|\Omega| \left( \sup_{m \in \Omega} \|\psi_m\|_m \right) \left( \sup_{m' \in \Omega} \|\phi_{m'}\|_{m'} \right). \tag{13.3.9}
\]
This shows that the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure. The Radon-Nikodym theorem (see Theorem 10.4.2) implies that there is a unique function $f_{\psi,\phi} \in L^1(I, dm)$ such that
\[
\mu_{\psi,\phi}(\Omega) = \int_{\Omega} f_{\psi,\phi}(m) \, dm. \tag{13.3.10}
\]
Using this representation in (13.3.9), we conclude that for any $\varphi \in \mathbb{R}$,
\[
\text{Re} \left( e^{i\varphi} \int_{\Omega} f_{\psi,\phi}(m) \, dm \right) \leq |\mu_{\psi,\phi}(\Omega)| \leq c|\Omega| \left( \sup_{m \in \Omega} \|\psi_m\|_m \right) \left( \sup_{m' \in \Omega} \|\phi_{m'}\|_{m'} \right).
\]
As a consequence, for almost all $m \in I$ (with respect to the Lebesgue measure $dm$),
\[
\text{Re} \left( e^{i\varphi} f_{\psi,\phi}(m) \right) \leq c \|\psi_m\|_m \|\phi_m\|_m.
\]
Since the phase factor is arbitrary, we obtain the pointwise bound
\[
|f_{\psi,\phi}(m)| \leq c \|\psi_m\|_m \|\phi_m\|_m \quad \text{for almost all } m \in I.
\]
Using this inequality, we can apply the Fréchet-Riesz theorem to obtain a unique operator $S_m \in L(H_m)$ such that
\[
f_{\psi,\phi}(m) = (\psi_m|S_m \phi_m)_m \quad \text{and} \quad \|S_m\| \leq c. \tag{13.3.11}
\]
Combining the above results, for any $\psi, \phi \in H^\infty$ we obtain
\[
\langle p_\psi | p_\phi \rangle = \langle \psi | S \phi \rangle = \int_{\sigma(S) \times \tilde{I}} \nu \, d(\psi | F_{\nu,m} \phi) = \int_{\tilde{I}} d\mu_{\psi,\phi} = \int_{\tilde{I}} f_{\psi,\phi}(m) \, dm = \int_{\tilde{I}} (\psi_m|S_m \phi_m)_m \, dm.
\]
This concludes the proof. □

Comparing the statement of Theorem 13.3.1 (ii) with Definition 13.2.2, we immediately obtain the following result.

**Corollary 13.3.2.** The strong mass oscillation property implies the weak mass oscillation property.

We next show uniqueness as well as the independence of the choice of the interval $I$.

**Proposition 13.3.3. (uniqueness of $S_m$)** The family $(S_m)_{m \in I}$ in the statement of Theorem 13.3.1 can be chosen such that for all $\psi, \phi \in H^\infty$, the expectation value $f_{\psi,\phi}(m) := (\psi_m|S_m \phi_m)_m$ is continuous in $m$,
\[
f_{\psi,\phi} \in C^0_0(I). \tag{13.3.12}
\]
The family $(S_m)_{m \in I}$ with the properties (13.3.4) and (13.3.12) is unique. Moreover, choosing two intervals $\tilde{I}$ and $I$ with $m \in \tilde{I} \subset I$ and $0 \notin \tilde{T}$, and denoting all the objects constructed in $\tilde{I}$ with an additional check, we have
\[
\tilde{S}_m = S_m. \tag{13.3.13}
\]
PROOF. Let us show that the function \( f_{\psi,\phi} \) is continuous. To this end, we choose a function \( \eta \in C^0_0(I) \). Then for any \( \varepsilon > 0 \) which is so small that \( B_\varepsilon(\text{supp} \eta) \subset I \), we obtain
\[
\int_I \left( f_{\psi,\phi}(m + \varepsilon) - f_{\psi,\phi}(m) \right) \eta(m) \, dm = \int_I f_{\psi,\phi}(m) \left( \eta(m - \varepsilon) - \eta(m) \right) \, dm
\]
which implies
\[
\left| \int_I \left( f_{\psi,\phi}(m + \varepsilon) - f_{\psi,\phi}(m) \right) \eta(m) \, dm \right| \leq \epsilon \| \psi_{m+\varepsilon} - \psi_m \| \sup_I |\eta|,
\]
where in (*) we used (13.3.7) and (13.3.8). Applying (13.3.2), we obtain
\[
\left| \int_I \left( f_{\psi,\phi}(m + \varepsilon) - f_{\psi,\phi}(m) \right) \eta(m) \, dm \right| \leq c \| \psi_{m+\varepsilon} - \psi_m \| \sup_I |\eta|,
\]
where the vector \( \psi_{m+\varepsilon} \) is defined by \( (\psi_{m+\varepsilon})_m := \psi_{m+\varepsilon} \). Since \( \lim_{\varepsilon \searrow 0} \| \psi_{m+\varepsilon} - \psi \| = 0 \) and \( \eta \) is arbitrary, we conclude that \( f_{\psi,\phi} \) is continuous (13.3.12). This continuity is important because it implies that the function \( f_{\psi,\phi} \) is uniquely defined pointwise (whereas in (13.3.10) this function could be modified arbitrarily on sets of measure zero).

In order to prove (13.3.13), we note that the representation (13.3.6) implies that
\[
(\psi|\hat S\phi) = (\psi|S\phi) \quad \text{for all } \psi, \phi \in \mathcal{H}^\infty.
\]
Using (13.3.8) and (13.3.10), it follows that
\[
\int_{\Omega} \hat f_{\psi,\phi}(m) \, dm = \int_{\Omega} f_{\psi,\phi}(m) \, dm \quad \text{for all } \Omega \subset \hat I.
\]
Choosing \( \hat f_{\psi,\phi}(m) \) and \( f_{\psi,\phi}(m) \) as continuous functions, we conclude that they coincide for every \( m \in \hat I \). It follows from (13.3.4) that the operators \( \hat S_m \) and \( S_m \) coincide. This concludes the proof.

**Exercises**

**Exercise 13.1.** This exercise illustrates the mass oscillation property. Let \( 0 < m_L < m_R \) and \( \eta \in C^\infty_0((m_L, m_R)) \). Show that the function \( f \) given by
\[
f(t) = \int_{m_L}^{m_R} \eta(m) e^{-i\sqrt{1+m^2} t} \, dm
\]
has rapid decay. Does this result remain valid if \( m_L \) and \( m_R \) are chosen to have opposite signs? Justify your finding by a proof or a counter example.

**Exercise 13.2.** Let \( \mathcal{M} \) be the “space-time strip”
\[
\mathcal{M} = \{(t, \vec{x}) \in \mathbb{R}^{1,3} \text{ with } 0 < t < T \}.
\]
Show that for any solution \( \psi \in C^\infty_{sc}(\mathcal{M}, S\mathcal{M}) \cap \mathcal{H}_m \) of the Dirac equation, the following inequality holds,
\[
|\langle \psi|\phi \rangle| \leq T \| \psi \|_m \| \phi \|_m.
\]
This estimate illustrates how in space-times of finite lifetime, the space-time inner product is a bounded sesquilinear form on \( \mathcal{H}_m \).

**Exercise 13.3.** Let \( \mathcal{M} \) again be the “space-time strip” of the previous exercise. As in the lecture, let \( \psi, \phi \in \mathcal{H} \cap C^\infty_{sc,0}(\mathcal{M} \times I, S\mathcal{M}) \) be families of smooth Dirac solutions of spatially compact support, with compact support in the mass parameter. Moreover, we again define the operators \( p, T : \mathcal{H} \to \mathcal{H} \) as in the lecture. Does the equation
\[
\langle pT\psi|p\phi \rangle = \langle p\psi|pT\phi \rangle
\]
(which appears in the weak mass oscillation property) in general hold? Justify your answer by a proof or a counter example.

Exercise 13.4. Let $\mathcal{M}$ again be the “space-time strip” of the previous exercises. Moreover, as in Exercise 4.6 we again let $\mathcal{H} \subset \mathcal{H}_m$ be a finite-dimensional subspace of the Dirac solution space $\mathcal{H}_m$, consisting of smooth wave functions of spatially compact support, i.e.

$$\mathcal{H} \subset C^\infty(\mathcal{M}, S\mathcal{M}) \cap H_m \text{ finite-dimensional}.$$ 

Show that the fermionic signature operator $S \in \text{L}(\mathcal{H})$ defined by

$$\langle \psi | \phi \rangle = (\psi | S \phi)_m \text{ for all } \psi, \phi \in \mathcal{H}$$

can be expressed within the causal fermion system by

$$S = - \int_M x \, d\rho(x)$$

(where $\rho$ is again the push-forward of $d\mu_M$).

Exercise 13.5. Let $E$ be the Banach space $E = C^0([0, 1], \mathbb{C})$ and $\Lambda : E \times E \to \mathbb{C}$ be sesquilinear, bounded and positive semi-definite.

(a) Assume that $\Lambda$ satisfies for a suitable constant $c > 0$ and all $f, g \in E$ the inequality

$$|\Lambda(f, g)| \leq c \sup_{x \in [0, 1]} |f(x) g(x)|.$$  \hfill (13.3.14)

Show that there is a regular bounded Borel measure $\mu$ such that

$$\Lambda(f, g) = \int_0^1 f(x) g(x) \, d\mu(x).$$

(b) Now make the stronger assumption that $\Lambda$ satisfies for a suitable constant $\tilde{c} > 0$ and all $f, g \in E$ the inequality

$$|\Lambda(f, g)| \leq \tilde{c} \int_0^1 |f(x) g(x)| \, dx.$$ \hfill (13.3.15)

Show that $\mu$ is absolutely continuous w.r.t. the Lebesgue measure. Show that there is a non-negative function $h \in L^1([0, 1], dx)$ such that

$$\Lambda(f, g) = \int_0^1 f(x) g(x) \, h(x) \, dx.$$ 

Show that $h$ is pointwise bounded by $c$.

(c) In order to clarify the different assumptions in this exercise, give an example for a sesquilinear, bounded and positive semi-definite functional $\Lambda$ which violates (13.3.14). Give an example which satisfies (13.3.14) but violates (13.3.15).
CHAPTER 14

Fourier Methods

14.1. Green’s Operators

The Green’s function $s_m(x,y)$ of the vacuum Dirac equation is defined by the distributional equation

\[(i\partial_x - m) s_m(x,y) = \delta^4(x-y),\]  

(14.1.1)

where $\delta^4(x,y)$ denotes the 4-dimensional Dirac distribution. Taking the Fourier transform of (14.1.1),

\[s_m(x,y) = \int \frac{d^4k}{(2\pi)^4} s_m(k) e^{-ik(x-y)},\]  

(14.1.2)

we obtain the algebraic equation

\[(\slashed{k} - m) s_m(k) = 1\]  

(14.1.3)

Multiplying by $\slashed{k} + m$ and using the identity $(\slashed{k} - m)(\slashed{k} + m) = k^2 - m^2$, one sees that if $k^2 \neq m^2$, the matrix $\slashed{k} - m$ is invertible. If conversely $k^2 = m^2$, we have $(\slashed{k} - m)^2 = -2m(\slashed{k} - m)$, which shows that the matrix $\slashed{k} - m$ is diagonalizable with eigenvalues $-2m$ and zero. Since the Dirac matrices (1.3.3) are trace-free, we have $\text{Tr}(\slashed{k} - m) = -4m$. It follows that the matrix $\slashed{k} - m$ has a two-dimensional kernel if $k$ is on the mass shell. This shows that the Green’s function of the Dirac equation is not unique. If we add to it any vector in the kernel of $\slashed{k} - m$, i.e. if we add to it a solution of the homogeneous Dirac equation, it still satisfies the defining equation (14.1.1) (for details see [11].)

A convenient method for solving the equation (14.1.3) for $s_m(k)$ is to use a $\pm i\epsilon$-regularization on the mass shell. Common choices are the advanced and the retarded Green’s functions, which are defined by

\[s_\vee^m(k) = \lim_{\epsilon \searrow 0} \frac{\slashed{k} + m}{k^2 - m^2 - \epsilon k^0}, \quad \text{and} \quad s_\wedge^m(k) = \lim_{\epsilon \searrow 0} \frac{\slashed{k} + m}{k^2 - m^2 + \epsilon k^0},\]  

(14.1.4)

respectively (with the limit $\epsilon \searrow 0$ taken in the distributional sense). Computing their Fourier transform (18.3.1), one sees that they are causal in the sense that their supports lie in the upper and lower light cone, respectively,

\[\text{supp } s_\vee^m(x,. \in J^+_x, \quad \text{supp } s_\wedge^m(x,. \subset J^-_x.\]  

(14.1.5)

Mathematically, the formulas in (14.1.4) define the Green’s functions in momentum space as tempered distributions. Taking their Fourier transform (18.3.1), the advanced and retarded Green’s functions are tempered distributions in the variable $\xi := y - x$. We also regard these distributions as integral kernels of corresponding operators on the wave functions, i.e.

\[(s_m(\psi))(x) := \int_M s_m(x,y) \psi(y) d^4y.\]  

We thus obtain operators

\[s_\wedge^m, s_\vee^m : C^0_\infty(M, SM) \to C^\infty_\text{sc}(M, SM).\]  

(14.1.6)
14. FOURIER METHODS

Here $C^\infty_0(\mathcal{M}, S\mathcal{M})$ denote the smooth functions with compact support in $\mathcal{M}$, taking values in the spinors, and $C^\infty_{sc}$ denotes the smooth functions with spatially compact support.

14.2. The Causal Fundamental Solution and Time Evolution

We now state a few properties of the Green’s operators and explain why they are useful. The considerations in this section are valid more generally in the presence of an external potential. Then the defining equation of the Green’s function (14.1.1) is modified similar to (13.3.3) to

$$
(i\hat{\partial}_x + \mathcal{B} - m)
\left[
\begin{array}{c}
s_m(x, y) \\
\hat{s}_m(x, y)
\end{array}
\right] = \delta^4(x - y)
$$

(14.2.1)

Then the existence of Green’s functions can no longer be proven by Fourier transformation. Instead, one can use methods of hyperbolic PDEs (symmetric hyperbolic systems) which we learned in the PDE2 course. Here we shall not enter these methods again. Instead, we simply assume that we are given advanced and retarded Green’s functions.

The causal fundamental solution $k_m$ is defined as the difference of the advanced and the retarded Green’s function,

$$
k_m(x, y) := \frac{1}{2\pi i} \left(s_m^\gamma(x, y) - \hat{s}_m^\gamma(x, y)\right).
$$

(14.2.2)

It is a distribution which is causal in the sense that it vanishes if $x$ and $y$ have space-like separation. Moreover, it is a distributional solution of the homogeneous Dirac equation,

$$
(i\hat{\partial}_x + \mathcal{B} - m)
\left[
\begin{array}{c}
s_m(x, y) \\
\hat{s}_m(x, y)
\end{array}
\right] = 0
$$

We choose Cauchy surfaces $\mathcal{N}_+$ and $\mathcal{N}_-$ lying in the future and past of supp $\phi$, respectively. Let $\Omega$ be the space-time region between these two Cauchy surfaces, i.e. $\partial\Omega = \mathcal{N}_+ \cup \mathcal{N}_-$. Then, according to (14.2.2),

$$
(\psi_m | k_m \phi)_m = (\psi_m | k_m \phi)_{\mathcal{N}_+} = \frac{i}{2\pi} \left(\psi_m | s_m^\gamma \phi\right)_{\mathcal{N}_+}
$$

where in the last line we applied the Gauß divergence theorem and used (13.1.2). Using that $\psi_m$ satisfies the Dirac equation, a calculation similar to (13.3.9) yields

$$
(\psi_m | k_m \phi)_m = \int_{\Omega} <\psi_m | \left(\mathcal{D} - m\right) \hat{s}_m^\gamma \phi>_x d\mu(x) = \int_{\Omega} <\psi_m | \phi>_x d\mu(x).
$$

As $\phi$ is supported in $\Omega$, we can extend the last integration to all of $\mathcal{M}$, giving the result.

In order to extend the result to general $\psi_m \in \mathcal{H}_m$, we use the following approximation argument. Let $\psi_m^{(n)} \in \mathcal{H}_m \cap C^\infty_{sc}(\mathcal{M}, S\mathcal{M})$ be a sequence which converges in $\mathcal{H}_m$ to $\psi_m$. Then obviously $(\psi_m^{(n)} | k_m \phi)_m \to (\psi_m | k_m \phi)_m$. In order to show that the right side...
of (14.2.3) also converges, it suffices to prove that $\psi_m^{(n)}$ converges in $L^2_{\text{loc}}(\mathcal{M}, S\mathcal{M})$ to $\psi_m$. Thus let $K \subset \mathcal{M}$ be a compact set contained in the domain of a chart $(x, U)$. Using Fubini’s theorem, we obtain for any $\psi \in \mathcal{H}_m \cap C^\infty_{\text{sc}}(\mathcal{M}, S\mathcal{M})$ the estimate
\[
\int_K \int d\mu_{\mathcal{M}} = \int dx^0 \int dx^1 \frac{\sqrt{g}}{\sqrt{-\gamma}} d^3x \leq C(K) (\psi|\psi)_m .
\]
Applying this estimate to the functions $\psi = \psi_m^{(n)} - \psi_m^{(n')}$, we see that $\psi_m^{(n)}$ converges in $L^2(K, S\mathcal{M})$ to a function $\tilde{\psi}$. This implies that $\psi_m^{(n)}$ converges to $\tilde{\psi}$ pointwise almost everywhere (with respect to the measure $d\mu_{\mathcal{M}}$). Moreover, the convergence of $\psi_m^{(n)}$ in $\mathcal{H}_m$ to $\psi_m$ implies that the restriction of $\psi_m^{(n)}$ to any Cauchy surface $\mathcal{N}$ converges to $\psi_m|_{\mathcal{N}}$ pointwise almost everywhere (with respect to the measure $d\mu_{\mathcal{N}}$). It follows that $\psi = \psi_m|_{K}$, concluding the proof. □

**Corollary 14.2.2.** The operator $k_m$, (14.2.2), is symmetric with respect to the inner product (13.1.3).

**Proof.** Using Proposition 14.2.1 we obtain for all $\phi, \psi \in C^\infty_0(\mathcal{M}, S\mathcal{M})$,
\[<k_m \phi | \psi> = (k_m \phi | k_m \psi)_m = <\phi | k_m \psi>_m ,\]
concluding the proof. □

Another application of the causal fundamental solution is that it can be used to solve the Cauchy problem. In the Cauchy problem one seeks solutions of the Dirac equation for given initial data on a Cauchy surface,
\[(i\partial + \mathcal{B} - m) \psi = 0 , \quad \psi|_{\mathcal{N}} = \psi_{\mathcal{N}} \in C^\infty(\mathcal{N}, S\mathcal{M}) . \tag{14.2.4}\]

**Proposition 14.2.3.** The solution of the Cauchy problem (14.2.4) has the representation
\[\psi(x) = 2\pi \int_{\mathcal{N}} k_m(x, y) \phi \psi_{\mathcal{N}}(y) d\mu_{\mathcal{N}}(y) , \tag{14.2.5}\]
where $k_m(x, y)$ is the causal fundamental solution (14.2.2).

**Proof.** Let us consider a point $x$ in the future of $\mathcal{N}$ (the case for the past is analogous). In this case, due to (14.2.2), the lemma simplifies to
\[\psi(x) = i \int_{\mathcal{N}} s_m^\wedge(x, y) \phi(y) \psi_{\mathcal{N}}(y) d\mu_{\mathcal{N}}(y) . \tag{14.2.6}\]

In preparation, we want to prove that for any $\phi \in C^\infty(\mathcal{M}, S\mathcal{M})$ which has compact support to the past of $\mathcal{N}$ and with the property that $(\mathcal{D} - m)\phi$ has compact support the equation
\[\phi = s_m^\wedge(\mathcal{D} - m)\phi \tag{14.2.7}\]
holds. To this end, we consider the function
\[\Xi(x) := \phi - s_m^\wedge(\mathcal{D} - m)\phi .\]
Applying the operator $(\mathcal{D} - m)$ and using the defining equation of the Green’s operators, one sees that $\Xi$ is a solution of the Dirac equation. Moreover, $\Xi$ obviously vanishes in the past of the support of $\phi$. The uniqueness of the solution of the Cauchy problem implies that $\Xi$ vanishes identically, proving (14.2.7).

In order to derive equation (14.2.6), we let $\eta \in C^\infty(\mathcal{M})$ be a function which is identically equal to one at $x$ and on $\mathcal{N}$, but such that the function $\eta\psi$ has compact
 support. (For example, in a foliation \( (\mathcal{N}_t)_{t \in \mathbb{R}} \) with \( \mathcal{N} = \mathcal{N}_0 \) one can take \( \eta = \eta(t) \) as a smooth function with \( \eta|_{[t_0, \infty]} \equiv 1 \) which vanishes if \( t < t_0 - 1 \). Then we can apply (14.2.7) to the wave function \( \phi = \eta \psi \). We thus obtain for any \( x \) in the future of \( \mathcal{N} \) the relations
\[
\psi(x) = (\eta \psi)(x) = \left( s_m^\Delta ((D - m)(\eta \psi)) \right)(x) = \left( s_m^\Delta (i\gamma^j (\partial_j \eta) \psi) \right)(x), \quad (14.2.8)
\]
where we have used that \( \psi \) is a solution of the Dirac equation.

To conclude the proof, for \( \eta \) in (14.2.8) we choose a sequence \( \eta_k \) which converges in the distributional sense to the function which in the future \( \mathcal{N} \) is equal to one and in the past of \( \mathcal{N} \) is equal to zero. This yields \( \partial_i \eta_k \to \nu \), and thus the right-hand-side of (14.2.8) is equal to the right-hand-side of (14.2.6).

The unique solvability of the Cauchy problem allows us to introduce the time evolution operator of the Dirac equation as follows. Solving the Cauchy problem with initial data at time \( t \) and evaluating the solution with the help of Proposition 14.2.3 at some other time \( t' \) gives rise to a mapping
\[ U^{t',t} : \mathcal{H}_t \to \mathcal{H}_{t'} , \]
which we call time-evolution operator. Since the scalar product (13.1.2) is time independent, the operator \( U^{t',t} \) is unitary, and since the Cauchy problem can be solved forwards and backwards in time, the unitary time evolution operators form a representation of the group \((\mathbb{R},+)\). More precisely,
\[ U^{t,t} = 1 \quad \text{and} \quad U^{t'',t} U^{t,t} = U^{t',t} . \]
Proposition 14.2.3 immediately gives the following representation of \( U^{t',t} \):
\[ (U^{t',t} \psi |_{t'}) (\gamma) = \int_{\mathbb{R}^3} U^{t',t}(\gamma, x) \psi(t, x) \, d^3 x , \quad (14.2.9) \]
where the kernel \( U^{t',t}(\gamma, x) \) is defined as
\[ U^{t',t}(\gamma, x) = 2\pi k_m((t', \gamma), (t, x)) \gamma^0 . \quad (14.2.10) \]

14.3. Proof of the Weak Mass Oscillation Property in the Minkowski Vacuum

In the remainder of this chapter, we return to the Dirac equation in Minkowski space (14.1.1). An external potential will be considered in the next chapter (Chapter 15).

The mass oscillation property in the Minkowski vacuum can be proved using Fourier methods. Here we shall give two different approaches in detail. The method of the first proof (in this section) is instructive because it gives an intuitive understanding of “mass oscillations”. However, this method only yields the weak mass oscillation property. The second proof (Section 14.4) is more abstract but also gives the strong mass oscillation property.

We again consider the foliation \( \mathcal{N}_t = \{(t, \bar{x}) \mid \bar{x} \in \mathbb{R}^3 \} \) of constant time Cauchy hypersurfaces in a fixed reference frame \((t, \bar{x})\) and a variable mass parameter \( m \) in the interval \( I = (m_L, m_R) \) with \( m_L, m_R > 0 \). The families of solutions \( \psi = (\psi_m)_{m \in I} \) of the Dirac equations \((i\partial - m)\psi_m = 0\) are contained in the Hilbert space \((\mathcal{H}, (\cdot, \cdot))\) with scalar product (13.2.3). The subspace \( \mathcal{H}^{\infty} \subset \mathcal{H} \) in Definition 13.2.1 is chosen as
\[ \mathcal{H}^{\infty} = C^{\infty}_{\text{sc},0}(\mathcal{M} \times I, S \mathcal{M}) \cap \mathcal{H} . \quad (14.3.1) \]
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For what follows, it is convenient to work with the Fourier transform in space, i.e.,
\[ \hat{\psi}(t, \vec{k}) = \int_{\mathbb{R}^3} \psi(t, \vec{x}) e^{-i \vec{k} \cdot \vec{x}} \, d^3x, \quad \hat{\psi}(t, \vec{k}) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \hat{\psi}(t, \vec{k}) e^{i \vec{k} \cdot \vec{x}}. \]

Then a family of solutions \( \psi \in \mathcal{H}^\infty \) has the representation
\[ \hat{\psi}_m(t, \vec{k}) = c_+(\vec{k}, m) e^{-i \omega(\vec{k}, m)t} + c_-(\vec{k}, m) e^{i \omega(\vec{k}, m)t} \quad \text{for all } m \in I \quad (14.3.2) \]
with suitable spinor-valued coefficients \( c_{\pm}(\vec{k}, m) \) and \( \omega(\vec{k}, m) := \sqrt{|\vec{k}|^2 + m^2} \). Integrating over the mass parameter, we obtain a superposition of waves oscillating at different frequencies. Intuitively speaking, this leads to destructive interference for large \( t \), giving rise to decay in time. This picture can be made precise using integration by parts in \( m \), as we now explain. Integrating (14.3.2) over the mass by applying the operator \( p \), (13.2.5), we obtain
\[ p\hat{\psi}(t, \vec{k}) = \int_I \left( c_+ e^{-i \omega t} + c_- e^{i \omega t} \right) dm \]
\[ = \int_I \frac{i}{t} \frac{d}{d}\omega \left( c_+ \frac{\partial}{\partial m} e^{-i \omega t} - c_- \frac{\partial}{\partial m} e^{i \omega t} \right) dm \]
\[ = -\frac{i}{t} \int_I \left[ \frac{\partial}{\partial m} \left( \frac{\omega}{m} c_+ e^{-i \omega t} - \frac{\omega}{m} c_- e^{i \omega t} \right) \right] dm. \]

Since the coefficients \( c_{\pm} \) depend smoothly on \( m \), the resulting integrand is bounded uniformly in time, giving a decay at least like \( 1/t \), i.e., \( |p\hat{\psi}(t, \vec{k})| \lesssim 1/t \). Iterating this procedure, one even can prove decay rates \( \lesssim 1/t^2, 1/t^3, \ldots \). The price one pays is that higher and higher powers in \( \omega \) come up in the integrand, which means that in order for the spatial Fourier integral to exist, one needs a faster decay of \( c_{\pm} \) in \(|\vec{k}|\). Expressed in terms of the initial data, this means that every factor \( 1/t \) gives rise to an additional spatial derivative acting on the initial data. This motivates the following basic estimate.

**Lemma 14.3.1.** For any \( \psi \in \mathcal{H}^\infty \), there is a constant \( C = C(m_L) \) such that
\[ \left\| (p\psi) |t| \right\|_{L^1} \lesssim \frac{C[4]}{1 + t^2} \sup_{m \in I} \sum_{b=0}^2 \left\| \left( \frac{\partial}{\partial_m} \right)^b \right\|_{L^2} \left( \mathcal{W}_2 \right), \quad (14.3.3) \]
where \( \| \cdot \|_{L^1} \) is the norm corresponding to the scalar product
\[ \langle \cdot, \cdot \rangle_{L^1} := 2\pi \int_{\mathbb{R}^3} \langle \cdot, \cdot \rangle \, d^3x : L^2(\mathcal{N}_1, S.M) \times L^2(\mathcal{N}_1, S.M) \to \mathbb{C} \]
(which is similar to (13.1.2), but now applied to wave functions which do not need to be solutions), and \( \| \cdot \|_{L^2} \) is the spatial Sobolev norm
\[ \| \phi \|_{L^2}^2 := \sum_{\alpha \text{ with } |\alpha| \leq 2} \int_{\mathbb{R}^3} |\nabla^\alpha \phi(\vec{x})|^2 \, d^3x, \quad (14.3.4) \]
where \( \alpha \) is a multi-index.

Braucht man denn hier die \( \mathcal{W}^2 \)-norm? Genügt es nicht, dass man für jedes \( \psi \in \mathcal{H}^\infty \) den quadratischen Abfall hat? Falls ja, vereinfache dies?
The absolute value in (14.3.4) is the norm $|.| := \sqrt{\langle \gamma^0 \rangle}$ on the spinors. If we again identify all spinor spaces in the Dirac representation with $\mathbb{C}^4$, this simply is the standard Euclidean norm on $\mathbb{C}^4$.

The proof of this lemma will be given later in this section. Before, we infer the weak mass oscillation property.

**Corollary 14.3.2.** The vacuum Dirac operator $i\partial$ in Minkowski space has the weak mass oscillation property with domain (14.3.1).

**Proof.** For every $\psi, \phi \in \mathcal{H}^\infty$, the Schwarz inequality gives

$$\langle p\psi|p\phi \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle (p\psi)||\gamma^0 (p\phi)||t \rangle dt \leq \int_{-\infty}^{\infty} \| (p\psi)|t \| \| (p\phi)|t \| dt.$$ (14.3.5)

Applying Lemma 14.3.1 together with the estimate

$$\| (p\phi)|t \|^2 = \int_{I \times I} (\phi_m|t \| \phi_{m'}|t \rangle \ dm \ dm'$$

$$\leq \frac{1}{2} \int_{I \times I} \left( \| \phi_m \|^2 + \| \phi_{m'} \|^2 \right) dm \ dm' = |I| \| \phi \|^2,$$

we obtain inequality (13.2.6) with

$$c = C \ |I|^2 \sup_{m \in I} \sum_{b=0}^2 \| \phi_m^b (\psi_m)|t = 0 \|_{W^{2,2}} \int_{-\infty}^{\infty} \frac{1}{1 + t^2} \ dt < \infty.$$ (14.3.6)

The identity (13.2.7) follows by integrating the Dirac operator by parts,

$$\langle pT\psi|p\phi \rangle = \langle pD\psi|p\phi \rangle = \langle Dp\psi|p\phi \rangle = \int_{\mathcal{M}} \langle Dp\psi|p\phi \rangle_x \ dx$$

$$\langle pT\psi|p\phi \rangle = \int_{\mathcal{M}} \langle pT\phi|p\phi \rangle_x \ dx = \langle p\psi|Dp\phi \rangle = \langle p\psi|pT\phi \rangle.$$ (14.3.7)

In (*), we used that the Dirac operator is formally self-adjoint with respect to the inner product $\langle .|. \rangle$. Moreover, we do not get boundary terms because of the time decay in Lemma 14.3.1. \qed

The remainder of this section is devoted to the proof of Lemma 14.3.1. Using the result of Proposition 14.2.3, we can express the solution $\psi_m$ of the Cauchy problem in terms of the causal fundamental solution $k_m$. In order to bring $k_m$ into a more explicit form, we use (14.2.2) together with formulas for the advanced and retarded Green’s functions. Indeed, these Green’s functions are the multiplication operators in momentum space

$$s_m^\wedge(p) = \lim_{\varepsilon \searrow 0} \frac{\phi + m}{k^2 - m^2 - i\varepsilon p^0} \quad \text{and} \quad s_m^\wedge(p) = \lim_{\varepsilon \searrow 0} \frac{\phi + m}{p^2 - m^2 + i\varepsilon k^0}$$

(with the limit $\varepsilon \searrow 0$ taken in the distributional sense, and where the vector $p$ is the four-momentum). We thus obtain in momentum space

$$k_m(p) = \frac{1}{2\pi} \langle \phi + m \rangle \lim_{\varepsilon \searrow 0} \left[ \frac{1}{p^2 - m^2 - i\varepsilon p^0} - \frac{1}{p^2 - m^2 + i\varepsilon p^0} \right]$$

$$= \frac{1}{2\pi} \langle \phi + m \rangle \lim_{\varepsilon \searrow 0} \left[ \frac{1}{p^2 - m^2 - i\varepsilon} - \frac{1}{p^2 - m^2 + i\varepsilon} \right] \epsilon(p^0)$$

...
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(where \( \epsilon \) denotes the step function, and for notational clarity we denoted the momentum variables by \( p \)). Employing the distributional equation

\[
\lim_{\epsilon \to 0} \left( \frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon} \right) = 2\pi i \delta(x),
\]

we obtain the simple formula

\[
k_m(p) = (\not{p} + m) \delta(p^2 - m^2) \epsilon(p^0).
\]

It is convenient to transform spatial coordinates of the time evolution operator to momentum space. First, in the Minkowski vacuum, the time evolution operator can be represented as in (14.2.9) with an integral kernel \( U_{t,t'}(\vec{y}, \vec{x}) \) which depends only on the difference vector \( \vec{y} - \vec{x} \). We set

\[
U_{t,t'}(\vec{k}) := \hat{R} \int_{\mathbb{R}^3} U_{t,t'}(\vec{y}, 0) e^{-i\vec{k}\cdot\vec{y}} d^3y.
\]

Combining (14.2.10) with (14.3.8) yields

\[
U_{t,t'}(\vec{k}) = \int_{-\infty}^{\infty} \frac{1}{\omega(\vec{k})} (\hat{k} + m) \gamma^0 \delta(\vec{k}^2 - m^2) |_{\vec{k}=\omega, \vec{k}} \epsilon(\omega) e^{-i\omega(t-t')} d\omega.
\]

Carrying out the \( \omega \)-integral, we get

\[
U_{t,t'}(\vec{k}) = \sum_{\pm} \Pi_\pm(\vec{k}) e^{\mp i\omega(t-t')},
\]

where we set

\[
\Pi_\pm(\vec{k}) := \pm \frac{1}{2\omega(\vec{k})} (\hat{k} + m) \gamma^0
\]

with \( \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2} \) and \( k_\pm = (\pm \omega, \vec{k}) \).

Moreover, applying Plancherel’s theorem, the scalar product (13.1.2) can be written in momentum space as

\[
(\psi_m | \phi_m)_m = (2\pi)^{-2} \int_{\mathbb{R}^3} \mathcal{A}_m(t, \vec{k}) | \gamma^0 \phi_m(t, \vec{k}) \rangle d^3k.
\]

The unitarity of the time evolution operator in position space implies that the matrix \( U_{t,t'}(\vec{k}) \) is unitary (with respect to the scalar product \( (\ldots, \cdot)_{L^2} \equiv \langle \cdot, \cdot \rangle \)), meaning that its eigenvalues are on the unit circle and the corresponding eigenspaces are orthogonal. It follows that the operators \( \Pi_\pm(\vec{k}) \) in (18.6) are the orthogonal projection operators to the eigenspaces corresponding to the eigenvalues \( e^{\pm i\omega(t-t')} \), i.e.

\[
\gamma^0 \Pi_s^* \gamma^0 = \Pi_s \quad \text{and} \quad \Pi_s(\vec{k}) \Pi_{s'}(\vec{k}) = \delta_{s,s'} \Pi_s(\vec{k}) \quad \text{for} \ s, s' \in \{+, -\}.
\]

(These relations can also be verified by straightforward computations using (14.3.10); see Exercise 14.9).

The next two lemmas involve derivatives with respect to the mass parameter \( m \). For clarity, we again denote the \( m \)-dependence of the operators by the subscript \( m \).

**Lemma 14.3.3.** The time evolution operator in the vacuum satisfies the relation

\[
(t - t') U_{t,t'}^m(\vec{k}) = \frac{\partial}{\partial m} V_{t,t'}^m(\vec{k}) + W_{t,t'}^m(\vec{k}),
\]

where
where

\[
V^{t,t'}_m(\vec{k}) = \sum_{\pm} \frac{i}{2m} (k^\pm + m) \gamma_0 e^{\mp i\omega(t-t')} \tag{14.3.13}
\]

\[
W^{t,t'}_m(\vec{k}) = \sum_{\pm} \frac{i}{2} \left( \frac{k^\pm + 1}{m^2} \mp \frac{\gamma_0}{\omega} \right) e^{\mp i\omega(t-t')} \tag{14.3.14}
\]

The operators \(V^{t,t'}_m\) and \(W^{t,t'}_m\) are estimated uniformly by

\[
\|V^{t,t'}_m(\vec{k})\| + \|W^{t,t'}_m(\vec{k})\| \leq C \left( 1 + \frac{|\vec{k}|}{m} \right) \tag{14.3.15}
\]

where the constant \(C\) is independent of \(m, \vec{k}, t\) and \(t'\) (and \(\|\cdot\|\) is any norm on the \(2 \times 2\)-matrices).

**Proof.** First, we generate the factor \(t-t'\) by differentiating the exponential in (18.6) with respect to \(\omega\),

\[
(t-t')U^{t,t'}_m(\vec{k}) = \sum_{\pm} \Pi_\pm (\vec{k}) \left( \pm i \frac{\partial}{\partial \omega} e^{\mp i\omega(t-t')} \right).
\]

Next, we want to rewrite the \(\omega\)-derivative as a derivative with respect to \(m\). Taking the total differential of the dispersion relation \(\omega^2 - |\vec{k}|^2 = m^2\) for fixed \(\vec{k}\), one finds that

\[
\frac{\partial}{\partial \omega} = \frac{\omega}{m} \frac{\partial}{\partial m}.
\]

Hence

\[
(t-t')U^{t,t'}_m = \sum_{\pm} \Pi_\pm \left( \pm i \frac{\omega}{m} \frac{\partial}{\partial m} e^{\mp i\omega(t-t')} \right)
\]

\[
= \frac{\partial}{\partial m} \sum_{\pm} \left( \pm i \frac{\omega}{m} \Pi_\pm e^{\mp i\omega(t-t')} \right) - \sum_{\pm} \left( \frac{\partial}{\partial m} \left[ \pm i \frac{\omega}{m} \Pi_\pm \right] \right) e^{\mp i\omega(t-t')}.
\]

Computing the operators in the round brackets using (14.3.10) gives the identities (14.3.13) and (14.3.14). Estimating these formulas, one obtains bounds which are at most linear in \(|\vec{k}|\), proving (14.3.15). \(\square\)

This method can be iterated to generate more factors of \(t-t'\). In the next lemma, we prove at least quadratic decay in time. For later use, it is preferable to formulate the result in position space.

**Lemma 14.3.4.** The time evolution operator in the vacuum has the representation

\[
U^{t,t'}_m = \frac{1}{(t-t')^2} \left( \frac{\partial^2}{\partial m^2} A^{t,t'}_m + \frac{\partial}{\partial m} B^{t,t'}_m + C^{t,t'}_m \right) \tag{14.3.17}
\]

with operators

\[
A^{t,t'}_m, B^{t,t'}_m, C^{t,t'}_m : W^{2,2}(\mathcal{N}_t, \mathcal{M}) \to L^2(\mathcal{N}_t, \mathcal{M})
\]

which are bounded uniformly in time by

\[
\|A^{t,t'}_m(\phi)\|_t + \|B^{t,t'}_m(\phi)\|_t + \|C^{t,t'}_m(\phi)\|_t \leq c \|\phi\|_{W^{2,2}}, \tag{14.3.18}
\]

where \(c\) is a constant which depends only on \(m\).
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Proof. A straightforward computation using exactly the same methods as in Lemma 14.3.3 yields the representation

\[ (t - t')^2 U_{m}^{t,t'}(\vec{k}) = \frac{\partial^2}{\partial m^2} A_{m}^{t,t'}(\vec{k}) + \frac{\partial}{\partial m} B_{m}^{t,t'}(\vec{k}) + C_{m}^{t,t'}(\vec{k}), \]

(14.3.19)

where the operators \( A_{m}^{t,t'}, B_{m}^{t,t'}, \) and \( C_{m}^{t,t'} \) are bounded by

\[ \| A_{m}^{t,t'}(\vec{k}) \| + \| B_{m}^{t,t'}(\vec{k}) \| + \| C_{m}^{t,t'}(\vec{k}) \| \leq C \left( 1 + \frac{|\vec{k}|}{m} + \frac{|\vec{k}|^2}{m^2} \right), \]

(14.3.20)

with a numerical constant \( C > 0. \) We remark that, compared to (14.3.12), the right of (14.3.20) involves an additional \( 1/m. \) This prefactor is necessary for dimensional reasons, because the additional factor \( t - t' \) in (14.3.19) (compared to (14.3.12)) brings in an additional dimension of length (and in natural units, the factor \( 1/m \) also has the dimension of length). The additional summand \( |\vec{k}|^2/m^2 \) in (14.3.20) can be understood from the fact that applying (14.3.16) generates a factor of \( \omega/m \) which for large \( |\vec{k}| \) scales like \( |\vec{k}|/m. \)

Translating this result to position space and keeping in mind that the vector \( \vec{k} \) corresponds to the derivative \(-i\vec{\nabla}, \) we obtain the result. \( \square \)

Proof of Lemma 14.3.1. First of all, the Schwarz inequality gives

\[ \|(p\psi)|_t\| \leq \int_I \|\psi_m\| dm \leq \sqrt{|I|} \|\psi\|. \]

Thus it remains to show the decay for large \( t, \) i.e.

\[ \|(p\psi)|_t\| \leq C \frac{|I|}{t^2} \sup_{m \in I} \sum_{b=0}^2 \|\partial_m^b (\psi_m)|_{t=0}\|_{W^{2,2}}. \]

(14.3.21)

We apply Lemma 14.3.3 and integrate by parts in \( m \) to obtain

\[ (p\psi)|_t = \int_I U_{m}^{t,0} \psi_m|_{t=0} dm = \frac{1}{t^2} \int_I (\partial_m^2 A_{m}^{t,0} + \partial_m B_{m}^{t,0} + C_{m}^{t,0}) \psi_m|_{t=0} dm \]

\[ = \frac{1}{t^2} \int_I \left( A_{m}^{t,0} (\partial_m^2 \psi_m|_{t=0}) - B_{m}^{t,0} (\partial_m \psi_m|_{t=0}) + C_{m}^{t,0} \psi_m|_{t=0} \right) dm. \]

Taking the norm and using (14.3.18) gives (14.3.21). \( \square \)

We finally note that the previous estimates are not optimal for two reasons. First, the pointwise quadratic decay in (14.3.3) is more than what is needed for the convergence of the integral in (14.3.6). Second and more importantly, the Schwarz inequality (14.3.5) does not catch the optimal scaling behavior in \( \vec{k}. \) This is the reason why the constant in (13.2.6) involves derivatives of \( \psi_m \) (cf. (14.3.6)), making it impossible to prove the inequality (13.2.8) which arises in the strong mass oscillation property. In order to improve the estimates, one needs to use Fourier methods both in space and time, as will be explained in the next section.
14.4. Proof of the Strong Mass Oscillation Property in the Minkowski Vacuum

**Theorem 14.4.1.** The vacuum Dirac operator in Minkowski space has the strong mass oscillation property with domain \( \text{(14.3.1)}. \)

Our proof relies on a Plancherel argument in space-time. It also provides an alternative method for establishing the weak mass oscillation property.

**Proof of Theorem 14.4.1.** Let \( \psi = (\psi_m)_{m \in I} \in \mathcal{H}^\infty \) be a family of solutions of the Dirac equation for a varying mass parameter in the Minkowski vacuum. Using Proposition 14.2.3, one can express \( \psi_m \) in terms of its values at time \( t = 0 \) by

\[
\psi_m(x) = 2\pi \int_{\mathbb{R}^3} k_m(x, (0, \vec{y})) \gamma^0 \psi_{m|t=0}(\vec{y}) \, d^3y.
\]

We now take the Fourier transform, denoting the four-momentum by \( \vec{k} \). Using (14.3.8), we obtain

\[
\psi_m(k) = 2\pi k_m(k) \gamma^0 \hat{\psi}_m^0(\vec{k}) = 2\pi \delta(k^2 - m^2) \epsilon(k^0) (\vec{k} \cdot \vec{m}) \gamma^0 \hat{\psi}_m^0(\vec{k}),
\]

where \( \hat{\psi}_m^0(\vec{k}) \) denotes the spatial Fourier transform of \( \psi_m|_{t=0} \) (in order to avoid an ambiguity of notation, the hat of the Fourier transform in space-time was omitted). Obviously, this is a distribution supported on the mass shell. In particular, it is not square integrable over \( \mathbb{R}^4 \).

Integrating over \( m \), we obtain the following function

\[
(p\psi)(k) = 2\pi \chi_I(m) \frac{1}{2m} \epsilon(k^0) (\vec{k} \cdot \vec{m}) \gamma^0 \hat{\psi}_m^0(\vec{k}) \bigg|_{m=\sqrt{k^2}} \tag{14.4.1}
\]

where \( m \) now is a function of the momentum variables. Since the function \( \psi_m|_{t=0} \) is compactly supported and smooth in the spatial variables, its Fourier transform \( \hat{\psi}_m^0(\vec{k}) \) has rapid decay. This shows that the function (14.4.1) is indeed square integrable. Using Plancherel, we see that condition (a) in Definition 13.2.2 is satisfied. Moreover, the operator \( T \) is simply the operator of multiplication by \( \sqrt{k^2} \), so that condition (b) obviously holds. This again shows the weak mass oscillation property.

In order to prove the strong mass oscillation property, we need to compute the inner product \( \langle p\psi | p\phi \rangle \). To this end, we first write this inner product in momentum space as

\[
\langle p\psi | p\phi \rangle = \int \frac{d^4k}{(2\pi)^4} \chi_I(m) \frac{1}{4m^2} \langle (\vec{k} \cdot \vec{m}) \gamma^0 \hat{\psi}_m^0(\vec{k}) | (\vec{k} \cdot \vec{m}) \gamma^0 \hat{\phi}_m^0(\vec{k}) \rangle \bigg|_{m=\sqrt{k^2}}
\]

\[
= \int \frac{d^4k}{4\pi^2} \chi_I(m) \frac{1}{2m} \langle \gamma^0 \hat{\psi}_m^0(\vec{k}) | (\vec{k} \cdot \vec{m}) \gamma^0 \hat{\phi}_m^0(\vec{k}) \rangle \bigg|_{m=\sqrt{k^2}}.
\]

Reparametrizing the \( k^0 \)-integral as an integral over \( m \), we obtain

\[
\langle p\psi | p\phi \rangle = \frac{1}{4\pi^2} \int_I dm \int_{\mathbb{R}^3} \frac{d^3k}{2|k^0|} \langle \gamma^0 \hat{\psi}_m^0(\vec{k}) | (\vec{k} \cdot \vec{m}) \gamma^0 \hat{\phi}_m^0(\vec{k}) \rangle \bigg|_{k^0=\pm \sqrt{|k|^2+m^2}} \tag{14.4.2}
\]

Estimating the inner product with the Schwarz inequality and applying Plancherel’s theorem, one finds

\[
|\langle p\psi | p\phi \rangle| \leq \frac{1}{4\pi^2} \int_I dm \int_{\mathbb{R}^3} ||\hat{\psi}_m^0(\vec{k})|| ||\hat{\phi}_m^0(\vec{k})|| \, d^3k \leq 2\pi \int_I ||\psi_m||_m ||\phi_m||_m \, dm.
\]

Thus the inequality (13.2.8) holds. \( \square \)
• Explain how to read off the eigenvalues and eigenspaces of the fermionic signature operator.

Exercises

Exercise 14.1. This exercise is devoted to a clean proof of the distributional relation \((14.4.5)\) in one dimension. More precisely, we want to prove the slightly more general statement that for any function \(\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})\),

\[
\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \eta(x) \left( \frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right) dx = 2\pi i \eta(0) .
\] (14.4.3)

(a) Let \(\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})\) with \(\eta(0) = 0\). Show with the help of Lebesgue’s dominated convergence theorem that (14.4.3) holds.

(b) Show with residues that (14.4.3) holds for the function \(\eta(x) = 1/(x^2 + 1)\).

(c) Combine the results of (a) and (b) to prove (14.4.3) for general \(\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})\).

Exercise 14.2. This exercise recalls basics on the principal value in one dimension

\[
\frac{1}{2} \lim_{\varepsilon \downarrow 0} \left( \frac{1}{x - i\varepsilon} + \frac{1}{x + i\varepsilon} \right) =: \frac{\text{PP}}{x} .
\] (14.4.4)

(a) Repeat the method in Exercise 14.1 to show that the limit of the left side of (14.4.4) exist for any \(\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})\). Derive a corresponding estimate which shows that \(\text{PP}\) is a well-defined tempered distribution.

(b) Show that for any \(\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})\),

\[
\text{PP}(\eta) = \lim_{\varepsilon \downarrow 0} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\eta(x)}{x} dx .
\]

Exercise 14.3. The goal of this exercise is to justify that the one-dimensional relations

\[
\lim_{\varepsilon \downarrow 0} \left( \frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right) = 2\pi i \delta(x) \] (14.4.5)

\[
\frac{1}{2} \lim_{\varepsilon \downarrow 0} \left( \frac{1}{x - i\varepsilon} + \frac{1}{x + i\varepsilon} \right) =: \frac{\text{PP}}{x} .
\] (14.4.6)

apply four-dimensional setting to obtain the relation

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{r^2 + (\varepsilon + it)^2} = \lim_{\varepsilon \downarrow 0} \frac{1}{r^2 - t^2 + i\varepsilon t} = -\frac{\text{PP}}{\xi^2} - i\pi \delta(\xi^2) \epsilon(\xi^0) ,
\] (14.4.7)

(a) Let \(T\) be a distribution on \(\mathbb{R}\), \(\Omega \subset M\) an open subset of Minkowski space and \(f : \Omega \rightarrow \mathbb{R}\) a smooth function with nowhere vanishing gradient. Show that the relation

\[
(f^* T)(\eta) := T(\phi_f(\eta)) , \eta \in C_0^\infty(\Omega)
\]

with

\[
\phi_f(\eta)(t) := \frac{\partial}{\partial t} \int_{\Omega} \Theta(t - f(x)) \eta(x) d^4x
\]

(where \(\Theta\) is the Heaviside function) defines \(f^* T\) as a distribution on \(\Omega\) (this is the so-called pullback of \(T\) under \(f\); for details see [64, Section 7.2]).
(b) Choosing \( \Omega \) as the half space in the future, \( \Omega = \{ x \in \mathcal{M}, x^0 > 0 \} \), one can rewrite the expression on the left of (14.4.7) as

\[
\lim_{\varepsilon \to 0} \frac{1}{r^2 - t^2 + i\varepsilon}.
\]

Use (a) to conclude that this expression is a well-defined distribution for any \( \varepsilon > 0 \). Show that the limit \( \varepsilon \searrow 0 \) exist in the distributional sense.

(c) Repeating the procedure of (b) for the half space in the past, one obtains a distribution on \( \mathcal{M} \setminus \{ t = 0 \} \). Show that this distribution coincides with the limit in (14.4.7).

**Hint:** Similar as in Exercise 14.1, one can estimate the behavior at the origin with Lebesgue’s dominated convergence theorem.

**Exercise 14.4.** This exercise is devoted to the advanced Green’s function \( s^A_m \).

(a) Assume that \( m > 0 \). Show that the limit \( \nu \searrow 0 \) in (14.1.4) exist in the distributional sense.

(b) Show that the limit \( \nu \searrow 0 \) in (14.1.4) also exists in the massless case \( m = 0 \) and that

\[
\lim_{m \searrow 0} s^A_m(k) = s^A_0(k) \quad \text{as a distribution}.
\]

**Hint:** Proceed similar as in Exercise 14.3.

(c) Consider the Fourier integral in the \( q^0 \)-variable

\[
\int_{-\infty}^{\infty} \frac{1}{q^2 - m^2 - i\nu q^0} e^{iq^0t} dq^0.
\]

Show with residues that this integral vanishes for sufficiently small \( \nu \) if \( t < 0 \).

(d) Argue with Lorentz invariance to prove the left side of (14.1.5).

**Exercise 14.5.** Modifying the location of the poles in (14.1.4) gives rise to the distribution

\[
s^F_m(k) := \lim_{\nu \searrow 0} \frac{k + m}{k^2 - m^2 + i\nu}.
\]

This is the well-known Feynman propagator, which is often described intuitively by saying that “positive frequencies move to the future and negative frequencies move to the past.” Make this sentence precise by a computation similar to that in Exercise 14.4 (c).

**Exercise 14.6.** Given \( \omega \in \mathbb{R} \), we consider the ordinary differential operator \( D = i\partial_t + \omega \).

(a) Construct the advanced and retarded Green’s functions, which satisfy in analogy to the kernels in the lecture the equation

\[
D_t s(t, t') = \delta(t - t')
\]

(b) Using the formula of the lecture, compute the resulting causal fundamental solution. How is it related to the time evolution operator \( U^{t, t'} \)? On which Hilbert space does the time evolution operator act as a unitary operator?

**Exercise 14.7.** Consider the massless Dirac equation \( D\psi = 0 \) in the two-dimensional space-time cylinder \( \mathbb{R} \times S^1 \), i.e.

\[
D = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_t + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_{\varphi}
\]

with \( t \in \mathbb{R} \) and \( \varphi \in (0, 2\pi) \).
(a) Choose the spin scalar product such that the Dirac matrices become symmetric. What is the resulting space-time inner product $<.,.>$? What is the scalar product $(.,.)$?

(b) Employ for $k \in \mathbb{Z}$ the separation ansatz 
$$\psi(t, \varphi) = e^{-ik\varphi} \chi(t) \quad \text{with} \quad \chi(t) \in \mathbb{C}^2.$$ 
Derive the resulting ODE for $\chi$. Compute the time evolution operator for this ODE. 

**Hint:** Use the result of Exercise 14.6.

(c) Use a Fourier series decomposition in order to deduce a series representation of the time evolution operator of the Dirac operator on $\mathbb{R} \times S^1$. Try to carry out the infinite series to obtain a closed expression for $U^{t,t'}$. How can one see finite propagation speed?

**Exercise 14.8.** As in Exercise 14.7, we consider the two-dimensional massless Dirac equation.

(a) Adapt the formulas of the lecture for the advanced and retarded Green’s operators in momentum space to the two-dimensional massless case.

(b) Compute the Fourier transform to obtain $s^\vee(x,y)$ and $s^\wedge(x,y)$.

(c) Use the result of (b) to compute the causal fundamental solution and the time evolution operator.

(d) How can one see finite propagation speed? How is the obtained formula related to the formula in Exercise 14.7 (c)?

**Exercise 14.9.** Verify the relations (14.3.11) by direct computation starting from the definition (14.3.10).

**Exercise 14.10.** Verify by formal computation that in the Minkowski vacuum, the fundamental solution $k_m$ and the Green’s operator $s_m$ defined by 
$$s_m := \frac{1}{2} (s_m^\vee + s_m^\wedge)$$ 
Satisfy the distributional relations in the mass parameters $m$ and $m'$

$$k_m k_m' = \delta(m - m') p_m$$
$$k_m s_m' = s_m' k_m = \frac{\text{PP}}{m - m'} k_m ,$$

where PP denotes the principal part, and $p_m$ is the distribution 
$$p_m(k) = (\slashed{k} + m) \delta(k^2 - m^2) .$$

**Hint:** By a “formal computation” we mean that you do not need to evaluate weakly in the mass with test functions.

**Exercise 14.11.** Proceed similar as in Exercise 14.10 to derive a relation for the operator product $s_m^\vee s_m'^\vee$. Derive the relation 
$$s_m s_m' = \frac{\text{PP}}{m - m'} (s_m - s_m') + \pi^2 \delta(m - m') p_m .$$
CHAPTER 15

Methods of Scattering Theory

We return to the Cauchy problem in the presence of an external potential,
\[(D - m)\psi = 0, \quad \psi|_{t_0} = \psi_0 \in C^\infty(N_0 \simeq \mathbb{R}^3, S.M), \tag{15.0.1}\]
with \(D\) as in (1.3.13). For notational clarity, we shall often denote the objects in the presence of the external potential by a tilde (the “interacting objects”), whereas the objects without tilde refer to the Minkowski vacuum.

15.1. The Lippmann-Schwinger Equation

The Dirac dynamics can be rewritten in terms of a symmetric operator \(\tilde{H}\). To this end, we multiply the Dirac equation (1.3.13) by \(\gamma^0\) and bring the \(t\)-derivative separately on one side of the equation,
\[i\partial_t \psi_m = \tilde{H}\psi_m, \quad \text{where} \quad \tilde{H} := -\gamma^0(\gamma^0 B + \gamma^0 m) \tag{15.1.1}\]
(note that \(\gamma^j \partial_j = \gamma^0 \partial_t + \gamma^0 \nabla\)). We refer to (15.1.1) as the Dirac equation in Hamiltonian form. The fact that the scalar product (13.1.2) is time independent implies that for any two solutions \(\phi_m, \psi_m \in C^\infty_{sc}(M, S.M) \cap \mathcal{H}_m\),
\[0 = \partial_t(\phi_m | \psi_m)_m = i((\tilde{H}\phi_m | \psi_m)_m - (\phi_m | \tilde{H}\psi_m)_m),\]
showing that the Hamiltonian is a symmetric operator on \(\mathcal{H}_m\). The Lippmann-Schwinger equation can be used to compare the dynamics in the Minkowski vacuum with the dynamics in the presence of an external potential. We denote the time evolution operator in the Minkowski vacuum by \(U^{t,t_0}_m\).

**Proposition 15.1.1.** The Cauchy problem (15.0.1) has a solution \(\psi_m\) which satisfies the equation
\[\psi_m|_t = U^{t,t_0}_m \psi_0 + i \int_{t_0}^t U^{\tau,t}_m (\gamma^0 B \psi_m)|_{\tau} d\tau, \tag{15.1.2}\]
referred to as the Lippmann-Schwinger equation.

**Proof.** Obviously, the wave function \(\psi_m|_t\) given by (15.1.2) has the correct initial values at \(t = t_0\). Thus it remains to show that \(\psi_m|_t\) satisfies the Dirac equation. To this end, we rewrite the Dirac equation in the Hamiltonian form (15.1.1), and separate the vacuum Hamiltonian \(H\) from the term involving the external potential,
\[(i\partial_t - H) \psi_m = -\gamma^0 B \psi_m \quad \text{with} \quad H = -i\gamma^0 \gamma^\nabla + \gamma^0 m. \tag{15.1.3}\]
Applying the operator \(i\partial_t - H\) to (15.1.2) and observing that the time evolution operator maps to solutions of the vacuum Dirac equation, only the derivative of the upper limit of integration contributes,
\[(i\partial_t - H) \psi_m|_t = -U^{t,t_\tau}_m (\gamma^0 B \psi_m)|_{\tau=t} = -\gamma^0 B \psi_m|_t,\]
so that (15.1.3) is indeed satisfied. 
\[\square\]
15.2. The Mass Oscillation Property in the Presence of an External Potential

The goal of this section is to prove the following result:

**Theorem 15.2.1.** Assume that the external potential $\mathcal{B}$ is smooth and for large times decays faster than quadratically in the sense that

$$|\mathcal{B}(t)|_{C^2} \leq \frac{c}{1 + |t|^{2+\varepsilon}}$$  \hspace{1cm} (15.2.1)

for suitable constants $\varepsilon, c > 0$. Then the strong mass oscillation property holds.

The $C^2$-norm in (15.2.1) is defined as follows. We denote spatial derivatives by $\nabla$ and use the notation with multi-indices, i.e. for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_p)$ we set $\nabla^\alpha = \partial_{\alpha_1} \cdots \partial_{\alpha_p}$ and denote the length of the multi-index by $|\alpha| = p$. Then the spatial $C^k$-norms of the potential are defined by

$$|\mathcal{B}(t)|_{C^k} := \max_{|\alpha| \leq k} \sup_{\vec{x} \in \mathbb{R}^3} |\nabla^\alpha \mathcal{B}(t, \vec{x})| ,$$  \hspace{1cm} (15.2.2)

where $|.|$ is the sup-norm corresponding to the norm $|\phi|^2 := \langle \phi | \gamma^0 \phi \rangle$ on the spinors.

**15.2.1. Proof of the Weak Mass Oscillation Property.** In this section, we prove the following theorem.

**Theorem 15.2.2.** Assume that the time-dependent external potential $\mathcal{B}$ is smooth and decays faster than quadratically for large times in the sense that (15.2.1) holds for suitable constants $c, \varepsilon > 0$. Then the Dirac operator $\mathcal{D} = i\partial /\partial t + \mathcal{B}$ has the weak mass oscillation property.

We expect that this theorem could be improved by weakening the decay assumptions on the potential. However, this would require refinements of our methods which would go beyond the scope of this paper. Also, using that Dirac solutions dissipate, the pointwise decay in time could probably be replaced or partially compensated by suitable spatial decay assumptions. Moreover, one could probably refine the result of the above theorem by working with other norms (like weighted $C^k$- or Sobolev norms).

The main step is the following basic estimate, which is the analog of Lemma 14.3.1 in the presence of an external potential.

**Proposition 15.2.3.** Under the decay assumptions (15.2.1) on the external potential $\mathcal{B}$, there are constants $c, \varepsilon > 0$ such that for every family $\psi \in H^\infty$ of solutions of the Dirac equation (1.3.13) with varying mass,

$$||(p\psi)|_t||_t \leq \frac{c}{1 + |t|^{1+\varepsilon}} \sup_{m \in I} \sum_{b=0}^2 \left\| (\partial^b_m \psi_m)|_{t=0} \right\|_{W^{2,2}} .$$  \hspace{1cm} (15.2.3)

We first show that this proposition implies the weak mass oscillation property.

**Proof of Theorem 15.2.2 Under the assumption that Proposition 15.2.3 holds.** In order to derive the inequality (13.2.6), we begin with the estimate

$$|\langle p\psi | p\phi \rangle| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| (p\psi|_t \mid p\phi|_t) \right| dt \leq \sup_{t \in \mathbb{R}} ||p\phi|_t|| \int_{-\infty}^{\infty} ||p\psi|_t|| dt .$$
The last integral is finite by Proposition 15.2.3. The supremum can be bounded by the Hilbert space norm using the Hölder inequality,

\[ \|p\phi\|_2 = \left\| \int_0^t \phi_m |t \right\| dm \leq \int_0^t \|\phi_m|_2 dm \leq \sqrt{|I|} \left( \int_0^t \|\phi_m|_2^2 dm \right)^{1/2} = \sqrt{|I|} \|\phi\|, \]

giving (13.2.6).

Using (1.3.12), the Dirac operator \( i\partial / \gamma + \mathcal{B} \) is formally self-adjoint with respect to the inner product \( \langle ., . \rangle \). Therefore, the identity (13.2.7) can be obtained just as in (14.3.7) by integrating the Dirac operator in space-time by parts, noting that we do not get boundary terms in view of the time decay in Proposition 15.2.3.

The remainder of this section is devoted to the proof of Proposition 15.2.3. We make use of the Lippmann-Schwinger equation (15.1.2),

\[ \psi_m |t = U_m^t \psi_m |_{t=0} + i \int_0^t U_{m,\tau}^t (\gamma^0 \mathcal{B} \psi_m) |_\tau dm. \]

Since the first summand of this equation is controlled by Lemma 14.3.1, it remains to estimate the second summand. Again using (14.3.17) and integrating by parts with respect to the mass, we obtain

\[ \int_I U_{m,\tau}^t (\gamma^0 \mathcal{B} \psi_m) |_\tau dm = \frac{1}{(t-\tau)^2} \int_I (A_{m,\tau}^t m^2 - B_{m,\tau}^t \partial_m + C_{m,\tau}^t) (\gamma^0 \mathcal{B} \psi_m) |_\tau dm \]

and thus

\[ \left\| \int_I U_{m,\tau}^t (\gamma^0 \mathcal{B} \psi_m) |_\tau dm \right\| \leq \frac{c |I|}{(t-\tau)^2} \sup_{m \in I} \sum_{\mu=0}^2 \|\mathcal{B}(\tau) (\partial_\mu \psi_m) |_\tau \|_{W^{2,2}} \leq \frac{c |I|}{(t-\tau)^2} \|\mathcal{B}(\tau)\|_{C^2} \sup_{m \in I} \sum_{\mu=0}^2 \|\partial_\mu \psi_m |_\tau \|_{W^{2,2}}. \]

We now bound \( \mathcal{B}(\tau) \) with the help of (15.2.1) and estimate the Sobolev norm \( \|\partial_\mu \psi_m |_\tau \|_{W^{2,2}} \) at time \( \tau \) by means of Lemma 11.6.1. This gives rise to the inequality

\[ \left\| \int_I U_{m,\tau}^t (\gamma^0 \mathcal{B} \psi_m) |_\tau dm \right\| \leq \frac{c^2 C |I|}{(t-\tau)^2} \left( 1 + |\tau|^2 \right)^{1/2} \sup_{m \in I} \sum_{\mu=0}^2 \|\partial_\mu \psi_m |_{t=0} \|_{W^{2,2}}, \]

which yields the desired decay provided that \( \tau \) and \( t \) are not close to each other. More precisely, we shall apply this inequality in the case \( |\tau| \leq |t|/2 \). Then the estimate simplifies to

\[ \left\| \int_I U_{m,\tau}^t (\gamma^0 \mathcal{B} \psi_m) |_\tau dm \right\| \leq \frac{\tilde{C}}{t^2 (1 + |\tau|^2)} \sup_{m \in I} \sum_{\mu=0}^2 \|\partial_\mu \psi_m |_{t=0} \|_{W^{2,2}} \quad \text{if } |\tau| \leq |t|/2 \quad (15.2.5) \]

with a new constant \( \tilde{C} > 0 \). In the remaining case \( |\tau| > |t|/2 \), we use the unitarity of \( U_{m,\tau}^t \) to obtain

\[ \left\| \int_I U_{m,\tau}^t (\gamma^0 \mathcal{B} \psi_m) |_\tau dm \right\| \leq |I| \|\mathcal{B}(\tau)\|_{C^0} \sup_{m \in I} \|\psi_m \|. \]
Applying (15.2.1) together with the inequality $|\tau| > |t|/2$, this gives
\[
\left\| \int_I U^t_{m} \left( \gamma^0 B \psi_m \right) |\tau\rangle \ dm \right\|_t \leq \tilde{C}' \sup_{m \in I} \|\psi_m\| \quad \text{if } |\tau| > |t|/2 . \tag{15.2.6}
\]
This again decays for large $t$ because $\tau$ is close to $t$ and $|B(\tau)|_{C^0}$ decays for large $\tau$.

Comparing (15.2.5) and (15.2.6), we find that the inequality in (15.2.5) even holds for all $\tau$. Thus integrating this inequality over $\tau \in [0, t]$, we obtain the following estimate for the second summand in (15.2.4),
\[
\left\| \int_I dm \int_0^t U^t_{m} \left( \gamma^0 B \psi_m \right) |\tau\rangle \ d\tau \right\|_t \leq C'' \sup_{m \in I} \sum_{b=0}^2 \|\partial^b_m \psi|_{t=0}\|_{W_{2,2}}
\]
(where $C'' > 0$ is a new constant). Combining this inequality with the estimate (14.3.3) of the first summand in (15.2.4), we obtain the desired inequality (15.2.3). This concludes the proof of Proposition 15.2.3.

15.2.2. Proof of the Strong Mass Oscillation Property. In this section, we prove the following result.

**Theorem 15.2.4.** Assume that the weak mass oscillation property holds and that the external potential $B$ satisfies the condition
\[
\int_{-\infty}^{\infty} |B(\tau)|_{C^0} \ d\tau < \infty . \tag{15.2.7}
\]
Then the Dirac operator $\mathcal{D} = i\partial + B$ has the strong mass oscillation property.

Combining this theorem with Theorem 15.2.2, one immediately obtains Theorem 15.2.1.

For the proof we shall derive an explicit formula for the fermionic signature operator (Proposition 15.2.5). This formula is obtained by comparing the dynamics in the presence of the external potential with that in the Minkowski vacuum using the Lippmann-Schwinger equation, and by employing distributional relations for products of fundamental solutions and Green’s functions (Lemma 15.2.8).

We first return to the formula (14.4.2) in the Minkowski vacuum. Applying Plancherel’s theorem and using (13.1.2), we conclude that
\[
\langle p\psi | p\phi \rangle = \int_I (\psi^0_m | S_m(\vec{k}) \phi^0_m) \ dm , \tag{15.2.8}
\]
where
\[
S_m(\vec{k}) := \sum_{k^0 = \pm \omega(\vec{k})} \frac{\vec{k} + m}{2 \omega(\vec{k})} \gamma^0 = \frac{\vec{k} \gamma + m}{\omega(\vec{k})} \gamma^0 . \tag{15.2.9}
\]
Comparing (15.2.8) with (13.3.4), one sees that the matrix $S_m(\vec{k})$ is indeed the fermionic signature operator, considered as a multiplication operator in momentum space. By direct computation, one verifies that the matrix $S_m(\vec{k})$ has eigenvalues $\pm 1$.

In order to compare the dynamics in the presence of the external potential with that in the Minkowski vacuum, we work with the Hamiltonian formulation. We decompose the Dirac Hamiltonian (15.1.1) into the Hamiltonian in the Minkowski vacuum (15.1.3) plus a potential,
\[
\tilde{H} = H + \mathcal{V} \quad \text{with} \quad \mathcal{V} := -\gamma^0 B .
\]
Proposition 15.2.5. Assume that the potential $\mathcal{B}$ satisfies the condition \textit{(15.2.7)}.

Then for every $\psi, \phi \in \mathcal{H}^\infty$, 

$$<p\psi|p\phi> = \int_I (\psi_m | \tilde{S}_m \phi_m)dm,$$  \hspace{1cm} \text{(15.2.10)}

where $\tilde{S}_m: \mathcal{H}_m \rightarrow \mathcal{H}_m$ are bounded linear operators which act on the wave functions at time $t_0$ by

$$\tilde{S}_m = S_m - \frac{i}{2} \int_0^\infty \epsilon(t - t_0) \left[ S_m U_{m,t} V(t) \tilde{U}_{m,0} - \tilde{U}_{m,0} V(t) S_m U_{m,t} \right] dt$$

$$+ \frac{1}{2} \left( \int_0^\infty \int_0^\infty + \int_{-\infty}^0 \int_{-\infty}^0 \right) \tilde{U}_{m,0} V(t) S_m U_{m,t'} V(t') \tilde{U}_{m,t} dt dt'$$

\hspace{1cm} \text{(15.2.11, 15.2.12)}

(and $S_m$ is again the fermionic signature operator of the vacuum \textit{(15.2.9)}).

Before entering the proof of this proposition, it is instructive to verify that the above formula for $\tilde{S}_m$ does not depend on the choice of $t_0$.

Remark 15.2.6. \textbf{(Independence of $\tilde{S}_m$ on $t_0$)} Our strategy is to differentiate the above formula for $\tilde{S}_m$ with respect to $t_0$ and to verify that we obtain zero. We first observe that taking a solution $\phi_m \in \mathcal{H}_m$ of the Dirac equation in the presence of $\mathcal{B}$, evaluating at time $t_0$ and applying the time evolution operator $\tilde{U}_{m,t_0}$ gives $\phi_m$ at time $t$, i.e. $\tilde{U}_{m,t_0} \phi_m|_{t_0} = \phi_m|_t$. Differentiating with respect to $t_0$ yields

$$\partial_{t_0} \tilde{U}_{m,t_0} \phi_m|_{t_0} = 0.$$  \hspace{1cm} \text{(15.2.11)}

The situation is different when one considers the time evolution operator of the vacuum. Namely, in the expression $U_{m,t_0} \phi_m|_{t_0}$, the wave function $\phi_m$ satisfies the Dirac equation $(i\partial_t - H) \phi_m = \mathcal{V} \phi_m$, whereas the time evolution operator solves the Dirac equation with $\mathcal{V} \equiv 0$. As a consequence,

$$\partial_{t_0} U_{m,t_0} \phi_m|_{t_0} = -i U_{m,t_0} (\mathcal{V} \phi_m)|_{t_0}.$$  \hspace{1cm} \text{(15.2.12)}

Using these formulas together with $U_{t_0,t_0} = 1 = \tilde{U}_{t_0,t_0}$, a straightforward computation gives

$$\partial_{t_0} (\psi_m | (15.2.11) \phi_m)|_{t_0} = -i(\psi_m | (S_m, \mathcal{V}) \phi_m)|_{t_0}$$

$$- \frac{i}{2} (-2) (\psi_m | (S_m \mathcal{V}(t_0) - \mathcal{V}(t_0) S_m) \phi_m)|_{t_0}$$

$$- \frac{i}{2} \int_0^\infty \epsilon(t - t_0) \left( (i\mathcal{V}(t_0)) \psi_m | (S_m U_{m,t_0} \mathcal{V}(t) \tilde{U}_{m,t_0} \phi_m)|_{t_0} dt$$

$$+ \frac{i}{2} \int_0^\infty \epsilon(t - t_0) \left( \psi_m | \tilde{U}_{m,0} \mathcal{V}(t) S_m U_{m,t} \mathcal{V}(t') \tilde{U}_{m,t} \phi_m)|_{t_0} dt$$

$$\partial_{t_0} (\psi_m | (15.2.12) \phi_m)|_{t_0} = - \frac{1}{2} \int_0^\infty \epsilon(t' - t_0) \left( \psi_m | \mathcal{V}(t_0) S_m U_{m,t_0} \mathcal{V}(t') \tilde{U}_{m,t} \phi_m)|_{t_0} dt'$$

$$- \frac{1}{2} \int_0^\infty \epsilon(t - t_0) \left( \psi_m | \tilde{U}_{m,0} \mathcal{V}(t) S_m U_{m,t} \mathcal{V}(t_0) \phi_m)|_{t_0} dt,$$

where for notational simplicity we here omitted the restrictions $|t_0$ for the solutions $\psi_m$ and $\phi_m$. Adding the terms gives zero.
The remainder of this section is devoted to the proof of Proposition 15.2.5. Our strategy is to combine the Lippmann-Schwinger equation with estimates in momentum space. We begin with two technical lemmas.

Lemma 15.2.7. Assume that the external potential \( \mathcal{B} \) satisfies condition (15.2.7). For any \( t_0 \in \mathbb{R} \), we denote the characteristic functions in the future respectively past of this hypersurface \( t = t_0 \) by \( \chi^\pm_{t_0}(x) \) (i.e. \( \chi^\pm_{t_0}(x) = \Theta(\pm(x^0 - t_0)) \), where \( \Theta \) is the Heaviside function). Then for any \( \psi_0 \in \mathcal{C}_c^\infty(\mathcal{M}_0, S\mathcal{M}_0) \cap \mathcal{H}_m \), the wave function \( k_m(\chi^\pm_{t_0}\mathcal{B}\psi_0) \) is a well-defined vector in \( \mathcal{H}_{t_0} \) and

\[
\|k_m(\chi^\pm_{t_0}\mathcal{B}\psi_0)\|_{t_0} \leq \frac{1}{2\pi} \|\psi_0\|_m \int_{-\infty}^\infty \chi^\pm_{t_0}(\tau) |\mathcal{B}(\tau)|_{C^0} d\tau.
\]

Proof. Using the integral kernel representation (14.2.9) and (14.2.10) together with the fact that the time evolution in the vacuum is unitary, we obtain

\[
2\pi \left\| \int \mathbb{R} \ k_m((t_0,\cdot),(\tau,\vec{y})) (\chi^\pm_{t_0}\mathcal{B}\psi_0)(\tau,\vec{y}) d^3y \right\|_{t_0}
= \left\| U^{t_0,\tau}_{m,\gamma^0}(\chi^\pm_{t_0}\mathcal{B}\psi_0)|\tau\right\|_{t_0} = \left\| \gamma^0 (\chi^\pm_{t_0}\mathcal{B}\psi_0)|\tau\right\|_{C^0} \leq |\mathcal{B}(\tau)|_{C^0} \|\psi_0\|_m.
\]

Integrating over \( \tau \) and using (15.2.7) gives the result.

The following lemma is proved in [41, Eqs. (2.13)–(2.17)] (see Exercises 14.10 and 14.11).

Lemma 15.2.8. In the Minkowski vacuum, the fundamental solution \( k_m \) and the Green’s function \( s_m \) defined by

\[
s_m := \frac{1}{2} \left( s^\vee_m + s^\wedge_m \right)
\]

satisfy the distributional relations in the mass parameters \( m \) and \( m' \)

\[
k_m k_{m'} = \delta(m - m') p_m
k_m s_{m'} = s_{m'} k_{m} = \frac{\text{PP}}{m - m'} k_m
s_m s_{m'} = \frac{\text{PP}}{m - m'} (s_m - s_{m'}) + \pi^2 \delta(m - m') p_m,
\]

where \( \text{PP} \) denotes the principal part, and \( p_m \) is the distribution

\[
p_m(k) = (\hat{k} + m) \delta(k^2 - m^2).
\]

Proof of Proposition 15.2.5. Let \( \psi \in \mathcal{H}^\infty \) be a family of solutions of the Dirac equation for varying mass. We denote the boundary values at time \( t_0 \) by \( \psi_0 \). Then we can write the Lippmann-Schwinger equation (15.1.2) as

\[
\psi_0 = U^{t_0,\tau}_{m,\gamma^0}(\chi^\pm \mathcal{B}\psi_0)|\tau\ d\tau.
\]

We now bring this equation into a more useful form. Expressing the time evolution operator with the help of (14.2.10) in terms of the fundamental solution, we obtain

\[
\psi_0(x) = 2\pi \int_{\mathbb{R}^3} k_m(x,(t_0,\vec{y})) \gamma^0 \psi_0(t_0,\vec{y}) d^3y + 2\pi i \int_{t_0}^t dy_0 \int_{\mathbb{R}^3} d^3y k_m(x,y)(\mathcal{B}\psi_0)(y).
\]
Applying (14.2.2) and using that the advanced and retarded Green's functions are supported in the future and past light cones, respectively, we can rewrite the last integral in terms of the advanced and retarded Green's functions,

$$
\psi_m = 2\pi k_m (\gamma^0 \delta_{t_0} \psi_m^0) - s_m^\wedge (\chi_{t_0}^+ B \psi_m) - s_m^\vee (\chi_{t_0}^- B \psi_m),
$$

where $\delta_{t_0}(x) := \delta(t_0 - x^0)$ is the Dirac distribution supported on the hypersurface $x^0 = t_0$. Next, we express the advanced and retarded Green's functions in terms of the Green's function (15.2.13): According to (14.2.2), we have the relations

$$
s_m = s_m^\vee - i\pi k_m = s_m^\wedge + i\pi k_m
$$

and thus

$$
\psi_m = k_m g_m - s_m B \psi_m \quad \text{with} \quad g_m := 2\pi \gamma^0 \delta_{t_0} \psi_m^0 + i\pi \epsilon_{t_0} B \psi_m, \quad (15.2.15)
$$

where $\epsilon_{t_0}$ is the step function $\epsilon_{t_0}(x) := \epsilon(x^0 - t_0)$ (and we omitted the brackets in expressions like $k_m g_m \equiv k_m(g_m)$). Note that the expression $k_m g_m$ is well-defined according to Lemma 15.2.7. We also remark that by applying the operator $(i\partial - m)$ to the distribution $g_m$ in (15.2.15), one immediately verifies that $\psi_m$ indeed satisfies the Dirac equation $(i\partial - m)\psi_m = -B\psi_m$.

Now we can compute the inner product $<p\psi|p\psi>$ with the help of Lemma 15.2.8. Namely, using (15.2.15),

$$
<p\psi|p\psi> = \int_{I \times I} <k_m g_m - s_m B \psi_m | k_m' g_m' - s_m' B \psi_m'> dm \, dm'
$$

$$
= \int_I \left( <g_m | p_m g_m> + \pi^2 <B \psi_m | p_m B \psi_m> \right) dm
$$

$$
+ \int_{I \times I} \frac{PP}{m - m'} \left( <B \psi_m | k_m g_m'> - <k_m g_m | B \psi_m'> \right. \\

$$

$$
\left. + <B \psi_m | (s_m - s_m') B \psi_m'> \right) \, dm \, dm'.
$$

Note that this computation is mathematically well-defined in the distributional sense because $\psi_m$ and $g_m$ are smooth and compactly supported in the mass parameter $m$. Employing the explicit formula for $g_m$ in (15.2.15), we obtain

$$
<p\psi|p\psi> = \int_I \left( <g_m | p_m g_m> + \pi^2 <B \psi_m | p_m B \psi_m> \right) dm.
$$

Comparing (14.3.3) with (15.2.14) and taking into account that the operator $S_m$ defined by (15.2.9) gives a minus sign for the states of negative frequency, we get

$$
p_m = S_m k_m.
$$

Using this identity together with Proposition 14.2.1 in the vacuum yields the relations

$$
<g_m | p_m g_m> = (k_m g_m | S_m k_m g_m)|_{t_0}
$$

$$
<B \psi_m | p_m B \psi_m> = (k_m B \psi_m | S_m k_m B \psi_m)|_{t_0}.
$$

We finally apply Proposition 14.2.3 to obtain the representation

$$
<p\psi|p\psi> = \int_I \left( (h_m | S_m h_m)|_{t_0} + \pi^2 (k_m B \psi_m | S_m k_m B \psi_m)|_{t_0} \right) \, dm, \quad (15.2.16)
$$

where

$$
h_m := \psi_m + i\pi k_m (\epsilon_{t_0} B \psi_m).
In this exercise we collect a few elementary properties of the ordered exponential. Expressing the operators $k_m$ according to (14.2.10) by the time evolution operator and writing $\psi_m$ in terms of the initial data as

$$\psi_m|_t = \tilde{U}^{t,t_0}\psi|_{t_0},$$

we obtain

$$(\psi_m | \tilde{S}_m \psi_m)_m = (\psi | S_m \psi)|_{t_0} - \frac{i}{2} \int_{-\infty}^{\infty} \epsilon(t - t_0) \left( S_m U^{t_0,t} \mathcal{V}(t) \tilde{U}^{t,t_0} \psi \right)|_{t_0} \, dt$$

$$+ \frac{i}{2} \int_{-\infty}^{\infty} \epsilon(t - t_0) \left( U^{t_0,t} \mathcal{V}(t) \tilde{U}^{t,t_0} \psi \right) S_m |_{t_0} \, dt$$

$$+ \frac{1}{4} \int_{\mathbb{R} \times \mathbb{R}} \epsilon(t - t_0) \epsilon(t' - t_0) \left( U^{t_0,t} \mathcal{V}(t) \tilde{U}^{t,t_0} \psi \right) S_m U^{t_0,t'} \mathcal{V}(t') \tilde{U}^{t',t_0} \psi |_{t_0} \, dt \, dt'$$

$$+ \frac{1}{4} \int_{\mathbb{R} \times \mathbb{R}} \left( U^{t_0,t} \mathcal{V}(t) \tilde{U}^{t,t_0} \psi \right) S_m U^{t_0,t'} \mathcal{V}(t') \tilde{U}^{t',t_0} \psi |_{t_0} \, dt \, dt'. $$

Rearranging the terms and polarizing gives the result.

**Proof of Theorem 15.2.4.** Since the time evolution operators are unitary and the operators $\tilde{S}_m$ have norm one (see (15.2.9)), the representation (15.2.11) and (15.2.12) gives rise to the following estimate for the sup-norm of $\tilde{S}_m$,

$$\|\tilde{S}_m\| \leq 1 + \int_{\mathbb{R}} |\mathcal{V}(t)|_{C^0} \, dt + \int_{\mathbb{R} \times \mathbb{R}} |\mathcal{V}(t)|_{C^0} |\mathcal{V}(t')|_{C^0} \, dt \, dt'.$$

The decay assumption (15.2.7) implies that the sup-norm of $\tilde{S}_m$ is bounded uniformly in $m$. Using this fact in (15.2.10) gives the inequality (13.2.8), thereby establishing the strong mass oscillation property.

We finally remark that the uniqueness statement in Proposition 13.3.3 implies that (15.2.11) and (15.2.12) yields an explicit representation of the fermionic signature operator in the presence of a time-dependent external potential.

**Exercises**

**Exercise 15.1.** For a smooth one-parameter family of matrices $F(\alpha)$, $\alpha \in \mathbb{R}$, the **ordered exponential** $\text{Pexp}(\int_a^b F(\alpha) \, d\alpha)$ is given by the Dyson series

$$\text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) = 1 + \int_a^b F(t_0) \, dt_0 + \int_a^b dt_0 F(t_0) \int_t^b F(t_1) \, dt_1$$

$$+ \int_a^b dt_0 F(t_0) \int_t^b dt_1 F(t_1) \int_{t_1}^b F(t_2) \, dt_2 + \cdots.$$

In this exercise we collect a few elementary properties of the ordered exponential.

(a) Assume that the matrix-valued function $F$ is commutative in the sense that $[F(\alpha), F(\beta)] = 0$ for all $\alpha, \beta \in [a, b]$.

Show that the ordered exponential reduces to the ordinary exponential,

$$\text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) = \exp \left( \int_a^b F(\alpha) \, d\alpha \right).$$
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**Hint:** Show inductively that

\[
\int_a^b dt_0 F(t_0) \int_t^b dt_1 F(t_1) \cdots \int_{t_{n-1}}^b dt_n F(t_n) = \frac{1}{(n+1)!} \left( \int_a^b F(t) \, dt \right)^{n+1}.
\]

(b) Assume that \( F \) is continuous on \([a, b]\). Show that the Dyson series converges absolutely and that

\[
\left\| \text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) \right\| \leq \exp \left( \int_a^b \| F(\alpha) \| \, d\alpha \right).
\]

**Hint:** Estimate the integrals and apply (a).

(c) Show by direct computation that the ordered exponential satisfies the equations

\[
\frac{d}{da} \text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) = -F(a) \text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) \quad (15.2.17)
\]

\[
\text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) = 1. \quad (15.2.18)
\]

Use the uniqueness theorem for solutions of ordinary differential equations to give an alternative definition in terms of the solution of an initial-value problem. Use this reformulation to show the group property

\[
\text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) \text{Pexp} \left( \int_b^c F(\alpha) \, d\alpha \right) = \text{Pexp} \left( \int_a^c F(\alpha) \, d\alpha \right). \quad (15.2.19)
\]

(d) Show that

\[
\frac{d}{db} \text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) = \text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right) F(b). \quad (15.2.20)
\]

**Hint:** Differentiate the identity \((15.2.19)\) in the case \(c = a\) and use the group properties \((15.2.18)\) and \((15.2.19)\).

(e) Show that

\[
\text{Pexp} \left( \int_a^b F(\alpha) \, d\alpha \right)^* = \text{Pexp} \left( \int_b^a (-F(\alpha)^*) \, d\alpha \right).
\]

Deduce that if \( F(\alpha) \) is an anti-Hermitian matrix, then the ordered exponential is a unitary matrix. **Hint:** There are two alternative methods. One method is to argue using the differential equations \((15.2.17)\) and \((15.2.20)\) or with the group property. A more computational approach is to take the adjoint of the Dyson series and reparametrize the integrals.

**EXERCISE 15.2.** Given \( \omega \in \mathbb{R} \) and a smooth function \( V(t) \), we consider the ordinary differential equation

\[
(i\partial_t + \omega) \phi(t) = V(t) \phi(t).
\]

(a) Write down the Lippmann-Schwinger equation, taking the right side of the equation as the perturbation. **Hint:** The free time evolution operator \( U^{t,t'} \) was computed in Exercise 14.6.

(b) Express the Lippmann-Schwinger equation in the case \( \omega = 0 \) explicitly as an integral equation. How is it related to the integral equation used in the Picard iteration (in the proof of the Picard-Lindelöf theorem)?
CHAPTER 16

Methods of Microlocal Analysis

16.1. A Brief Overview

In the previous chapter we constructed the fermionic signature operator

$$S_m : \mathcal{H}_m \to \mathcal{H}_m,$$

being a bounded and symmetric operator. We now briefly recall why this construction is useful.

Basically, there are two ways to proceed:

- Construction of a corresponding causal fermion system:
  The subspace $\mathcal{H}$ is introduced typically as the negative spectral subspace of the fermionic signature operator,

  $$\mathcal{H} := (\chi_{(-\infty,0)}(S_m))(\mathcal{H}_m) \subset \mathcal{H}_m.$$  

  We saw that in the Minkowski vacuum, $\mathcal{H}$ corresponds precisely to the subspace of all negative-frequency solutions of the Dirac equation. In the presence of an external potential, $\mathcal{H}$ can be regarded as a generalization of the Dirac sea picture to the time-dependent setting.

  Now we can proceed as described in Section 4.3: For $0 < \varepsilon < \varepsilon_{\text{max}}$ we again introduce regularization operators $(\mathcal{R}_\varepsilon)$

  $$\mathcal{R}_\varepsilon : \mathcal{H} \to C^0(\mathcal{M}, \mathcal{S}\mathcal{M})$$

  and define the local correlation operators $F^\varepsilon(x) \in L(\mathcal{H})$ by

  $$\langle u | F^\varepsilon(x) v \rangle_{\mathcal{H}} = -\langle (\mathcal{R}_\varepsilon u)(x) | (\mathcal{R}_\varepsilon v)(x) \rangle,$$

  for all $u, v \in \mathcal{H}$.

  Taking the push-forward measure,

  $$\rho = (F^\varepsilon)_* \mu$$

  we obtain a causal fermion system $(\mathcal{H}, F, \rho)$.

  Now one can analyze the structures of the obtained causal fermion systems and relate them to the causal structure of Minkowski space, the potentials contained in $\mathcal{B}$, etc. One can also introduce particles and/or anti-particles by “occupying additional state” or by “creating holes in the Dirac sea.” The most important and difficult task is to analyze the causal action for such systems in the limit $\varepsilon \searrow 0$. For which potentials and particle/anti-particle configurations does one have a critical point of the causal action? This question is analyzed in [3].

- Alternatively, one can prefer not to introduce the regularization at this stage. This is a good procedure as long as the interaction is purely by an external field. As soon as one wants to describe the full interaction (no matter if using the causal action principle or in the framework of standard quantum field theory), an ultraviolet regularization is essential, giving rise to the subtle question of what happens in the limit $\varepsilon \searrow 0$.  

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Let us outline the procedure without regularization. Then the right object to consider is the unregularized kernel of the fermionic projector $P(x, y)$ defined as follows. Recall that in Section 14.2 the causal fundamental solution was defined as an operator to the solution space

$$k_m := \frac{1}{2\pi i} \left( s^\vee - s^\wedge \right) : C^\infty_0(\mathcal{M}, S\mathcal{M}) \to \mathcal{H}_m \cap C^\infty_{sc}(\mathcal{M}, S\mathcal{M}).$$

Multiplying by the projection operator to the negative spectral subspace of the fermionic signature operator gives the unregularized fermionic projector $P = -\chi_{(-\infty, 0)}(S)k_m : C^\infty_0(\mathcal{M}, S\mathcal{M}) \to \mathcal{H}_m \cap C^\infty_{sc}(\mathcal{M}, S\mathcal{M}).$

Next, one represents this operator as an integral operator with a distributional kernel:

**Theorem 16.1.1.** There is a unique distribution $P \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ such that for all $\phi, \psi \in C^\infty_0(\mathcal{M}, S\mathcal{M}),$

$$\langle \phi | P \psi \rangle = P(\phi \otimes \psi).$$

The proof uses the Schwartz kernel theorem; see [56, Theorem 3.12].

Next, the connection to quantum field theory is made by the following theorem, which is based on an abstract construction due to Araki [1] (see [55, Theorem 1.4] or [36, Theorem 1.5.1]):

**Theorem 16.1.2.** There are fermionic field operators $\hat{\Psi}^\alpha(x)$ and $\hat{\Psi}^\beta(y)^*$ together with a ground state $|0\rangle$ with the following properties:

(a) The canonical anti-commutation relations hold:

$$\{ \hat{\Psi}^\alpha(x), \hat{\Psi}^\beta(y)^* \} = k_m(x, y)^\beta, \quad \{ \hat{\Psi}^\alpha(x), \hat{\Psi}^\beta(y) \} = 0 = \{ \hat{\Psi}^\alpha(x)^*, \hat{\Psi}^\beta(y)^* \}.$$

(b) The two-point function is given by

$$\langle 0 | \hat{\Psi}^\alpha(x) \hat{\Psi}^\beta(y)^* | 0 \rangle = -P(x, y)^\alpha.$$ 

We again point out that this theorem defines a quasi-free (i.e. non-interacting) quantum field theory. Introducing an interaction is a difficult task. A first step is to introduce Wick ordering and time-ordered products. It turns out that in order to make mathematical sense of these operations, the two-point distribution must be of Hadamard form. If this is the case, one can even define the Clearly, here we cannot enter the constructions of perturbative quantum field theory (the interested reader is referred for example to [87]). But the Hadamard property can be understood simply in the context of distribution theory, as we now outline.

### 16.2. The Wave Front Set

The general idea of microlocal analysis is to extend Fourier methods such as to obtain information simultaneously both in position and in momentum space. We here explain this idea in the example of the notion of the wave front set and the Hadamard property.

Good references are [74, 93].

We work in an open subset $U \subset \mathbb{R}^n$. We denote the distributions in $U$ by $\mathcal{D}'(U)$ (being the dual space of $C^\infty(\mathcal{U}, \mathbb{C})$ with the topology induced by the $C^k$-norms). An open conic neighborhood of a point $\xi \in \mathbb{R}^n$ is defined to be an open neighborhood which is invariant under the action of $\mathbb{R}^+$ by multiplication. Thus an open conic neighborhood is of the form

$$\{ \lambda x \mid x \in S, \lambda \in \mathbb{R}^+ \},$$

where $S$ is an open subset of $S^{n-1} \subset \mathbb{R}^n$. 
16.3. The Hadamard Expansion

We now consider the wave front set the unregularized kernel of the fermionic projector \( P(x, y) \). It is a distribution on \( \mathcal{M} \times \mathcal{M} \simeq \mathbb{R}^4 \times \mathbb{R}^4 \). Hence its wave front set lies in the set

\[
WF P \subset (\mathbb{R}^4 \times \mathbb{R}^4) \times (\mathbb{R}^4 \times \mathbb{R}^4) \setminus \{0\}.
\]

**Definition 16.3.1.** \( P(x, y) \) is said to be of **Hadamard form** if its wave front set has the property

\[
WF P \subset \{(x_1, x_2, \xi, -\xi) \mid \xi^2 = 0, \xi^0 < 0 \text{ and } y - x \sim \xi\}.
\]

- Intuitively, this means that there are only singularities on the light cone, and that these singularities are formed only of negative frequencies.

The Hadamard property can be expressed in much more detail by writing down the singularity structure explicitly in position space. Indeed, based on the work of Radzikowski \[85\], it can be shown that for solutions of the Dirac equation, the above definition is equivalent to a *Hadamard expansion* of the form (see \[91\] or \[69\], page 156)

\[
P(x, y) = \lim_{\epsilon \searrow 0} i\partial_x \left( \frac{U(x, y)}{\sigma_\epsilon(x, y)} + V(x, y) \log \sigma_\epsilon(x, y) + W(x, y) \right), \tag{16.3.1}
\]

where

\[
\sigma_\epsilon(x, y) := (y - x)_j (y - x)_j - i\epsilon (y - x)^0, \tag{16.3.2}
\]

and \( U, V \) and \( W \) are smooth functions on \( \mathcal{M} \times \mathcal{M} \) taking values in the \( 4 \times 4 \)-matrices acting on the spinors (we always denote space-time indices by latin letters running from 0, \ldots, 3).

This local expansion was indeed the starting point by Hadamard (see \[70\], \[63\] or \[5\]). In Minkowski space, the *light-cone expansion* \[26\], \[27\] (see also \[36\] Section 2.2) gives a
systematic procedure for computing an infinite number of Hadamard coefficients in one step.

- explain how the Hadamard expansion can be obtained by “solving the equation iteratively”

It turns out that for an external potential in Minkowski space, the kernel of the fermionic projector is indeed of Hadamard form.

**Theorem 16.3.2.** Assume that the external potential $B$ is smooth, and that its time derivatives decay at infinity in the sense that \((15.2.1)\) holds and in addition that

$$
\int_{-\infty}^{\infty} |\partial^p_t B(t)|_{C^0} dt < \infty \quad \text{for all } p \in \mathbb{N}
$$

(with the $C^0$-norm as defined in \((15.2.2)\)). Moreover, assume that the potential satisfies the bound

$$
\int_{-\infty}^{\infty} B(t)|_{C^0} dt < \sqrt{2} - 1.
$$

Then the fermionic projector $P(x,y)$ is of Hadamard form.

**16.4. The Unregularized Kernel of the Fermionic Projector as a Distribution**

From Definition 13.3.4, the operator $S_m$ is obviously symmetric. Thus the spectral theorem gives rise to the spectral decomposition

$$
S_m = \int_{\sigma(S_m)} \nu dE_\nu,
$$

where $E_\nu$ is the spectral measure (see for example [86]). The spectral measure gives rise to the spectral calculus

$$
f(S_m) = \int_{\sigma(S_m)} f(\nu) dE_\nu,
$$

where $f$ is a bounded Borel function.

**Definition 16.4.1.** Assume that the Dirac operator $D$ on $(M,g)$ satisfies the strong mass oscillation property (see Definition 13.2.3). We define the operators $P_\pm : C^\infty_0(M,S\mathcal{M}) \to \mathcal{H}_m$ by

$$
P_+ = \chi_{[0,\infty)}(S_m) k_m \quad \text{and} \quad P_- = -\chi_{(-\infty,0)}(S_m) k_m
$$

(where $\chi$ denotes the characteristic function). The fermionic projector $P$ is defined by $P = P_-$.

**Proposition 16.4.2.** For all $\phi, \psi \in C^\infty_0(M,S\mathcal{M})$, the operators $P_\pm$ are symmetric, \( < P_\pm \phi | \psi > = < \phi | P_\pm \psi > \).

Moreover, the image of $P_\pm$ is the positive respectively negative spectral subspace of $S_m$, i.e.

$$
P_+(C^\infty_0(M,S\mathcal{M})) = E_{(0,\infty)}(\mathcal{H}_m), \quad P_-(C^\infty_0(M,S\mathcal{M})) = E_{(-\infty,0)}(\mathcal{H}_m).
$$

**Proof.** According to Proposition 14.2.1

$$
<P_\mp \phi \mid \psi> = (P_\mp \phi \mid k_m \psi)_m = -(\chi_{(-\infty,0)}(S_m) k_m \phi \mid k_m \psi)_m
$$

$$
= -(k_m \phi \mid \chi_{(-\infty,0)}(S_m) k_m \psi)_m = <\phi \mid P_- \psi>.
$$

The proof for $P_+$ is similar. The relations \((16.4.2)\) follow immediately from the fact that $k_m(C^\infty_0(M,S\mathcal{M}))$ is dense in $\mathcal{H}_m$. □
16.4. THE UNREGULARIZED KERNEL OF THE FERMIONIC PROJECTOR AS A DISTRIBUTION

Similar as in [56, Theorem 3.12], the fermionic projector can be represented by a two-point distribution on $\mathcal{M}$. As usual, we denote the space of test functions (with the Fréchet topology) by $\mathcal{D}$ and define the space of distributions $\mathcal{D}'$ as its dual space.

**Theorem 16.4.3.** Assume that the strong mass oscillation property holds. Then there is a unique distribution $P \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ such that for all $\phi, \psi \in C^\infty_0(\mathcal{M}, S\mathcal{M})$,

$$<\phi|P\psi> = P(\phi \otimes \psi).$$

**Proof.** According to Proposition 14.2.1 and Definition 16.4.1,

$$<\phi|P\psi> = (k_m \phi | P \psi) = -(k_m \phi | \chi_{(-\infty,0)}(S_m) k_m \psi).$$

Since the norm of the operator $\chi_{(-\infty,0)}(S_m)$ is bounded by one, we conclude that

$$|<\phi|P\psi>| \leq ||k_m \phi|| ||k_m \psi|| = (<\phi|k_m \phi> <\psi|k_m \psi>)^{\frac{1}{2}},$$

where in the last step we again applied Proposition 14.2.1. As $k_m \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$, the right side is continuous on $\mathcal{D}(\mathcal{M} \times \mathcal{M})$. We conclude that also $<\phi|P\psi>$ is continuous on $\mathcal{D}(\mathcal{M} \times \mathcal{M})$. The result now follows from the Schwartz kernel theorem (see [74, Theorem 5.2.1], keeping in mind that this theorem applies just as well to bundle-valued distributions on a manifold simply by working with the components in local coordinates and a local trivialization). □

Exactly as explained in [56, Section 3.5], it is convenient to use the standard notation with an integral kernel $P(x,y)$,

$$<\phi|P\psi> = \int_{\mathcal{M} \times \mathcal{M}} <\phi(x)| P(x,y) \psi(y)>_x d\mu_\mathcal{M}(x) d\mu_\mathcal{M}(y)$$

$$(P\psi)(x) = \int_{\mathcal{M}} P(x,y) \psi(y) d\mu_\mathcal{M}(y)$$

(where $P(.,.)$ coincides with the distribution $P$ above). In view of Proposition 16.4.2, we know that the last integral is not only a distribution, but a function which is square integrable over every Cauchy surface. Moreover, the symmetry of $P$ shown in Proposition 16.4.2 implies that

$$P(x,y)^* = P(y,x),$$

where the star denotes the adjoint with respect to the spin scalar product. Finally, exactly as shown in [56, Proposition 3.13], the spatial normalization property of Proposition 16.4.4 makes it possible to obtain a representation of the fermionic projector in terms of one-particle states. To this end, one chooses an orthonormal basis $(\psi_j)_{j \in \mathbb{N}}$ of the subspace $\chi_{(-\infty,0)}(S_m) \subset \mathcal{H}_m$. Then

$$P(x,y) = -\sum_{j=1}^{\infty} |\psi_j(x)> <\psi_j(y)|$$

with convergence in $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$.

We now specify the normalization of the fermionic projector. We introduce an operator $\Pi$ by

$$\Pi : \mathcal{H}_m \rightarrow \mathcal{H}_m, \quad (\Pi \psi_m)(x) = -2\pi \int_{\mathcal{N}} P(x,y) \psi_m(y) d\mu_\mathcal{N}(y),$$

where $\mathcal{N}$ is any Cauchy surface.
Proposition 16.4.4. (spatial normalization) The operator $\Pi$ is a projection operator on $\mathcal{H}_m$.

Proof. According to Proposition 14.2.3, the spatial integral in (16.4.3) can be combined with the factor $k_m$ in (16.4.1) to give the solution of the corresponding Cauchy problem. Thus

$$\Pi : \mathcal{H}_m \to \mathcal{H}_m, \quad (\Pi \psi_m)(x) = \chi_{(-\infty,0)}(S_m) \psi_m,$$

showing that $\Pi$ is a projection operator. □

Instead of the spatial normalization, one could also consider the mass normalization (for details on the different normalization methods see [61]). To this end, one needs to consider families of fermionic projectors $P_m$ indexed by the mass parameter. Then for all $\phi, \psi \in C_0^\infty(\mathcal{M}, \mathcal{S}_M)$, we can use (13.3.4) and Proposition 14.2.1 to obtain

$$<p(P_m \phi) | p(P_m' \psi)> = \int_I (P_m \phi | S_m P_m \psi)_m dm = \int_I (k_m \phi | S_m \chi_{(-\infty,0)}(S_m) k_m \psi)_m dm$$

$$= \int_I <\phi | S_m \chi_{(-\infty,0)}(S_m) k_m \psi> dm = -<\phi | p(S_m P_m \psi)>,$$

which can be written in a compact formal notation similar to (??) as

$$P_m P_{m'} = \delta(m - m') (-S_m) P_m.$$ 

Due to the factor $(-S_m)$ on the right, in general the fermionic projector does not satisfy the mass normalization condition. The mass normalization condition could be arranged by modifying the definition (16.4.1) to

$$S_m^{-1} \chi_{(-\infty,0)}(S_m) k_m.$$ 

Here we prefer to work with the spatial normalization. For a detailed discussion of the different normalization methods we refer to [61] Section 2.

We finally remark that corresponding causal fermion systems can be constructed exactly as in [56] Section 4] by introducing regularization operators $(\mathcal{R}_\varepsilon)_{\varepsilon > 0}$, computing the local correlation operators $F^\varepsilon(x)$ and defining the universal measure by $d\rho = F^\varepsilon d\mu_\mathcal{M}$.

16.5. Proof of the Hadamard Property

In this section, we give the proof of Theorem 16.3.2. We closely follow the presentation in [55]. In preparation, we derive so-called frequency splitting estimates which give control of the “mixing” of the positive and negative frequencies in the solutions of the Dirac equation as caused by the time-dependent external potential (Theorem 16.5.1). Based on these estimates, we will complete the proof of Theorem 16.3.2 at the end of Section 16.5.2.

16.5.1. Frequency Mixing Estimates. For the following constructions, we again choose the hypersurface $\mathcal{N} := \mathcal{N}_{t_0}$ at some given time $t_0$. Moreover, we always fix the mass parameter $m > 0$. Since we are no longer considering families of solutions, for ease in notation we omit the index $m$ at the Dirac wave functions, the scalar products and the corresponding norms. We also identify the solution space $\mathcal{H}_m$ with the Hilbert space $\mathcal{H}_{t_0}$ of square integrable wave functions on $\mathcal{N}$. On $\mathcal{H}_{t_0}$, we can act with the Hamiltonian $H$ of the vacuum, and using the above identification, the operator $H$ becomes an operator on $\mathcal{H}_m$ (which clearly depends on the choice of $t_0$).
We work with a so-called \textit{frequency splitting} with respect to the vacuum dynamics. To this end, we decompose the Hilbert space $\mathcal{H}_m$ as

$$\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^-$$

with $\mathcal{H}^\pm = \chi^\pm(H)\mathcal{H}_m$, where $\chi^\pm$ are the characteristic functions

$$\chi^+ := \chi_{[0,\infty)} \quad \text{and} \quad \chi^- := \chi_{(-\infty,0)} \quad (16.5.1)$$

For convenience, we write this decomposition in components and use a block matrix notation for operators, i.e.

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A^+_+ & A^+_- \\ A^-_+ & A^-_- \end{pmatrix},$$

where $A^s_{s'} = \chi^s(H)A\chi^{s'}(H)$ and $s, s' \in \{\pm\}$.

The representation in Proposition 15.2.5 makes it possible to let the fermionic signature operator $\tilde{S}_m$ act on the Hilbert space $\mathcal{H}_m$ (for fixed $m$). We decompose this operator with respect to the above frequency splitting,

$$\tilde{S}_m = S^D + \Delta \tilde{S}, \quad \text{where} \quad S^D := \tilde{S}_m^+ + \tilde{S}_m^- \quad \text{and} \quad \Delta \tilde{S} := \tilde{S}_m^+ - \tilde{S}_m^-.$$ 

Thus the operator $S^D$ maps positive to positive and negative to negative frequencies. The operator $\Delta \tilde{S}$, on the other hand, mixes positive and negative frequencies. In the next theorem, it is shown under a suitable smallness assumption on $B$ that the operators $\chi^\pm(\tilde{S}_m)$ coincide with the projections $\chi^\pm(H)$, up to smooth contributions. The main task in the proof is to control the “frequency mixing” as described by the operator $\Delta \tilde{S}$.

**Theorem 16.5.1.** Under the assumptions of Theorem 16.3.2, the operators $\chi^\pm(\tilde{S}_m)$ have the representations

$$\chi^\pm(\tilde{S}_m) = \chi^\pm(H) + \frac{1}{2\pi i} \int_{\partial B_{\frac{1}{2}}(\pm1)} (\tilde{S}_m - \lambda)^{-1} \Delta \tilde{S} (S^D - \lambda)^{-1} d\lambda, \quad (16.5.2)$$

where the contour integral is an integral operator with a smooth integral kernel.

Here $B_{\frac{1}{2}}$ denotes the open ball of radius $1/2$. The operator $(\tilde{S}_m - \lambda)^{-1}$ is also referred to as the \textit{resolvent} of $\tilde{S}_m$.

This theorem will be proved in several steps. We begin with a preparatory lemma.

**Lemma 16.5.2.** Under the assumptions (15.2.1) and (16.3.3), the spectrum of $S^D$ is located in the set

$$\sigma(S^D) \subset \left[ -\frac{3}{2}, -\frac{1}{2} \right] \cup \left[ \frac{1}{2}, \frac{3}{2} \right]. \quad (16.5.3)$$

Moreover,

$$\chi^\pm(S^D) = \chi^\pm(H), \quad (16.5.4)$$

and the operators $\chi^\pm(\tilde{S}_m)$ have the representations (16.5.2).

**Proof.** Since the subspaces $\mathcal{H}^\pm$ are invariant under the action of $S^D$, our task is to show that the spectrum of $S^D|_{\mathcal{H}^\pm}$ is positive and negative, respectively. This statement would certainly be true if we replaced $S^D$ by $S_m$, because the operator $S_m$ has the eigenvalues $\pm1$ with $\mathcal{H}^\pm$ as the corresponding eigenspaces. Estimating the representation in Proposition 15.2.5 with the Schwarz inequality, we obtain

$$\left| \langle \psi | S^D \phi \rangle - \langle \psi | S_m \phi \rangle \right| \leq \left( c + \frac{c^2}{2} \right) \|\psi\| \|\phi\| \quad \text{with} \quad c := \int_{-\infty}^{\infty} |B(\tau)|_{C^0} d\tau.$$
Using the assumption \((16.3.3)\), we conclude that
\[
\left| (\psi | S^D \phi) - (\psi | \tilde{S}_m \phi) \right| \leq \frac{1}{2} \| \psi \| \| \phi \| \quad \text{for all } \psi, \phi \in \mathcal{H}_m.
\]

Standard estimates on the continuity of the spectrum (see for example \([76] \), §IV.3) yield that the spectrum of \(S^D\) differs by that of the operator \(\tilde{S}_m\) at most by \(1/2\). This gives \((16.5.3)\) and \((16.5.4)\).

In order to prove the representation \((16.5.2)\), we take the resolvent identity
\[
(\tilde{S}_m - \lambda)^{-1} = (S^D - \lambda)^{-1} - (\tilde{S}_m - \lambda)^{-1} \Delta \tilde{S} (S^D - \lambda)^{-1},
\]
form the contour integral and apply \((16.5.4)\). This gives the result. \(\square\)

The next lemma relates the smoothness of an integral kernel to the boundedness of the product of the operator with powers of the vacuum Hamiltonian.

**Lemma 16.5.3.** Let \(A \in L(\mathcal{H}_m)\) be an operator which maps smooth functions to smooth functions and has the property that for all \(p, q \in \mathbb{N}\), the operator product
\[
H^q A H^p : C^\infty(\mathcal{N}, \mathcal{M}) \to C^\infty(\mathcal{N}, \mathcal{M})
\]
extends to a bounded linear operator on \(\mathcal{H}_m\). Then, considering \(A\) as an operator on \(\mathcal{H}_m\), this operator can be represented as an integral operator with a smooth integral kernel, i.e.
\[
(A\psi)(x) = \int_{\mathcal{N}} A(x, (t_0, \vec{y})) \gamma^0 \psi(t_0, \vec{y}) d^3 y \quad \text{with} \quad A \in C^\infty(\mathcal{M} \times \mathcal{M}).
\]

**Proof.** Since in momentum space, the square of the Hamiltonian takes the form
\[
H(\vec{k})^2 = \left( \gamma^0 (\vec{\gamma} \vec{k} + m) \right)^2 = ( - \vec{\gamma} \vec{k} + m) (\vec{\gamma} \vec{k} + m) = |\vec{k}|^2 + m^2,
\]
the wave function \(\hat{\psi}\) defined by
\[
\hat{\psi}(\vec{k}) := \frac{1}{|\vec{k}|^2 + m^2} e^{i\vec{k}\vec{x}_0} \Xi
\]
for a constant spinor \(\Xi\) and \(\vec{x}_0 \in \mathbb{R}^3\), satisfies the equation
\[
H^2 \hat{\psi}(\vec{x}) = \delta^3(\vec{x} - \vec{x}_0) \Xi.
\]
Moreover, one verifies immediately that \(\psi \in \mathcal{H}_{t_0}\) is square-integrable. Using the last equation together with \((16.5.5)\), we conclude that
\[
H^q A(\delta^3(\vec{x} - \vec{x}_0) \Xi) = H^q A H^2 \psi \in \mathcal{H}_{t_0}.
\]
Since \(q\) is arbitrary, it follows that \(A\) has an integral representation in the spatial variables,
\[
(A\phi)(\vec{x}) = \int_{\mathcal{N}} A(\vec{x}, \vec{y}) \gamma^0 \phi(\vec{y}) d^3 y \quad \text{with} \quad A \in C^\infty(\mathcal{N} \times \mathcal{N}).
\]
We now extend this integral kernel to \(\mathcal{M} \times \mathcal{M}\) by solving the Cauchy problem in the variables \(x\) and \(y\). This preserves smoothness by the global existence and regularity results for linear hyperbolic equations, giving the result. \(\square\)

**Lemma 16.5.4.** Under the assumptions of Theorem \((16.3.3)\) for all \(p \in \mathbb{N}\) the iterated commutator
\[
[S^{(p)} := \left[ H, [H, \ldots, [H, \tilde{S}_m] \ldots] \right]_{p \text{ factors}}
\]
is a bounded operator on \(\mathcal{H}_m\).
Proof. In the vacuum, the Hamiltonian clearly commutes with the time evolution operator,
\[ [H, U_m^{t,t'}] = 0. \] (16.5.6)

In order to derive a corresponding commutator relation in the presence of the external potential, one must take into account that \( \tilde{H} \) is time-dependent. For ease in notation, we do not write out this dependence, but instead understand that the Hamiltonian is to be evaluated at the correct time, i.e.

\[ \tilde{U}_{m}^{t,t'} \tilde{H} = \tilde{U}_{m}^{t,t'} \tilde{H}(t') \quad \text{and} \quad \tilde{H} \tilde{U}_{m}^{t,t'} = \tilde{H}(t) \tilde{U}_{m}^{t,t'}. \]

Then

\[ (i\partial_t - \tilde{H})(\tilde{H} \tilde{U}_{m}^{t,t'} - \tilde{U}_{m}^{t,t'} \tilde{H}) = i\tilde{H} \tilde{U}_{m}^{t,t'} \quad \text{and} \quad \tilde{H} \tilde{U}_{m}^{t,t'} - \tilde{U}_{m}^{t,t'} \tilde{H} \big|_{t=t'} = 0 \]

(here and in what follows the dot denotes the partial derivative with respect to \( t \)). Solving the corresponding Cauchy problem gives

\[ [\tilde{H}, \tilde{U}_{m}^{t,t'}] = \int_{t'}^{t} \tilde{U}_{m}^{\tau,t'} \dot{\tilde{H}} \tilde{U}_{m}^{\tau,t'} d\tau. \] (16.5.7)

In order to compute the commutator of \( H \) with the operator products in (15.2.11) and (15.2.12), we first differentiate the expression \( U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'} \) with respect to \( t \),

\[ i\partial_t (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) = iU_{m}^{t',t} \dot{\triangledown} \tilde{U}_{m}^{t,t'} + U_{m}^{t',t} \triangledown \tilde{H} \tilde{U}_{m}^{t,t'} - U_{m}^{t',t} \triangledown H \triangledown \tilde{U}_{m}^{t,t'}. \] (16.5.8)

Moreover, using the commutation relations (16.5.6) and (16.5.7), we obtain

\[
H (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) - (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) \tilde{H} \\
= U_{m}^{t',t} \triangledown \tilde{H} \tilde{U}_{m}^{t,t'} - U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'} + U_{m}^{t',t} \triangledown \tilde{H} \tilde{U}_{m}^{t,t'} \\
= iU_{m}^{t',t} \dot{\triangledown} \tilde{U}_{m}^{t,t'} - i\partial_t (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) + \int_{t'}^{t} U_{m}^{t',t} \triangledown \tilde{U}_{m}^{\tau,t'} \dot{\tilde{H}} \tilde{U}_{m}^{\tau,t'} d\tau, \\
\]

where in the last step we applied (16.5.8). It follows that

\[
[H, U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}] = H (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) - (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) \tilde{H} + (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) \triangledown \\
= iU_{m}^{t',t} \dot{\triangledown} \tilde{U}_{m}^{t,t'} + (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) \triangledown - i\partial_t (U_{m}^{t',t} \triangledown \tilde{U}_{m}^{t,t'}) + \int_{t'}^{t} U_{m}^{t',t} \triangledown \tilde{U}_{m}^{\tau,t'} \dot{\tilde{H}} \tilde{U}_{m}^{\tau,t'} d\tau. \\
\]

Proceeding in this way, one can calculate the commutator of \( H \) with all the terms in (15.2.11) and (15.2.12). We write the result symbolically as

\[ [H, \tilde{S}_m] = \mathcal{S}^{(1)}, \]

where \( \mathcal{S}^{(1)} \) is a bounded operator. Higher commutators can be computed inductively, giving the result.

We point out that this lemma only makes a statement on the iterative commutators. Expressions like \([HP, \tilde{S}_m] \) or \( H^q \tilde{S}_m HP \) will not be bounded operators in general. However, the next lemma shows that the operator \( \Delta \mathcal{S} \) has the remarkable property that multiplying by powers of \( H \) from the left and/or right again gives a bounded operator.

Lemma 16.5.5. Under the assumptions of Theorem 16.3.2 for all \( p, q \in \mathbb{N} \cup \{0\} \) the product \( H^p \Delta \mathcal{S} H^p \) is a bounded operator on \( \mathcal{H}_m \).
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\[ \sigma(H) \]

\[ \gamma \]

\[ \sigma(H) \]

\[ -m \]

\[ m \]

\[ \text{Figure 16.1. The contour } \gamma. \]

**Proof.** We only consider the products \( H^q S_-^+ \) because the operator \( S_-^+ \) can be treated similarly. Multiplying \((16.5.7)\) from the left and right by the resolvent of \( H \), we obtain

\[
[(H - \mu)^{-1} \tilde{S}_m] = -(H - \mu)^{-1} S^{(1)} (H - \mu)^{-1}.
\]

Writing the result of Lemma 16.5.4 as

\[
[H, S^{(p)}] = S^{(p+1)} \quad \text{with} \quad S^{(p+1)} \in \mathcal{L}(\mathcal{H})
\]

yields more generally the commutation relations

\[
[(H - \mu)^{-1}, S^{(p)}] = -(H - \mu)^{-1} S^{(p+1)} (H - \mu)^{-1} \quad \text{for } p \in \mathbb{N}.
\] (16.5.9)

Choosing a contour \( \gamma \) which encloses the interval \((-\infty, -m]\) as shown in Figure 16.1, one finds

\[
H S_- = \frac{1}{2\pi i} \int_{\gamma} \mu (H - \mu)^{-1} \tilde{S}_m \chi^+(H) \, d\mu
\]

\[
= S H \chi^-(H) \chi^+(H) + \frac{1}{2\pi i} \int_{\gamma} \mu (H - \mu)^{-1} S^{(1)} (H - \mu)^{-1} \chi^+(H) \, d\mu
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \mu (H - \mu)^{-1} S^{(1)} (H - \mu)^{-1} \chi^+(H) \, d\mu,
\]

where in the last step we used that \( \chi^-(H) \chi^+(H) = 0 \). In order to show that this operator product is bounded, it is useful to employ the spectral theorem for \( H \), which we write as

\[
f(H) = \int_{\mathbb{R} \setminus [-m,m]} f(\lambda) \, dE_\lambda,
\] (16.5.10)

where \( dE_\lambda \) is the spectral measure of \( H \). This gives

\[
H S_- = \iint_{\mathbb{R} \times \mathbb{R}} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{\mu}{\lambda - \mu} \frac{1}{\lambda' - \mu} \chi^+(\lambda') \, dE_\lambda \right) S^{(1)} \, dE_{\lambda'} \, d\mu
\]

\[
= -\iint_{\mathbb{R} \times \mathbb{R}} \frac{\lambda}{\lambda - \lambda'} \chi^-(\lambda) \chi^+(\lambda') \, dE_\lambda \, S^{(1)} \, dE_{\lambda'}.
\] (16.5.11)

Note that the term \( \lambda - \lambda' \) is bounded away from zero. Thus the factor \( \lambda/(\lambda - \lambda') \) is bounded, showing that the operator \( H S_- \) is in \( \mathcal{L}(\mathcal{H}_m) \).

This method can be iterated. To this end, we first rewrite the product with commutators,

\[
H^q S_- = \chi^-(H) (H^- \chi^-(H))^p \tilde{S}_m \chi^+(H)
\]

\[
= \chi^-(H) [H^-, [H^-, \ldots, [H^-, S] \ldots]] \chi^+(H),
\]
where we used the abbreviation \( H^- := H \chi^-(H) \). Multiplying from the right by \( H^p \), we can commute factors \( H^+ := H \chi^+(H) \) to the left to obtain
\[
H^q S^+ H^p = (-1)^p \chi^-(H) \left[ H^+, \ldots, [H^+, [H^-, [H^-, S_m] \ldots]] \ldots \right] \chi^+(H).
\]

Representing each factor \( H^\pm \) by a contour integral, one can compute the commutators inductively with the help \((16.5.9)\). Applying the spectral theorem \((16.5.10)\) to the left and right of the resulting factor \( S(p+q) \) yields a constant times the expression
\[
\int_{\mathbb{R} \times \mathbb{R}} \chi^-(\lambda) \chi^+(\lambda') dE_\lambda S^{(p+q)} dE_{\lambda'}
\]
\[
\times \int_{\gamma_1} \frac{\mu_1 \, d\mu_1}{(\lambda - \mu_1)(\lambda - \mu_1)} \cdots \int_{\gamma_{p+q}} \frac{\mu_{p+q} \, d\mu_{p+q}}{(\lambda - \mu_{p+q})(\lambda - \mu_{p+q})}.
\]

Carrying out the contour integrals with residues, we obtain similar to \((16.5.11)\) an expression of the form
\[
H^q S^+ H^p = \int_{\mathbb{R} \times \mathbb{R}} f(\lambda, \lambda') \chi^-(\lambda) \chi^+(\lambda') dE_\lambda S^{(p+q)} dE_{\lambda'}
\]
with a bounded function \( f \). This concludes the proof.

**Proof of Theorem** \([16.5.1]\) It remains to be shown that the contour integral in \((16.5.2)\) has a smooth integral kernel. To this end, we multiply the integrand from the left by \( H^q \) and from the right by \( H^p \) and commute the factors \( H \) iteratively to the inside. More precisely, we use the formula
\[
H^q(\tilde{S}_m - \lambda)^{-1} = \sum_{a=0}^{q} \left[ H, \ldots, [H, H, (\tilde{S}_m - \lambda)^{-1}] \ldots \right] H^{q-a}
\]
(note that the sum is telescopic; here we use the convention that the summand for \( a = 0 \) is simply \((\tilde{S}_m - \lambda)^{-1} H^q\)). Hence
\[
H^q (\tilde{S}_m - \lambda)^{-1} \Delta \tilde{S} (S^D - \lambda)^{-1} H^p
\]
\[
= \sum_{a=0}^{q} \sum_{b=0}^{p} \left[ H, \ldots, [H, (\tilde{S}_m - \lambda)^{-1}] \ldots \right] H^{q-a} \Delta \tilde{S} H^{p-b} \left[ \ldots [S^D - \lambda)^{-1}, H], \ldots, H] \right].
\]

According to Lemma \([16.5.5]\) the intermediate product \( H^{q-a} \Delta \tilde{S} H^{p-b} \) is a bounded operator. Moreover, the commutators can be computed inductively with the help of Lemma \([16.5.4]\) and the formula
\[
[H, (\tilde{S}_m - \lambda)^{-1}] = - (\tilde{S}_m - \lambda^{-1}) [H, \tilde{S}_m] (\tilde{S}_m - \lambda^{-1})
\]
(and similarly for \( S^D \)). This gives operators which are all bounded for \( \lambda \in \partial B_1(\pm 1) \). Since the integration contour is compact, the result follows.
16.5.2. Proof of the Hadamard Form. Relying on the frequency mixing estimates of the previous section, we can now give the proof of Theorem 16.3.2. Recall that the fermionic projector is given by (see (16.4.1))

\[ P = -\chi^{-}(\tilde{S}_m) \tilde{k}_m, \quad (16.5.12) \]

where we again used the short notation (16.5.1). Here again the operator \( \chi^{-}(\tilde{S}_m) \) acts on the solution space \( \mathcal{H}_m \) of the Dirac equation, which can be identified with the space \( \mathcal{H}_{t_0} \) of square integrable wave functions at time \( t_0 \) (see the beginning of Section 16.5.1). For the following arguments, it is important to note that this identification can be made at any time \( t_0 \).

In order to prove that the bi-distribution corresponding to \( P \) is of Hadamard form, we compare the fermionic projectors for three different Dirac operators and use the theorem on the propagation of singularities in [91]. More precisely, we consider the following three fermionic projectors:

1. The fermionic projector \( P_{\text{vac}} \) in the Minkowski vacuum.
2. The fermionic projector \( \tilde{P} \) in the presence of the external potential \( \tilde{B}(x) := \eta(x^0) B(x) \), where \( \eta \geq 0 \) is a smooth function with \( \eta|_{(-\infty,0)} \equiv 0 \) and \( \eta|_{(1,\infty)} \equiv 1 \).
3. The fermionic projector \( P \) in the presence of the external potential \( B(x) \).

The potential \( \tilde{B} \) vanishes for negative times, whereas for times \( x^0 > 1 \) it coincides with \( B \). Thus it smoothly interpolates between the dynamics with and without external potential. The specific form of the potential \( \tilde{B} \) in the transition region \( 0 \leq x^0 \leq 1 \) is of no relevance for our arguments.

In the Minkowski vacuum, the relation (16.5.12) gives the usual two-point function composed of all negative-frequency solutions of the Dirac equation. It is therefore obvious that the bi-distribution \( P_{\text{vac}}(x,y) \) is of Hadamard form.

We now compare \( P_{\text{vac}} \) with \( \tilde{P} \). To this end, we choose an arbitrary time \( t_0 < 0 \). Then, applying the result of Theorem 16.5.1 to (16.5.12), we get

\[ P_{\text{vac}} = -\chi^{-}(H) k_m \quad \text{and} \quad \tilde{P} = -\chi^{-}(H) \tilde{k}_m + \text{(smooth)}, \]

where \( \tilde{k}_m \) is the causal fundamental solution in the presence of the potential \( \tilde{B} \). Since \( \tilde{B} \) vanishes in a neighborhood of the Cauchy surface at time \( t_0 \), we conclude that \( P_{\text{vac}} \) and \( \tilde{P} \) coincide in this neighborhood up to a smooth contribution. It follows that also \( \tilde{P}(x,y) \) is of Hadamard form in this neighborhood. Using the theorem on the propagation of singularities [91 Theorem 5.5], we conclude that \( \tilde{P}(x,y) \) is of Hadamard form for all \( x,y \in \mathcal{M} \).

Next, we compare \( \tilde{P} \) with \( P \). Thus we choose an arbitrary time \( t_0 > 1 \). Using again the result of Theorem 16.5.1 in (16.5.12), we obtain

\[ \tilde{P} = -\chi^{-}(H) \tilde{k}_m + \text{(smooth)} \quad \text{and} \quad P = -\chi^{-}(H) k_m + \text{(smooth)} \]

(where the smooth contributions may of course be different). Since \( \tilde{B} \) and \( B \) coincide in a neighborhood of the Cauchy surface at time \( t_0 \), we infer that \( \tilde{P} \) and \( P \) coincide in this neighborhood up to a smooth contribution. As a consequence, \( P(x,y) \) is of Hadamard form in this neighborhood. Again applying [91 Theorem 5.5], it follows that \( P(x,y) \) is of Hadamard form for all \( x,y \in \mathcal{M} \). This concludes the proof of Theorem 16.3.2.
CHAPTER 17

Fock Space Constructions

• Sollte man das noch hinzufügen?

17.1. The Fermionic Fock Space and Hartree-Fock States

Usually, a many-fermion state is described by a vector in the fermionic Fock space, which we now introduce (see also [86, Section II.4] or [92, Section I.1]). We let $\mathcal{H}^n = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ be the $n$-fold tensor product, endowed with the natural scalar product

$$\langle \psi_1 \otimes \cdots \otimes \psi_n | \phi_1 \otimes \cdots \otimes \phi_n \rangle := \langle \psi_1 | \phi_1 \rangle \cdots \langle \psi_n | \phi_n \rangle \quad (17.1.1)$$

Totally anti-symmetrizing the tensor product gives the wedge product

$$\psi_1 \wedge \cdots \wedge \psi_n := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)} \quad (17.1.2)$$

(here $S_n$ denotes the set of all permutations and $\text{sign}(\sigma)$ is the sign of the permutation $\sigma$).

The wedge product gives rise to a mapping

$$\Lambda_n : \underbrace{\mathcal{H} \times \cdots \times \mathcal{H}}_{n \text{ factors}} \rightarrow \mathcal{H}^n : (\psi_1, \ldots, \psi_n) \mapsto \psi_1 \wedge \cdots \wedge \psi_n.$$ 

We denote the image of this mapping by $\mathcal{F}_n$. The vectors in $\mathcal{F}_n$ are called $n$-particle Hartree-Fock states or factorizable states.

Let us examine in which sense a projector in the one-particle Hilbert space characterizes a many-particle quantum state. Thus let $P$ be a projector in the Hilbert space $(\mathcal{H}, \langle . | . \rangle)$, for simplicity of finite rank $f$, i.e.

$$P^* = P = P^2 \quad \text{and} \quad \dim P(\mathcal{H}) = f.$$ 

In order to get a connection to the fermionic Fock space formalism, we choose an orthonormal basis $\psi_1, \ldots, \psi_f$ of $P(\mathcal{H})$ and form the Hartree-Fock state

$$\Psi := \psi_1 \wedge \cdots \wedge \psi_f \in \mathcal{F}_f. \quad (17.1.3)$$

The choice of our orthonormal basis was unique only up to the unitary transformations

$$\psi_i \rightarrow \tilde{\psi}_i = \sum_{j=1}^f U_{ij} \psi_j \quad \text{with} \quad U \in U(f). \quad (17.1.4)$$

Due to the anti-symmetrization, this transformation changes the corresponding Hartree-Fock state only by a phase factor,

$$\tilde{\psi}_1 \wedge \cdots \wedge \tilde{\psi}_f = \det U \; \psi_1 \wedge \cdots \wedge \psi_f. \quad (17.1.5)$$

Thus we can indeed associate to the projector $P$ a Hartree-Fock state, which is well-defined up to a phase. As the phase of $\Psi$ has no physical significance, the physical
system is described equivalently by a projector \( P_f \) on the many-particle state \( \Psi \), i.e. in bra-/ket notation\(^2\)

\[
P_f = \frac{1}{\|\Psi\|_F^2} \langle \Psi \rangle \langle \Psi | = f! |\psi_1 \wedge \cdots \wedge \psi_f \rangle \langle \psi_1 \wedge \cdots \wedge \psi_f | : \mathcal{F}_f \to \mathcal{F}_f . \tag{17.1.6}
\]

Since the phase freedom drops out when forming the projector (17.1.6), this operator is well-defined. The next proposition gives an alternative definition of \( P_f \) which does not involve a choice of basis.

**Proposition 17.1.1.** For any projector \( P \) in \((\mathcal{H}, \langle \cdot | \cdot \rangle)\) of rank \( f \), the corresponding operator

\[
P_f : \mathcal{F}_f \to \mathcal{F}_f : \psi_1 \wedge \cdots \wedge \psi_f \to (P\psi_1) \wedge \cdots \wedge (P\psi_f) \tag{17.1.7}
\]

is a projector onto an \( f \)-particle Hartree-Fock state. The mapping \( P \to P_f \) gives a one-to-one correspondence between projectors in \( \mathcal{H} \) and projectors on Hartree-Fock states in \( \mathcal{F} \).

**Proof.** It follows immediately from the definitions that \( P_f \) is symmetric and idempotent, and is thus a projector. To compute the rank of \( P_f \), we choose an orthonormal basis \( \psi_1, \ldots, \psi_f \) of \( P(\mathcal{H}) \) and extend it to an orthonormal basis of \( \mathcal{H} \). As is obvious from (17.1.7), the operator \( P_f \) applied to any wedge product of basis vectors vanishes unless all basis vectors are elements of the set \( \{\psi_1, \ldots, \psi_f\} \). Hence the vector \( \Psi := \psi_1 \wedge \cdots \wedge \psi_f \) is a basis of the image of \( P_f \). We conclude that \( P_f \) has indeed rank one and is thus a projector onto the Hartree-Fock state \( \Psi \).

Now suppose conversely that \( P_f \) is a projector onto a Hartree-Fock state. Representing this operator in the form (17.1.6), we let \( P \) be the projector in \( \mathcal{H} \) on the subspace \( \langle \psi_1, \ldots, \psi_f \rangle \). Then the operator \( P_f \) has the representation (17.1.7), concluding the proof. \( \square \)

\(\text{\(f!\) comes about because, according to our conventions (17.1.2) and (17.1.1),}
\[
\langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \wedge \cdots \wedge \psi_f \rangle = \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \otimes \cdots \otimes \psi_f \rangle
\]
\[
= \frac{1}{f!} \sum_{\sigma \in \mathcal{S}_f} (-1)^{\text{sign}(\sigma)} \langle \psi_{\sigma(1)} | \psi_1 \rangle \cdots \langle \psi_{\sigma(f)} | \psi_f \rangle = \frac{1}{f!} .
\]
Part 4

Applications and Outlook
Overview of the Continuum Limit

The continuum limit is a limit which allows to deduce quantum theory, general relativity and quantum field theory from the theory of causal fermion systems. In this chapter, we give an overview about this limit. A thorough explanation of all details can be found in the simultaneously appearing book [36].

Among other things, in this overview, we intend to address worries about the legitimacy of this limit. Therefore, we have carefully chosen the presentation such that the reader can clearly identify what is “put into” the analysis, and what comes out. Furthermore, this chapter only builds on the concepts introduced in Ch. ??, no further knowledge is required at this point.

18.1. General Idea

The continuum limit allows to connect the theory of causal fermion systems to quantum theory, general relativity and quantum field theory. However, we want to make it very clear from the beginning that the continuum limit does not merely consist of changing the value of one constant of the theory. Indeed, the theory, as outlined, e.g., in Ch. ??, does not possess constants on the very fundamental level. Constants, such as the charge or mass of particles, only appear when specifying the minimizers of the theory. To see what this means consider again the example of section ??, where we have constructed the causal fermion system representing Minkowski vacuum. If, in this case, we change the mass of the Dirac sea particles $\psi_k$, from $m$ to $m'$, and carry out the same construction as in Sec. ??, we obtain a different causal fermion system. (This is so since different mass implies different solutions of the Dirac eq. and therefore different local correlation matrices in general.) This means that if one changes the constants characterizing a causal fermion system one really changes the causal fermion system itself. And in general, in cases where the constants are not merely the masses of Dirac sea particles, it is not clear that a change of constant will yields a causal fermion system which is again a minimizer of the causal action principle.

This shows that in the context of this theory, it is natural that the continuum limit is indeed a limit of causal fermion systems. This means that one considers a family of causal fermion systems $(\mathcal{H}_\tau, \mathcal{F}_\tau, \rho_\tau)_{\tau \in (-\varepsilon, \varepsilon)}$ and the limit

$$ (\mathcal{H}_\tau, \mathcal{F}_\tau, \rho_\tau) \rightarrow (\mathcal{H}_0, \mathcal{F}_0, \rho_0), $$

where $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$ is a causal fermion system which is in some sense “distinguished”. In practise, both the number $n$ and $f$ in the definition of causal fermion systems (Def. ??),
and thus $\mathcal{H}$ and $\mathcal{F}$, are fixed. Therefore, Eq. (??) reduces to

$$\rho_\tau \to \rho_0.$$  \hfill (18.1.2)

In this sense, the continuum limit is a limit.

At this point, the reader probably wonders how something as concrete as quantum theory or general relativity can “come out” of something of the sort of Eq. (18.1.2). What lies behind this is a fact which we appeared already in Ch. ?? and Ch. ??: Causal fermion systems can be described with objects analogous to objects of contemporary physical theories, e.g. wave-functions, tangent spaces, connections, etc., c.f. sections ??, ?? and ??.

Therefore, the limit (18.1.2) is at the same time a limit of all those objects:

$$\psi_k^\tau \to \psi_k^0$$

$$\nabla_\tau \to \nabla_0$$

... The essential question in the continuum limit is thus which equations relate those objects and the central statement is that in the limit (18.1.2), those are the equations of contemporary physics. Taking a limit is necessary since the “distinguished” causal fermion systems $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$ have a very singular behavior, c.f. below, and thus the limit corresponds to a regularization of $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$.

This is the general concept behind the continuum limit, which we will explain in more detail in this chapter. Note that the name **continuum limit** is of historical origin: The concepts which led to the theory of causal fermion systems were first built out of discrete spacetimes, and the continuum limit (18.1.2) was indeed a limit which consisted out of a family of such discrete systems which approximated a continuous spacetime. This is not so any more, both the theory of causal fermion systems and the continuum limit are of course fully compatible with continuous spacetimes.

### 18.2. The Fermionic Projector

The fermionic projector, which was the central object of the investigations of Chs. ??, ??, and ??, also plays a key role in the continuum limit. This is so since it projects upon all occupied wave-functions, i.e. all wave-functions which are “really there”. (C.f. the explanation of this on p. ??.) - bra-ket notation - link zu wellenfunktionen -

The general idea of the continuum limit is the following. We have seen in Ch. ?? that given a causal fermion system, the fermionic projector $P$ can be introduced, and we have seen that it projects upon all occupied wave functions. Furthermore, we have seen in Ch. ?? that the physical situation of Minkowski space (and Dirac “particles” thereon) can be reexpressed as a causal fermion system, and in Ch. ?? that the fermionic projector of
the causal fermion system describing Minkowski space takes the form

\[ P(x, y) = \frac{1}{(2\pi)^4} \int (\delta(k^2 - m^2) \theta(-k^0) e^{-i(x-y)k} d^4k. \]  

(18.2.1)

It is easy to check that \( P \) satisfies the vacuum Dirac equation

\[ (i\partial_x - m)P(x, y) = 0, \]  

(18.2.2)

The physical picture is that this causal fermion system can be described as though Dirac particles were propagating in vacuum, i.e. without external potential.

The central question behind the continuum limit is whether there is also a causal fermion system which can be described as though there were Dirac particles propagating in an external potential. Of course, since one is only interested in “physical” causal fermion systems, according to the postulates of the theory, this causal fermion system should be a solution of the causal variational principle. In mathematical terms, the question is therefore: Is there a causal fermion system which solves the causal variational principle and whose fermionic projector \( \tilde{P} \) satisfies the Dirac equation with external potential \( B - \):

\[ (i\partial_x + B - m)\tilde{P}(x, y) = 0. \]  

(18.2.3)

The essential conceptual point is that the requirement of it being a solution of the causal variational principle leads to constraint equations which involve \( B \) and the fermionic projector. Only if those equations are fulfilled can the causal fermion system be a minimizer of the causal variational principle. The analysis, which we sketch in some more detail in this chapter, shows that those equations turn out to be the equations known from contemporary physics, e.g. the Yang-Mills equation coupled to the Dirac current.

The external potential can a priori have any form. It could be, e.g., a vector potential, i.e. \( B(x) = A(x) \), where \( A \) is a 4-vector on Minkowski space. Within this chapter, we will always assume that \( B \) is of the form

\[ B = \chi_L A_L + \chi_R A_R. \]  

(18.2.4)

where \( \chi_L \) and \( \chi_R \) are as described on p. ?? and \( A_L, A_R \) are called left-handed and right-handed component, respectively. In order to obtain the limit for the gravitational dynamics, \( B \) can be chosen as a differential operator, c.f. [36].

18.3. Overview of the Methods

In order to plug the causal fermion system corresponding to \( \tilde{P} \) into the causal variational principle, we first need to express \( \tilde{P} \) explicitly for a given potential \( B \). We have already seen a non-perturbative construction thereof in Ch. ??, but for the purpose of the continuum limit it is more helpful to do so perturbatively. Once this is done, we reformulate the causal variational principle in terms of the eigenvalues of the quantity \( A_{xy} = \tilde{P}(x,y)\tilde{P}(y,x) \), and insert the result of the perturbative construction.

18.3.1. Perturbative construction of \( \tilde{P} \).

The perturbative construction of \( \tilde{P} \) is carried out in two steps:
Step (1) Construct the Greens functions of the Dirac equation with external potential \((18.2.3)\) perturbatively.

Step (2) Carry the result of step (1) over to \(\tilde{P}\).

**Step (1):** We already know the advanced and retarded Greens functions of the vacuum Dirac equation from Ch. ?? These are the functions

\[
s_\wedge^\wedge_{\text{adv}}(x,y) = \int \frac{d^4k}{(2\pi)^4} s_\wedge^\wedge(k) e^{-ik(x-y)},
\]

with

\[
s_\wedge(k) = \lim_{\varepsilon \downarrow 0} \frac{k + m}{k^2 - m^2 - i\varepsilon k^0} \quad \text{and} \quad s_\wedge(k) = \lim_{\varepsilon \downarrow 0} \frac{k + m}{k^2 - m^2 + i\varepsilon k^0}.
\]

For convenience, we include the mass \(m\) of the Dirac equation \((18.2.3)\) into the external potential. This means that we consider the case where the vacuum Dirac equation is massless, and we have as Dirac equation with external potential \((18.3.3)\)

\[
(i\partial_x + B) \tilde{P}(x,y) = 0,
\]

where \(B = \mathcal{B} + m\). The advanced and retarded Green’s functions thereof can be obtained from \(s_\wedge\) and \(s_\wedge\) perturbatively by

\[
\tilde{s}_\wedge^\wedge = \sum_{n=0}^{\infty} (-s_\wedge B)^n s_\wedge^\wedge, \quad \tilde{s}_\wedge = \sum_{n=0}^{\infty} (-s_\wedge B)^n s_\wedge^\wedge.
\]

This perturbation expansion is called *causal perturbation expansion*.

In order to be able to analyze and utilize the terms \((-s_\wedge B)^n s_\wedge\) of the perturbation expansion, we introduce a further expansion ...

**Step (2):**

In order to answer this question, we calculate the fermionic projector \(\tilde{P}\) perturbatively in powers of \(\mathcal{B}\) (still without specifying the dependence of \(\mathcal{B}\) on \(x\)) and then plug \(\tilde{P}\) into the causal variational principle, or to be more precise into the Euler-Lagrange equations of the causal variational principle. What one finds then is that the causal fermion system which \(\tilde{P}\) belongs to can only by a minimizer of the causal variational principle (and hence, can “realizable in nature” (c.f. Sec. ??)) if the external potential \(\mathcal{B}\) and the wave functions \(\psi\) are related in a particular way. If, e.g., we choose \(\mathcal{B}\) as in (??), we find that the causal fermion system is a minimizer only if \(A_L, A_R\) and \(\psi\) are related by

Todo: Ψ sinnvoll erklären.
18.4. Analysis of the Causal Action Principle

\[(i\partial + \gamma^5 A_a - m)\Psi = 0, \quad C_0 (\partial^k A_a^l - \Box A_a^k) - C_2 A_a^k = 12\pi^2 \overline{\Psi} \gamma^5 \gamma^k \Psi, \quad (18.3.5)\]

where \(A_a = A_L + A_R\). Here, \(C_0\) and \(C_2\) are constants which are explained in Sec. ?? below. Those are the Dirac and Yang-Mills equation, i.e. the equations which govern the interaction of fermions and the electromagnetic field.

This shows that indeed the right physical equations appear in the continuum limit for this potential. In the remainder of this chapter we explain in more detail how the analysis is performed and how the result is obtained. For the treatment of other potentials and the deduction of the Einstein equations we refer to [36].

We now explain in more detail how the analysis works. Mathematically, the central point of the analysis is that the fermionic projector \(P(x,y)\) diverges on the light-cone, i.e. if \((x - y)^2 = 0\). In order to cope with this, one initially expands not only in powers of \(B\) but also in powers of \(\xi^2\), according to the following definition.

We now mention a few directions which cannot be covered in the lectures and give a brief outlook.

18.4. Analysis of the Causal Action Principle

The causal action principle was introduced at the beginning of these lectures (see Section ??). But we never came back to it. Indeed, entering the analysis of this variational principle would have been a separate topic. Here we can only mention a few works in this direction. In the papers [30, 32], the existence of minimizers is proved in various situation. In simple terms, these results show that the causal action principle makes mathematical sense.

The important question is how the minimizers look like. In [19] a few simple numerical examples are discussed. In [29] an effect of spontaneous symmetry breaking elaborated. This can be interpreted as a manifestation of a more general effect of “spontaneous structure formation” (see [44, Section 3]). In [59, 10] the Euler-Lagrange equations corresponding to the causal action principle are worked out. The analysis in [10] reveals a tendency of minimizers to be discrete. For a physical discussion of this effect we refer to [44, Section 4]. Finally, the paper [43] is devoted to the question how an initial value problem can be posed for causal variational principles, and whether it has a unique solution.

Except for these few results, nothing is known on the structure of the minimizers. It is an interesting open problem to get a better understanding of the structure of minimizers of causal variational principles. It is planned to study the causal action principle in more advanced explicit examples in the near future.

\[\text{Todo: Fußnote anpassen}\]

---

\(^1\)This can be understood from the fact that if \((x - y)^2 = 0\), the exponential term vanishes and therefore the integral diverges.
18.5. The Continuum Limit of Causal Fermion Systems

It was the main motivation early on to get an alternative description of quantum field theory. This connection can indeed be made by considering the so-called *continuum limit*. Similar as in Section 9.1 one introduces an ultraviolet regularization on a microscopic scale $\varepsilon$. But then one analyzes the Euler-Lagrange equations corresponding to the causal action principle in the limit as $\varepsilon \searrow 0$ when the regularization is removed. A subtle point is that the results do depend on how the regularization is introduced. In order to analyze this effect in detail, one considers a large class of regularizations (“method of variable regularization”). It turns out that the structure of the effective equations obtained in the continuum limit are indeed regularization independent; only the coupling constants and the bosonic masses depend on the details of the regularizations.

Due to these complications, it does not seem suitable to enter the details of the continuum limit analysis in a lecture series. But we the interested reader to the survey article [33] and the references therein. In the more recent paper [34], the continuum limit is worked out for a system involving neutrinos, giving rise to an effective interaction described by a left-handed, massive SU(2) gauge field and a gravitational field. The next step is to work out a model involving leptons and quarks [35], which should combine gravity with all the interactions of the standard model. The plan is to publish the three papers [31, 34, 35] together with a nice general introduction as a book.

We finally point out that it is an important open problem to work out the detailed expansion of the effective interaction in terms of Feynman diagrams (including all loop diagrams) and to compare it to the standard expansion of perturbative quantum field theory. This is a major project for the future.

Exercises
Remaining Exercises

Exercise 18.1. (Lorentz transformation of spinors) In the lecture, you have been introduced to the Lorentz transformation, to the Dirac equation and to spinors. If you are not trained as a physicists, you might wonder about the physical meaning of those quantities, and about the relation between them. The goal of this exercise is to help you understand this better.

1. First of all, explain briefly what Lorentz transformations are used for in physics. Make an example connected to you driving your car to university.

2. Electrons can be described by the Dirac equation, i.e., they are spinors. What happens to spinors when you make a Lorentz transformation, i.e., when you look at them from a moving reference frame? Why does anything have to happen at all?

3. In the lecture, you have seen the following lemma: For any orthochronous proper Lorentz transformation $\Lambda$ there is a unitary matrix $U(\Lambda)$ such that

$$U(\Lambda) \Lambda^l_j \gamma^j U(\Lambda)^{-1} = \gamma^l.$$ 

Explain its meaning and give a proof.

Hints: Since $\Lambda$ is orthochronous and proper, you can write it in the form $\Lambda = \exp(\lambda)$, where $\Lambda$ denotes matrix $(\Lambda^l_j)_{l,j}$. Take a look the family $\Lambda(t) = \exp(\lambda t)$ of Lorentz transformations and differentiate it at $t = 0$ to obtain

$$\lambda^l_j g_{lk} = -g_{jm} \lambda^m_k,$$

where $g_{jm}$ is the Lorentz metric. (Use that Lorentz transformation leave the Lorentz metric invariant.)

Use this equation to evaluate that the matrix

$$u := \frac{1}{4} \lambda^l_k \gamma^l \gamma^k$$

is anti-selfadjoint (with respect to the spin scalar product) and thus that the matrix $U(t) := \exp(tu)$ is unitary. Finally, take a look at the family of matrices

$$A(t) := U(t) \Lambda(t)^l_j \gamma^j U(t)^{-1}.$$ 

Use the commutation relations $[\gamma^l \gamma^k, \gamma^j] = 2 \left( \gamma^l g^{kj} \delta^k_i - \delta^k_l \gamma^k \right)$ to evaluate $\frac{d}{dt} A(t)$, and prove the lemma by comparing $A(0)$ and $A(1)$.

Exercise 18.2. (Getting familiar with the Dirac equation) In this exercise you will familiarize yourself with some of the objects in the context of the Dirac equation, which will be used throughout this lecture.

2 Please limit all text answers during this tutorial to 2-3 sentences (quarter of a page) per question.
(1) Take a look at the Dirac equation with external field, \((i\gamma^k(\partial_k - iA_k) - m)\psi = 0\). Multiply it with the operator \((i\gamma^j(\partial_j - iA_j) + m)\) and compare the result to the Klein-Gordon equation with external field. Explain why the additional term describes the coupling of spin to the electromagnetic field.

(2) Check that in the Dirac representation, the Gamma matrices are self-adjoint w.r.t. the spin scalar product. In light of exercise 1, do you have an idea why this self-adjointness is important?

(3) Take a look at the pseudoscalar matrix \(\rho = i\gamma^0\gamma^1\gamma^2\gamma^3\). Evaluate whether it is self-adjoint and whether it commutes with the Gamma matrices. What is it used for?

(4) In order to define an operator whose square is the wave operator, Dirac introduced the Gamma matrices and their anti-commutation relations. (They form a so-called Clifford algebra.) As you know, Dirac introduced them as \(4 \times 4\) matrices. Why so? What is the minimal dimension of a representation of this algebra? (Please proof your answer.)

(5) Show that the Gamma matrices satisfy the relation \([\gamma_i\gamma^j, \gamma^k] = 2(\gamma_i g^{kj} - \delta_i^j \gamma^k)\) which you have (probably) used in exercise 1.

Exercise 18.3. (1) Show that the space \(C^\infty_0(\mathbb{R}^n)\) of smooth real functions on \(\mathbb{R}^n\) with compact support is dense in \(\mathcal{S}(\mathbb{R}^n)\).

(2) For two continuous functions \(f, g\) on \(\mathbb{R}^n\) we define their convolution \(f * g\) by \(f * g(x) := \int_{\mathbb{R}^n} f(y) \cdot g(x - y) \, dy\). Show that \(*\) restricts to a continuous map \(* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\). Hint: It might shorten the proof considerably if you use symmetry and linearity of \(*\) to reduce the problem to showing that \(f_k \to_k 0\) in \(\mathcal{S}(\mathbb{R}^n)\) implies \(f_k * g \to_k 0\) in \(\mathcal{S}(\mathbb{R}^n)\).

Exercise 18.4. Let \(n \geq 1\), \(f \in \mathcal{S}(\mathbb{R}^n)\), \(A \in \mathcal{S}(\mathbb{R}^n)^*\).

Show that for all \(g \in \mathcal{S}(\mathbb{R}^n)\) we have
\[
A\left(x \mapsto \int_{\mathbb{R}^n} f(y - x)g(y) \, dy\right) = \int_{\mathbb{R}^n} A\left(x \mapsto f(y - x)\right)g(y) \, dy.
\]

To that purpose, proceed as follows:

1) Show that both sides of the equation are well-defined.

2) Let \(g \in C^\infty_0(\mathbb{R}^n)\) and \(R > 0\) such that \(\text{supp}(g) \subset B_R(0)\). Prove the claim for \(g\) by approximating the integral \(\int f(y - x)g(y) \, dy\) by a Riemann sum
\[
S_N(x) := \sum_{k \in \mathbb{Z}^n \atop |k| \leq R \cdot N} \frac{1}{N^n} f(k/N - x)g(k/N)
\]
and showing that \(S_N(\cdot)\) converges in the topology of \(\mathcal{S}(\mathbb{R}^n)\) to \(\int f(y - \cdot)g(y) \, dy\) (for \(N \to \infty\)).

3) Finally, use a denseness argument.

Exercise 18.5. In this exercise, we will take a look at the perturbative construction of the causal Green’s functions of the Dirac equation \((i\partial + B - m)\psi(x) = 0\) in an external field,
\[
\tilde{s}_m^\vee = \sum_{n=0}^\infty (-s_m^\vee B)^n s_m^\vee, \quad \tilde{s}_m^\wedge = \sum_{n=0}^\infty (-s_m^\wedge B)^n s_m^\wedge. \quad (18.5.1)
\]
and check why this construction is unique. To this end, make the ansatz
\[ \tilde{s}_m^\vee = \sum_{n=0}^{\infty} s^{(n)}_m, \]
where \( s^{(n)}_m \) is exactly of order \( B^n \) and show that \( \forall n > 1 \)
\[ (i\partial - m) s^{(n)}_m = -B s^{(n-1)}_m \]
(where operator products are defined as in the lecture). Use the uniqueness of solutions of the homogeneous Dirac equation to show that, if we require
\[ \text{supp } \tilde{s}_{m,x}^\vee \subset J_x^\vee(x), \]
then \( s^{(0)}_m = s_m^\vee \) and \( s^{(n)}_m = -s_m^\vee B s^{(n-1)}_m \) hold uniquely.

**Exercise 18.6.** Recall the definition of the Hilbert space \( H_t \) and of the time evolution operator \( U^{t,t'} : H_t \to H_t' \) from the lecture. Perform a partial Fourier transformation to show that
\[ U^{t,t'}(\vec{k}) = \int_{-\infty}^{\infty} \left( \vec{k} + m \right) \gamma^0 \delta(k^2 - m^2) \mid_{k=(\omega,\vec{k})} \epsilon(\omega) e^{-i\omega(t-t')} d\omega. \]
Carry out the \( \omega \) Integration to obtain
\[ U^{t,t'}(\vec{k}) = \sum_{\pm} \Pi_\pm(\vec{k}) e^{\mp i\omega(\vec{k})(t-t')}, \]
where \( \Pi_\pm(\vec{k}) \) are projection operators and
\[ \omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2} \quad \text{and} \quad k_\pm = (\pm \omega(\vec{k}),\vec{k}). \]
Use this to conclude that \( U^{t,t'} \) is unitary. (Note that it is part of the exercise to show that the \( \Pi_s(\vec{k}) \) are projection operators.)

**Exercise 18.7.** From the lecture you know the fermionic projector of the vacuum in momentum space
\[ P_m(k) = (\vec{k} + m) \delta(k^2 - m^2) \Theta(-k^0). \]
In this exercise, you will calculate its (inverse) Fourier transform,
\[ P_m(x,y) = \int \frac{d^4k}{(2\pi)^4} P_m(k) e^{-ik(x-y)}, \]
which is the (kernel of) the fermionic projector of the vacuum in “position space”. To do so,
1. calculate the inverse Fourier transform of \( \delta(k^2 - m^2) \Theta(-k^0) \)
\[ T_{m^2}(x,y) := \int \frac{d^4k}{(2\pi)^4} \delta(k^2 - m^2) \Theta(-k^0) e^{-ik(x-y)} \]
using Bessel functions.

Hint: Use spherical coordinates for \( \xi = (y-x) \) and \( k \) in “spatial directions” (i.e., use 3-dim. spherical coordinates and keep the \( t / k^0 \)-component). Carry out the integral over the angles (as compared to time- and radial coordinates) first. Then, carry out the \( k^0 \) integration and look up the resulting formula in a textbook or website which contains information about Fourier transforms of
Bessel functions (possibly you might have to rearrange the integral s.t. you integrate \((-\infty, \infty)\) rather than \((0, \infty)\)). It might prove helpful to set \(r = 0\) and \(t = 0\) before integrating, respectively.

If you have trouble revisit GRIPS at the end of the week to see if more hints for this exercise have been added.

(2) Expand the resulting Bessel functions to obtain the formula for \(T_{m^2}\) which you have seen in the lecture.

(3) Argue why \(P_m(x, y) = (i\delta_x + m)T_{m^2}\) and calculate \(P_m(x, y)\) by applying the derivative.

Ps: This exercise is not so easy. Don’t get your hair teared ;)

Exercise 18.8. Give a review in words what you have been taught in the lecture so far. Make sure you explain which objects were introduced, which purpose they serve and why they were introduced. Do not repeat definitions.

(Your list should include at least the following: fermionic projector, causal pertubation expansion, \(k_m, p_m, \tilde{k}_m, \tilde{p}_m\), mass normalization, momentum normalization, \(P^\text{sea}, P^\text{sea}_\text{res}\), \(\tilde{R}_\lambda\), light cone expansion, inductive light cone expansion, resummation, phase-free contribution, Theorem 4.6, residual argument, residual fermionic projector, phase-inserted line integrals, \(T^{(n)}, T^\text{reg}\).)

Hint: You might have problem summarizing the last few objects, e.g. the residual fermionic projector. If so, ask in the beginning of the next lecture about it!

Exercise 18.9. Assume you want to tell your good old friend Kermit the frog about this lecture. In the last years, he has done some quite extensive studies in physics and mathematics, so he is familiar with most of the technical tools used in the lecture. Write him a letter (4-5 handwritten pages) in which you explain the following points:

- the overall goal of the lecture,
- the physical content,
- what one has to do to achieve the overall goal,
- some explanation about every chapter in the script and the most important mathematical structures therein.

He will tell us how good an impression he got about the lecture from you letter and advice us how many points we should give for the letter.

Exercise 18.10. In the lecture, you have been presented with the following corollary:

Let \(C \in \{p_m, k_m\}\) and \(C' \in \{p_{m'}, k_{m'}\}\) and \(b_m^<, b_m^>\) as in the lecture. Then the following calculation rule holds:

\[
Cb_m^<=b_m^<C' = CC' + \delta(m-m')\pi^2Cp_mp_mb_mC'.
\]  
(18.5.2)

Proof it. (This corollary is necessary in order to construct a perturbation expansion for \(\tilde{k}_m\), which is in turn necessary for many calculations in this lecture.)

Hint: First use the calculation rules of Lemma 3.1 to verify that

\[
C\left(\sum_{l=0}^{1}(B_{s_m})^l(s_{m'}B)^{n-l}\right)C' = 0.
\]
Next, verify that for any \( n \geq 2 \),
\[
C\left( \sum_{l=0}^{n} (B s_m)^l (s_m B)^{-l} \right) C' = \delta(m - m') \pi^2 C \sum_{l=0}^{n-2} (B s_m)^l B p_m B (s_m B)^{-l-2} C'.
\]
(split up the sum into an \( l = 0 \) term, an \( l = n \) term and \( \sum_{l=1}^{n-1} \)). With the help of this, calculate
\[
C b_m^\wedge b_m^\wedge C'
\]
by using a suitable index-shift. (This quantity is equal to \( C \sum_{n=0}^{\infty} (-B s_m)^n \sum_{n'=0}^{\infty} (-s_m B)^{n'} C' \). Define your new indices \( k, l \) such that this expression takes the form \( C \sum_{k=0}^{\infty} \sum_{l=0}^{n} (-B s_m)^l (-s_m B)^{k-l} C' \).)

**Exercise 18.11.** (Causal Green’s functions of the Dirac equation) In the lecture you were introduced to the Dirac equation and its causal Green’s functions
\[
s_m^\wedge(k) = \lim_{\epsilon \downarrow 0} \frac{k + m}{k^2 - m^2 - i\epsilon k^0} \quad \text{and} \quad s_m^\vee(k) = \lim_{\epsilon \downarrow 0} \frac{k + m}{k^2 - m^2 + i\epsilon k^0},
\]
as well as to the Feynman propagator
\[
s_m^F(k) := \lim_{\epsilon \downarrow 0} \frac{k + m}{k^2 - m^2 + i\epsilon}.
\]
(1) Argue that these limits exist as distributions, and show that if one takes the inverse Fourier transform of \( s_m^\vee(k) \) and \( s_m^\wedge(k) \),
\[
s_m(x, y) = \int \frac{d^4k}{(2\pi)^4} s_m(k) e^{+ik(x-y)},
\]
one obtains
\[
\text{supp } s_m^\vee(x,.) \subset J^\vee_x \quad \text{and} \quad \text{supp } s_m^\wedge(x,.) \subset J^\wedge_x,
\]
i.e. \( s_m^\vee / \wedge(x, y) \) are causal. Give a brief interpretation of what this means.

(2) Show that the Fourier transform of the Feynman propagator is
\[
s_m^F(x, y) = \frac{i}{(2\pi)^3} \int_{\mathbb{R}^3} (\frac{k + m}{k^2 + m^2}) e^{-ik(x-y)} \mid_{k=\epsilon(t,\omega(\rho),\rho)} d\mu_\rho,
\]
where \( t := (y - x)^0, \omega(\rho) := \sqrt{\rho^2 + m^2} \) and \( d\mu_\rho := \frac{d^3\rho}{2\omega(\rho)} \) (\( \epsilon \) denotes the sign function \( \epsilon(x) = 1 \) for \( x \geq 0 \) and \( \epsilon(x) = -1 \) otherwise).

Try to interpret the result: Which kind of frequencies appear for positive times, which kind for negative times? What could this mean physically?

Hints: ad i) You can show this by proving first that \( s_m^\vee(x - y) = 0 \) if \( (x^0 - y^0) > 0 \) and then invoking Lorentz invariance to show that \( \text{supp } s_m^\vee(x,.) \subset J^\wedge_x \). (The latter argument can be given in words.) But feel free to look for a different solution! Use residues to calculate the integrals!

**Exercise 18.12.** (Multiplication operators) Let \( n \in \mathbb{N} \setminus \{0\} \).
(1) Let $g$ be a (not necessarily continuous) function $g : \mathbb{R}^n \to \mathbb{R}$ such that there is an $r \in (0, \infty)$ with $x \mapsto g(x)/(1 + |x|^r)$ in $L^1(\mathbb{R}^n)$. Show that the assignment $f \mapsto T_0(f) := \int_{\mathbb{R}^n} f(x) \cdot g(x) \, dx$ defines a tempered distribution $T_0 \in \mathcal{S}(\mathbb{R}^n)^*$. 

(2) Construct a function $g \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ (such that $T_0 \in \mathcal{S}(\mathbb{R}^n)^*$) not pointwise bounded by any polynomial, i.e., such that there is no polynomial function $P : \mathbb{R} \to \mathbb{R}$ with $|g(x)| \leq P(|x|)$ $\forall x \in \mathbb{R}^n$.

**Exercise 18.13. (A special distribution)** For $f \in \mathcal{S}(\mathbb{R})$ we define

$$A(f) := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{f(x)}{x} \, dx.$$ 

Prove that $A$ is a tempered distribution, that is, $T \in \mathcal{S}(\mathbb{R})^*$.

**Exercise 18.14. (External field problem)** In physics, the notion of “particle” and “anti-particle” is often introduced as follows: Solutions of the Dirac equation with positive frequency are called “particles” and solutions with negative frequency “anti-particles”. In this exercise, we will check in how far this makes sense.

To this end, take a look at the Dirac equation in an external field:

$$(i \slashed{\partial} + B - m) \psi = 0. \quad (18.5.4)$$

Assume that $B$ is time-dependent and has the following form:

$$B(t, x) = V \Theta(t - t_0)\Theta(t_1 - t),$$

where $V \in \mathbb{R}$ and $t_0 = 0$, $t_1 = 1$. In order to construct a solution thereof, for a given momentum $\vec{k}$, we use plane wave solutions of the Dirac equation,

$$\psi(t, \vec{x}) = e^{-i\omega t + i\vec{k}\vec{x}} \chi_{\vec{k}},$$

where $\chi_{\vec{k}}$ is a spinor $\in \mathbb{C}^4$, and patch them together suitably. (The quantity $\omega$ is called the “frequency” or “energy”, and $\vec{k}$ the “momentum”.) To simplify the calculation, we set $\vec{k} = (k_1, 0, 0)^T$. Proceed as follows:

1. First, take a look at the region $t < t_0$. Reformulate Eq. (18.5.4) s.t. there is only the time derivative on the left hand side. (Hint: Multiply by $\gamma^0$.)

2. Insert the plane wave ansatz with $\vec{k} = (k_1, 0, 0)^T$ into the equation. Your equation now has the form $\omega \psi = H(k_1) \psi$. Show that the eigenvalues of $H(k_1)$ are $\pm \omega_0$ with $\omega_0 := \sqrt{(k_1)^2 + m^2}$.

3. Show that one eigenvector belonging to $\pm \omega_0$ is $\chi_0^\pm := \left(\frac{m+\omega_0}{k_1}, 0, 0, 1\right)^T$ and that one eigenvector belonging to $-\omega_0$ is $\chi_0^- := \left(\frac{m-\omega_0}{k_1}, 0, 0, 1\right)^T$. (Both eigenvalues have multiplicity 2, but we don’t need the other two eigenvectors here.)

4. With this, you have constructed plane wave solutions $e^{-i(\pm \omega_0)t + i\vec{k}\vec{x}} \chi_0^\pm$ for $t < t_0$ and also for $t > t_1$. By transforming $t \rightarrow (m - V)$, you immediately obtain plane wave solutions also for $t_0 < t < t_1$. Denote the respective quantities by $\omega_1$ and $\chi_1^\pm$.

5. Assume that for $t < t_0$ there is one “particle” present, i.e. set

$$\psi(t, \vec{x}) = e^{-i\omega_0 t + i\vec{k}\vec{x}} \chi_0^+ \quad \text{for } t < t_0.$$

Assume that the solution for $t_0 < t < t_1$ takes the form

$$Ae^{-i\omega_1 t + i\vec{k}\vec{x}} \chi_1^+ + Be^{-i(-\omega_1)t + i\vec{k}\vec{x}} \chi_1^- \quad \text{with } A, B \in \mathbb{R}.$$
Calculate $A$ and $B$ for the case $k_1 = 1$ and $V = m$ by demanding continuity of the solution at $t = t_0$.

(6) Assume that for $t > t_1$ the solution takes the form

$$C e^{-i\omega_0 t + i\hat{k} \cdot \hat{x}_0^+} + D e^{-i(\omega_0 t + i\hat{k} \cdot \hat{x}_0^-)} \quad \text{with } C, D \in \mathbb{C}.$$ 

Calculate $C$ and $D$ for $m = 2$ by demanding continuity of the solution at $t = t_1$ (you may want to use Mathematica here).

(7) Interpret what you have found. Why could this be called the “external field problem”?

EXERCISE 18.15. (Understanding the light cone expansion) This exercise aims to familiarize you with some of the particularities of the light cone expansion.

1. Let $A(x, y) := (x - y)^{2k_0}$ with $k_0 \in \mathbb{Z}$. Which order(s) on the light cone is this? (Prove your answer.) Construct a light-cone expansion of $A(x, y)$ and prove that it is one.

2. Let $B(x, y) := (x - y)^{2k_0} + (x - y)^{2k_1}$, where $k_0, k_1 \in \mathbb{Z}$ and $k_0 < k_1$. Which order(s) on the light cone is this? (Prove your answer.) Construct a light-cone expansion of $B(x, y)$ and prove that it is one.

3. Let $C(x, y) := (x - y)^{2k_0} f(x, y) + (x - y)^{2k_1} g(x, y)$, where $f$ and $g$ are smooth functions in $x$ and $y$ and $k_0, k_1$ as above. Construct a light-cone expansion of $C(x, y)$ and prove that it is one.

4. Let $D(x, y) := \sin \left( (x - y)^2 \right) (x - y)^2$. Use your results from ii.) and iii.) to construct two different light-cone expansions of $D(x, y)$. Why might this non-uniqueness not be a problem for the scope of the lecture?

5. Finally, consider the function

$$E(x, y) = \sin \left( (y - x)^2 \right) + \begin{cases} e^{-\frac{1}{(y-x)^2}} & \text{if } (y-x)^2 \geq 0, \\ 0 & \text{else} \end{cases}$$

Determine its order(s) on the light cone and a light cone expansion. (Prove your answer.)

Hint: For iv.) and v.): Expand the sine function. First, consider the function where $(y - x)^2$ is the Lorentz inner product of $(y - x)$ with itself.

1. Which order $O \left( (x - y)^{2p} \right)$ does it have on the light-cone? (Prove your answer.)

2. Calculate the formal Taylor expansion of the function around $(x - y)^2 = 0$ and prove that this is a light cone expansion of $A(x, y)$ as defined in the lecture. Interpret your result.

3. Is the following series a light-cone expansion of $A(x, y)$? Prove your answer and interpret it.

$$A^{[g]}(x, y) = A(x, y), \quad A^{[j]}(x, y) = 0 \quad \forall \ j > g$$

Second, consider the function

$$B(x, y) = \lim_{\varepsilon \to 0} \frac{\sin \left( (y - x)^2 \right)}{(y - x)^4 + i\varepsilon}$$

4. Determine its order on the light cone. Substitute the sine function by its series expansion and show that the resulting series is a light cone expansion of $B$. 

(5) Can you find a different light cone expansion of $B$? Why is this non-uniqueness of the expansion no problem for our purposes?

(6) Which order on the light cone does the function

$$\lim_{\varepsilon \to 0} \frac{\log((y-x)^2)}{(y-x)^4 + i\varepsilon}$$

have?

**Exercise 18.16.** (The Dyson series (5+1 points)) In this exercise, we will study the so-called Dyson series and the ordered exponential. We will take a look at how both appear in quantum field theory. In exercise 4.3 you will then see how this appears in the context of the light cone expansion of the fermionic projector.

In quantum field theory, one is given some Hamiltonian $H$ (a linear though, unfortunately, not bounded operator on some Hilbert (Fock) space $\mathcal{H}$). It splits up into a “free” part $H_0$ and an interaction part $H_1$:

$$H = H_0 + H_1.$$  

What one is interested in is the time evolution operator belonging to $H$, i.e. $U(t) = e^{-iHt}$. Usually, the time evolution operator belonging to $H_0$, i.e. $U_0(t) = e^{-iH_0 t}$, has been understood to a sufficient extend, and we therefore assume that it is known. What remains to be understood is the “effect” of $H_1$. To this end, one studies the operator $V(t) := U_0(-t)U(t)$.

(1) Show that $V(t)$ satisfies the differential equation

$$\frac{dV(t)}{dt} = -iH_1(t)V(t),$$

where $H_1(t) = U_0(-t)H_1U_0(t)$. (Note that $H_0$ and $H_1$ may not commute.)

(2) Re-express this differential equation as an integral equation taking the form

$$V(t) = 1 + \int_0^t \ldots$$

(3) On the right-hand-side of your integral equation, there is a factor $V(\tau)$ (in case you denoted your integration variable by $\tau$). You can now find an expression for $V(t)$ up to a given order in $H_1$ by plugging in the integral equation into itself several times. Write down the expression up to second order in $H_1$.

(4) Generalize this to arbitrary order in $H_1$, i.e. show that $V(t) \doteq 1 + \sum_{n=1}^{\infty} V_n(t)$ for a suitable $V_n(t)$ of the form

$$V_n(t) = \frac{1}{i^n} \int_0^t \int_0^{\tau_n} \ldots \int_0^{\tau_2} \ldots d\tau_1 \ldots d\tau_{n-1} d\tau_n,$$

where $\doteq$ denotes that this is a formal expansion. This is the so-called Dyson series, which is in the heart of QFT.

**Bonus question:** Does the series converge in some topology if $H_1$ is a bounded operator? (Prove your answer.)

(5) Let $t_1, \ldots, t_n$ be distinct elements of $\mathbb{R}$. The time-ordered product of operators $A_1, \ldots, A_n$ is defined as

$$\mathcal{T}[A_1(t_1) \cdot \ldots \cdot A_n(t_n)] := A_{i_1}(t_{i_1}) \cdot \ldots \cdot A_{i_n}(t_n) \quad \text{for} \quad t_{i_1} > t_{i_2} > \ldots > t_{i_n}.$$
(This means just that one orders the product \( A_1(t_1) \cdots A_n(t_n) \) s.t. operators evaluated at highest times are on the left.) Show that

\[
V(t) = \mathcal{T} \exp \left[ \frac{1}{i} \int_0^t H_1(\tau) d\tau \right],
\]

where the exponential is defined via its series expansion. This is the definition time-ordered exponential.

(6) Show that it is exactly the ordered exponential which you have defined in the lecture up to peculiarities in the choice of integration borders.

**Exercise 18.17.** (Understanding Theorem 4.3) Lemma 4.2 of the lecture (script) gives you the means to calculate operator products of the form \( S(l) V S(-1) \), where \( V \) is a multiplication operator such as \( V(x) = \gamma^k A_k(x) = \hat{A}(x) \). In the last exercise, you have extended this lemma to hold also for the case \( S(l) V S(-1) \). This is good since now you can also calculate operator products of the form \( s V s \), where \( s \) is one of the Dirac Green’s functions, \( s = s^\wedge \) or \( s = s^\vee \). In this exercise, you will see how this works.

(1) Remind yourself: Why do we want to calculate operator products of the form \( s V s \) to a given order on the light cone? What’s the big goal?

(2) Calculate the operator product \(-s/\hat{A}s\) to leading order on the light cone. Show that your result is of the form (4.32) (Theorem 4.3).

Instructions: Use Eq. (4.12) of the script to rewrite the factors \( s \) in the operator product in terms of \( S \). Consider only terms including \( S(l) \) with \( l \leq 0 \) and argue why it is sufficient to do so. Use the result of the last exercise to obtain (with abbreviations as introduced in the lecture (script))

\[
(-s \hat{A}s)(x,y) = i \int_x^y (y-x)^l A_l(z_1) dz_1 s(x,y) + \mathcal{O}((x-y)^{-2}) .
\]

(3) Use this result to obtain \( s \hat{A}s \hat{A}s \) to leading order. Generalize this to obtain \((-s \hat{A})^n s\) to leading order. Show that all terms are of the form (4.32). Hint: Look at the iteration-example (4.25) given in the lecture and do it analogously: Redefine \( A^k(z) \) to include also the terms which you already iterated.

(4) Use this to show that

\[
\tilde{s}(x,y) = P \exp \left( -i \int_x^y (y-x)_j A^j(z) dz \right) s(x,y) + \mathcal{O}((x-y)^{-2}).
\]

What does this mean for you? Why does the ordered exponential (and thus the Dyson series) appear here? Do you think that this is a hint that quantum field theory might be a first or second order effective theory of something more fundamental or do you not think so?

(5) Bonus question: In QFT, what researchers do is to calculate the leading terms of the Dyson series, i.e. e.g. \( V_1(t), V_2(t) \) and possibly even \( V_3(t) \). This is called “perturbative QFT”. Why is the name “non-perturbative” justified for the things which you have learned so far?

Hint: Think about which quantity the word “perturbative” refers to and use the result of question (iv) for your answer.

**Exercise 18.18.** (Mass and spatial normalization of the fermionic projector*) In the lecture you have seen that the fermionic projector can be defined with mass- and spatial
normalization. Also, in exercise 3.2 you have seen that this indeed makes a difference if one considers the fermionic projector with external potential. In this exercise, you will learn about a symmetry between the two normalizations which was only discovered very recently.

Look for the article “Perturbative Description of the Fermionic Projector: Normalization, Causality and Furry’s Theorem” by Felix Finster and Jürgen Tolksdorf on http://arxiv.org. Read section 3.4 carefully and carry out all the intermediate steps which were not carried out explicitly in the publication.

**Exercise 18.19.** (Mass and spatial normalization of the fermionic projector) In the lecture you have seen that the fermionic projector can be defined with mass- and spatial normalization. Also, in exercise 3.2 you have seen that this indeed makes a difference if one considers the fermionic projector with external potential. In this exercise, you will learn about a symmetry between the two normalizations which was only discovered very recently.

Recall the following definitions from the lecture:

\[
(A \mid B)(x, z) := 2\pi \int_{\mathbb{R}^3} A(x, (t, \vec{y})) \gamma^0 B((t, \vec{y}), z) \, d^3 y ,
\]

\[
A \cdot B = C \quad \text{stands for} \quad A_m B_{m'} = \delta(m - m') C_m .
\]

(1) Prove the following equations:

\[
k_m \mid t k_m = k_m = p_m \mid t p_m \mid t k_m = p_m = p_m \mid t k_m
\]

**Exercise 18.20.** (Yang-Mills tensor) One of the aims of the lecture is to show how every-day physical equations appear in the continuum limits. In this exercise, we will see a first hint for that. To this end, consider the following terms in the light cone expansion of \( s(\chi_R \bar{A}) \):

\[
\chi_L \left( -s \chi_R \bar{A} \right)(x, y) \overset{\text{phase-free}}{\approx} \chi_L S^{(0)}(x, y) \xi^i \int_x^y dz \mid 0, 1 \mid 0 \right) (\bar{\phi} A_i)(z)
\]

\[- \chi_L S^{(0)}(x, y) \int_x^y dz \mid 0, 0 \mid 0 \right) \bar{A}(z) + \chi_L S^{(0)}(x, y) \bar{A}(x) + \ldots,
\]

where \( \chi_L \) is one of the chiral projectors and abbreviations were used as in the lecture.

(1) Show that each of those terms is phase-free as defined in the lecture, and explain why no calculation is necessary to write down Eq. (18.5.9) to a given order on the light cone.

(2) Show that the three terms are equal to just one term which contains the Yang-Mills tensor \( F_{ik} = \partial_i A_k - \partial_k A_i \).

(3) Explain what one has to do in order to obtain the contributions to the light cone expansion of the residual fermionic projector \( \bar{P}^{res} \) from an equation such as Eq. (18.5.9). Transform the three terms accordingly.

**Exercise 18.21.** (Signature operator) Prove the following proposition, which shows that the causal fundamental solution is the signature operator of the dual pairing \( < \cdot | \cdot > \).

**Proposition 5.1:** For any \( \psi_m \in \mathcal{H}_m \) and \( \phi \in C^\infty_0(M, SM) \),

\[
(\psi_m \mid k_m \phi) = <\psi_m|\phi> .
\]

**Hint:** Proceed as follows:
(1) First, assume that $\psi_m \in C^\infty_c(M, SM)$. Chose Cauchy surfaces $N_+$ and $N_-$ which lie the future and past of supp $\phi$, respectively. Show that then

$$ (\psi_m | k_m \phi) = i \int_\Omega \nabla_j <\psi_m | \gamma^j s_m^x \phi> \, d\mu(x), \quad (18.5.11) $$

where $\Omega$ is the space-time region between the two surfaces.

(2) Use that $\psi_m$ satisfies the Dirac equation to show that

$$ (\psi_m | k_m \phi) = \hat{\Omega} \nabla_j \phi \, d\mu(x) \quad (18.5.12) $$

(3) In order to extend the result to general $\psi_m \in H_m$, let $\psi^{(n)}_m \in H_m \cap C^\infty_c(M, SM)$ be a sequence which converges in $H_m$ to $\psi_m$. Show that $(\psi^{(n)}_m | k_m \phi) \to (\psi_m | k_m \phi)$ and that $<\psi^{(n)}_m | \phi> \to <\psi_m | \phi>$. (For the last step, use that $\phi$ is compactly supported and Fubini’s theorem. Also, for the whole exercise, do not forget about our good friends Mr. Cauchy and Mr. Schwarz.)

**Exercise 18.22.** (Regularization operator) Prove that the family $(\mathcal{R}_\varepsilon)_{\varepsilon > 0}$ of operators

$$ (\mathcal{R}_\varepsilon \psi)(t, \vec{x}) := (U^{t,t_0}(\eta_\varepsilon * \psi|_{t=t_0})) (\vec{x}), \quad (18.5.13) $$

is a family of regularization operators, where $U^{t,t_0}$ is the time evolution operator defined in the lecture,

$$ \eta_\varepsilon(\vec{x}) = e^{-\frac{|\vec{x}|^2}{\varepsilon^2}} \frac{1}{\varepsilon^3} $$

and

$$ (\eta_\varepsilon * \psi)(\vec{x}) := \int \eta_\varepsilon(\vec{x} - \vec{y}) \psi(\vec{y}) \, d^3 y. $$

You may use the following inequality:

$$ ||(U^{t,t_0} \phi)(t, \vec{x})||_{C^1} \leq C ||\phi||_{W^{k,p}(\mathbb{R}^3)} \quad \forall \phi \in H_m, \quad (18.5.14) $$

where $W^{k,p}$ denotes the Sobolev spaces (sometimes also denoted by $H^{k,p}$).

**Hint:** Within this exercise, do not think of $H_m$ as representing the identification of the $H_t$ which were introduced in the context of the time evolution operators. Note that on the right hand side of Eq. (5.2) in the the lecture, the norm denotes the norm on $H_m$. For the proof of the latter, use Eq. (18.5.14), a Fourier transformation in spatial directions, Plancherel’s theorem and the behavior of convolutions under the Fourier transform.

**Exercise 18.23.** (Bonus exercise) Prove inequality (18.5.14).

**Exercise 18.24.** (Causal fermion systems) Let $(H = l^2(\mathbb{N}), \langle . | . \rangle)$ be the Hilbert space of square integrable valued sequences, i.e.,

$$ \langle u | v \rangle := \sum_{l=1}^\infty \bar{u}_l \, v_l. $$
Let $x_k$ be an operator on $H$ which is defined as
\[
(x_k u)_k = -u_{k+1} \\
(x_k u)_{k+1} = -u_k \\
(x_k u)_l = 0 \quad \forall \quad l \in \mathbb{N} \setminus \{k, k+1\},
\]
i.e. for $u = (u_1, u_2, \ldots, u_k, u_{k+1}, \ldots)$,
\[
x_k u := (0, \ldots, 0, -u_{k+1}, -u_k, 0, \ldots)_{k-1 \text{ entries}}.
\]

(1) Construct a causal fermion system $(H, F, \rho)$ where $H$ is as above and where $\text{supp} \; \rho = \{x_k\}_{k \in \mathbb{N}}$.
Hint: Use a discrete counting measure, i.e. a sum of Dirac measures.

(2) Derive the following objects for this causal fermion system and analyze them:
(a) Space-time $M$
(b) Spin spaces $(S_x, <\cdot|\cdot>_x)$
(c) Wave functions $\psi$
(d) Krein space $K$ (including topology!)

(3) Answer the following questions: Is the map $\Psi : H \to K$ well-defined and continuous? Is it an embedding? Are $H$ and $K$ isomorphic?

(4) Try to give a physical interpretation of what this causal fermion system could represent.

Exercise 18.25. (Eigenvalues of $xy$) Let $(H, F, \rho)$ be a causal fermion system. Show that for $x,y \in F$, the operator product $xy$ has the same non-trivial eigenvalues as the closed chain $P(x,y)P(y,x)$, where $P(x,y)$ is defined in Eq. (5.5).
Hint: Remember that you cannot use the characteristic polynomial on infinite dimensional vector spaces! Therefore, argue why, for this purpose, you can restrict $xy$ to $S_y$. Prove the identity $\det(BC - \lambda I) = \det(CB - \lambda I)$ and use it.

Exercise 18.26. (The fermionic projector in bra-ket notation) Familiarize yourself with the bra-ket notation (e.g. read the English Wikipedia article on this subject). Afterwards, prove that
\[
P(x,y) = -\sum_{k \in \mathbb{N}} |\psi(u_k(x))\rangle \langle \psi(u_k(y))|.
\]

Exercise 18.27. ($\gamma$-matrices) Prove the following identities:

1. $\gamma^5 = \frac{i}{4!} \epsilon_{\mu \nu \alpha \beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta$
2. $\gamma^\mu \gamma^\mu = 4I_4$
3. $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$
4. $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu \rho} I_4$
5. $\text{tr}(\gamma^\mu) = 0$
6. trace of any product of an odd number of $\gamma^\mu$ is zero
7. trace of $\gamma^5$ times a product of an odd number of $\gamma^\mu$ is still zero
8. $\text{tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu \nu}$
9. $\text{tr}(\gamma^{\mu_1} \ldots \gamma^{\mu_m}) = \text{tr}(\gamma^{\mu_m} \ldots \gamma^{\mu_1})$
Exercise 18.28. (Eigenvalues of the closed chain) Prove Lemma 7.1 of the lecture, i.e. prove that the characteristic polynomial of $A^\varepsilon_{xy}$ has the roots

$$\lambda_\pm = g\bar{g} + h\bar{h} \pm \sqrt{(g\bar{g})^2 - g^2 \bar{g}^2 + (g\bar{h} + h\bar{g})^2}.$$

Show that they either form a complex conjugate pair or else they are both real and have the same sign.

Hint: From the lecture you know that you can write the (regularized) fermionic projector as

$$P^\varepsilon(x,y) = g_j(x,y) \gamma^j + h(x,y), P^\varepsilon(y,x) = g_j(x,y) \gamma^j + h(x,y).$$

With this, calculate the matrix $A^\varepsilon_{xy} =: A$ and decompose it into the form

$$A = A_1 + A_2 + \mu,$$

where

$$A_1 = \frac{1}{2} [g, \bar{g}], \quad A_2 = h\bar{g} + g\bar{h}.$$

Show that the matrices $A_1$ and $A_2$ anti-commute and use this to show that

$$(A - \mu)^2 = (g\bar{g})^2 - g^2 \bar{g}^2 + (g\bar{h} + h\bar{g})^2.$$

Use this polynomial equation in $A$ to find the zeros of the characteristic polynomial of $A$.

Exercise 18.29. (Understanding the $i\varepsilon$ - Regularization) Note: The next exercise sheet will contain a “reloaded” version of this exercise with more explanations. But go ahead and try it already if you like.

Prove the following statements from the lecture:

(1) \( T^{(0)}_{[p]} \rightarrow -\frac{1}{8\pi^3} \frac{1}{2r (t - r - i\varepsilon)} \), \( T^{(-1)}_{[p]} \rightarrow -\frac{2}{r} \frac{\partial}{\partial t} T^{(0)}_{[p]} = -\frac{1}{8\pi^3} \frac{1}{r^2 (t - r - i\varepsilon)^2} \). \hfill (18.5.15)

(2) \( A^\varepsilon_{xy} = \frac{1}{256\pi^6 r^4} \frac{t^2 - r^2 + \varepsilon^2}{|t-r+i\varepsilon|^4} \left[ \frac{\varepsilon \gamma^0}{|\varepsilon|} \right] \). \hfill (18.5.16)

(3) The bilinear contribution to $A^\varepsilon_{xy}$ makes the eigenvalues $\lambda_\pm$ complex.

(4) The leading divergence of the integral (7.11) in $\varepsilon$ can be described by the scaling

$$T^{(n)} \sim (\varepsilon |\xi|)^{n-1} \quad \text{and} \quad dt \sim \varepsilon.$$

(5) The scaling of the other divergent contributions is obtained by multiplying with powers of

$$\frac{\varepsilon}{|\xi|}.$$

(6) \( \frac{1}{4} \text{Tr} ((\bar{g} - i\varepsilon \gamma^0)(g + i\varepsilon \gamma^0)) = \frac{1}{2} (\bar{g} - i\varepsilon \gamma^0)^2 + \frac{1}{2} (g + i\varepsilon \gamma^0)^2 \). \hfill (18.5.17)
Here is a list of formulae from the lecture which you might find helpful in proving this exercise. (Of course, you can find all of them at various places in the script.)

\[ T_{m^2}(x, y) := \int \frac{d^4 k}{(2\pi)^4} \delta(k^2 - m^2) \Theta(-k^0) e^{-ik(x-y)} \]

\[ T_{m^2}(x, y) = -\frac{1}{8\pi^3} \left( \frac{\text{PP}}{(y-x)^2} + i\pi \delta((y-x)^2) \epsilon((y-x)^0) \right) \tag{18.5.18} \]

\[ + \frac{m^2}{32\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1)!} \frac{(m^2(y-x)^2)^j}{4^j} \times \left( \log |m^2(y-x)^2| + cj + i\pi \Theta((y-x)^2) \epsilon((y-x)^0) \right) \tag{18.5.19} \]

\[ T^{(l)} = \left( \frac{d}{da} \right)^l T_a |_{a=0} . \tag{18.5.20} \]

\[ P(x, y) = (i\partial - m) T_{m^2}(x, y) \tag{18.5.21} \]

\[ \hat{P}^\varepsilon(k) = (\hat{k} + m) \delta(k^2 - m^2) \Theta(-k^0) \exp(\varepsilon k^0) . \]

\[ \lim_{\varepsilon \to 0} \log(a\xi^2 - i\varepsilon \xi^0) + i\pi = \log |a\xi^2| + i\pi \Theta(\xi^2) \epsilon(\xi^0) \]

**Exercise 18.30. (Understanding the i\varepsilon - Regularization Reloaded (1))**

1. Look up the formula for the Greensfunction of the Klein-Gordon-Operator in position space, \( T_a(x, y) \), in the script (\( a = m^2 \)). Drop the \( \log(a) \)-Term (this corresponds to the “regularization” introduced in Ch. 4) and the smooth contributions (since we are presently only interested in the behavior of the singular components). Then, calculate \( T^{(n)} \) for \( n \leq 0 \).

2. Explain the idea behind the \( i\varepsilon \)-regularization and explain why it amounts to setting \( t \to t - i\varepsilon \), where \( \xi^0 \equiv t \).

3. Plug in the \( i\varepsilon \)-regularization into \( T^{(0)} \). Use that the only goal we presently have is to evaluate any composite expressions of \( T \)’s weakly on the light-cone away from the origin, which means that \( t \approx |\vec{\xi}| =: r \) to show that

\[ T^{(0)}_{[0]} \to -\frac{1}{8\pi^3} \frac{1}{2r(t - r - i\varepsilon)} . \]

4. Use the defining formula for \( T^{(-1)} \),

\[ \xi_k T^{(-1)}(x, y) := 2\frac{\partial}{\partial x^k} T^{(0)}(x, y) \tag{18.5.22} \]

for \( k = 0 \) to obtain

\[ T^{(-1)} \to T^{(-1)}_{[0]} = -\frac{1}{8\pi^3} \frac{1}{r^2(t - r - i\varepsilon)^2} . \]

(Note that Eq. \[ 18.5.22 \] follows from the corresponding equation for \( S \) due to the residual argument.)
(5) Given the Greens-functions for the Klein-Gordon operator, $T$, you know how to construct the fermionic operator. Do so for the regularized and massless case, i.e., set

$$P^c(x,y) = \frac{i}{2} \xi T^{(-1)}_{[0]}.$$ 

With this, show that

$$A^c_{xy} := P^c(x,y) P^c(y,x) = \frac{1}{256 \pi^6 r^4} \frac{t^2 - r^2 + \varepsilon^2}{|t - r + i\varepsilon|^4} - \frac{1}{256 \pi^6 r^4} \frac{i\varepsilon [\gamma^0, \xi]}{|t - r + i\varepsilon|^4}$$

(18.5.23)

(6) Show that if $x$ is on the light cone of $y$ (or vice versa), which means that $t = r$, then for every $\varepsilon > 0$, the last term of Eq. (18.5.23) makes the eigenvalues $\lambda_\pm$ of $A_{xy}$ complex. What does this mean physically?

Hint: In the last exercise session, we have discussed that if one sees the eigenvalues of a matrix containing $\gamma^k$’s, it is a good idea to check if the matrix fulfills a nice polynomial equation by squaring it. Try this here, it works.

$$(i\varepsilon [\gamma^0, \xi])^2 = -4\varepsilon^2 |\xi|^2 < 0$$

**Exercise 18.31.** (Understanding the $\varepsilon$ - Regularization Reloaded (2)) Next, we want to understand the scaling behavior of $T^{(n)}_{[n]}$ if the regularization is removed. (We need to know this in order to evaluate which terms to keep and which terms to “throw away” when analysing the continuum limits.) To this end, we take a look at simple fractions which are evaluated weakly on the light cone,

$$\int_{|\xi| - \varepsilon}^{|\xi| + \varepsilon} dt \eta(t, \xi) \frac{\bar{T}_{\gamma_1}^{(a_1)} \cdots \bar{T}_{\gamma_n}^{(a_n)} T_{\gamma_1}^{(b_1)} \cdots T_{\gamma_n}^{(b_n)}}{T_{\gamma_1}^{(c_1)} \cdots T_{\gamma_n}^{(c_n)} T_{\gamma_1}^{(d_1)} \cdots T_{\gamma_n}^{(d_n)}}$$

(18.5.24)

and investigate their divergence with $\varepsilon \to 0$.

**Assumption:** $\eta(t)$ “varies on the scale $l_{\text{macro}}$”, i.e.:

$$\frac{\partial}{\partial t} \eta|_{t'} \approx \frac{1}{l_{\text{macro}}} \eta(t')$$

(18.5.25)

when $t' \approx r$.

Proceed as follows:

(1) First of all, use 8.2(i) to show that $T^{(n)} \sim ((t - i\varepsilon - r)(t - i\varepsilon + r))^{n-1}$, where $\sim$ denotes proportionality. (Think about why you can drop the log($\xi$)-Term.)

(2) Assume that the support of $\eta(t)$ is contained in the interval $(r - \varepsilon, r + \varepsilon)$, where $r = |\xi|$. Show that

$$\int_{-\infty}^{\infty} \eta(t) \left(\frac{(t - i\varepsilon - r)(t - i\varepsilon + r)}{(t - i\varepsilon - r)(t + i\varepsilon + r)}\right)^N \left(\frac{(t + i\varepsilon - r)(t + i\varepsilon + r)}{(t + i\varepsilon - r)(t + i\varepsilon + r)}\right)^Q dt$$

for suitable $N, P, Q, M \in \mathbb{R}^+$. 

(3) Assume that $N - P < 0$. The integrand of the last equation has a pole at $t = r + i\varepsilon$. Close the integration contour (utilizing the support-condition of $\eta(t)$) and evaluate the integral with the residue theorem. (Assume that $\eta$ is only supported around this pole, i.e. neglect the other poles.)
(4) The residuum which you get contains a factor
\[ \frac{d^{-N+P-1}}{dt^{-N+P-1}} \left( \eta(t) \left( \frac{(t - i\varepsilon - r)(t - i\varepsilon + r)}{(t - i\varepsilon - r)(t - i\varepsilon + r)} \right)^{M} \right) \bigg|_{t=r+i\varepsilon}. \]

Apply one \( t \)-derivative to the term in brackets and evaluate it at \( t = r + i\varepsilon \). Evaluate how the four different terms which you obtain behave when \( \varepsilon \to 0 \). (Assume \( M - Q < 0 \).) With this, prove that
\[ (18.5.24) \sim r N - P + M - Q, \]

(5) Finally, use part (ii) to show that if one associates one of the \( \varepsilon \) of Eq. (18.5.26) to \( dt \),
\[ dt \sim \varepsilon, \]
then one has indeed
\[ T_{(n)}^{(n)} \sim (\varepsilon |\xi|)^{n-1} (1 + O(\varepsilon/r))(1 + O(\varepsilon/\ell_{macro})), \]
as stated in the lecture.

Exercise 18.32. (Spectral projectors of the regularized vacuum) Show that
\[ F_{\pm} = \frac{1}{2} \left( 1 \pm \frac{[\xi, \xi]}{z - \bar{z}} \right) + \xi (\text{deg} \leq 0) + (\text{deg} < 0). \quad (18.5.27) \]
and
\[ F_{\pm} P(x, y) = \begin{cases} 0 & \text{for } "+" \\ \frac{i}{2} \xi T_{[0]}^{(-1)} & \text{for } "-" \end{cases} + (\text{deg} < 2). \quad (18.5.28) \]

Exercise 18.33. (Resolvent formalism) In the lecture, we have defined
\[ G_k = \frac{1}{2\pi i} \oint_{\Gamma_k} (z - A)^{-1} dz, \quad (18.5.29) \]
where \( \Gamma_k \) is a small contour around the \( k \)th eigenvalue of the matrix \( A \) which does not enclose any other eigenvalue. If you are not familiar with the resolvent formalism, have a look e.g. into Kato, “Perturbation Theory for Linear Operators”, p. 36-40.

(1) Determine the spectrum of the following matrices and the corresponding operators \( G_k \):
\[ A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \]
\[ B := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \]

(2) Prove the resolvent identity.

(3) Show that \( G_k G_l = \delta_{k,l} G_l \). Hint: For the case \( k = l \), use the fact that the two integration contours \( \Gamma_k \) and \( \Gamma_l \) lie outside one another. For the case \( k = l \) use that you can deform the contour continuously (so long as it does not “swipe over another pole”) and deform \( \Gamma_k \) and \( \Gamma_l \) such that one encloses the other.

Exercise 18.34. (Understanding the contraction rules) This exercise intends to make you familiar with the contraction rules in the formalism of the continuum limit. The goal is to prove the contraction rules for the \( i\varepsilon \)-regularization.
(1) Explain what the general idea is behind the method of variable regularization and behind the formalism of the continuum limit.

In oder to prove the contraction rules for the $\varepsilon$-regularization, we first need to know explicitly the functions $\xi^{(n)}_{[p]}, \bar{\xi}^{(n)}_{[p]}$ and $T^{(n)}_{[p]}$. (Set $T^{(n)}_{(p)} = 0$.) Since we are working in the $\varepsilon$-regularization, we define

$$\xi^{(n)}_{[p]} := \xi\big|_{\varepsilon^0 \to \varepsilon^0 - \varepsilon} \quad \forall \, n, p$$

(2) Prove that

$$T^{(0)}_{[p]} = -\frac{1}{8\pi^2} \frac{1}{2r(t + i\varepsilon - \tau)} (1 + O(\varepsilon)) \quad (18.5.30)$$

$$T^{(n)}_{[p]} = c_n (2r(t + i\varepsilon - \tau))^{n-1} \left( \log(2r(t + i\varepsilon - \tau)) + d_n \right) (1 + O(\varepsilon)) \quad \forall \, n \geq 1 \quad (18.5.31)$$

where $c_n, d_n \in \mathbb{C}$.

(3) Use the equation

$$\frac{\partial}{\partial x_k} T^{(l)} = \frac{1}{2} \xi_k T^{(l-1)} \quad (18.5.32)$$

to fix all coefficients $c_n, d_n$. Hint: Only consider the case $k = 0$ and use again that $t = r + \mathcal{O}(\varepsilon)$. Mind the hidden minus sign $(\xi = (y - x))$.

We can now prove the contraction rules

$$\left(\xi^{(n)}_{[p]}\right)_j \left(\xi^{(n')}_{[p']}\right)_j = \frac{1}{2} \left(z^{(n)}_{[p]} + z^{(n')}_{[p']}\right) \quad (18.5.33)$$

$$\left(\bar{\xi}^{(n)}_{[p]}\right)_j \left(\bar{\xi}^{(n')}_{[p']}\right)_j = \frac{1}{2} \left(\bar{z}^{(n)}_{[p]} + \bar{z}^{(n')}_{[p']}\right) \quad (18.5.34)$$

$$\bar{z}^{(n)}_{[p]} T^{(n)}_{[p]} = -4 \left( n T^{(n+1)}_{[p]} + T^{(n+2)}_{[p]} \right). \quad (18.5.35)$$

(4) In the context of the $\varepsilon$-regularization, it suffices to define $z^{(n)}_{[p]} := \left(\xi^{(n)}_{[p]}\right)_j \left(\xi^{(n)}_{[p]}\right)_j$.

Prove that with this definition, eqs. (18.5.33) and (18.5.34) hold up to $\mathcal{O}(\varepsilon^2)$.

(5) Prove that up to smooth contributions we have for the $T^{(n)}_{[p]}$ derived in part (iii)

$$\left(\xi^{(n)}_{[p]}\right)^2 T^{(n)}_{[p]} = -4 \, n \, T^{(n+1)}_{[p]}$$

and show that this corresponds exactly to eq. (18.5.35).

**Exercise 18.35.** (Bra-ket notation II) Prove that in the context of the Bra-ket notation, for $u, v \in \mathcal{H}$,

$$\text{Tr} \left( |v\rangle \langle u| A \right) = \text{Tr} \left( \langle u| A v \right) = \langle u| A v \rangle.$$  

Hint: Use an orthonormal basis of $\mathcal{H}$.

**Exercise 18.36.** (Double null spinor frame) Given the $\varepsilon$-regularization, calculate explicitly the double null spinor frame $(f^L_x, f^L_y, f^R_x, f^R_y)$ at a point $x$ with respect to a point $y$. (Up to $\mathcal{O}(\varepsilon)$.). Verify that $\xi f^L_x = f^R_y$.

Hint: Remember that the double null spinor frame is composed of a particular set of eigenvectors of the matrix $A_{xy}$ (hence the term “w.r.t. a point $y$”). Assume that the spatial component of $\xi := x - y$ is normalized and points into the z-direction, $\xi = (0, 0, 1)$. In oder to determine $f^L_x$, calculate the matrix $\chi^0 F_s$ explicitly and determine its eigenvector. (And remember that the $\xi$ in Eq. (8.9) in the script is really $\xi^{(n)}_{[p]}$.)
EXERCISE 18.37. (Matrix elements in the double null spinor frame) In $T^{(-1)}_{(0)}$ be an arbitrary regularization of $T^{(-1)}$ and let

$$P_0(x,y) := \frac{i}{2} \chi_L \xi T^{(-1)}_{(0)}.$$  \hspace{1cm} (18.5.36)

Determine the matrix elements $P_0(x,y)^{LL+ +}$ and $P_0(x,y)^{LL+ -}$ (denoted by $F^{LL+ +}(P_0(x,y))$ and $F^{LL+ -}(P_0(x,y))$ in the script). Simplify the expressions as far as possible.

Hint: Use the cyclic property of the trace, the anti-commutation relations of the gamma matrices and the contraction rules.

EXERCISE 18.38. (Gauge potentials) In this exercise, you will see how the gauge freedom of electrodynamics is contained in causal fermion systems.

1. Show that if $A_\mu$ is a solution of the homogeneous Maxwell equations on Minkowski space, so is $A'_\mu := A_\mu + \partial_\mu \Lambda$, where $\Lambda$ is a real-valued function on Minkowski space.

2. The Dirac equation coupled to electromagnetism, $(i\partial + A - m)\psi = 0$, is not invariant under the transformation $A'_\mu := A_\mu + \partial_\mu \Lambda$, hence this is not a gauge transformation. Show that if one suitably transforms $\psi$ together with $A_\mu$,

$$A_\mu(x) \rightarrow A'_\mu(x),$$  \hspace{1cm} (18.5.37)

$$\psi(x) \rightarrow U(x)\psi$$  \hspace{1cm} (18.5.38)

this is a gauge transformation. (Determine $U(x)$.)

3. Use exercise 6.3 to show that in the context of causal fermion systems, the transformation (18.5.38) is exactly a transformation

$$P(x,y) \rightarrow U(x)P(x,y)U(y)^*.$$  \hspace{1cm} (18.5.39)

4. Show that the transformation (18.5.39) leaves the causal action $S$ invariant. This means that if a causal fermion system can be described by the vector potential $A_\mu$, there is another causal fermion system which is a minimizer of the causal variational principle which can be described by the vector potential $A'_\mu$.

EXERCISE 18.39. (Perturbation of the eigenvalues of the closed chain) Assume that you want to study if a causal fermion system can be described by Dirac spinors $\psi$ coupled to some quantity $\mathcal{X}$. If $\mathcal{X} = \gamma^5 A$, you know already from the lecture or talks that the causal fermion system can only be a minimizer of the causal action principle (and hence “physically admissible”) if $A$ and $\psi$ are related by the Dirac equation coupled to the Yang-Mills equation.

Now assume that you wanted to study this for some (possibly very exotic) $\mathcal{X}$ and that you find that the corresponding perturbation of the fermionic projector to first order $\delta P$ (sometimes also denoted by $\Delta P$ in the script) is

$$\delta P(x,y) = \chi_L \psi,$$

where $\chi_L$ is the left-handed chiral projector and $\nu$ is some arbitrary vector in Minkowski space.

1. Calculate the corresponding variation of the eigenvalues of the closed chain to first order in the variation, $\delta \lambda_k^{xy}$, up to leading order in the degree. Which eigenvalues change, which not?
(2) Assume that \( \phi = \xi + \bar{\xi} \) and calculate \( \delta \lambda_k^{xy} \) explicitly (i.e. your result should not contain any Gamma matrices and traces thereof any more).

(3) Assume that you have derived the more general perturbation

\[ \delta P(x,y) = \chi_L \psi_L + \chi_R \psi_R. \]  \hspace{1cm} (18.5.40)

where \( \nu_L \) and \( \nu_R \) are vectors in Minkowski space. Based on your results in part (i), answer the following question: For which vectors \( \nu_L \) and \( \nu_R \) does

\[ |\lambda_k^{xy}| = |\lambda_k^{xy}| \quad \forall k, l \]

hold?

(4) What does this mean for potential \( X \) which you (hypothetically) derived \( (18.5.40) \) from?

Hint: Use equation (8.35) from the script, and remember that the \( F^x_{xy} \) are given by \( \chi_{L/R} F_{\pm} \).
APPENDIX A

The Spin Coefficients

In this appendix, we verify by explicit computation that the matrices $E_j$ containing the spin coefficients as given by (3.2.24),

$$E_j = \frac{i}{2} \Gamma \partial_j \Gamma - \frac{i}{16} \text{Tr} (G^m \nabla_j G^n) G_m G_n + \frac{i}{8} \text{Tr} (\Gamma G_j \nabla_m G^n) \Gamma,$$  

(A.0.1)

have the following behavior under gauge transformations:

$$E_j \to UE_j U^{-1}$$ for U(1) gauge transformations (A.0.2)

$$E_j \to UE_j U^{-1} + iU(\partial_j U^{-1})$$ for SU(2, 2) gauge transformations (A.0.3)

Under U(1) gauge transformations, all the terms in (A.0.1) remain unchanged because $U$ and its partial derivatives commute with $\Gamma$ as well as with the $G^j$. Therefore, the relation (A.0.2) is obvious. Thus it remains to consider SU(2, 2) gauge transformations. Our goal is to verify (A.0.3) for a fixed space-time point $p$.

We decompose the gauge transformation $U$ as $U = U_2 U_1$ with

$$U_1(x) = U(p)$$

$$U_2(x) = U(x) U^{-1}(p).$$

Being constant, the first transformation clearly satisfies the transformation law (A.0.3). Therefore, it suffices to consider a gauge transformation $U$ with $U(p) = 1$. Then (A.0.3) can be written as

$$\tilde{E}_j = E_j - i\partial_j U.$$  

(A.0.4)

We now compute the transformation law of each of the summands in (A.0.1) after each other:

(i) odd transformations:

$$\frac{i}{2} \tilde{\Gamma} \partial_j \tilde{\Gamma} = \frac{i}{2} \Gamma (\partial_j \Gamma + [\partial_j U, \Gamma])$$

$$= \frac{i}{2} \Gamma \partial_j \Gamma + \frac{i}{2} \Gamma \left( (\partial_j U) \Gamma - \Gamma (\partial_j U) \right)$$

$$= \frac{i}{2} \Gamma \partial_j \Gamma - i\Gamma \Gamma \partial_j U$$

$$= \frac{i}{2} \Gamma \partial_j \Gamma - i\partial_j U$$

(ii) even transformations:

$$\frac{i}{2} \tilde{\Gamma} \partial_j \tilde{\Gamma} = \frac{i}{2} \Gamma \left( \partial_j \Gamma + [\partial_j U, \Gamma] \right) = \frac{i}{2} \Gamma \partial_j \Gamma$$

Thus for odd transformations, we get the correct transformation law, whereas for even transformations the desired term $i\partial_j U$ is still missing.
(2) \(-\frac{i}{16} \text{Tr}(G^m \nabla_j G^n) G_m G_n\)

(i) odd transformations:
\[-\frac{i}{16} \text{Tr} \left( \tilde{G}^m \nabla_j \tilde{G}^n \right) \tilde{G}_m \tilde{G}_n \]
\[= - \frac{i}{16} \text{Tr} \left( G^m \nabla_j G^n \right) G_m G_n - \frac{i}{16} \text{Tr} \left( G^m \left[ \partial_j U, G^n \right] \right) G_m G_n \quad (A.0.5)\]
\[= - \frac{i}{16} \text{Tr} \left( G^m \nabla_j G^n \right) G_m G_n , \quad (A.0.6)\]

where we used that \(G^m, G^n\) and \(\partial_j U\) are odd, implying that the trace in the last summand in (A.0.5) vanishes.

(ii) \(\partial_j U = i\sigma_{kl}\) für for arbitrary indices \(k,l\):
\[-\frac{i}{16} \text{Tr} \left( G^m \nabla_j G^n \right) G_m G_n - i \partial_j U \]

(iii) \(\partial_j U = \tilde{\Gamma}\):
\[-\frac{i}{16} \text{Tr} \left( \tilde{G}^m \nabla_j \tilde{G}^n \right) \tilde{G}_m \tilde{G}_n \]
\[= - \frac{i}{16} \text{Tr} \left( G^m \nabla_j G^n \right) G_m G_n - \frac{i}{16} \text{Tr} \left( G^m \left[ \tilde{\Gamma}, G^n \right] \right) G_m G_n \]
\[= - \frac{i}{16} \text{Tr} \left( G^m \nabla_j G^n \right) G_m G_n - \frac{i}{8} \text{Tr} \left( \sigma_{mn} i\sigma_{kl} \right) \sigma_{mn} \]
\[= - \frac{i}{16} \text{Tr} \left( G^m \nabla_j G^n \right) G_m G_n - i \partial_j U \]

Thus we get the correct transformation law for bilinear transformations \(\partial_j U = i\sigma_{kl}\).

(3) \(\frac{i}{8} \text{Tr} \left( \Gamma G_j \nabla_m G^m \right) \Gamma\)
\[\frac{i}{8} \text{Tr} \left( \tilde{\Gamma} \tilde{G}_j \nabla_m \tilde{G}^m \right) \tilde{\Gamma} = \frac{i}{8} \text{Tr} \left( \Gamma G_j \nabla_m G^m \right) \Gamma + \frac{i}{8} \text{Tr} \left( \Gamma G_j \left[ \partial_m U, G^m \right] \right) \Gamma \]
\[= \frac{i}{8} \text{Tr} \left( \Gamma G_j \nabla_m G^m \right) \Gamma + \frac{i}{8} \text{Tr} \left( \partial_m U \left[ G^m, \Gamma G_j \right] \right) \Gamma \]
\[= \frac{i}{8} \text{Tr} \left( \Gamma G_j \nabla_m G^m \right) \Gamma - \frac{i}{4} \text{Tr} \left( \partial_m U \Gamma \delta_j^m \right) \Gamma \]
\[= \frac{i}{8} \text{Tr} \left( \Gamma G_j \nabla_m G^m \right) \Gamma - \frac{i}{4} \text{Tr} \left( \left( \partial_j U \right) \Gamma \right) \Gamma , \]

where we used the relations
\([G^j, \Gamma G^k] = -\Gamma \{G^j, G^k\}\)
as well as the anti-commutation relations for Dirac matrices. We again distinguish different cases:

(i) $\partial_j U$ is odd or $\partial_j U = i\sigma_{kl}$:

$$\frac{i}{8} \text{Tr} \left( \tilde{\Gamma} \tilde{G}_j \nabla_m \tilde{G}^m \right) \tilde{\Gamma} = \frac{i}{8} \text{Tr} (\Gamma G_j \nabla_m G^m) \Gamma$$

(ii) $\partial_j U = \Gamma$:

$$\frac{i}{8} \text{Tr} \left( \tilde{\Gamma} \tilde{G}_j \nabla_m \tilde{G}^m \right) \tilde{\Gamma} = \frac{i}{8} \text{Tr} (\Gamma G_j \nabla_m G^m) \Gamma - i\partial_j U$$

Hence we get the correct transformation law if $\partial_j U = \Gamma$.

Adding all the terms gives the desired transformation law (A.0.4).
Bibliography


