Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 9

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is the difference between the causal action principle and the causal variational principle? Is the latter a generalization or a simplification of the former?

(ii) What role is played by the local trace in point (i)?

(iii) How is the boundedness constraint treated?

(iv) Why is $\mathscr{F}$ not a manifold? Why is $\mathscr{F}^{\mathrm{reg}}$ a manifold? How can charts be constructed?

Exercises

Exercise 9.1: The support of a measure

In order to illustrate how to encode geometric information in the support of a measure, let $\mathscr{M} \subset \mathbb{R}^3$ be a smooth surface described in a parametrization Φ. Thus given an open subset $\Omega \subset \mathbb{R}^2$, we consider a smooth injective map

$$\Phi \,:\, \Omega \to \mathbb{R}^3$$

with the property that $D\Phi|_p : \mathbb{R}^2 \to \mathbb{R}^3$ has rank two for all $p \in \Omega$. Then the surface $\mathscr{M}$ is defined as the image $\Phi(\Omega) \subset \mathbb{R}^3$. We now introduce the measure ρ as the *push-forward measure* of the Lebesgue measure on $\mathbb{R}^2$: Let μ be the Lebesgue measure on $\mathbb{R}^2$. We define a set $U \subset \mathbb{R}^3$ to be ρ-measurable if and only if its preimage $\Phi^{-1}(U) \subset \mathbb{R}^2$ is μ-measurable. On the ρ-measurable sets we define the measure ρ by

$$\rho(U) = \mu\big(\Phi^{-1}(U)\big) \,.$$

Verify that the ρ-measurable sets form a σ-algebra, and that ρ is a measure. What are the sets of ρ-measure zero? What is the support of the measure ρ?

Suppose that Φ is no longer assumed to be injective. Is ρ still a well-defined measure? Is ρ well-defined if Φ is only assumed to be continuous? What are the minimal regularity assumptions on Φ needed for the push-forward measure to be well-defined? What is the support of ρ in this general setting?

Exercise 9.2: Derivation of the causal variational principle on the sphere

We consider the causal fermion systems in the case $n = 1$ and $f = 2$. For a given parameter $\tau > 1$ we introduce the mapping $F : M \to \mathscr{F}$ by

$$F(\vec{x}) = \tau\, \vec{x}\vec{\sigma} + \mathbb{1} \,. \tag{1}$$

(i) Compute the eigenvalues of the matrix $F(\vec{x})$ and verify that it has one positive and one negative eigenvalue.

(ii) Use the identities between Pauli matrices

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\, \sigma^k \,, \tag{2}$$

to compute the matrix product,

$$F(\vec{x})\, F(\vec{y}) = \big(1 + \tau^2\, \vec{x}\vec{y}\big)\, \mathbb{1} + \tau\, (\vec{x} + \vec{y})\vec{\sigma} + i\tau^2\, (\vec{x} \wedge \vec{y})\vec{\sigma}.$$

(iv) compute the eigenvalues of this matrix product to obtain

$$\lambda_{1/2} = 1 + \tau^2 \cos\vartheta \pm \tau\sqrt{1 + \cos\vartheta}\, \sqrt{2 - \tau^2\, (1 - \cos\vartheta)}\,, \tag{3}$$

where ϑ denotes the angle ϑ between $\vec{x}$ and $\vec{y}$.

(v) Verify that if $\vartheta \leq \vartheta_{\max}$ with

$$\vartheta_{\max} := \arccos\left(1 - \frac{2}{\tau^2}\right),$$

then the eigenvalues $\lambda_{1/2}$ are both real. Conversely, if $\vartheta > \vartheta_{\max}$, then the eigenvalues form a complex conjugate pair.

(vi) Use the formula

$$\lambda_1 \lambda_2 = \det(F(\vec{x})F(\vec{y})) = \det(F(\vec{x}))\, \det(F(\vec{y})) = (1+\tau)^2(1-\tau)^2 > 0$$

ton conclude that if the eigenvalues $\lambda_{1/2}$ are both real, then they have the same sign.

(vii) Combine the above findings to conclude that the causal Lagrangian can be simplified to

$$\begin{aligned} \mathcal{L}(x,y) &= \max\big(0, \mathscr{D}(x,y)\big) \qquad \text{with} \\ \mathscr{D}(x,y) &= 2\tau^2\, \big(1 + \langle x,y \rangle\big) \Big(2 - \tau^2\, \big(1 - \langle x,y \rangle\big)\Big)\,. \end{aligned}$$

Exercise 9.3: The action and boundedness constraint of the Lebesgue measure on the sphere

We consider the causal variational principle on the sphere as introduced in the previous exercise. We let $d\mu$ be the surface area measure, normalized such that $\mu(S^2) = 1$.

(i) Use the formula for the causal Lagrangian on the sphere to compute the causal action. Verify that

$$\mathcal{S}[F] = \frac{1}{2} \int_0^{\vartheta_{\max}} \mathcal{L}(\cos\vartheta)\, \sin\vartheta\, d\vartheta = 4 - \frac{4}{3\tau^2}\,. \tag{4}$$

(ii) Show that the functional $\mathcal{T}$ is given by

$$\mathcal{T}[F] = 4\tau^2(\tau^2 - 2) + 12 - \frac{8}{3\tau^2}\,. \tag{5}$$

Hence the causal action (4) is bounded uniformly in τ, although the function F, (1), as well as the functional $\mathcal{T}$, (5), diverge as $\tau \to \infty$.

Exercise 9.4: A minimizer with singular support

We again consider the causal variational principle on the sphere as introduced in Exercise 9.2. Verify by direct computation that in the case $\tau = \sqrt{2}$, the causal action of the normalized counting measure supported on the *octahedron* is smaller the action of μ. *Hint:* For $\tau = \sqrt{2}$ the opening angle of the light cone is given by $\vartheta = 90°$, so that all distinct spacetime points are spacelike separated. Moreover, the causal action of the normalized Lebesgue measure is given in Exercise 9.3 (i).

Exercise 9.5: A causal variational principle on $\mathbb{R}$

We let $\mathscr{F} = \mathbb{R}$ and consider the Lagrangians

$$\mathcal{L}_2(x,y) = (1+x^2)(1+y^2) \qquad \text{and} \qquad \mathcal{L}_4(x,y) = (1+x^4)(1+y^4)\,. \tag{6}$$

We minimize the corresponding causal actions within the class of all normalized regular Borel measures on $\mathbb{R}$. Show with a direct estimate that the Dirac measure δ supported at the origin is the unique minimizer of these causal variational principles.

Exercise 9.6: A causal variational principle on S^1

We let $\mathscr{F} = S^1$ be the unit circle parametrized as $e^{i\varphi}$ with $\varphi \in \mathbb{R} \bmod 2\pi$ and consider the Lagrangian

$$\mathcal{L}(\varphi, \varphi') = 1 + \sin^2\big(\varphi - \varphi' \bmod 2\pi\big)\,. \tag{7}$$

We minimize the corresponding causal action within the class of all normalized regular Borel measures on S^1. Show by direct computation and estimates that every minimizer is of the form

$$\rho = \tau\,\delta\big(\varphi - \varphi' - \varphi_0 \bmod 2\pi\big) + (1-\tau)\,\delta\big(\varphi - \varphi' - \varphi_0 + \pi \bmod 2\pi\big) \tag{8}$$

for suitable values of the parameters $\tau \in [0,1]$ and $\varphi_0 \in \mathbb{R} \bmod 2\pi$.

Exercise 9.7: Non-smooth EL equations

We return to the example of the counting measure on the octahedron as considered in Exercise 9.4.

(i) Compute the function $\ell(x)$. Show that the EL equations are satisfied.

(ii) Show that the function ℓ is *not* differentiable at any point x of the octahedron. Therefore, it is not possible to formulate the restricted EL equations $\nabla_{\mathfrak{u}}\ell|_M = 0$.

This example illustrates why in the research papers one carefully keeps track of differentiability properties by introducing suitable jet spaces.