Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 9

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is the difference between the causal action principle and the causal variational principle? Is the latter a generalization or a simplification of the former?

(ii) What role is played by the local trace in point (i)?

(iii) How is the boundedness constraint treated?

(iv) Why is $\mathscr{F}$ not a manifold? Why is $\mathscr{F}^{\text{reg}}$ a manifold? How can charts be constructed?

Exercises

Exercise 9.1: The time-direction functional in Minkowski space (6 points)

Away from the light-cone the kernel of the fermionic projector P^ε converges to a smooth function P. More precisely, for $\xi \in \mathbb{R}^3 \setminus L_0$,

$$P^\varepsilon(\xi) \to P(\xi) = (i\partial\!\!\!/ + m)\, T_{m^2}(\xi), \quad T_{m^2}(\xi) := \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik\cdot\xi}.$$

where T_{m^2} is smooth on $\mathbb{R}^4 \setminus L_0$.

(i) Show with a symmetry argument (without explicit computation of Fourier integrals!) that the imaginary part of the function T vanishes for spacelike vectors ξ.

(ii) Referring to Exercise 5.1, deduce that, for spacelike separation, $\alpha \in i\mathbb{R}$, $\beta \in \mathbb{R}$ and $a = 0$.

A causal fermion system distinguishes a *direction of time* by means of the anti-symmetric real functional

$$\mathcal{C} : M \times M \ni (A, B) \mapsto i \operatorname{tr}(B\, A\, \pi_B\, \pi_A - A\, B\, \pi_A\, \pi_B) \in \mathbb{R}.$$

From Exercise 5.1-(ii) we know that $P^\varepsilon(0)$ is invertible. Let us define $\nu := P^\varepsilon(0)^{-1} \in \operatorname{Mat}(4, \mathbb{C})$. Using the identifications as in Exercise 5.3-4 ($x \equiv F^\varepsilon(x)$, $S_x \cong \mathbb{C}^4$), prove the following identities (up to global constants)

(i) $\pi_x\, y\, x\, \pi_y\, \pi_x|_{S_x} = P^\varepsilon(x, y)\, P^\varepsilon(y, x)\, P^\varepsilon(x, y)\, \nu\, P^\varepsilon(y, x)\, \nu$.

(ii) $\mathcal{C}(x, y) = i \operatorname{Tr}_{\mathbb{C}^4}\big(P^\varepsilon(x, y)\, \nu\, P^\varepsilon(y, x)\, \big[\nu, A^\varepsilon_{xy}\big]\big)$.

Let x, y be spacelike separated. Using (2) and (ii), what can you infer about the size of the functional $\mathcal{C}(x, y)$ in the limit $\varepsilon \searrow 0$? *Hint: Discuss the commutator in (ii). The scaling in ε from Exercise 5.1-(ii) may be useful.*

Exercise 9.2: Closedness of the local correlation function (6 points)

The goal of this exercise is to show that the local correlation operators, as defined in Exercise 5.3, realizes a one-to-one topological identification of Minkowski space with a closed subset of $\mathscr{F}$. Let us define the *local correlation function* by

$$F^\varepsilon : \mathbb{R}^4 \to \mathscr{F}, \quad \langle u, F^\varepsilon(x)v\rangle := -\prec\mathfrak{R}_\varepsilon u(x), \mathfrak{R}v(x)\succ. \tag{1}$$

Thanks to the translation invariance fo the Dirac sea, it can be proved that all the $F^\varepsilon(x)$ are unitarily equivalent, in particular they have the same norm.

(i) *Continuity:* The regularization operator as in Exercise 5.3 can be chosen to fulfill:

   (a) There is $C > 0$ such that $|\mathfrak{R}u(x)| \leq C\|u\|$ for all $x \in \mathbb{R}^4$

   (b) For all $x \in \mathbb{R}^4$ and $\delta > 0$ there is $r > 0$ such that $|\mathfrak{R}u(x) - \mathfrak{R}u(y)| \leq \delta\|u\|$ for all $u \in \mathscr{H}_m^-$ and all $y \in B_r(x)$.

   Use these properties to show that F^ε is continuous in the operator topology.

(ii) *Injectivity:* Let $F^\varepsilon(x) = F^\varepsilon(y)$. We need to show that $x = y$. Referring to Exercise 1.2, consider the elements $u_n^{(\boldsymbol{p})} \in \mathscr{H}_m^-$ whose regularization reads ($\mathfrak{e}_4 = (0,0,0,1)$)

$$(\mathcal{R}_\varepsilon u_n^{(\boldsymbol{p})})(z) = \int_{\mathbb{R}^3} (p_-(\mathbf{k})\mathfrak{e}_4)\, h_n(\mathbf{k}-\mathbf{p})\, e^{-\varepsilon\omega(\mathbf{k})}\, e^{i(\omega(\mathbf{k})t_z + \mathbf{k}\cdot\mathbf{z})}\, d^3\boldsymbol{k},$$

   where h_n is a Dirac delta sequence. Apply definition (1) to the vectors $u_n^{(0)}, u_n^{(\mathbf{p})}$ and take the limit $n \to \infty$. How can the arbitrariness of $\mathbf{p}$ be exploited in order to infer that $x = y$? Motivate your answer. *Hint: Note that* $p_-(\mathbf{p})$ *depends continuously on* $\mathbf{p}$.

(iii) *Closedness:* The final step consists in proving that the local correlation function is closed, i.e. it maps closed sets to closed sets. In particular, it follows that $F^\varepsilon(\mathbb{R}^4)$ is closed and that the inverse $(F^\varepsilon)^{-1}|_{F^\varepsilon(\mathbb{R}^4)}$ is continuous. The identification is then complete. Here we need a general result from topology: Every *proper* function (i.e. such that the preimage of any compact set is compact) between metric spaces is also closed.

   (a) Let $K \subset \mathscr{F}$ be compact and let $\{x_n\}_n \subset H := (F^\varepsilon)^{-1}(K)$. By compactness of K there exists a subsequence $\{y_n\}_n$ such that $F^\varepsilon(y_n) \to A \in K$. Show that A is self-adjoint and different from zero. *Hint: How could the comment after* (1) *be exploited?*

   (b) Show that the subspace of $\mathscr{H}_m^-$ of solutions of the form

$$u_\varphi(x) := \int_{\mathbb{R}^3} d^3\mathbf{k}\,(p_-(\mathbf{k})\varphi(\mathbf{k}))\, e^{i(\omega(\mathbf{k})t_x + \mathbf{k}\cdot\mathbf{x})}, \quad \text{with } \varphi \in S(\mathbb{R}^4, \mathbb{C}^4),$$

   is dense. Deduce that there exists at least one $\varphi \in S(\mathbb{R}^4, \mathbb{C}^4)$ such that $\langle u_\varphi, Au_\varphi\rangle \neq 0$.

   (c) Convince yourself that $e^{-\varepsilon\omega}\, p_-\, \varphi \in S(\mathbb{R}^3, \mathbb{C}^4)$ for any $\varphi \in S(\mathbb{R}^3, \mathbb{C}^4)$.

   (d) It can be proved that the solutions of the Dirac equation with initial data in $S(\mathbb{R}^3, \mathbb{C}^4)$ decay polynomially in both space and time direction. Use (2) and (3) to show that the sequence $\{y_n\}_n$ cannot be unbounded.

   (e) Conclude that $\{x_n\}_n$ has a converging subsequence in H.

Exercise 9.3: On the differentiable manifold structure of regular points (4 points)

Let $\mathscr{H}$ be a Hilbert space of finite dimension N. The set $\mathscr{F}^{\text{reg}}$ of regular points can be endowed with a differentiable structure. Precisely, let $x \in \mathscr{F}^{\text{reg}}$. Choosing a Hilbert basis and using a block matrix representation in $\mathscr{H} = S_x \oplus S_x^{\perp} \cong \mathbb{C}^{2n} \oplus \mathbb{C}^{N-2n}$, the operator x can be rewritten as

$$x \equiv \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \operatorname{Mat}(N-2n, \mathbb{C}), \quad X \in \operatorname{Symm}(2n, \mathbb{C}) := \left\{ A \in \operatorname{Mat}(2n, \mathbb{C}),\ A^{\dagger} = A \right\}, \tag{2}$$

where † refers to the Euclidean scalar product. With this idenitification in mind, we now define

$$\begin{aligned} \Phi : \big(\operatorname{Symm}(2n, \mathbb{C}) \oplus \mathrm{L}(\mathbb{C}^{2n}, \mathbb{C}^{N-2n})\big) \cap B_{\varepsilon}(0) &\to \mathscr{F}^{\text{reg}} \\ (A, B) &\mapsto \begin{pmatrix} X + A & B \\ B^{\dagger} & B^{\dagger}\,(X+A)^{-1}\,B \end{pmatrix} \end{aligned} \tag{3}$$

Prove the following statements.

(i) For sufficiently small ε, Φ is well-defined, continuous and injective.

(ii) For sufficiently small δ, $B_{\delta}(x) \subset \operatorname{im}\Phi$ and the restriction $\Phi : \Phi^{-1}(B_{\delta}(x)) \to B_{\delta}(x)$ is homeomorphic.

Hint: With the help of a unitary operator U diagonalize any $y \in \mathscr{F}^{\text{reg}}$ as in (2). Exploit this to show that y must take the form (3), if its distance from x is sufficiently small.

(iii) Could one repeat the exercise if the Euclidean inner product is replaced by the canonical spin inner product on $\mathbb{C}^{2n}$? Motivate you answer.