Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 8

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) How can Minkowski space be represented by a causal fermion system?

(ii) What is the physical and mathematical role played by the ultraviolet regularization?

(iii) What correspondence can be established between the spin spaces and the spinor space?

(iv) How do the physical wave equations and the kernel of the fermionic projector look like under this correspondence?

(v) In which sense does the causal structure of the causal fermion system correspond to that of Minkowski space?

Exercises

Exercise 8.1: The regularized fermionic projector in Minkowski space

Consider the kernel of the fermionic projector regularized in momentum space by a convergence-generating factor $e^{-\varepsilon|k^0|}$, i.e.

$$P^\varepsilon(x,y) = \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\, e^{-\varepsilon|k^0|}. \tag{1}$$

(i) Show that $P^\varepsilon(x,y)$ can be written as $\not{v}^\varepsilon + \beta^\varepsilon$, for $v^\varepsilon_j, \beta^\varepsilon$ smooth functions of $\xi = y - x$.

(ii) Compute $P^\varepsilon(x,x)$. Is this matrix invertible? How does it scale in ε?

For ξ spacelike or timelike, i.e. away from the lightcone, the limit $\varepsilon \searrow 0$ of (1) is well-defined. More precisely, it can be shown that $v^\varepsilon_j \to \alpha\, \xi_j$ and $\beta^\varepsilon \to \beta$ pointwise, for α, β smooth complex functions. Find smooth real functions a, b such that

$$\lim_{\varepsilon\to 0} A^\varepsilon_{xy} = a\not{\xi} + b. \tag{2}$$

How do the eigenvalues of (2) look like? Discuss them in relation to the notion of causality in the setting of causal fermion systems.

Exercise 8.2: Understanding the connection between causal structure and closed chain

Let $x, y \in \mathbb{R}^4$ be timelike separated vectors and assume that $\xi := y - x$ is normalized to $\eta(\xi, \xi) = 1$. As explained in Exercise 8.1, the limit $\varepsilon \searrow 0$ of the closed chain A^ε_{xy} takes the form $A = a\,\not\xi + b$. Consider the matrices

$$F_\pm := \frac{1}{2}\left(\mathbb{I} \pm \not\xi\right) \in \mathrm{M}(4, \mathbb{C}).$$

Prove the following statements.

(i) $F_\pm$ have rank two and map to eigenspaces of A. What are the corresponding eigenvalues?

(ii) $F_\pm$ are idempotent and symmetric with respect to the spin inner product $\prec\cdot\,,\,\cdot\succ$ on $\mathbb{C}^4$.

(iii) The image of the matrices $F_\pm$ is positive or negative definite.

(iv) The image of F_+ is orthogonal to that of F_- (with respect to the spin inner product)

The result of this exercise can be summarized by saying that the $F_\pm$ are the spectral projection operators of A.

Exercise 8.3: Spin spaces in Minkowski space - part 1

Let $\mathscr{H}_m^-$ denote the Hilbert space of negative-energy solutions of the Dirac equation as introduced in the lecture. By means of a convergence-generating factor as in Exercise 8.1 it is possible to define a bounded *regularization operator*

$$\mathfrak{R}_\varepsilon : \mathscr{H}_m^- \to \mathscr{H}_m^- \cap C^\infty(\mathbb{R}^4, \mathbb{C}^4),$$

which can be proved to be injective. As you know from the lecture, this allows us to define *local correlation operators* $F^\varepsilon(x)$ on $\mathscr{H}_m^-$ via

$$\langle u | F^\varepsilon(x) v\rangle := -\prec \mathfrak{R}_\varepsilon u(x), \mathfrak{R}_\varepsilon v(x) \succ. \tag{3}$$

This gives rise to a causal fermion system, called the regularized Dirac sea vacuum.

(i) Let Σ_0 denote the Cauchy surface at time $t = 0$. Show that, for any $x \in \mathbb{R}^4$ and $\chi \in \mathbb{C}^4$,

$$(i\partial\!\!\!/ - m) P^\varepsilon(\,\cdot\,, x)\chi = 0 \quad \text{and} \quad P^\varepsilon(\,\cdot\,, x)\chi\big|_{\Sigma_0} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4).$$

Conclude that $P^\varepsilon(\,\cdot\,, x)\chi \in \mathscr{H}_m^- \cap C^\infty(\mathbb{R}^4, \mathbb{C}^4)$.

(ii) Convince youself that

$$\mathfrak{R}_\varepsilon(P^\varepsilon(\,\cdot\,, x)\chi) = P^{2\varepsilon}(\,\cdot\,, x)\chi.$$

(iii) Let $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_4\}$ denote the canonical basis of $\mathbb{C}^4$. Using (ii) of Exercise 8.1, show that the wave functions $P^\varepsilon(\,\cdot\,, x)\mathfrak{e}_\mu$, for $\mu = 1, 2, 3, 4$, are linearly independent.

(iv) Let $S_x := F^\varepsilon(x)(\mathscr{H}_m^-)$ endowed with $\prec u, v\succ_x := -\langle u|F^\varepsilon(x)v\rangle$ be the *spin space* at $x \in \mathbb{R}^4$. Show that the following mapping is an isometry of indefinite inner products (i.e. injective and product preserving),

$$\Phi_x : S_x \ni u \mapsto \mathfrak{R}_\varepsilon u(x) \in \mathbb{C}^4.$$

Conclude that the causal fermion system is regular at $x \in \mathbb{R}^4$, i.e. $\dim S_x = 4$, if and only if there exist vectors $u_\mu \in \mathscr{H}_m^-$, for $\mu = 1, 2, 3, 4$, such that the $\mathfrak{R}_\varepsilon u_\mu(x) \in \mathbb{C}^4$ are linearly independent.

(v) Conclude that the causal fermion system is regular at every $x \in \mathbb{R}^4$.

Exercise 8.4: Spin spaces in Minkowski space - part 2

Let $x, y \in \mathbb{R}^4$ and $\{u_n\}_n$ be a Hilbert basis of $\mathscr{H}_m^-$. It can be shown that

$$P^{2\varepsilon}(x,y) = -\frac{1}{2\pi}\sum_{n=1}^{\infty} |\mathfrak{R}_\varepsilon u_n(x) \succ\prec \mathfrak{R}_\varepsilon u_n(y)|.$$

Let $\mathrm{P}(A,B) := \pi_A\, B|_{S_B}$ denote the fermionic projector. Prove the following statements.

(i) $\mathfrak{R}_\varepsilon(\pi_{F^\varepsilon(x)} u)(x) = \mathfrak{R}_\varepsilon u(x)$ for all $u \in \mathscr{H}_m^-$.

(ii) $\Phi_x\, \mathrm{P}(F^\varepsilon(x), F^\varepsilon(y))\, \Phi_y^{-1} = 2\pi\, P^{2\varepsilon}(x,y)$.

(iii) $P^\varepsilon(\,\cdot\,, x)a \in S_x$ for all $a \in \mathbb{C}^4$.
Hint: Use (i)-(ii), regularity and the injectivity of $\mathfrak{R}_\varepsilon$.

Conclude that $S_x = \{P^\varepsilon(\,\cdot\,,x)a \mid a \in \mathbb{C}^4\}$.

Exercise 8.5: The time-direction functional in Minkowski space (6 points)

Away from the light-cone the kernel of the fermionic projector P^ε converges to a smooth function P. More precisely, for $\xi \in \mathbb{R}^3 \setminus L_0$,

$$P^\varepsilon(\xi) \to P(\xi) = (i\partial\!\!\!/ + m)\, T_{m^2}(\xi), \quad T_{m^2}(\xi) := \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik\cdot\xi}.$$

where T_{m^2} is smooth on $\mathbb{R}^4 \setminus L_0$.

(i) Show with a symmetry argument (without explicit computation of Fourier integrals!) that the imaginary part of the function T vanishes for spacelike vectors ξ.

(ii) Referring to Exercise 8.1, deduce that, for spacelike separation, $\alpha \in i\mathbb{R}$, $\beta \in \mathbb{R}$ and $a = 0$.

A causal fermion system distinguishes a *direction of time* by means of the anti-symmetric real functional

$$\mathcal{C} : M \times M \ni (A,B) \mapsto i\,\mathrm{tr}(B\,A\,\pi_B\,\pi_A - A\,B\,\pi_A\,\pi_B) \in \mathbb{R}.$$

From Exercise 8.1-(ii) we know that $P^\varepsilon(0)$ is invertible. Let us define $\nu := P^\varepsilon(0)^{-1} \in \mathrm{Mat}(4,\mathbb{C})$. Using the identifications as in Exercise 8.3-4 ($x \equiv F^\varepsilon(x)$, $S_x \cong \mathbb{C}^4$), prove the following identities (up to global constants)

(i) $\pi_x\, y\, x\, \pi_y\, \pi_x|_{S_x} = P^\varepsilon(x,y)\, P^\varepsilon(y,x)\, P^\varepsilon(x,y)\, \nu\, P^\varepsilon(y,x)\, \nu$.

(ii) $\mathcal{C}(x,y) = i\,\mathrm{Tr}_{\mathbb{C}^4}\big(P^\varepsilon(x,y)\, \nu\, P^\varepsilon(y,x)\, [\nu, A^\varepsilon_{xy}]\big)$.

Let x, y be spacelike separated. Using (2) and (ii), what can you infer about the size of the functional $\mathcal{C}(x,y)$ in the limit $\varepsilon \searrow 0$? *Hint: Discuss the commutator in (ii). The scaling in ε from Exercise 8.1-(ii) may be useful.*

Exercise 8.6: Closedness of the local correlation function (6 points)

The goal of this exercise is to show that the local correlation operators, as defined in Exercise 8.3, realize a one-to-one topological identification of Minkowski space with a closed subset of $\mathscr{F}$. Let us define the *local correlation function* by

$$F^\varepsilon : \mathbb{R}^4 \to \mathscr{F}, \quad \langle u, F^\varepsilon(x) v\rangle := -\prec\mathfrak{R}_\varepsilon u(x), \mathfrak{R} v(x)\succ. \tag{4}$$

Thanks to the translation invariance fo the Dirac sea, it can be proved that all the $F^\varepsilon(x)$ are unitarily equivalent, in particular they have the same norm.

(i) *Continuity:* The regularization operator as in Exercise 8.3 can be chosen to fulfill:

  (a) There is $C > 0$ such that $|\mathfrak{R}u(x)| \leq C\|u\|$ for all $x \in \mathbb{R}^4$

  (b) For all $x \in \mathbb{R}^4$ and $\delta > 0$ there is $r > 0$ such that $|\mathfrak{R}u(x) - \mathfrak{R}u(y)| \leq \delta\|u\|$ for all $u \in \mathscr{H}_m^-$ and all $y \in B_r(x)$.

  Use these properties to show that F^ε is continuous in the operator topology.

(ii) *Injectivity:* Let $F^\varepsilon(x) = F^\varepsilon(y)$. We need to show that $x = y$. As in previous exercises, consider the elements $u_n^{(\boldsymbol{p})} \in \mathscr{H}_m^-$ whose regularization reads ($\mathfrak{e}_4 = (0,0,0,1)$)

$$(\mathcal{R}_\varepsilon u_n^{(\boldsymbol{p})})(z) = \int_{\mathbb{R}^3} (p_-(\mathbf{k})\mathfrak{e}_4)\, h_n(\mathbf{k}-\mathbf{p})\, e^{-\varepsilon\omega(\mathbf{k})}\, e^{i(\omega(\mathbf{k})t_z + \mathbf{k}\cdot\mathbf{z})}\, d^3\boldsymbol{k},$$

  where h_n is a Dirac delta sequence. Apply definition (4) to the vectors $u_n^{(0)}, u_n^{(\mathbf{p})}$ and take the limit $n \to \infty$. How can the arbitrariness of $\mathbf{p}$ be exploited in order to infer that $x = y$? Motivate your answer. *Hint: Note that $p_-(\mathbf{p})$ depends continuously on $\mathbf{p}$.*

(iii) *Closedness:* The final step consists in proving that the local correlation function is closed, i.e. it maps closed sets to closed sets. In particular, it follows that $F^\varepsilon(\mathbb{R}^4)$ is closed and that the inverse $(F^\varepsilon)^{-1}|_{F^\varepsilon(\mathbb{R}^4)}$ is continuous. The identification is then complete. Here we need a general result from topology: Every *proper* function (i.e. such that the preimage of any compact set is compact) between metric spaces is also closed.

  (a) Let $K \subset \mathscr{F}$ be compact and let $\{x_n\}_n \subset H := (F^\varepsilon)^{-1}(K)$. By compactness of K there exists a subsequence $\{y_n\}_n$ such that $F^\varepsilon(y_n) \to A \in K$. Show that A is self-adjoint and different from zero. *Hint: How could the comment after* (4) *be exploited?*

  (b) Show that the subspace of $\mathscr{H}_m^-$ of solutions of the form

$$u_\varphi(x) := \int_{\mathbb{R}^3} d^3\mathbf{k}\, (p_-(\mathbf{k})\varphi(\mathbf{k}))\, e^{i(\omega(\mathbf{k})t_x + \mathbf{k}\cdot\mathbf{x})}, \quad \text{with } \varphi \in S(\mathbb{R}^4, \mathbb{C}^4),$$

  is dense. Deduce that there exists at least one $\varphi \in S(\mathbb{R}^4, \mathbb{C}^4)$ such that $\langle u_\varphi, Au_\varphi\rangle \neq 0$.

  (c) Convince yourself that $e^{-\varepsilon\omega}\, p_-\, \varphi \in S(\mathbb{R}^3, \mathbb{C}^4)$ for any $\varphi \in S(\mathbb{R}^3, \mathbb{C}^4)$.

  (d) It can be proved that the solutions of the Dirac equation with initial data in $S(\mathbb{R}^3, \mathbb{C}^4)$ decay polynomially in both space and time direction. Use (2) and (3) to show that the sequence $\{y_n\}_n$ cannot be unbounded.

  (e) Conclude that $\{x_n\}_n$ has a converging subsequence in H.

Exercise 8.7: On the differentiable manifold structure of regular points (4 points)

Let $\mathscr{H}$ be a Hilbert space of finite dimension N. The set $\mathscr{F}^{\mathrm{reg}}$ of regular points can be endowed with a differentiable structure. Precisely, let $x \in \mathscr{F}^{\mathrm{reg}}$. Choosing a Hilbert basis and using a block matrix representation in $\mathscr{H} = S_x \oplus S_x^{\perp} \cong \mathbb{C}^{2n} \oplus \mathbb{C}^{N-2n}$, the operator x can be rewritten as

$$x \equiv \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}(N-2n, \mathbb{C}), \quad X \in \mathrm{Symm}(2n, \mathbb{C}) := \left\{ A \in \mathrm{Mat}(2n, \mathbb{C}),\ A^{\dagger} = A \right\}, \tag{5}$$

where † refers to the Euclidean scalar product. With this idenitification in mind, we now define

$$\begin{aligned} \Phi : \big(\mathrm{Symm}(2n, \mathbb{C}) \oplus \mathrm{L}(\mathbb{C}^{2n}, \mathbb{C}^{N-2n})\big) \cap B_{\varepsilon}(0) &\to \mathscr{F}^{\mathrm{reg}} \\ (A, B) &\mapsto \begin{pmatrix} X + A & B \\ B^{\dagger} & B^{\dagger}\,(X+A)^{-1}\,B \end{pmatrix} \end{aligned} \tag{6}$$

Prove the following statements.

(i) For sufficiently small ε, Φ is well-defined, continuous and injective.

(ii) For sufficiently small δ, $B_{\delta}(x) \subset \mathrm{im}\Phi$ and the restriction $\Phi : \Phi^{-1}(B_{\delta}(x)) \to B_{\delta}(x)$ is homeomorphic.

Hint: With the help of a unitary operator U diagonalize any $y \in \mathscr{F}^{\mathrm{reg}}$ as in (5)*. Exploit this to show that y must take the form* (6)*, if its distance from x is sufficiently small.*

(iii) Could one repeat the exercise if the Euclidean inner product is replaced by the canonical spin inner product on $\mathbb{C}^{2n}$? Motivate you answer.