Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 7

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What are the spin spaces? In which sense do they form a topological bundle? Why is it of advantage to endow them with an indefinite inner product?

(ii) What is a wave function in spacetime? What is the physical meaning of the so-called physical wave functions?

(iii) What is the physical and geometrical content of the kernel of the fermionic projector? What is the true advantage of using the closed chain instead of the operator products?

(iv) In which space can the kernel of the fermionic projector be in fact understood as the kernel of a integral linear operator? Is it a Hilbert space?

Exercises

Exercise 7.1: On the spectrum of the closed chain - part 2

Let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system and $x, y \in \mathscr{F}$. For the closed chain

$$A_{xy} := P(x,y)P(y,x) : (S_x, \prec \cdot\,, \cdot \succ_x) \to (S_x, \prec \cdot\,, \cdot \succ_x),$$

the mathematical setting analyzed in Exercise 3.3 is somewhat different, because A_{xy} is a symmetric operator on an *indefinite inner product space*. On the other hand, we know that A_{xy} is isospectral to xy. Indeed, the symmetry result in Exercise 3.3-(iv) can be used to prove a corresponding statement for the closed chain:

$$\det(A_{xy} - \lambda\,\mathbb{I}) = 0 \iff \det(A_{xy} - \overline{\lambda}\,\mathbb{I}) = 0.$$

This result is well-known in the theory of indefinite inner product spaces. In order to derive it from Exercise 3.3-(iv), one can proceed as follows: First, represent the indefinite inner product in the form $\prec \cdot\,, \cdot \succ = \langle \cdot\,, S\, \cdot \rangle$, where $\langle \cdot\,, \cdot \rangle$ is a scalar product and S is an invertible operator which is symmetric (with respect to this scalar product). Next, show that the operator $B := A_{xy}S$ is symmetric (again with respect to this scalar product). Finally, write the closed chain as $A_{xy} = BS^{-1}$ and apply Exercise 3.3-(iv).

Exercise 7.2: A causal causal fermion system on ℓ_2 - part 3

We return to the example of Exercises 2.3 and 3.2. This time we equip it with a *Krein structure.*

(i) For any $k \in \mathbb{N}$, construct the spin space S_{x_k} and its spin scalar product.

(ii) Given a vector $u \in \mathscr{H}$, what is the corresponding wave function ψ^u? What is the Krein inner product $< \cdot, \cdot >$?

(iii) What is the topology on the Krein space $\mathscr{K}$? Does the wave evaluation operator $\Psi : u \mapsto \psi^u$ give rise to a well-defined and continuous mapping $\Psi : \mathscr{H} \to \mathscr{K}$? If yes, is it an embedding? Is it surjective?

(iv) Repeat part *(iii)* of this exercise for the causal fermion system obtained if the operators x_k are multiplied by k, i.e.

$$x_k u := (0, \ldots, 0, k\, u_{k+1}, k\, u_k\, 0, \ldots).$$

Exercise 7.3: Stability of the causal structure

A binary relation P on $\mathscr{F}$ is said to be *stable under perturbations* if

$$(x_0, y_0) \in P \implies \exists\, r > 0 \;:\; B_r(x_0) \times B_r(y_0) \subset P.$$

Two points $x, y \in \mathscr{F}$ are said to be *properly timelike* separated if the closed chain A_{xy} has a strictly positive spectrum and if all eigenspaces are definite subspaces of $(S_x, \prec \cdot, \cdot \succ)$.

(i) Show that proper timelike separation implies timelike separation.

(ii) Show by a counterexample with 3×3 matrices that the notion of timelike separation is *not* stable under perturbations.

(iii) Show that the notion of properly timelike separation is stable under perturbations.

Exercise 7.4: Time direction

The ability to distinguish between past and future can be described in mathematical terms by the existence of an antisymmetric functional $\mathcal{T} : M \times M \to \mathbb{R}$. One then says that

$$\left\{ \begin{array}{ll} y \text{ lies in the } \textit{future} \text{ of } x & \text{if } \mathcal{T}(x,y) > 0 \\ y \text{ lies in the } \textit{past} \text{ of } x & \text{if } \mathcal{T}(x,y) < 0\,. \end{array} \right.$$

Can you think of a simple non-trivial example of such a functional which involves only products and linear combinations of the spacetime operators and the orthogonal projections on the corresponding spin spaces?