Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 8

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) How can Minkowski space be represented by a causal fermion system?

(ii) What is the physical and mathematical role played by the ultraviolet regularization?

(iii) What correspondence can be established between the spin spaces and the spinor space?

(iv) How do the physical wave equations and the kernel of the fermionic projector look like under this correspondence?

(v) In which sense does the causal structure of the causal fermion system correspond to that of Minkowski space?

Exercises

Exercise 8.1: The regularized fermionic projector in Minkowski space

Consider the kernel of the fermionic projector regularized in momentum space by a convergence-generating factor $e^{-\varepsilon|k^0|}$, i.e.

$$P^\varepsilon(x,y) = \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4}\,(\not{k}+m)\,\delta(k^2-m^2)\,\Theta(-k^0)\,e^{-ik(x-y)}\,e^{-\varepsilon|k^0|}. \tag{1}$$

(i) Show that $P^\varepsilon(x,y)$ can be written as $\not{v}^\varepsilon + \beta^\varepsilon$, for $v_j^\varepsilon, \beta^\varepsilon$ smooth functions of $\xi = y - x$.

(ii) Compute $P^\varepsilon(x,x)$. Is this matrix invertible? How does it scale in ε?

For ξ spacelike or timelike, i.e. away from the lightcone, the limit $\varepsilon \searrow 0$ of (1) is well-defined. More precisely, it can be shown that $v_j^\varepsilon \to \alpha\,\xi_j$ and $\beta^\varepsilon \to \beta$ pointwise, for α, β smooth complex functions. Find smooth real functions a, b such that

$$\lim_{\varepsilon\to 0} A^\varepsilon_{xy} = a\not{\xi} + b. \tag{2}$$

How do the eigenvalues of (2) look like? Discuss them in relation to the notion of causality in the setting of causal fermion systems.

Exercise 8.2: Understanding the connection between causal structure and closed chain

Let $x, y \in \mathbb{R}^4$ be timelike separated vectors and assume that $\xi := y - x$ is normalized to $\eta(\xi, \xi) = 1$. As explained in Exercise 5.1, the limit $\varepsilon \searrow 0$ of the closed chain A^ε_{xy} takes the form $A = a \not{\xi} + b$. Consider the matrices

$$F_\pm := \frac{1}{2}\left(\mathbb{I} \pm \not{\xi}\right) \in \mathrm{M}(4, \mathbb{C}).$$

Prove the following statements.

(i) $F_\pm$ have rank two and map to eigenspaces of A. What are the corresponding eigenvalues?

(ii) $F_\pm$ are idempotent and symmetric with respect to the spin inner product $\prec \cdot\,, \cdot \succ$ on $\mathbb{C}^4$.

(iii) The image of the matrices $F_\pm$ is positive or negative definite.

(iv) The image of F_+ is orthogonal to that of F_- (with respect to the spin inner product)

The result of this exercise can be summarized by saying that the $F_\pm$ are the spectral projection operators of A.

Exercise 8.3: Spin spaces in Minkowski space - part 1

Let $\mathscr{H}_m^-$ denote the Hilbert space of negative-energy solutions of the Dirac equation as introduced in the lecture. By means of a convergence-generating factor as in Exercise 5.1 it is possible to define a bounded *regularization operator*

$$\mathfrak{R}_\varepsilon : \mathscr{H}_m^- \to \mathscr{H}_m^- \cap C^\infty(\mathbb{R}^4, \mathbb{C}^4),$$

which can be proved to be injective. As you know from the lecture, this allows us to define *local correlation operators* $F^\varepsilon(x)$ on $\mathscr{H}_m^-$ via

$$\langle u | F^\varepsilon(x) v \rangle := -\prec \mathfrak{R}_\varepsilon u(x), \mathfrak{R}_\varepsilon v(x) \succ. \tag{3}$$

This gives rise to a causal fermion system, called the regularized Dirac sea vacuum.

(i) Let Σ_0 denote the Cauchy surface at time $t = 0$. Show that, for any $x \in \mathbb{R}^4$ and $\chi \in \mathbb{C}^4$,

$$(i\not{\partial} - m) P^\varepsilon(\,\cdot\,, x)\chi = 0 \quad \text{and} \quad P^\varepsilon(\,\cdot\,, x)\chi\big|_{\Sigma_0} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4).$$

Conclude that $P^\varepsilon(\,\cdot\,, x)\chi \in \mathscr{H}_m^- \cap C^\infty(\mathbb{R}^4, \mathbb{C}^4)$.

(ii) Convince youself that

$$\mathfrak{R}_\varepsilon(P^\varepsilon(\,\cdot\,, x)\chi) = P^{2\varepsilon}(\,\cdot\,, x)\chi.$$

(iii) Let $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_4\}$ denote the canonical basis of $\mathbb{C}^4$. Using (ii) of Exercise 5.1, show that the wave functions $P^\varepsilon(\,\cdot\,, x)\mathfrak{e}_\mu$, for $\mu = 1, 2, 3, 4$, are linearly independent.

(iv) Let $S_x := F^\varepsilon(x)(\mathscr{H}_m^-)$ endowed with $\prec u, v \succ_x := -\langle u | F^\varepsilon(x) v \rangle$ be the *spin space* at $x \in \mathbb{R}^4$. Show that the following mapping is an isometry of indefinite inner products (i.e. injective and product preserving),

$$\Phi_x : S_x \ni u \mapsto \mathfrak{R}_\varepsilon u(x) \in \mathbb{C}^4.$$

Conclude that the causal fermion system is regular at $x \in \mathbb{R}^4$, i.e. $\dim S_x = 4$, if and only if there exist vectors $u_\mu \in \mathscr{H}_m^-$, for $\mu = 1, 2, 3, 4$, such that the $\mathfrak{R}_\varepsilon u_\mu(x) \in \mathbb{C}^4$ are linearly independent.

(v) Conclude that the causal fermion system is regular at every $x \in \mathbb{R}^4$.

Exercise 8.4: Spin spaces in Minkowski space - part 2

Let $x, y \in \mathbb{R}^4$ and $\{u_n\}_n$ be a Hilbert basis of $\mathscr{H}_m^-$. It can be shown that

$$P^{2\varepsilon}(x,y) = -\frac{1}{2\pi} \sum_{n=1}^{\infty} |\mathfrak{R}_\varepsilon u_n(x) \succ\!\prec \mathfrak{R}_\varepsilon u_n(y)|.$$

Let $\mathrm{P}(A,B) := \pi_A \, B|_{S_B}$ denote the fermionic projector. Prove the following statements.

(i) $\mathfrak{R}_\varepsilon(\pi_{F^\varepsilon(x)} u)(x) = \mathfrak{R}_\varepsilon u(x)$ for all $u \in \mathscr{H}_m^-$.

(ii) $\Phi_x \, \mathrm{P}(F^\varepsilon(x), F^\varepsilon(y)) \, \Phi_y^{-1} = 2\pi \, P^{2\varepsilon}(x,y)$.

(iii) $P^\varepsilon(\,\cdot\,, x)a \in S_x$ for all $a \in \mathbb{C}^4$.
Hint: Use (i)-(ii), regularity and the injectivity of $\mathfrak{R}_\varepsilon$.

Conclude that $S_x = \{P^\varepsilon(\,\cdot\,, x)a \mid a \in \mathbb{C}^4\}$.